JIRSS (2027)
Vol. 26, No. 1, pp 1-25
DOI: 10.22034/jirss.2026.2083523.1177

Comparison of Standardized and Adjusted Working Residuals for Gamma-Pareto Regression and Gamma-Pareto Ridge Regression Models

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Received: 2026-01-08, Accepted: 2026-10-01, Published online: 2026-10-05

Abstract. Influential observations can substantially affect the estimation, interpretation, and predictive performance of Gamma-Pareto regression models, particularly when multicollinearity is present. This study investigates the detection of influential observations in the Gamma-Pareto regression model and its ridge-based counterpart, the Gamma-Pareto ridge regression model. Several established diagnostic measures, including the Covariance Ratio, Welch Distance, and Hadi's Measure, together with their ridge-regression counterparts, are considered for assessing the influence of individual observations. To enhance the reliability of influence diagnostics in the presence of dispersion-related effects, we used an extended Hadi's Measure for the Gamma-Pareto ridge regression model based on standardized and adjusted working residuals. The diagnostic measures are evaluated through comprehensive analyses of both real and simulated datasets under varying levels of dispersion. The results demonstrate that the proposed Hadi's Measure in Ridge provides a more effective and stable mechanism for identifying influential observations than the conventional diagnostics, with different dispersion settings. These findings highlight the usefulness of incorporating standardized and adjusted working residuals into influence diagnostics and provide a practical framework for detecting influential observations in Gamma-Pareto ridge regression model.

Keywords. Diagnostic Measures, G-PRM, G-PRRM, Influential Observations, Multicollinearity.

MSC: 62-XX, 62Rxx, 62R10.

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1 Introduction

The linear regression model has been extended in the form of generalized linear model (GLM), which uses a link function to model a non-normal response variable. Several statistical models including the Linear regression model (LRM), the Gamma-Pareto regression model (G-PRM) and (G-PRRM), the Poisson regression model, the beta regression model, and others were unified by GLM. The introduction of GLM allows for the investigation of response variable dependence on two explanatory variables. Another variation of the GLM was discussed by (McCullagh and Nelder, 1989; Saleem and Akbar, 2024). Data points that differ significantly from other data points and lack neighboring values are considered influential observations. Influential observations are essentially unusual data values. Index plots, Covariance Ratio (CVR_i), Welch Distance (WD_i), Hadi's Measure (H_{di}), Covariance Ratio in Ridge (CVR_{Ri}), Welch Distance in Ridge (WD_{Ri}) and Hadi's Measure in Ridge (H_{dRi}) and potential residual plots are a few statistical techniques used to identify influential observations.

An essential step in working with the regression model is influence analysis. However, identifying any odd effects on model fitting and inferences is the main goal of influence diagnostics. Global influence diagnostics and local influence diagnostics are the two general methods used to study influence diagnostics. There are numerous methods for these diagnostics in the literature (Cook; 1970, Atkinson; 1981, Cook and Weisberg; 1982, Cook; 1986). The diagonal elements are utilized for influence diagnostics and are also referred to as leverages. In order to influence additional diagnostic measures, the leverages serve as an indicator. If the data is small, then an observation is considered influential if the DFFITS value is greater than one, Chatterjee and Hadi (1988).

The current article focuses on the global influence diagnostics for the G-PRM and the G-PRRM. The global influence measures are the case deletion diagnostics. One variant of the GLM is the G-PRM. The literature contains a few studies on influence diagnostics for the GLM using various probability models. The influence diagnostics for logistic regression were first proposed by Pregibon (1981). A technique used to evaluate the partial influence on the LR estimates was put suggested by Lee (1988). By using the deviance and single deletion case methods for the LR, Williams (1987) offered the influence diagnostics. A technique used to evaluate the impact of the GLM regression coefficients was put forth by Thomas and Cook (1989). To identify the influential observations, Landwehr and Pregibon (1993) employed partial residual plots for the GLM with various canonical linkages. See Preisser and Qaqish (1996), Zhu and Lee (2001), Xu et al. (2006), Venezuela et al. (2007), and Xie and Wei (2007) for additional information on GLM influence analysis. In order to find influential observations, Amin et al. (2016) compared various GLM residuals in the inverse Gaussian regression. Amin et al. (2017) discussed the issue of influence diagnostics in the gamma regression model, utilizing Pearson and adjusted deviance residuals with diagnostic measures such as Cook's Distance and DFFITS. Mittlbock and Heinzl (2002) explained the common R2 measure provides a useful means to quantify the degree to which variation in the dependent variable can be explained by the covariates in a linear regression model. Using Pearson, Deviance, and Likelihood residuals Saleem and Akbar (2004) provided the influence diagnostics for the G-PRM. We now examine these diagnostics using ridge estimates for the G-PRM. We first study the G-PRM overview and applications prior to the discussion.

Ozkale et al. (2018) deal adverse effects of multicollinearity and unusual observations are seen in logistic regression and attention had been given in the literature to each of these problems separately. However, multicollinearity and unusual observations can arise simultaneously in logistic regression. The Gamma-Pareto distribution (G-PD) was generated by Alzaatreh et al. (2012). The G-PD represents one member of the exponential family distribution Hanum et al.

(2016). Thus, the regression modeling for the G-PRM could be developed using GLM. Hanum et al. (2016) developed GLM and G-PD analytically. The commonly used GLM gamma is based on the gamma distribution (GD). It is common for the right skew data to fit when GLM gamma is used for analysis. Alzaatreh et al. (2012) mentioned the mathematical connection between G-PD and GD. Given that the G-PD developed from the GD, it makes sense that Hanum et al. (2015) used it to model and forecast extreme monthly rainfall. The regression model is based on G-PD. GLM are typically used as regression models for non-normal response variables. In a simulated G-PD using GLM gamma, Hanum et al. (2016) investigated the connection between the explanatory variable and the distributed response variable. Hanum et al. (2017) discussed the use of GLM gamma to model Gamma-Pareto (G-P) distributed data in monthly rainfall estimation using TRMM data. The Shifted Gamma-Generalized Pareto Distribution model was discussed by Zheng et al. (2013) in order to map the safety continuum and estimate crashes. According to Ashour et al. (2014), the new Log-Gamma-Pareto Distribution was developed. Alzaatreh and Ghosh (2016) introduced a new Gamma-Pareto (IV) distribution and its applications. They define a new distribution, namely Gamma-Pareto (IV) distribution, here after called as GPD (IV) distribution. Various properties of the GPD (IV) are investigated such as limiting behavior, moments, mode, and Shannon entropy.

