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A Flexible Logit-*t* Distribution with Adaptable Tail Behavior

Abbas Mahdavi[1], **Javier E. Contreras-Reyes**[2]
[1]Department of Statistics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.
[2]Instituto de Matemática, Física y Estadística, Facultad de Ingeniería y Negocios, Universidad de Las Américas, 7 Norte 1348, Viña del Mar, Chile.

Abstract. In this study, we introduce a novel unit flexible-*t* distribution specifically designed for continuous data constrained to the unit interval (0, 1), a common scenario in many applied fields such as medicine, economics, and environmental sciences. By integrating the flexible-*t* distribution's capacity to handle heavy tails and multimodality with the logit transformation, the proposed model provides a versatile and robust framework for analyzing rates, proportions, and similar bounded outcomes. To ensure accurate and stable parameter estimation, we employ an EM-type algorithm, which proves efficient even with complex likelihood structures. The effectiveness of our approach is assessed through a comprehensive simulation study, the analysis of two real data sets, and a regression application that illustrates the practical use of the proposed model. Empirical results consistently indicate that the proposed flexible logit-*t* model outperforms existing unit-interval models in terms of goodness-of-fit, adaptability to diverse data shapes, and predictive precision. Our findings underscore the model's practical relevance and establish it as a valuable addition to the analytical toolkit for both researchers and practitioners dealing with bounded continuous responses.

Keywords. EM-type algorithms, flexible-*t* distribution, logit-transformation, unit distribution.

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Corresponding Author: Abbas Mahdavi (a.mahdavi@vru.ac.ir).
Javier E. Contreras-Reyes (jecontrr@uc.cl).

1 Introduction

The beta distribution is commonly utilized to model percentages and proportions within the interval $(0,1)$. Nevertheless, various alternative models have been introduced in the literature that may provide more appropriate frameworks than the beta distribution. For example, the Topp–Leone distribution (Topp and Leone, 1955), Kumaraswamy distribution (Kumaraswamy, 1980), arcsine distribution (Arnold and Groeneveld, 1980), generalized beta type distribution (McDonald, 1984), standard two-sided power distribution (Van Dorp and Kotz, 2002), Mc Donald arcsine distribution (Cordeiro and Lemonte, 2014), exponentiated Topp–Leone distribution (Pourdarvish et al., 2015), generalized beta distribution (Gómez-Déniz and Sarabia, 2018) and unit Lindley distribution (Mazucheli et al., 2019). Other unit models include the unit gamma distribution (Grassia, 1977), the unit Weibull distribution (Mazucheli et al., 2020), and the logit slash distribution (Korkmaz, 2020). However, many of these models cannot accommodate multimodality or heavy tails, which are common features of bounded data in medicine, economics, environmental sciences, and psychology.

Recently, several researchers have proposed distribution classes that provide more flexibility in modeling asymmetric data. One of these models is the flexible normal (FN) distribution, represented as by $FN(\mu, \sigma^2, \delta)$ (refer to Martínez-Flórez et al. (2018)). Its density function is defined as follows:

$$f_{FN}(y;\mu,\sigma^2,\delta) = \frac{\phi\left(\left|\frac{y-\mu}{\sigma}\right| - \delta\right)}{2\sigma\Phi(\delta)}, \quad y \in \mathbb{R}, \tag{1.1}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ correspond the probability density function (pdf) and cumulative density function (cdf) of standard normal distribution. Note that the standard normal distribution is a special case of the FN distribution when $\delta = 0$.

In general, researchers often consider student-t distribution instead of normal one, particularly in scenarios where sample sizes are small or when data exhibit heavier tails than those predicted by the normal distribution. A flexible-t type distribution (Mahdavi et al., 2024), in similar manner with (1.1), is represented by $FT(\mu, \sigma^2, \delta, \nu)$ and has the following pdf:

$$f_{FT}(y;\mu,\sigma^2,\delta,\nu) = \frac{t\left(\left|\frac{y-\mu}{\sigma}\right| - \delta;\nu\right)}{2\sigma T(\delta;\nu)}, \quad y \in \mathbb{R}, \tag{1.2}$$

where $t(\cdot;\nu)$ and $T(\cdot;\nu)$ denote the pdf and cdf of a t Student distribution with ν degrees of freedom.

To set a distribution on interval (a,b), it is possible to apply a transformation or conditioning the random variable (Quintero and Contreras-Reyes, 2018). An example of this transformation is the logistic function, which derived a unit distribution over $(0,1)$ from a random variable $Z \in (-\infty, \infty)$. The logit transformation, stochasticly represented as $X \overset{d}{=} e^Z/(1+e^Z)$ has been deeply studied in several contexts (Andersen et al., 2012; Liang et al., 2014; Korkmaz, 2020; Pierrot and Pinson, 2021; Mahdavi and Contreras-Reyes, 2025; Mahdavi et al., 2026). In particular, the logit-normal (LN) distribution is

the logit transformation of a normal variable. The pdf of the LN distribution is given by

$$f_{LN}(x;\mu,\sigma^2) = \frac{\exp\left\{-\frac{1}{2\sigma^2}\big(\text{logit}(x)-\mu\big)^2\right\}}{x(1-x)\sqrt{2\pi\sigma^2}}, \quad 0<x<1, \tag{1.3}$$

where

$$\text{logit}(x) = \log\left(\frac{x}{1-x}\right). \tag{1.4}$$

The LN distribution involves asymmetry and turns unimodal or bimodal for some specific values of σ.

Despite the growing number of unit distributions, a model that simultaneously accommodates multimodality, heavy tails, and a flexible kurtosis parameter is still lacking. The present paper fills this gap by introducing the flexible logit-t (FLT) distribution, obtained by applying the logit transformation to the flexible-t distribution. The resulting four-parameter model includes the flexible logit-normal and logit-normal distributions as special cases, and it can be unimodal, bimodal, or trimodal.

The proposed distribution is characterized by its heavy tails, elevated kurtosis, and potential multimodality, making it particularly well-suited for modeling data sets that are constrained within the $(0,1)$ range. To facilitate the practical application of the FLT distribution, we propose an efficient Expectation-Maximization (EM)-type algorithm for estimating its parameters. The EM algorithm, originally introduced by Dempster et al. (1977) for maximum likelihood (ML) estimation in scenarios involving censored and truncated data, serves as a robust framework for our parameter estimation process. We also derive the main distributional properties of the proposed model, including the cumulative distribution function, quantile function, moments, modal structure, and hazard rate. Furthermore, we conducted a series of experiments, including a simulation study and an analysis of two real-world data sets. Through this comprehensive exploration, we aim to contribute valuable insights into statistical modeling within bounded intervals and provide a powerful tool to researchers for their analytical endeavors.

The remainder of this paper is structured as follows. Section 2 introduces the FLT distribution and its distributional properties and discusses several mathematical properties that are essential for developing the EM algorithm. In Section 3, we outline the EM-type algorithm used for the iterative computation of ML estimates. Section 4 examines the finite sample properties of the ML estimators through a simulation study. Next in Section 5 offers a comparison with existing models by analyzing two real data sets and illustrates the regression application of the FLT distribution. Finally, Section 6 discusses possible multivariate extensions, and Section 7 provides concluding remarks.