De Andrade et al. (2017) discussed the gamma generalized Pareto distribution and its uses in survival analysis. Alzaghal (2020) applied the exponentiated Gamma-Pareto distribution to bladder cancer susceptibility. Dar et al. (2020) discussed the weighted Gamma-Pareto distribution and its applications.

The G-PRM is used to model response variables that are positively skewed have a constant coefficient of variation and have a mean proportionate to variance. Reliability and survival studies Lawless (2003) and numerous industries, such as the chemical and semiconductor sectors, whose production data adhere to G-PD Runger et al. (2005) are examples of uses for such models. Rainfall forecasting is another application of the G-PRM Wheater et al. (2006).

Recently, the G-PRM was used in the medical sciences by Karl et al. (2018) and Park et al. (2018). With varying connection functions, several researchers examine the G-PRM in various dimensions. The generalized log-GRM was also created by Ortega et al. (2009) using the cure fraction. In their work, Ortega et al. (2012) presented the log-GRM with censored. Similar to other GLM, the Gamma-based model has some negative outcomes. Numerous researcher (Atkinson; 1985, Firth; 1988, McCullagh and Nelder; 1989, Das; 2014, Qasim et al. 2018) have claimed that the analysis provided by the G-P and log-normal distributions is identical for positively skewed data. They have also demonstrated that the regression estimates for the log-normal distribution and the G-P distribution may diverge when the variance is not constant. For the lognormal model, the variance is constant but for the G-PRM it is not. According to Jiao and Chen (2004), the estimates from the G-PRM outperform those from the log-normal model for sequential populations.

The problem of multicollinearity occurs in many regressions analysis research because the explanatory variables are interrelated. This problem has direct unfavorable implications on the estimates and other related metrics. In order to address this issue, biased estimating techniques are favored over the conventional approach. The most widely used technique is the ridge estimation method which was first presented by Hoerl and Kennard (1970) and is used to lessen the impact of multicollinearity on the model estimates.

The GLM with ridge estimators under the various exponential families of distributions has recently been the subject of some research. To bit ridge regression (Sjolander; 2014), Gamma ridge regression (Amin et al. 2017), and logistic regression with ridge estimator (Schaeffer et al; 1984, Lee; 1984, Cessie and Houwelingen; 1992, Lesaffre; 1993, Kibria ; 2012) are all included in the primary study.

The issues of multicollinearity and influencing observations can occasionally arise in regression models at the same time. Some studies have already looked at influence analysis using ridge estimates for linear models (LM) (Walker and Birch; 1988, Belsley et al; 1980, Billor; 1999, Labra et al; 2007, Jahufer and Jianbo; 2009, Ullah et al; 2013, Emami; 2015, Emami; 2016)

In the currently available literature, it is found that Covariance Ratio (CVR_i), Welch Distance (WD_i), Hadi's Measure (H_{di}), Covariance Ratio in Ridge (CVR_{Ri}), Welch Distance in Ridge (WD_{Ri}) and Hadi's Measure in Ridge (H_{dRi}) has not yet been specifically used to study the behavior of standardized Pearson residuals in different diagnostics measures. This article fills that gap by going over how standardized and adjusted working residuals affect different diagnostics measures' performance in the G-PRM and G-PRRM. For the influence diagnostics of the G-PRM and G-PRRM, we will employ the standardized and adjusted working residuals.

2 Material and Methods

2.1 The notation, estimation techniques, for Gamma-Pareto regression model

Alzaatreh et al. (2012) state that the G-PD PDF is provided by:

$$f(y;\alpha,\beta,\gamma) = \frac{\gamma^{-1}}{\beta^{\alpha}\Gamma(\alpha)}\left(\log\left(\frac{y}{\gamma}\right)\right)^{\alpha-1}\left(\frac{y}{\gamma}\right)^{-\left(\frac{1}{\beta}+1\right)} \tag{2.1}$$

with $\alpha,\beta,\gamma > 0$ and $y > \gamma$.

The mean and variance of G-P distribution are, $E(a(y)) = \alpha\beta$, $V(a(y)) = \alpha\beta^2$ respectively and $a(y)$ is a function of $f(y)$. According to Hanum et al. (2016), parameters α and β in Eq. (2.1) can be replaced with ϕ^{-1} and $\mu\phi$, respectively. The Gamma-Pareto density for y under these conditions is given by

$$f(y;\mu,\phi) = \frac{\gamma^{-1}}{(\mu\phi)^{\frac{1}{\phi}}\Gamma\left(\frac{1}{\phi}\right)}\left(\log\left(\frac{y}{\gamma}\right)\right)^{\frac{1}{\phi}-1}\left(\frac{y}{\gamma}\right)^{-\left(\frac{1}{\mu\phi}+1\right)} \tag{2.2}$$

with $y > \gamma$, $\gamma > 0$, $\mu > 0$ and $\phi > 0$. Here the mean and variance of y are $E(a(y_i)) = \mu_i$ and $V(a(y_i)) = \phi V(a(\mu_i)) = \phi{\mu_i}^2$. For the ith observation, let $x_{i1}, x_{i2}, \ldots, x_{ip}$ represent the p non-stochastic regressors. Following that, the G-PRM for the response variable y a mean is provided by, According to Hanum et al. (2016), Link function g in GLM is $g(\mu_i) = gE(a(y_i)) = X_i^T\beta = \eta_i$ where $\mu_i = E(a(y_i))$, which are presented in Table 1.