2 The FLT distribution

2.1 The model

The random variable Y follows FLT distribution, denoted as $FLT(\mu, \sigma, \delta, \nu)$ and defined on $(0, 1)$, if its pdf is

$$f_{FLT}(y; \mu, \sigma^2, \delta, \nu) = \frac{t\Big(|\eta| - \delta; \nu\Big)}{2y(1-y)\sigma T(\delta; \nu)}, \quad 0 < y < 1, \tag{2.1}$$

where $\eta = \Big(\text{logit}(y) - \mu\Big)/\sigma$. Note that FLT distribution yields the flexible logit normal (FLN) one when $\nu \to \infty$, with the following pdf:

$$f_{FLN}(y; \mu, \sigma^2, \delta) = \frac{\phi\Big(|\eta| - \delta\Big)}{2y(1-y)\sigma\Phi(\delta)}, \quad 0 < y < 1, \tag{2.2}$$

and it reduced to the LN distribution when $\delta = 0$ and $\nu \to \infty$.

Several curves of FLT density are shown in Figure 1, whose are compared with FLN and LN ones. Plots indicate that FLT distribution exhibits several shapes characterized by multimodality and flexible tails.

It is easy to show that the FLT distribution can be represented stochastically in the following way:

$$\begin{aligned} Y|\tau &\sim FLN(\mu, \tau^{-1}\sigma^2, \tau^{1/2}\delta), \\ \tau &\sim \frac{g(\tau; \nu/2, \nu/2)\Phi(\tau^{1/2}\delta)}{T(\delta; \nu)}, \end{aligned} \tag{2.3}$$

where $g(\cdot; \alpha, \beta)$ represents the pdf of the gamma distribution with mean α/β.

From (2.3) the joint pdf of Y and τ, $0 < y < 1$, $\tau > 0$, can be expressed as follows:

$$\begin{aligned} f_{Y,\tau}(y, \tau) =& f_\tau(\tau) f_{Y|\tau}(y), \\ =& \frac{\tau^{1/2}}{2y(1-y)\sigma T(\delta; \nu)} g(\tau; \nu/2, \nu/2)\phi\Big(\tau^{1/2}(|\eta| - \delta)\Big). \end{aligned} \tag{2.4}$$

The conditional distribution of τ given Y, resulting from (2.4), is given by

$$\tau \mid (Y = y) \sim \Gamma\Big(\frac{\nu+1}{2}, \frac{\nu + (|\eta| - \delta)^2}{2}\Big), \tag{2.5}$$

where $\Gamma(a, b)$ denotes the Gamma density with shape and scale parameters a and b.

Next, we proposed an EM-type algorithm to compute the ML estimates of the parameters of $FLT(\mu, \sigma, \nu)$ using the joint density (2.4).

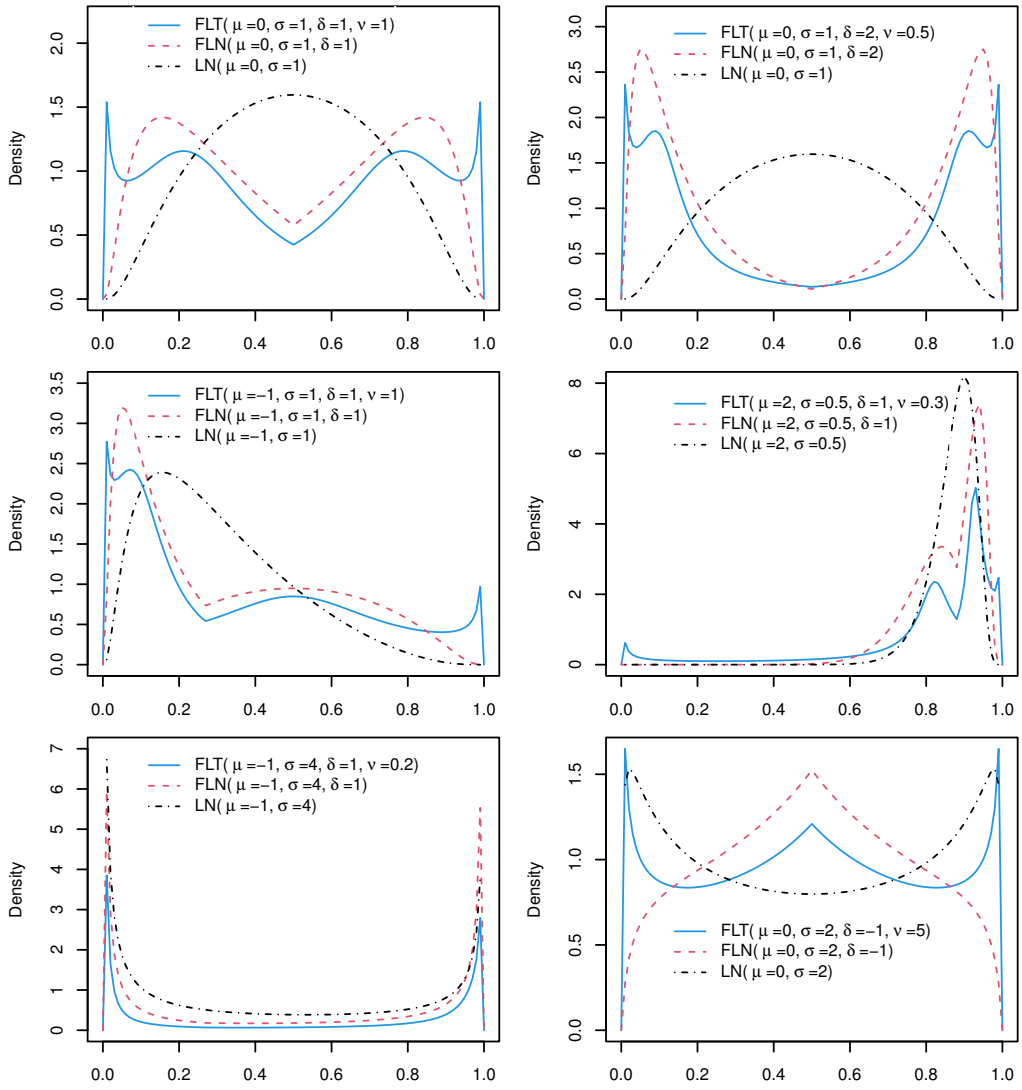


Figure 1: Various density curves of the FLT distribution.

2.2 Distributional properties

The stochastic representation in (2.3) provides a convenient basis for studying the distributional properties of the FLT distribution. In particular, the moments of Y can be obtained by first considering the conditional FLN distribution given the mixing variable τ, and then averaging the resulting conditional moments with respect to the distribution of τ.

For a given value of τ, the conditional distribution in (2.3) gives

$$E(Y^r \mid \tau) = \frac{1}{2\Phi(\sqrt{\tau}\delta)} \int_{-\infty}^{\infty} \left[\text{logistic}\left(\mu + \frac{\sigma z}{\sqrt{\tau}}\right)\right]^r \phi(|z| - \sqrt{\tau}\delta)\, dz, \qquad r = 1, 2, \ldots,$$

where

$$\text{logit}(x) = \log\left(\frac{x}{1-x}\right), \qquad 0 < x < 1, \tag{2.6}$$

and logistic$(\cdot)$ denotes its inverse,

$$\text{logistic}(u) = \frac{e^u}{1+e^u} = \frac{1}{1+e^{-u}}, \qquad u \in \mathbb{R}.$$

By the law of total expectation,

$$E(Y^r) = \int_0^{\infty} E(Y^r \mid \tau)\, f_\tau(\tau)\, d\tau. \tag{2.7}$$

Using the mixing density in (2.3), yielding

$$E(Y^r) = \frac{1}{2T(\delta;\nu)} \int_0^{\infty} g(\tau; \nu/2, \nu/2) \int_{-\infty}^{\infty} \left[\text{logistic}\left(\mu + \frac{\sigma z}{\sqrt{\tau}}\right)\right]^r \phi(|z| - \sqrt{\tau}\delta)\, dz\, d\tau. \tag{2.8}$$

Thus, the stochastic representation provides a numerical representation of all positive integer moments of the FLT distribution. Although a general closed-form expression is not available, the conditional moment can be evaluated numerically and then averaged over the mixing distribution.