Table 1: Different link function of Gamma-Pareto Distribution

Link function	Form of link function
Inverse link function	$\mu = \frac{1}{X_i^T\beta}$
Identity link function	$\mu = X_i^T\beta$
Log link function	$\mu = \log(X_i^T\beta)$
	$\mu = e^{X_i^T\beta}$

2.2 Estimation the GLM Gamma-Pareto Regression Model Parameters

Finding the likelihood functions derivative with respect to β_j is the first step in estimating the parameter β_j using maximum likelihood and τ_i is the function of $f(y)$. By using Eq. (2.2)

$$\frac{\partial l}{\partial \beta_j} = U_j = \sum_{i=1}^{N} \left[\frac{\partial l_i}{\partial \beta_j} \right] = \sum_{i=1}^{N} \left[\frac{\partial l_i}{\partial \tau_i} \frac{\partial \tau_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \beta_j} \right] \tag{2.3}$$

Now

$$\frac{\partial l_i}{\partial \tau_i} = a(y) b^T(\tau) + c^T(\tau) = \beta^{-2} \left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i \right) \tag{2.4}$$

$$\frac{\partial \tau_i}{\partial \mu_i} = \frac{1}{\frac{\partial \mu_i}{\partial \tau_i}} = \frac{1}{\frac{\partial \alpha\beta}{\partial \beta}} = \frac{1}{\alpha} \tag{2.5}$$

$$\frac{\partial \mu_i}{\partial \beta_j} = \frac{\partial \mu_i}{\partial \eta_i} x_{ij} \tag{2.6}$$

Where $\frac{\partial \mu_i}{\partial \eta_i}$ based on the GLM link function. So, the score for β_j in GLM Gamma-Pareto is

$$\frac{\partial l}{\partial \beta_j} = U_j = \sum_{i=1}^{N} \alpha^{-1} \beta^{-2} \left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i \right) \frac{\partial \mu_i}{\partial \eta_i} x_{ij} \tag{2.7}$$

$$U_j = \sum_{i=1}^{N} \left[var\left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i \right) \right]^{-1} \left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i \right) \frac{\partial \mu_i}{\partial \eta_i} x_{ij} \tag{2.8}$$

The variance U_j is

$$var(U_j) = \zeta_{jk} = \sum_{i=1}^{N} \frac{x_{ij} x_{ik}}{\left[var\left(\log\left(\frac{y_i}{\gamma}\right) \right) \right]} \left(\frac{\partial \mu_i}{\partial \eta_i} \right)^2 = \mathbf{X^T \widehat{W} X} \tag{2.9}$$

Where,

$$\widehat{\mathbf{W}} = \frac{1}{\left[var\left(\log\left(\frac{y_i}{\gamma}\right) \right) \right]} \left(\frac{\partial \mu_i}{\partial \eta_i} \right)^2 \tag{2.10}$$

Here the estimators of β_j is not in close form.

Where $\mathbf{x_i^T} = (x_{i1}, x_{i2}, \ldots, x_{ip})$, $\beta^T = (\beta_0, \beta_1, \ldots, \beta_p)$ is a vector of regression coefficient including intercept which is generally unknown and $g(.)$ is the link function. For the G-PRM, this link function is either used as identity, reciprocal or log and we consider the reciprocal link function. Iterative weighted least squares (IWLS) are a statistical technique used in regression analysis particularly in generalized linear models. IWLS is an extension of Weighted least squares (WLS). IWLS is used to find maximum likelihood estimates and standard deviations in generalized linear models. IWLS can be used to address issues in G-PRM such as multicollinearity and influential observations. IWLS can help mitigate multicollinearity by weighting observations assigning weights to observations can reduce the impact of correlated predictors. To stabilize estimates, IWLS can provide more stable estimates by down-weighting observations with high leverage. IWLS can also address influential observations by down-weighting outliers assigning lower weights to observations with large residuals or leverage. In G-PRM, IWLS can be particularly useful due to the distribution's heavy-tailed nature. By addressing multicollinearity and influential observations, IWLS can improve model fit and predictive accuracy. The

benefits using IWLS in G-PR can lead to improved model fit by addressing multicollinearity and influential observations. More accurate predictions by reducing the impact of influential observation and correlated predictors.

Let $\hat{\beta}$, $\hat{\mu}$ and $\hat{\phi}$ be the maximum likelihood estimate (MLE) obtained by maximizing the log-likelihood Eq. (2.7) using the Newton-Raphson iterative algorithm. The MLE of β is the solution of the system of Eq. by setting first derivative of Eq. (2.7) that equals to zero, then we have

$$U(\beta) = \frac{\partial l_i}{\partial \beta} = \frac{1}{\phi}\left(\mathbf{y} - \frac{1}{X\beta}\right)\mathbf{X} = 0, \tag{2.11}$$

where $U(\beta)$ is the score vector with dimension $(p+1)\times 1$. Since solution of the system of equations in Eq. (2.8) is nonlinear, so the iterative fisher scoring procedure is used to estimate the unknown parameter. For iterative procedure of the G-PRM, initial values and full algorithm for the estimation of unknown parameter can be found in Hardin and Hilbe (2012). Let $\beta^{(k)}$ be the approximated MLE of β with k iterations, then according to Green (1984). Iterative weighted least squares (IWLS) were proposed by Dobson and Barnett (2002) as a method for estimating β_j.

$$\beta^{(k+1)} = \beta^{(k)} + \{I(\beta^{(k)})\}^{-1} U(\beta^{(k)}) \tag{2.12}$$

Where $I(\beta^{(k)})$ is the $(p+1)\times(p+1)$ information matrix and both information and score vectors are evaluated at $\beta^{(k)}$. At convergence, Eq. (2.5), the unknown parameters can be evaluated by using Eq. (2.7) and (2.8) and applying IWLS, we have;

$$\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X^T}\widehat{\mathbf{W}}\mathbf{z}_i^* \tag{2.13}$$

which gives,

$$\hat{\beta} = \left(\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}\right)^{-1}\left(\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{z}_i^*\right) \tag{2.14}$$