Since $0 < Y < 1$, all positive integer moments of Y are finite for every $\nu > 0$. Hence, the mean, variance, skewness, and Pearson kurtosis can be obtained from the first four raw moments. In particular,

$$E(Y) = m_1, \tag{2.9}$$

$$\text{Var}(Y) = m_2 - m_1^2, \tag{2.10}$$

$$\text{Skew}(Y) = \frac{m_3 - 3m_1m_2 + 2m_1^3}{(m_2 - m_1^2)^{3/2}}, \tag{2.11}$$

$$\text{Kurt}(Y) = \frac{m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4}{(m_2 - m_1^2)^2}, \tag{2.12}$$

where $m_r = E(Y^r)$.

Several properties follow directly from the symmetry of the transformed variable. The random variable

$$X = \frac{\text{logit}(Y) - \mu}{\sigma}$$

has a distribution symmetric about zero. Therefore,

$$\text{logistic}(\mu + \sigma X) + \text{logistic}(\mu - \sigma X) = 1.$$

It follows that, when $\mu = 0$,

$$E(Y) = \frac{1}{2}. \tag{2.13}$$

More generally, the median of Y is

$$\text{Med}(Y) = \text{logistic}(\mu). \tag{2.14}$$

Thus, μ determines the median on the unit scale, while σ, δ, and ν control the dispersion and shape of the distribution.

The symmetry also implies a useful relationship between distributions with opposite values of μ. If

$$Y \sim FLT(\mu, \sigma^2, \delta, \nu),$$

then

$$1 - Y \sim FLT(-\mu, \sigma^2, \delta, \nu).$$

Consequently,

$$E_{-\mu}(Y) = 1 - E_{\mu}(Y),$$

while the variance and Pearson kurtosis are unchanged and the skewness changes sign. In particular, when $\mu = 0$, the distribution is symmetric about 1/2, and its skewness is zero.

The cdf of Y is

$$F_{FLT}(y) = \frac{1}{2} + \text{sign}(\eta)\frac{T(|\eta| - \delta; \nu) - T(-\delta; \nu)}{2T(\delta; \nu)}, \qquad 0 < y < 1. \tag{2.15}$$

Consequently, the quantile function can be obtained by first calculating

$$\eta_p = \begin{cases} \delta + T^{-1}(1 - 2(1-p)T(\delta;\nu);\nu)\,, & p > \frac{1}{2}, \\ -\delta - T^{-1}(1 - 2pT(\delta;\nu);\nu)\,, & p < \frac{1}{2}, \\ 0, & p = \frac{1}{2}, \end{cases}$$

and then

$$Q_Y(p) = \text{logistic}(\mu + \sigma\eta_p). \tag{2.16}$$

This representation also provides a direct method for generating random observations from the FLT distribution.

The modal structure of the FLT distribution can be examined from its density. For $y \neq \text{logistic}(\mu)$, differentiation of the log-density in (2.1) gives

$$-\frac{(\nu+1)(|\eta| - \delta)}{\nu + (|\eta| - \delta)^2}\,\text{sign}(\eta)\frac{1}{\sigma y(1-y)} - \frac{1}{y} + \frac{1}{1-y} = 0. \tag{2.17}$$

Depending on the values of σ, δ, and ν, this equation may have multiple solutions, corresponding to local modes and antimodes of the density. The point $y = \text{logistic}(\mu)$

Table 1: Numerical distributional characteristics of the FLT distribution.

Case	μ	σ	δ	ν	Mean	Variance	Skewness	Kurtosis
Bimodal	0.00	1.00	1.00	3.00	0.50	0.09	0.00	1.59
Trimodal	-1.00	1.00	2.00	1.00	0.39	0.14	0.31	1.41

should be examined separately because of the absolute-value term in (2.1). This feature allows the FLT distribution to accommodate bimodal and trimodal shapes, as illustrated in Figure 1.

Table 1 reports the numerical mean, variance, skewness, and kurtosis for the two parameter settings used in the simulation study. In the bimodal case $(\mu, \sigma, \delta, \nu) = (0, 1, 1, 3)$, the mean is 0.50 and the skewness is 0.00, which is consistent with the symmetry property $E(Y) = 1/2$ when $\mu = 0$. The variance is 0.09 and the kurtosis is 1.59, indicating a platykurtic, bimodal shape with lighter tails than the normal distribution. In the trimodal case $(\mu, \sigma, \delta, \nu) = (-1, 1, 2, 1)$, the positive skewness reflects the shift induced by $\mu = -1$, while the larger variance and lower kurtosis are consistent with the three modes and the heavy tails associated with $\nu = 1$. It shows that the FLT distribution can produce a wide range of means, variances, skewness, and kurtosis values, confirming its flexibility for bounded data.

Finally, the hazard rate function (hrf) is directly obtained from the density and cdf:

$$h(y) = \frac{f_{FLT}(y; \mu, \sigma^2, \delta, \nu)}{1 - F_{FLT}(y)}, \qquad 0 < y < 1. \tag{2.18}$$

Figure 2 illustrates the wide variety of shapes that the hrf of the FLT distribution can take. Depending on the parameter values, the hazard rate may increase monotonically from zero to a sharp peak near the upper boundary, or decrease initially before rising again, producing a bathtub-like shape. The distribution can also generate more complex, non-monotone patterns characterized by interior peaks or local bumps, as well as curves that remain low for most of the interval and then rise steeply as y approaches one. These curves highlight the flexibility of the model, which is particularly well suited to settings where event times or failure processes exhibit hazards that change substantially over a finite interval.

The cdf, quantile function, moments, modal structure, and hazard rate can therefore be evaluated from the FLT distribution. The numerical moment calculations presented in Table 1 illustrate the effects of the model parameters on its main distributional characteristics.

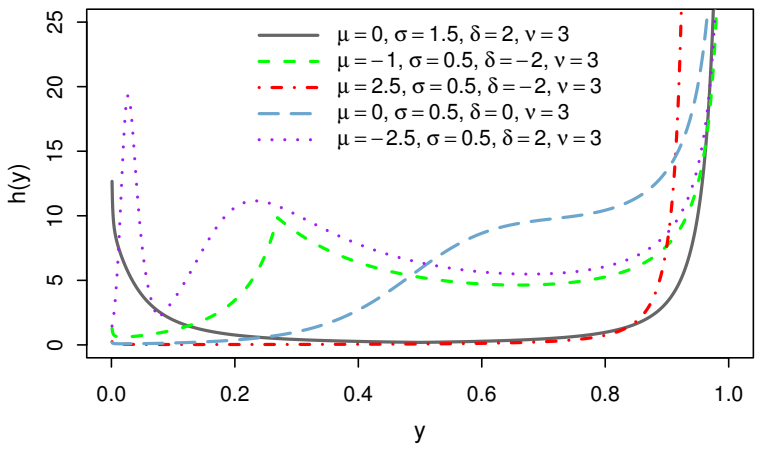


Figure 2: The hrf of the FLT distribution under various parameter settings, illustrating monotone, bathtub, and non-monotone shapes.

3 Maximum likelihood estimation of the parameters

The Expectation-Maximization (EM) algorithm (Dempster et al., 1977) and its variants the Expectation-Conditional Maximization (ECM; Meng and Rubin, 1993) and Expectation-Conditional Maximization Either (ECME; Liu and Rubin, 1994) algorithms are powerful statistical tools used for parameter estimation in models with latent variables, particularly in cases where the data is incomplete or has missing values. The ECM algorithm enhances the traditional EM approach by allowing for a more flexible conditional maximization step, which can be computationally advantageous when maximizing the complete data log-likelihood is complex. The ECME algorithm further generalizes this by introducing a framework that permits alternating between E and C (conditional maximization) steps, thereby improving convergence properties and efficiency. Both extensions aim to provide greater flexibility and robustness in handling a wider range of statistical models and data sets (Contreras-Reyes et al., 2014). Next, we show an application of ECME algorithm for ML estimation of parameters associated with the FLT distribution.

Let $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$ represent the observed data, while $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)^\top$ denote the latent variable. The complete dataset is denoted as $\boldsymbol{y}_c = (\boldsymbol{y}^\top, \boldsymbol{\tau}^\top)^\top$. Based on (2.4), the log-likelihood function of parameter vector $\boldsymbol{\theta} = (\mu, \sigma, \delta, \nu)^\top$ corresponding to the complete-data $\boldsymbol{y}_c$, omitting any additive constants and terms that do not involve the model parameters, is given by

$$\begin{aligned} \ell_c(\boldsymbol{\theta} \mid \boldsymbol{y}_c) &= -n\log\sigma - n\log T(\delta;\nu) - n\log\Gamma\Big(\frac{\nu}{2}\Big) + \frac{n\nu}{2}\log\Big(\frac{\nu}{2}\Big) \\ &\quad -\frac{1}{2}\sum_{i=1}^{n}\Big\{\tau_i(|\eta_i| - \delta)^2 + \nu(\tau_i - \log\tau_i)\Big\}, \end{aligned} \tag{3.1}$$

where $\eta_i = \Big(\text{logit}(y_i) - \mu\Big)/\sigma$.

Considering the present approximation of θ in the kth iteration, denoted as $\hat{\theta}^{(k)} = (\hat{\mu}^{(k)}, \hat{\sigma}^{(k)}, \hat{\delta}^{(k)}, \hat{\nu}^{(k)})^\top$. The E-step of the EM algorithm requires the conditional expectations derived from the conditional distribution as in (2.5):

$$\begin{aligned}\hat{\tau}_i^{(k)} &= E(\tau_i \mid y_i, \hat{\boldsymbol{\theta}}^{(k)}) \\ &= \frac{\hat{\nu}^{(k)} + 1}{\hat{\nu}^{(k)} + \left(|\hat{\eta}_i^{(k)}| - \hat{\delta}^{(k)}\right)^2}, \qquad (3.2) \\ \hat{e}_i^{(k)} &= E(\log \tau_i \mid y_i, \hat{\boldsymbol{\theta}}^{(k)}) \\ &= \psi\left(\frac{\hat{\nu}^{(k)} + 1}{2}\right) - \log\left(\frac{\hat{\nu}^{(k)} + \left(|\hat{\eta}_i^{(k)}| - \hat{\delta}^{(k)}\right)^2}{2}\right), \qquad (3.3)\end{aligned}$$

where $\hat{\eta}_i^{(k)} = (\text{logit}(y_i) - \hat{\mu}^{(k)})/\hat{\sigma}^{(k)}$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

We can then form the objective function $Q(\boldsymbol{\theta} \mid \hat{\boldsymbol{\theta}}^{(k)})$ as:

$$\begin{aligned}Q(\boldsymbol{\theta} \mid \hat{\boldsymbol{\theta}}^{(k)}) &= -n\log\sigma - n\log T(\delta;\nu) - n\log\Gamma\left(\frac{\nu}{2}\right) + \frac{n\nu}{2}\log\left(\frac{\nu}{2}\right) \\ &\quad -\frac{1}{2}\sum_{i=1}^{n}\left\{\hat{\tau}_i^{(k)}(|\eta_i| - \delta)^2 + \nu(\hat{\tau}_i^{(k)} - \hat{e}_i^{(k)})\right\}. \qquad (3.4)\end{aligned}$$

The CM-step subsequently optimizes $Q(\boldsymbol{\theta} \mid \hat{\boldsymbol{\theta}}^{(k)})$ with respect to $\boldsymbol{\theta}$, resulting in an updated estimate denoted as $\hat{\theta}^{(k+1)}$, as described in the following:

CMQ-step 1: Update $\hat{\mu}^{(k+1)}$ by

$$\hat{\mu}^{(k+1)} = \frac{1}{\sum_{i=1}^{n}\hat{\tau}_i^{(k)}}\sum_{i=1}^{n}\hat{\tau}_i^{(k)}\left\{\text{logit}(y_i) - \hat{\sigma}^{(k)}\hat{\delta}^{(k)}\text{sign}\left(\text{logit}(y_i) - \hat{\mu}_i^{(k)}\right)\right\}. \qquad (3.5)$$

CMQ-step 2: Fix $\mu = \hat{\mu}^{(k+1)}$ and update $\hat{\sigma}^{(k)}$ by maximizing (3.4) over σ, which is equivalent to solving a second-order linear equation, resulting in a single positive root, given by

$$\hat{\sigma}^{(k+1)} = \frac{-a \pm \sqrt{a^2 + 4nb}}{2n},$$

where

$$\begin{aligned}a &= \hat{\delta}^{(k)}\sum_{i=1}^{n}\hat{\tau}_i^{(k)}|\text{logit}(y_i) - \hat{\mu}^{(k+1)}|, \\ b &= \sum_{i=1}^{n}\hat{\tau}_i^{(k)}\left(\text{logit}(y_i) - \hat{\mu}^{(k+1)}\right)^2.\end{aligned}$$

CMQ-step 3: Obtain $\hat{\delta}^{(k+1)}$ by maximizing (3.4) over δ to have

$$\hat{\delta}^{(k+1)} = \frac{1}{\sum_{i=1}^{n}\hat{\tau}_i^{(k)}}\left\{\sum_{i=1}^{n}\hat{\tau}_i^{(k)}|\hat{\eta}_i^{(k+1)}| - n\frac{t(\hat{\delta}^{(k)};\hat{\nu}^{(k)})}{T(\hat{\delta}^{(k)};\hat{\nu}^{(k)})}\right\},$$

where $\hat{\eta}_i^{(k+1)} = (\text{logit}(y_i) - \hat{\mu}^{(k+1)})/\hat{\sigma}^{(k+1)}$.

To update $\hat{\nu}^{(k+1)}$, it is necessary to determine the root of ν. As was mentioned in Liu and Rubin (1994), this process could take much computational cost. Thus, a suitable approach is the maximization of constrained log-likelihood function, which derived the CML step of ECME algorithm as follows.