Using $\widehat{W}$ and $var(U_j)$ for G-PD and obtained the iteration for β_j as, i is a number of observations $i = 1, 2, 3, \ldots, n$. and j are a number of parameters $j = 1, 2, 3, \ldots, p$ and $k \neq j$

$$\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}\hat{\beta} = \sum_{k=1}^{p}\sum_{i=1}^{N}\frac{x_{ij}x_{ik}}{\left[var\left(\log\left(\frac{y_i}{\gamma}\right)\right)\right]}\left(\frac{\partial\mu_i}{\partial\eta_i}\right)^2 b_k + \frac{\left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i\right)x_{ij}}{\left[var\left(\log\left(\frac{y_i}{\gamma}\right)\right)\right]}\left(\frac{\partial\mu_i}{\partial\eta_i}\right) \tag{2.15}$$

$$z_i^* = \sum_{i=1}^{N} x_{ij}b_k + \left(\log\left(\frac{y_i}{\gamma}\right) - \mu_i\right)\frac{\partial\mu_i}{\partial\eta_i}$$

where $z_i^* = \hat{\eta}_i + \left((y_i - \hat{\mu}_i)/\hat{\mu}_i^2\right)$ is the adjusted response variable (Hardin and Hilbe; 2012), $\widehat{\mathbf{W}} = diag(\hat{\mu}_1^2, \ldots, \hat{\mu}_n^2)$ is the matrix of weight function and $\hat{\mu}_i = \left(1/\mathbf{x_i^T}\hat{\beta}\right)$. $\hat{\beta}$ is estimated by minimizing the weighted sum of squared errors under the assumption that there is no multicollinearity in the cross product of $\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}$. Both z_i^* and $\widehat{\mathbf{W}}$ are computed by fisher scoring in fisher scoring iterative method (for more information, see Hardin and Hilbe; 2012). Here we use the change in deviance criterion for the iterative procedure. If there is multicollinearity among the explanatory variables then we use the ridge estimation for the G-PRM. We then standardized the explanatory variables and denoted them by X^* in such a way that $\mathbf{X^{*T}}\widehat{\mathbf{W}}\mathbf{X^*}$ = Correlation $(\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X})$, $\sum X^* = 0$ and $\sum x^{*2} = 1$.

Let $g(\mu_i) = 1/X_i^T\hat{\beta} = \eta_i$ be the reciprocal link function of the G-PRM, where $\eta_i = \mathbf{x_i^T}\beta^*$ be the linear predictor and β^* is a $(p+1)\times 1$ vector of unknown parameters that includes intercept. Let

$Z = (1, X^*)^T$ be the design matrix, and $E(y_i) = \mu_i$ be the mean. The MLE of $\hat{\beta}^*$ is then determined using the procedure we covered in Eq. (2.4-2.13) and is provided by

$$\hat{\beta}^* = \left(\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z}\right)^{-1}\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z}^*_{\mathbf{i}} \tag{2.16}$$

We now turn our attention to the G-PRM influence diagnostics. The estimated dispersion parameter, leverage, and Pearson residuals serve as the foundation for these diagnostics.

2.3 Hat matrix, leverage and residuals for the G-PRM

Hat matrix for the G-PRM is defined by

$$\mathbf{H} = \widehat{\mathbf{W}}^{\frac{1}{2}}\mathbf{Z}(\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z})^{-1}\mathbf{Z^T}\widehat{\mathbf{W}}^{\frac{1}{2}} \tag{2.17}$$

In contrast, leverages are **H**, diagonal elements, or $h_{ii} = diag(H)$. Because of the combination of the response variable and the explanatory variables, the GLM's leverages are more useful than the LM in detecting influence on the parameters.

2.4 The G-PRM Working Residuals

This type of the GLM residuals by Hardin and Hilbe (2012). The GLM residuals is used in convergence theory and in well-fitting of the statistical model (Gill, 2001).

The Working residuals is defined as

$$r_{w_i} = (y_i - \hat{\mu}_i)\left(\frac{\partial \eta_i}{\partial \mu_i}\right) \tag{2.18}$$

Where for the G-PRM $\frac{\partial \eta_i}{\partial \mu_i} = -\frac{1}{\mu_i^2}$. So, the estimated Working residuals for the G-PRM Eq. (2.18) become is given below.

$$r_{w_i} = (y_i - \hat{\mu}_i)\left(-\frac{1}{\mu_i^2}\right) \tag{2.19}$$

Where $\hat{\mu}_i = \left(1/\mathbf{z_i^T}\hat{\beta}^*\right)$ and $var(\hat{\mu}_i) = \hat{\mu}_i^2$.

Where dispersion parameter $\hat{\phi} = (1/(n-p-1))\sum_{i=1}^{n} r_{w_i}^2$

Similarly, using equation (16) the standardized Working residuals is defined by

$$st_{w_i} = \frac{r_{w_i}}{\sqrt{\hat{\phi}\, m_{ii}}} \tag{2.20}$$

Here, $m_{ii} = 1 - h_{ii}$

Using equation (16) the Adjusted Working residuals is defined as

$$a_{w_i} = \frac{r_{w_i} - E(r_{w_i})}{\sqrt{V(r_{w_i})}} \tag{2.21}$$

3 Influence Diagnostic Measures

3.1 Covariance Ratio in the G-PRM

The DFFITS statistic shows how each influential observation affects estimated values and fitted values. The statistic supplies no information about the overall precision of calculated estimates. To measure such influence, we have another technique measuring the effect of ith influential observation on the overall precision of the estimation, this technique is known as covariance ratio CVR_i. CVR_i is used to measure the ith observation influence on the variances of the estimates (Ullah and Pasha, 2009) and is defined for the GLM, CVR_i can also be used as an influence diagnostic statistic in the G-PRM. CVR_i for the G-PRM under the assumption that the independent variables are not multicollinear is define by

$$CVR_i = \frac{\left[\frac{(n-p'-1)+r^{2*}_{w(i)}}{n-p'}\right]^{p'}}{\left[\frac{\alpha-r^{2*}_i}{m_{ii}}\right]} \tag{3.1}$$

If $CVR_i > 1$, this indicated that the ith observation improves the precision otherwise the inclusion of ith observation degrades the precision of estimation. Belsley et al. (2004) has pointed out that the ith observation is influential if $CVR_i > 1 + 3\frac{p'}{n}$ or if $CVR_i < 1 - 3\frac{p'}{n}$.

Chatterjee and Hadi (2012) indicate that CVR_i may exceed one, if standardized residuals are small and leverages are large and CVR me be smaller than one, if standardized residuals are large and leverages are small. The CVR methodology leads to high values when both standardized residuals and leverages show large (small) quantities. The CVR diagnostic approach faces two influencing elements which may compromise its performance. The same calculation method is applied to CVR through alternative G-PRM residuals. The CVR_i for standardized Working residuals used Eq. (2.20) and (3.1)

$$CVR_{SWi} = \frac{\left[\frac{(n-p'-1)+r^{2*}_{w(i)}}{n-p'}\right]^{p'}}{\left[\frac{\alpha-r^{2*}_i}{m_{ii}}\right]} \tag{3.2}$$

The CVR_i for adjusted Working residuals used Equation (2.21) and (3.1)

$$CVR_{AWi} = \frac{\left[\frac{(n-p'-1)+r^{2*}_{w(i)}}{n-p'}\right]^{p'}}{\left[\frac{\alpha-r^{2*}_i}{m_{ii}}\right]} \tag{3.3}$$

3.2 Welch Distance in the G-PRM

Welsch (1982) modify the DFFITS to Welsch distance (WD) defined as

$$WD_i = DFFITS_i \sqrt{\frac{n-1}{m_{ii}}} \tag{3.4}$$

According to the Eq. (3.4), WD for the G-PRM defined as

$$WD_i = (n-1)\, t_i^2 \frac{h_{ii}}{m_{ii}^2} \tag{3.5}$$

$$WD_i = (n-1)\left(sr_{p_i}\sqrt{\frac{n-p'}{n-p'-sr_{p_i}^2}}\right)^2 \frac{h_{ii}}{m_{ii}^2} \tag{3.6}$$

Where $t_i = sr_{p_i}\sqrt{\frac{n-p'}{n-p'-sr_{p_i}^2}}$ and WD is computed with other forms of the G-PRM residuals. The declared as influential observation by the WD methods, if $WD_i > 3\sqrt{p'}$.

The WD_i for standardized Working residuals used Eq. (2.20) and (3.6)

$$WD_{swi} = (n-1)\left(sr_{w_i}^2\sqrt{\frac{n-p'}{n-p'-sr_{w_i}^2}}\right)^2 \frac{h_{ii}}{m_{ii}^2} \tag{3.7}$$

The WD_i for adjusted Working residuals used Eq. (2.21) and (3.6)

$$WD_{awi} = (n-1)\left(ar_{w_i}^2\sqrt{\frac{n-p'}{n-p'-ar_{w_i}^2}}\right)^2 \frac{h_{ii}}{m_{ii}^2} \tag{3.8}$$

3.3 Hadi's Measure in the G-PRM

Hadi (1992) introduced a Hadi's measure for the detection of influential observation.

$$H_{di} = \frac{p'}{m_{ii}}\frac{d_i^2}{1-d_i^2} + \frac{h_{ii}}{m_{ii}} \tag{3.9}$$

$$H_{di} = \frac{p'}{m_{ii}}\frac{\left(\frac{sr_{p_i}^2}{\sum_{i=1}^n sr_{p_i}^2}\right)^2}{1-\left(\frac{sr_{p_i}^2}{\sum_{i=1}^n sr_{p_i}^2}\right)} + \frac{h_{ii}}{m_{ii}} \tag{3.10}$$

Where $d_i = \frac{sr_{p_i}^2}{\sum_{i=1}^n sr_{p_i}^2}$ based on the standardized Pearson residuals in the G-PRM. On the basis of H_d measure, Hadi (1992) has pointed out that the observation is influential observation, if for the ith case, $H_{di} > Mean(H_{di}) + c\sqrt{var(H_{di})}$. Similarly, the H_d statistic is computed with other forms of the G-PRM residuals and with similar cutoff point.

The AP_i for standardized Working residuals used Eq. (2.20) and (3.10)

$$H_{dswi} = \frac{p'}{m_{ii}}\frac{\left(\frac{sr_{swi}^2}{\sum_{i=1}^n sr_{swi}^2}\right)^2}{1-\left(\frac{sr_{swi}^2}{\sum_{i=1}^n sr_{swi}^2}\right)} + \frac{h_{ii}}{m_{ii}} \tag{3.11}$$

The AP_i for adjusted Working residuals used Eq. (2.21) and (3.10)

$$H_{dawi} = \frac{p'}{m_{ii}}\frac{\left(\frac{ar_{w_i}^2}{\sum_{i=1}^n ar_{w_i}^2}\right)^2}{1-\left(\frac{ar_{w_i}^2}{\sum_{i=1}^n ar_{w_i}^2}\right)} + \frac{h_{ii}}{m_{ii}} \tag{3.12}$$

4 Gamma-Pareto ridge regression model

The variance of $\hat{\beta}^*$, i.e. $V(\hat{\beta}^*) = \hat{\phi}(\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z})^{-1}$ increases when severe multicollinearity is present and the results become deceptive (Segerstedt, 1992). The ridge estimator for the GLM which is based on the Hoerl and Kennard (1970) method was introduced by Segerstedt (1992) in order to solve this issue. Following Segerstedt (1992), Saleem and Akbar (2024) calculated the G-PRRM estimates using the relation, and the G-PRM estimator using the ridge estimator becomes the G-PRRM.

$$\hat{\beta}_R = (\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z} + KI)^{-1}\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z}\hat{\beta}^*, \tag{4.1}$$

Where, I is the identity matrix of order $(p+1)\times(p+1)$ and $K(>0)$ is the ridge parameter. When estimating ridge regression coefficients, the selection of the ridge parameter is crucial. The optimal ridge parameter for the G-PRRM was determined by Saleem and Akbar (2024) and is given by $K = \left(\hat{\phi}/\left(\prod_{j=1}^{p+1}\alpha_j^2\right)^{1/p+1}\right)$ where α_j^2 is the jth element of $\gamma'\hat{\beta}^*$ and γ is the eigen vector defined such that that $(\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}) = \gamma'\Lambda\gamma$ where $\Lambda = diag(\lambda_j)$ and λ_j are the eigen vales of $\mathbf{X^T}\widehat{\mathbf{W}}\mathbf{X}$. Since this is based on the variance function and the variance function varies for each exponential family of distributions, W varies for each of them. Keep in mind that the condition number index is used to test for multicollinearity in the GLM; see (Kibria; 2012) for more information.