CML-step: Update $\hat{\nu}^{(k+1)}$ by optimizing the following constrained log-likelihood function:

$$\hat{\nu}^{(k+1)} = \arg\max_{\nu} \sum_{i=1}^{n} \log f_{FLT}\Big(y_i; \hat{\mu}^{(k+1)}, \hat{\sigma}^{2(k+1)}, \hat{\delta}^{(k+1)}, \nu\Big).$$

Because EM algorithm is iterative, it is required several iterations until a convergence criterion is accomplished. This criterion is the relative change of the log-likelihood function, defined as

$$\left\| \frac{\ell(\hat{\boldsymbol{\theta}}^{(k+1)})}{\ell(\hat{\boldsymbol{\theta}}^{(k)}) - 1} \right\| < \epsilon,$$

where

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f_{FLT}(y_i; \boldsymbol{\theta}).$$

In order to obtain convergence, it is necessary to establish a enogh small value of ε. Therefore, a threshold of $\varepsilon = 10^{-6}$ is acceptable for the relative change criterion.

4 Simulation study

A simulation study is conducted with varying sample sizes to assess the performance of the proposed algorithm. Our goal is determine if the proposed ECME algorithm can accurately estimate the true parameters. We generate 1000 Monte Carlo (MC) samples of sizes n = 50, 100, 200, 400, 600, 800, and 1000 from a FLT distribution, as is respectively illustrated in Figures 3 (a) and (b) for one simulated case with following true parameters:

Bimodal case: Data are simulated from the FLT distribution with the following parameters: $\boldsymbol{\theta}_1 = (\mu, \sigma, \delta, \nu)^\top = (0, 1, 1, 3)^\top$.

Trimodal case: Data are simulated from the FLT distribution with the following parameters: $\boldsymbol{\theta}_2 = (\mu, \sigma, \delta, \nu)^\top = (-1, 1, 2, 1)^\top$.

The precision of the parameter estimates is evaluated by calculating the mean of 1000 standard errors derived from the root mean square error (RMSE), which is expressed as follows:

$$RMSE = \sqrt{\frac{1}{1000} \sum_{r=1}^{1000} (\hat{\theta}_k - \theta_T)^2},$$

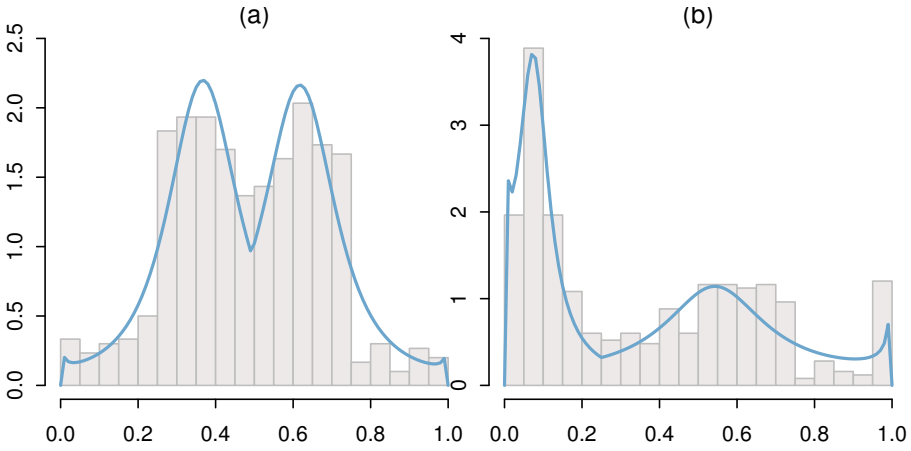


Figure 3: Simulated data of $n = 500$ from LFT distribution along with the fitted pdf plot. (a) Bimodal case and (b) Trimodal case.

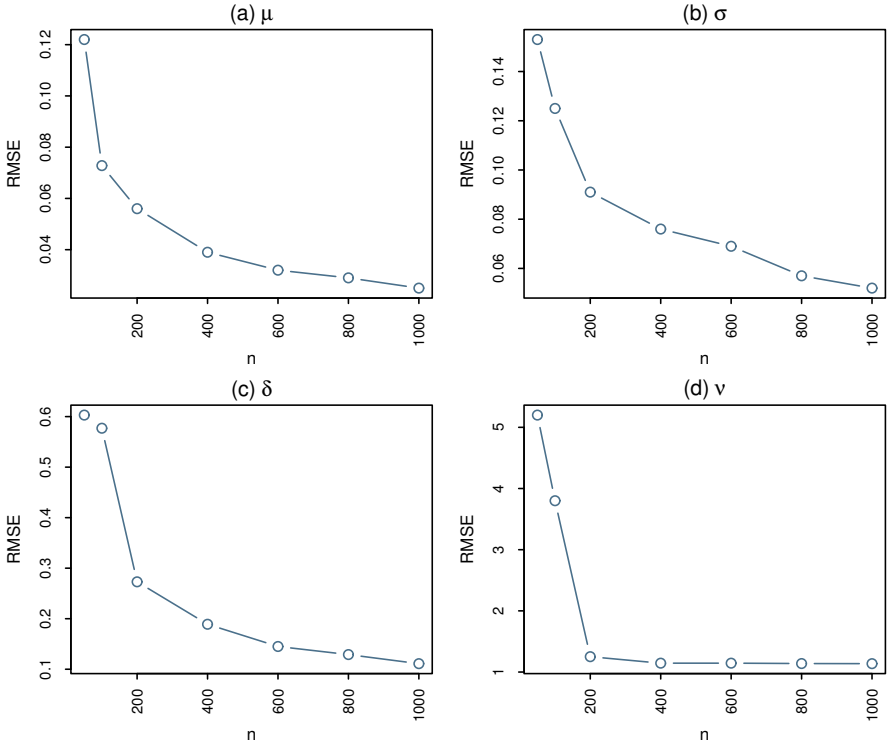


Figure 4: Average RMSE of the model parameters under the bimodal case

where $\hat{\theta}_r$ represent the ML estimate of a a specific parameter at the kth replication, and θ_T denotes the true parameter.

The relationship between RMSE and sample sizes for each parameter across both

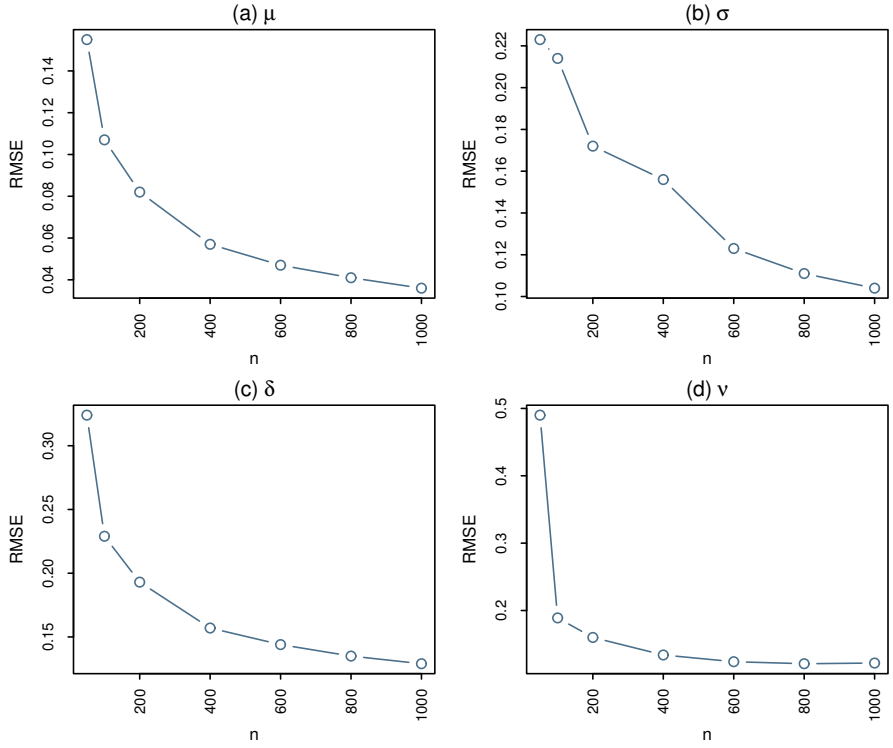


Figure 5: Average RMSE of the model parameters under the trimodal case.

scenarios is illustrated in Figures 4 and 5. The results indicate that RMSE values tend to decrease as sample size increases, suggesting improved estimation accuracy. It can be seen that the resulting RMSE values gradually decrease with the growth of the sample size, signifying that the estimation results become better and better. It is noteworthy that the shape parameters δ and ν exhibit slightly higher RMSE values, attributed to the likelihood's flatness concerning these parameters.