4.1 The G-PRRM hat matrix, leverages and residuals

$$\mathbf{H}_R = \widehat{\mathbf{W}}^{1/2}\mathbf{Z}(\mathbf{Z^T}\widehat{\mathbf{W}}\mathbf{Z} + KI)^{-1}\mathbf{Z^T}\widehat{\mathbf{W}}^{1/2} \tag{4.2}$$

Conversely, the diagonal components of the hat matrix H_R, which is provided by Eq. (3.8), i.e., $h_{Rii} = diag(H_R)$, are the leverages with ridge estimator. Since the leverages with ridge estimator are smaller than the leverages of the G-PRM, i.e., $h_{Rii} \leq h_{ii}$, some adjustments are required, but the leverages with ridge estimator can also be useful for the detection of influential observations. For the G-PRRM, the working residuals using the ridge estimator are defined by

$$r_{w(Ri)} = (y_i - \hat{\mu}_{Ri})\left(\frac{\partial\eta_{Ri}}{\partial\mu_{Ri}}\right) \tag{4.3}$$

Where for the G-PRM $\frac{\partial\eta_{Ri}}{\partial\mu_{Ri}} = -\frac{1}{\mu_{Ri}^2}$ So, the estimated Working residuals for the G-PRRM Eq. (4.3) become is given below.

$$r_{w(Ri)} = (y_i - \hat{\mu}_{Ri})\left(-\frac{1}{\mu_{Ri}^2}\right) \tag{4.4}$$

Where $\hat{\mu}_{Ri} = \left(1/\mathbf{z_i^T}\hat{\beta}_R\right)$ and $V(\hat{\mu}_{Ri}) = \left(1/\mathbf{z_i^T}\hat{\beta}_R\right)^2$ are the mean and variance functions, respectively for the G-PRRM.

Similarly, using Eq. (4.4) the standardized Working residuals is defined by

$$sr_{w(Ri)} = \frac{r_{w_{Ri}}}{\sqrt{\hat{\phi}_R\, m_{Rii}}} \tag{4.5}$$

Eq. (4.5) defines h_{Rii}, the ith diagonal elements of the hat matrix with ridge estimator H_{Rii}, and $\hat{\phi}_R = (1/(n-p-1))\sum_{i=1}^{n} r_{w(Ri)}^2$ is the dispersion parameter with ridge estimate.

Here, $m_{Rii} = 1 - h_{Rii}$

Using Eq. (4.4) the Adjusted Working residuals is defined as

$$ar_{w(Ri)} = \frac{r_{w_{Ri}} - E(r_{w_{Ri}})}{\sqrt{V(r_{w_{Ri}})}} \tag{4.6}$$

5 Influence diagnostics in G-PRRM

The variances of the G-PRM estimates are so high when the explanatory variables are correlated that the results are deceptive for evaluating influence. The estimation and influence assessment are done using the G-PRRM in order to solve this issue. The cut points of the G-PRRM influence measures are comparable to those of the G-PRM and the influence measures with ridge estimates are now formulated as follows.

5.1 Covariance Ratio in the G-PRRM

The DFFITS statistic shows how each influential observation affects estimated values and fitted values. The statistic supplies no information about the overall precision of calculated estimates. To measure such influence, we have another technique measuring the effect of ith influential observation on the overall precision of the estimation, this technique is known as covariance ratio CVR_{Ri}. CVR_{Ri} is used to measure the ith observation influence on the variances of the estimates (Ullah and Pasha, 2009) and is defined for the GLM, CVR_{Ri} can also be used as an influence diagnostic statistic in the G-PRM. CVR_{Ri} for the G-PRM under the assumption that the independent variables are not multicollinear is define by

$$CVR_{Ri} = \frac{\left[\frac{(n-p'-sr^2_{Rpi})}{n-p'}\right]^{p'}}{m_{Rii}} \tag{5.1}$$

If $CVR_{Ri} > 1$, this indicated that the ith observation improves the precision otherwise the inclusion of ith observation degrades the precision of estimation. Belsley et al. (2004) has pointed out that the ith observation is influential if $CVR_{Ri} > 1 + 3\frac{p'}{n}$ or if $CVR_{Ri} < 1 - 3\frac{p'}{n}$.

Chatterjee and Hadi (2012) indicate that CVR_{Ri} may exceed one, if standardized residuals are small and leverages are large and CVR me be smaller than one, if standardized residuals are large and leverages are small. The CVR methodology leads to high values when both standardized residuals and leverages show large (small) quantities. The CVR diagnostic approach faces two influencing elements which may compromise its performance. The same calculation method is applied to CVR through alternative G-PRM residuals. The CVR_{Ri} for standardized Working residuals used Eq. (4.5) and (5.1)

$$CVR_{SW(Ri)} = \frac{\left[\frac{(n-p'-sr^2_{w(Ri)})}{n-p'}\right]^{p'}}{m_{Rii}} \tag{5.2}$$

The CVR_{Ri} for adjusted Working residuals used Eq. (4.6) and (5.1)

$$CVR_{AW(Ri)} = \frac{\left[\frac{(n-p'-ar^2_{w(Ri)})}{n-p'}\right]^{p'}}{m_{Rii}} \tag{5.3}$$

5.2 Welsch Distance in the G-PRRM

Welsch (1982) modify the DFFITS to Welsch distance (WD) defined as

$$WD_{i(Ri)} = DFFITS_i \sqrt{\frac{n-1}{m_{Rii}}} \tag{5.4}$$

According to the Eq. (5.4), WD for the G-PRM defined as

$$WD_{i(Ri)} = (n-1)\, t_i^2 \frac{h_{Rii}}{m_{Rii}^2} \tag{5.5}$$

$$WD_{i(Ri)} = (n-1)\left(sr_{PRi}\sqrt{\frac{n-p'}{n-p'-sr_{PRi}^2}}\right)^2 \frac{h_{Rii}}{m_{Rii}^2} \tag{5.6}$$

Where $t_i = sr_{PRi}\sqrt{\frac{n-p'}{n-p'-sr_{PRi}^2}}$ and WD is computed with other forms of the G-PRM residuals. The declared as influential observation by the WD methods, if $WD_{Ri} > 3\sqrt{p'}$.