5 Illustrative examples

In this study, we illustrate our proposed methodology by analyzing two real-world data sets. To facilitate a comprehensive comparison, we fit several statistical distributions to these data sets, including

a) Beta distribution :

$$f(y:\alpha,\beta) = \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad \alpha>0,\ \beta>0,$$

where $B(\alpha,\beta)$ is the beta function.

b) Kumaraswamy (KU) distribution (Kumaraswamy, 1980):

$$f_{KU}(y:\alpha,\beta) = \alpha\beta y^{\alpha-1}(1-y^{\alpha})^{\beta-1}, \quad \alpha>0,\ \beta>0.$$

c) Generalized beta of the first kind (GBI) (McDonald, 1984):

$$f_{GBI}(y:\alpha,\beta,\delta) = \frac{\delta}{B(\alpha\delta^{-1},\beta)} y^{\alpha-1}(1-y^{\delta})^{\beta-1}, \quad \alpha>0,\ \beta>0,\ \delta>0.$$

d) McDonald arcsine (McA) distribution (Cordeiro and Lemonte, 2014):

$$f_{McA}(y:\alpha,\beta,\delta) = \frac{\delta}{B(\alpha\delta^{-1},\beta)\pi\sqrt{y-y^2}}\left[\frac{2}{\pi}arcsin(\sqrt{y})\right]^{\alpha-1} \times\left(1-\left[\frac{2}{\pi}arcsin(\sqrt{y})\right]^{\delta}\right)^{\beta-1}, \quad \alpha>0,\ \beta>0,\ \delta>0.$$

Furthermore, the reduced FLN and LN distributions are fitted as the sub-models of FLT distribution using the EM-algorithm given in Section 3. Given that the competing models vary in the number of parameters they encompass, we employ standard statistical criteria to assess their goodness-of-fit. Specifically, we utilize the Akaike Information Criterion (AIC) (Akaike, 1973) and the Bayesian Information Criterion (BIC) (Schwarz, 1978). Through the model comparison considering the penalized criteria, the Kolmogorov–Smirnov (K-S) test (Smirnov, 1948) is considered to evaluate the goodness of fit of all models.

5.1 Anxiety data

This application considered a dataset derived from a study conducted with 166 women in Australia, aimed at exploring the intricate relationships between stress, anxiety, and depression. This dataset serves as a valuable resource for understanding how these psychological factors interact and influence one another within this population. The data is accessible through the `betareg` `R` package, which facilitates the fitting of beta regression models, and further details can be found in the work of Smithson and Verkuilen (2006).

In this analysis, we focus specifically on the anxiety variable, which is critical for assessing the mental health status of the patients. The summary results of the fitted models are detailed in Table 2. The findings indicate that the proposed FLT distribution demonstrates a significantly superior fit compared to competing models, as evidenced by its notably lower values of both AIC and BIC. Besides, the smallest K-S test statistic (0.062) suggests that the FLT model yields the shortest distance between the empirical and parametric distributions of experimental data. Moreover, the estimated parameters of the model, specifically the shape parameter $\hat{\delta}$ and the flat parameter $\hat{\nu}$, reveal that the data possesses skewed and leptokurtic characteristics. To illustrate these findings, Figure 6 presents a histogram of the anxiety data overlaid with the fitted FLT, FLN,

Table 2: Summary results from fitting various models to the Anxiety data

Parameter	Beta	KU	GBI	McA	LN	FLN	FLT
$\hat{\mu}(\hat{\alpha})$	3.21	3.39	3.10	13.83	1.16	1.21	1.28
$\hat{\sigma}(\hat{\beta})$	1.31	1.55	4.41	0.32	1.22	1.53	0.45
$\hat{\delta}$	–	–	25.58	18.04	–	–0.87	0.51
$\hat{\nu}$	–	–	–	–	–	–	1.75
$\hat{\ell}$	35.97	37.78	44.02	52.27	33.14	46.16	66.70
AIC	–67.95	–71.56	–82.05	–98.54	–62.28	–86.32	–125.40
BIC	–62.68	–66.29	–74.14	–90.63	-57.01	–78.41	–114.86
K-S statistic	0.183	0.177	0.171	0.125	0.163	0.107	0.062

and LN distributions. A closer examination reveals that the FLT distribution more accurately captures the heavy tail and leptokurtic features inherent in the data when compared to the other models.

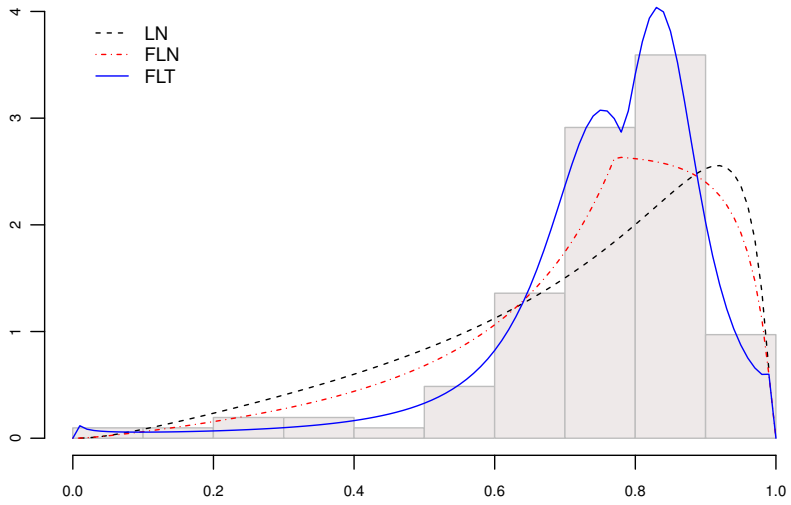


Figure 6: Histogram of the anxiety samples. The lines represent fitted FLT, FLN and LN densities.

5.2 Pain data

The pain data, derived from a comprehensive survey conducted between 2010 and 2014 involving nearly 10,000 patients in UK health clinics, offers valuable insights into the quality of care received by individuals experiencing pain. This dataset is available in the DBR R package and consists of 1,318 observations across three key variables that capture essential aspects of patient experiences. By analyzing this data frame, researchers can explore correlations between patient-reported outcomes and the care provided, with the goal of identifying areas for improvement in healthcare services. In this analysis,

Table 3: Summary results from fitting various models to the pain data

Parameter	Beta	KU	GBI	McA	LN	FLN	FLT
$\hat{\mu}(\hat{\alpha})$	1.86	1.85	1.86	1.02	–1.41	–34.24	0.57
$\hat{\sigma}(\hat{\beta})$	1.04	1.04	1.04	3.35	11.70	0.95	0.46
$\hat{\delta}$	–	–	1.35	2.26	–	36.65	0.73
$\hat{\nu}$	–	–	–	–	–	–	1.08
$\hat{\ell}$	190.22	190.19	190.25	148.05	–364.91	2059.80	2535.79
AIC	-376.44	–376.39	–374.50	–290.10	733.82	–4113.61	–5063.57
BIC	–366.07	–366.02	–358.95	–274.55	744.1901	–4098.05	–5042.84
K-S statistic	0.149	0.149	0.149	0.156	0.105	0.500	0.049

we specifically focus on the severity variable, which measures patients' perceptions of pain severity. The severity is originally measured on a scale from 0 to 10. Since the FLT distribution is defined on $(0, 1)$, we first scaled the scores by dividing by 10 and then replaced the boundary values 0 and 1 by 10^{-30} and $1 - 10^{-30}$, respectively, so that all observations remain strictly inside $(0, 1)$ and the logit transformation remains finite.

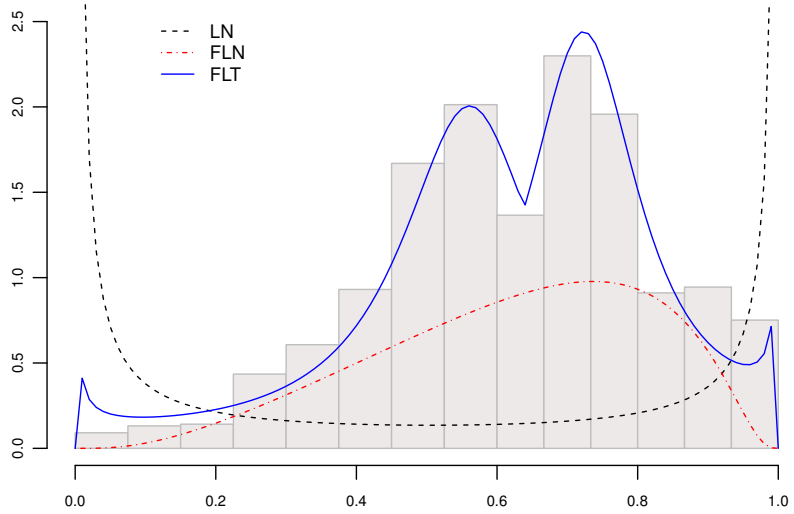


Figure 7: Histogram of the pain samples. The lines represent fitted FLT, FLN and LN densities.

Table 3 presents the ML estimates for the parameters across various competing models. The findings reveal that the LN distribution provides the poorest fit, as it is not suitable for bimodal data. In contrast, the FLN distribution offers a more satisfactory fit due to its inherent flexibility, although it still falls short of being an ideal match. The proposed FLT distribution, which includes an additional shape parameter ν, achieves the best fit based on the AIC, BIC, and K-S test statistics. Figure 7 shows the histogram of the pain data with fitted curves for the LN, FLN, and FLT models superimposed. It is clear that the FLT model effectively captures the long kurtosis, skewness, and tail

characteristics of the data, demonstrating superior accuracy compared to both the FLN and LN models.

5.3 FLT regression model

The FLT distribution can also be used as the response distribution in a regression model for bounded outcomes. Let $Y_i \in (0,1)$ denote the response and let $\boldsymbol{x}_i = (1, x_{i1}, \ldots, x_{ip})^\top$ be the vector of covariates for observation i, $i = 1, \ldots, n$. We consider the model

$$Y_i \sim FLT(\mu_i, \sigma^2, \delta, \nu), \qquad \text{logit}(\mu_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}, \tag{5.1}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^\top$ is the vector of regression coefficients. Since the median of an FLT random variable is logistic(μ), the conditional median of Y_i is

$$\text{Med}(Y_i \mid \boldsymbol{x}_i) = \text{logistic}(\boldsymbol{x}_i^\top \boldsymbol{\beta}). \tag{5.2}$$

Thus, the regression coefficients describe covariate effects on the conditional median on the logit scale. In particular, a one-unit increase in x_{ij}, with the other covariates held fixed, changes the log-odds of the conditional median by β_j. The parameters σ, δ, and ν are common to all observations and control the dispersion, modality, and tail behaviour of the response distribution, respectively.

Estimation of

$$\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \sigma, \delta, \nu)^\top$$

can be carried out using the ECME algorithm described in Section 3. In the regression setting,

$$\eta_i = \frac{\text{logit}(y_i) - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\sigma},$$

and the E-step is obtained by replacing the location parameter in the corresponding expressions of Section 3 with $\boldsymbol{x}_i^\top \boldsymbol{\beta}$. In particular,

$$\hat{\tau}_i^{(k)} = \frac{\hat{\nu}^{(k)} + 1}{\hat{\nu}^{(k)} + \left(|\hat{\eta}_i^{(k)}| - \hat{\delta}^{(k)}\right)^2}, \tag{5.3}$$

where

$$\hat{\eta}_i^{(k)} = \frac{\text{logit}(y_i) - \boldsymbol{x}_i^\top \hat{\boldsymbol{\beta}}^{(k)}}{\hat{\sigma}^{(k)}}.$$

Conditional on the current estimates of the remaining parameters, the CM-step for $\boldsymbol{\beta}$ has a weighted least-squares form. Specifically,

$$\hat{\boldsymbol{\beta}}^{(k+1)} = \left(\sum_{i=1}^{n} \hat{\tau}_i^{(k)} \boldsymbol{x}_i \boldsymbol{x}_i^\top \right)^{-1} \sum_{i=1}^{n} \hat{\tau}_i^{(k)} \boldsymbol{x}_i \left\{ \text{logit}(y_i) - \hat{\sigma}^{(k)} \hat{\delta}^{(k)} \, \text{sign}\left[\text{logit}(y_i) - \boldsymbol{x}_i^\top \hat{\boldsymbol{\beta}}^{(k)} \right] \right\}. \tag{5.4}$$

The sign term in (5.4) is evaluated at the current value $\hat{\boldsymbol{\beta}}^{(k)}$. The updates for σ, δ, and ν are obtained in the same manner as in Section 3, with μ replaced by the observation-specific linear predictor $\boldsymbol{x}_i^\top \boldsymbol{\beta}$.

Table 4: Estimated regression coefficients and goodness-of-fit measures for the pain data.

	Beta regression	LN regression	FLN regression	FLT regression
Intercept	−1.88	−2.55	−1.68	−1.63
Age	0.01	0.01	0.01	0.01
Interference	0.31	0.41	0.29	0.28
σ	–	1.49	5.89	0.43
δ	–	–	−7.88	0.42
ν	–	–	–	1.61
$\hat{\ell}$	610.43	79.07	586.52	783.08
AIC	−1212.86	−150.14	−1163.05	−1554.16
BIC	−1192.13	−129.40	−1137.13	−1523.05

To illustrate the regression application, we consider the pain data set introduced in preceding section. The response is the scaled pain severity score, while age and pain interference are used as explanatory variables. We fit the FLT regression model in (5.1) and compare its results with those obtained from beta regression (Ferrari and Cribari-Neto, 2004) and LN regression (Liang et al., 2014). The beta regression model is a standard approach for continuous responses on the unit interval, with covariates commonly linked to the conditional mean through a specified link function (Ferrari and Cribari-Neto, 2004). The LN model instead specifies a normal distribution for the logit-transformed response (Liang et al., 2014). Furthermore, the FLN regression is fitted to the data as a sub-model of the FLT regression by fixing $\nu = 300$.