The WD_i for standardized Working residuals used Eq. (4.5) and (4.6)

$$WD_{SW(Ri)} = (n-1)\left(sr_{W(Ri)}^2\sqrt{\frac{n-p'}{n-p'-sr_{W(Ri)}^2}}\right)^2 \frac{h_{Rii}}{m_{Rii}^2} \tag{5.7}$$

The WD_{Ri} for adjusted Working residuals used Eq. (4.6) and (5.6)

$$WD_{AW(Ri)} = (n-1)\left(ar_{W(Ri)}^2\sqrt{\frac{n-p'}{n-p'-ar_{W(Ri)}^2}}\right)^2 \frac{h_{Rii}}{m_{Rii}^2} \tag{5.8}$$

5.3 Hadi's Measure in the G-PRRM

Hadi (1992) introduced a Hadi's measure for the detection of influential observation.

$$H_{d(Ri)} = \frac{p'}{m_{Rii}}\frac{d_{Ri}^2}{1-d_{Ri}^2} + \frac{h_{Rii}}{m_{Rii}} \tag{5.9}$$

$$H_{d(Ri)} = \frac{p'}{m_{Rii}}\frac{\left(\frac{sr_{PRi}^2}{\sum_{i=1}^n sr_{PRi}^2}\right)^2}{1-\left(\frac{sr_{PRi}^2}{\sum_{i=1}^n sr_{PRi}^2}\right)} + \frac{h_{Rii}}{m_{Rii}} \tag{5.10}$$

Where $d_{(Ri)} = \frac{sr_{PRi}^2}{\sum_{i=1}^n sr_{PRi}^2}$ based on the standardized Pearson residuals in the G-PRRM. On the basis of H_{dR} measure, Hadi (1992) has pointed out that the observation is influential observation, if for the ith case, $H_{dRi} > Mean(H_{dRi}) + c\sqrt{var(H_{dRi})}$. Similarly, the H_d statistic is computed with other forms of the G-PRM residuals and with similar cutoff point. The AP_{Ri} for standardized Working residuals used Eq. (4.5) and (5.10)

$$H_{dSW(Ri)} = \frac{p'}{m_{Rii}}\frac{\left(\frac{sr_{W(Ri)}^2}{\sum_{i=1}^n sr_{W(Ri)}^2}\right)^2}{1-\left(\frac{sr_{W(Ri)}^2}{\sum_{i=1}^n sr_{W(Ri)}^2}\right)} + \frac{h_{Rii}}{m_{Rii}} \tag{5.11}$$

The AP_{Ri} for adjusted Working residuals used Equation. (4.6) and (5.10)

$$H_{dAW(Ri)} = \frac{p'}{m_{Rii}} \frac{\left(\frac{ar^2_{W(Ri)}}{\sum_{i=1}^{n} ar^2_{W(Ri)}}\right)^2}{1 - \left(\frac{ar^2_{W(Ri)}}{\sum_{i=1}^{n} ar^2_{W(Ri)}}\right)} + \frac{h_{Rii}}{m_{Rii}} \tag{5.12}$$

6 Numerical evaluations

With the aid of a simulation study and an actual data set, we will describe the G-PRM and G-PRRM influence diagnostics results in this section.

6.1 The simulation Study

To compare the effectiveness of the G-PRM and G-PRRM influence diagnostic techniques, we take into account the following Monte Carlo scheme. We generated a response variable that has a Gamma Pareto distribution using the algorithm of Hanum et al. (2016). With the reciprocal link function of the G-PRM, the data is generated as follows: $y_i \sim G-P(\alpha, \beta, \gamma)$, where $\mu_i = E(y_i) = (1/z_i\beta)$. We choose arbitrary values of the true β vector so that $\beta^T\beta = 1$. $\beta_0 = 0.05$ is fixed intercept and excluded from norm. All vectors satisfy $\sum_{j=1}^{p} \beta_o^2 = 1$.

$p = 2$: $\beta^T\beta = 0.6^2 + 0.8^2 = 1$
Where $\hat{\mu} = E(y_i) = \frac{1}{\beta_0+\beta_1 X_{1i}+\beta_2 X_{2i}}$.
Where $\hat{\mu} = E(y_i) = \frac{1}{0.05+0.6000X_{1i}+0.8000X_{2i}}$.
$p = 4$: $\beta^T\beta = 0.7^2 + 0.6^2 + 0.5^2 + 0.4^2 = 1.122$ normalized
Where $\hat{\mu} = E(y_i) = \frac{1}{0.05+0.6236X_{1i}+0.5345X_{2i}+0.4454X_{3i}+0.3563X_{4i}}$
$p = 6$: $\beta^T\beta = (0.8, 0.6, 0.5, 0.4, 0.3, 0.2) = 1.241$ normalized
Where $\hat{\mu} = E(y_i) = \frac{1}{0.05+0.6447X_{1i}+0.4835X_{2i}+0.4029X_{3i}+0.3223X_{4i}+0.2417X_{5i}+0.1611X_{6i}}$.
$p = 8$: $\beta^T\beta = (0.7, 0.6, 0.5, 0.4, 0.4, 0.3, 0.3, 0.2) = 1.2806$ normalized
$\hat{\mu} = E(y_i) = \frac{1}{0.05+0.5466X_{1i}+0.4685X_{2i}+0.3904X_{3i}+0.3123X_{4i}+0.3123X_{5i}+0.2342X_{6i}+0.2342X_{7i}+0.1562X_{8i}}$ $p = 10$: $\beta^T\beta = (0.6, 0.5, 0.4, 0.4, 0.3, 0.3, 0.3, 0.2, 0.2, 0.2) = 1.1489$ *normalized*
$\hat{\mu} = E(y_i) =$

$$\frac{1}{0.05+0.5222X_{1i}+0.4352X_{2i}+0.3481X_{3i}+0.3481X_{4i}+0.2611X_{5i}+0.2611X_{6i}+0.2611X_{7i}+0.1740X_{8i}+0.1740X_{9i}+0.1740X_{10}}.$$

The following arbitrary values for the dispersion parameter $\phi = 0.04, 0.11, 0.33, 0.67, 2$ and 5. And $10,000$ replications are planned for each of the sizes $n = 25, 50, 100, 200, 300, 400$ and 500. The explanatory variables without any influential observations were produced as follows, per Amin et al. (2017) and X_1, X_2, X_3 and X_4 are generated randomly from Uniform distribution $X_{ij} \sim U(0, 1)$ $i = 1, 2, \ldots, n$; and $j = 1, 2, 3, 4$.

$$X_{ij} = \sqrt{(1 - a_0^2)}Z_{ij} + a_0 Z_{i,j+1} \text{ and } Z_{ij} \sim U(0, 1),\ i = 1, 2, \ldots, n;\ \text{and } j = 1, 2, \ldots, p \tag{6.1}$$