Table 4 reports the parameter estimates and goodness-of-fit measures for the four models. The FLT regression provides the largest maximized log-likelihood and consequently the smallest AIC and BIC among the fitted models. The estimated regression coefficients have the same signs across the four models. In the FLT regression, the positive coefficient for age ($\hat{\beta}_{\text{age}} = 0.01$) indicates that the conditional median pain score increases slightly with age, while the positive coefficient for interference ($\hat{\beta}_{\text{interference}} = 0.28$) indicates higher conditional median pain scores for patients who report greater pain interference in daily life.

The estimated value $\hat{\nu} = 1.61$ indicates that the fitted FLT model retains substantial tail flexibility, while $\hat{\delta} = 0.42$ plays a role in determining the modality of the conditional response distribution. These additional parameters allow the regression model to account for distributional features that are not represented by the regression coefficients alone. Thus, the regression formulation extends the FLT distribution from a marginal model for bounded data to a framework in which covariates can be incorporated directly into the conditional distribution.

6 Possible multivariate extensions

Introducing a multivariate version of the FLT distribution is a natural direction for further research. However, such an extension is not obtained by simply replacing the scalar standardized variable in the univariate density by a vector. In particular, the proposed FLT distribution is constructed through the absolute standardized quantity

$$|\eta| = \left|\frac{\text{logit}(y) - \mu}{\sigma}\right|,$$

which plays a central role in determining both the shape of the density and its normalization. In a multivariate setting, there is no unique componentwise analogue of this construction. A suitable extension therefore requires a radial representation that preserves the main features of the univariate model while allowing dependence among the components.

One possible approach is to first construct a multivariate flexible-t distribution on the real linearly transformed scale and then apply the logit transformation componentwise. Let $\mathbf{X} = (X_1, \ldots, X_p)^\top$ be a p-dimensional random vector. A possible radial extension can be defined through a density of the form

$$f_{\mathbf{Z}}(\mathbf{z}) = C_p(\delta, \nu)\, h_p(|r - \delta|\,; \nu)\,, \qquad r = \|\mathbf{z}\|,$$

where

$$\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}), \qquad r = \|\mathbf{Z}\| = (\mathbf{Z}^\top \mathbf{Z})^{1/2},$$

$\boldsymbol{\Sigma}$ is a positive-definite scale matrix, $h_p(\cdot; \nu)$ is an appropriate radial kernel, and $C_p(\delta, \nu)$ is the normalizing constant. The absolute value $|r - \delta|$ is used so that the kernel remains well defined for all $r \geq 0$, including $r < \delta$, and so that the construction reduces to the univariate FLT when $p = 1$.

For example, a kernel based on the multivariate-t radial form can be considered,

$$h_p(u; \nu) = \left(1 + \frac{u^2}{\nu}\right)^{-(\nu+p)/2},$$

with the normalizing constant determined from

$$C_p(\delta, \nu)^{-1} = S_{p-1} \int_0^\infty r^{p-1}\, h_p(|r - \delta|\,; \nu)\; dr,$$

where $S_{p-1} = 2\pi^{p/2}/\Gamma(p/2)$ denotes the surface area of the unit sphere in $\mathbb{R}^p$. For $\delta = 0$, this integral reduces to the standard multivariate-t normalizing constant. For $\delta \neq 0$, it does not admit a closed form and must be evaluated numerically. The resulting normalizing constant then depends on δ, ν, and p in a nontrivial way.

This construction illustrates an important feature of the multivariate problem. In the univariate case, the radial coordinate is simply $r = |z|$, and the integration over the two half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$ leads to the factor $2T(\delta; \nu)$ appearing in the proposed FLT

density. In higher dimensions, the corresponding normalizing constant depends on the dimension through the factor r^{p-1} and generally does not reduce to the simple expression $2T(\delta;\nu)$. Consequently, the multivariate extension requires a new normalization and cannot be obtained by a direct replacement of $|\eta|$ with a Mahalanobis distance.

Once a suitable distribution for $\mathbf{X}$ has been established, a multivariate FLT distribution on $(0,1)^p$ can be obtained by applying the logistic transformation componentwise:

$$Y_j = \frac{\exp(X_j)}{1+\exp(X_j)}, \qquad j = 1,\dots,p,$$

or, in vector form,

$$\mathbf{Y} = (Y_1,\dots,Y_p)^\top, \qquad \mathbf{X} = \operatorname{logit}(\mathbf{Y}).$$

The Jacobian of this transformation is

$$\left|\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\right| = \prod_{j=1}^{p} \frac{1}{y_j(1-y_j)}.$$

Therefore, any properly normalized multivariate flexible-t density $f_{\mathbf{X}}$ on $\mathbb{R}^p$ induces the bounded-data density

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}\Big(\operatorname{logit}(\mathbf{y})\Big) \prod_{j=1}^{p} \frac{1}{y_j(1-y_j)}, \qquad 0 < y_j < 1, \quad j = 1,\dots,p.$$

Such a construction would retain an important feature of the present model: the logistic transformation would determine the support $(0,1)^p$, while the flexible-t component would control the dependence, central concentration, and tail behaviour on the transformed scale. The positive-definite matrix $\boldsymbol{\Sigma}$ would additionally provide a mechanism for modelling dependence among the transformed variables. However, it should be noted that, as is typical for radial distributions, the marginal distributions of the components are not members of the same family. Consequently, the MFLT model should be viewed as a joint model for dependent bounded responses rather than as a collection of univariate FLT marginals. Moreover, the shape parameter δ and the tail parameter ν are shared by all components, so the model cannot accommodate component-specific tail behaviour.

Several theoretical and computational issues would need to be addressed before such a model could be regarded as a complete multivariate FLT distribution. These include the derivation of a tractable normalizing constant, the identification of a useful stochastic representation, the investigation of marginal and conditional distributions, the development of an estimation procedure, and the construction of an efficient EM- or ECME-type algorithm. The choice of the radial kernel is also important, because it should provide a meaningful multivariate analogue of the univariate FLT while retaining interpretable limiting cases.

Therefore, the multivariate extension is not mathematically precluded by the absolute-value term in the univariate FLT density. Rather, the absolute-value construction indicates that a multivariate formulation should be based on a radial representation

rather than a componentwise substitution. A complete development of such a model, together with its distributional properties and estimation methodology, is beyond the scope of the present work and is left for future research.

7 Concluding remarks

This paper proposed the novel FLT distribution, which combines structures from the flexible-t distribution and the logit function. This innovative approach is particularly effective for modeling unit data that often displays complex patterns, such as heavy tails and multiple peaks. As a result, it's a valuable resource for researchers who work with these types of data sets. To improve the estimation of parameters for this model, we developed an ECME algorithm. We also assessed the model's performance through a simulation study and by analyzing two real-world data sets, comparing our results to several existing models in the literature. Our findings indicate that the FLT distribution can be a strong alternative to traditional unit distributions, especially for modeling rates and proportions. The regression model developed in this paper expands the scope of statistical modeling based on the FLT distribution to settings where the dependent variable is proportional.

A multivariate extension of the FLT distribution is an interesting direction for future research. Such an extension is not obtained by a direct componentwise replacement of the absolute-value term in the univariate density. The absolute standardized variable is central to the flexible-t construction, whereas in the multivariate setting it would need to be replaced by an appropriate radial and directional representation. Developing such a model would require a separate treatment of its density, stochastic representation, distributional properties, and estimation procedure. We therefore leave the development of a fully specified multivariate FLT distribution for future work.

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