Here, we assume the following arbitrary values for a_0^2: 0.75, 0.80, 0.85, 0.90, 0.95 and 0.99, where a_0^2 is the collinearity level between two explanatory variables. Then we make a significant change in observation in X's and made some observations as influential, i.e. $5^{th}, 10^{th}, 15^{th}, 20^{th}$ and 25^{th}, such that $X_{5j} = X_{5j} + A_0$, $X_{10j} = X_{10j} + A_0$, $X_{15j} = X_{15j} + A_0$, $X_{20j} = X_{20j} + A_0$ and $X_{25j} = X_{25j} + A_0$, $j = 1, 2, \ldots, p$, where $A_0 = \bar{X}_j + 6$. The diagnostics are carried out on the basis of the Gamma-Pareto generated samples in order to identify generated influential observations with different levels of collinearity and dispersions. The scheme is introduced

Table 2: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 0.04$, $\alpha_0^2 = 0.75$ and $p = 2$

		G-PRM						G-PRRM					
		$\phi = 0.04$ and $\alpha_0^2 = 0.75$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
25	5	93.2	93.2	98.7	57.4	65.6	92.7	92.6	92.6	99.5	58.0	64.5	92.1
	10	84.7	85.1	98.3	80.3	80.4	88.1	84.4	84.2	97.3	78.9	78.6	84.8
	15	75.1	75.7	95.6	70.4	69.6	75.9	74.9	74.8	93.8	70.4	69.7	75.2
	20	59.8	61.2	91.0	25.6	35.4	61.1	60.3	63.8	89.7	58.6	57.8	63.9
	25	46.6	52.4	81.7	49.6	46.7	51.9	50.8	57.0	85.3	47.0	50.3	56.2
50	5	94.9	94.9	99.9	69.8	74.6	94.5	93.5	93.5	99.5	69.7	72.4	93.3
	10	87.6	87.5	97.4	82.8	82.6	87.8	88.5	88.7	98.0	84.6	84.6	88.8
	15	83.9	83.8	97.1	80.7	79.6	83.8	82.3	82.7	96.3	79.5	79.0	81.8
	20	76.7	77.3	94.1	73.6	73.6	76.9	77.8	78.2	94.7	74.5	73.2	77.9
	25	70.1	71.7	92.7	68.2	68.1	71.1	73.5	74.3	92.3	70.6	68.9	74.7
100	5	93.7	93.7	99.7	79.6	82.3	93.7	95.9	95.9	99.9	78.8	80.6	95.2
	10	91.0	91.0	98.8	87.6	87.8	90.0	91.4	91.4	97.8	88.6	88.3	91.7
	15	85.5	85.5	98.6	83.8	83.7	86.1	88.8	88.8	97.9	85.8	85.5	88.5
	20	84.6	85.1	95.1	82.3	81.5	84.1	84.0	83.8	96.1	82.0	81.7	84.2
	25	82.7	83.3	95.3	81.0	80.2	82.2	83.1	83.7	95.6	81.3	79.3	83.1
200	5	95.7	95.7	99.7	88.0	88.1	95.1	96.4	96.4	99.8	86.6	87.8	95.6
	10	94.7	94.7	98.6	92.1	92.8	93.9	93.7	93.8	99.0	91.9	91.5	94.1
	15	91.6	91.6	98.1	89.9	89.4	91.5	92.1	92.3	98.8	89.5	89.7	91.6
	20	89.4	89.9	97.6	88.3	88.4	89.6	90.3	90.6	98.2	89.0	89.1	90.4
	25	89.7	89.7	96.4	88.5	88.1	89.5	88.3	88.5	97.5	87.2	86.5	88.0
300	5	95.6	95.6	99.7	89.4	90.6	95.8	95.8	95.8	99.5	88.3	89.3	96.5
	10	93.2	93.2	98.8	93.2	92.4	93.9	94.0	94.0	98.2	92.1	92.5	93.8
	15	92.6	92.7	98.8	90.5	91.0	93.2	93.4	93.4	98.8	92.0	91.3	92.9
	20	91.3	91.3	97.7	90.5	89.9	91.1	92.0	92.3	99.1	91.5	91.0	92.6
	25	90.0	90.1	98.1	89.1	88.8	88.8	91.0	90.8	98.1	90.2	90.4	91.3
400	5	96.2	96.2	99.5	92.3	93.0	96.5	95.8	95.6	99.6	90.9	91.2	95.4
	10	94.9	94.9	98.5	93.5	93.2	95.1	95.2	95.3	98.8	94.0	93.6	95.0
	15	93.2	93.2	98.2	92.8	93.0	93.7	95.7	95.8	98.8	94.9	94.3	95.6
	20	92.6	92.4	98.1	91.9	92.2	92.7	91.6	91.7	98.7	90.8	90.1	91.6
	25	91.8	91.8	98.3	91.6	91.2	92.8	90.4	90.2	97.1	89.6	89.6	90.3
500	5	96.7	96.8	99.6	92.8	93.2	96.5	96.9	97.0	99.3	93.1	92.7	97.0
	10	96.5	96.5	99.4	95.3	95.0	96.4	96.3	96.4	99.5	95.1	94.5	95.6
	15	93.4	93.3	98.3	93.0	92.6	93.6	94.6	94.5	99.5	93.3	93.5	94.9
	20	93.0	93.4	98.9	91.9	92.2	93.7	93.6	93.9	98.7	93.3	93.0	93.8
	25	90.5	90.4	97.6	89.0	89.8	91.1	92.6	92.6	98.6	92.0	92.0	92.9

in such a wat that we can observe impact of finite sample impact ($n = 25, 50, 100, 200, 300, 400$ and 500), impact of dispersion ($\phi = 0.04, 0.11, 0.33, 0.67, 2$ and 5), impact of collinearity levels ($a_0^2 = 0.75, 0.80, 0.85, 0.90, 0.95$ and 0.99) and impact of influential observations ($5^{th}, 10^{th}, 15^{th}, 20^{th}$ and 25^{th}). R software is used to perform these simulation results.

Comparison of behavior standardized and adjusted working residuals in different diagnostics measures in G-PRM and G-PRRM is displayed in Tables 2-7. When the various dispersion parameters listed above are taken into account, they have a significant impact on the detection percentage for all methods with the exception of WD_i. It's interesting to note that sample size has a significant impact on the rate of identifying influential observations as sample size increases, the percentage increases and at very large sample sizes all influential observations are detected. All of the diagnostics measures exhibit nearly the same behavior. It is evident from analyzing the results that the detection rate is impacted by ϕ (levels of dispersion) and

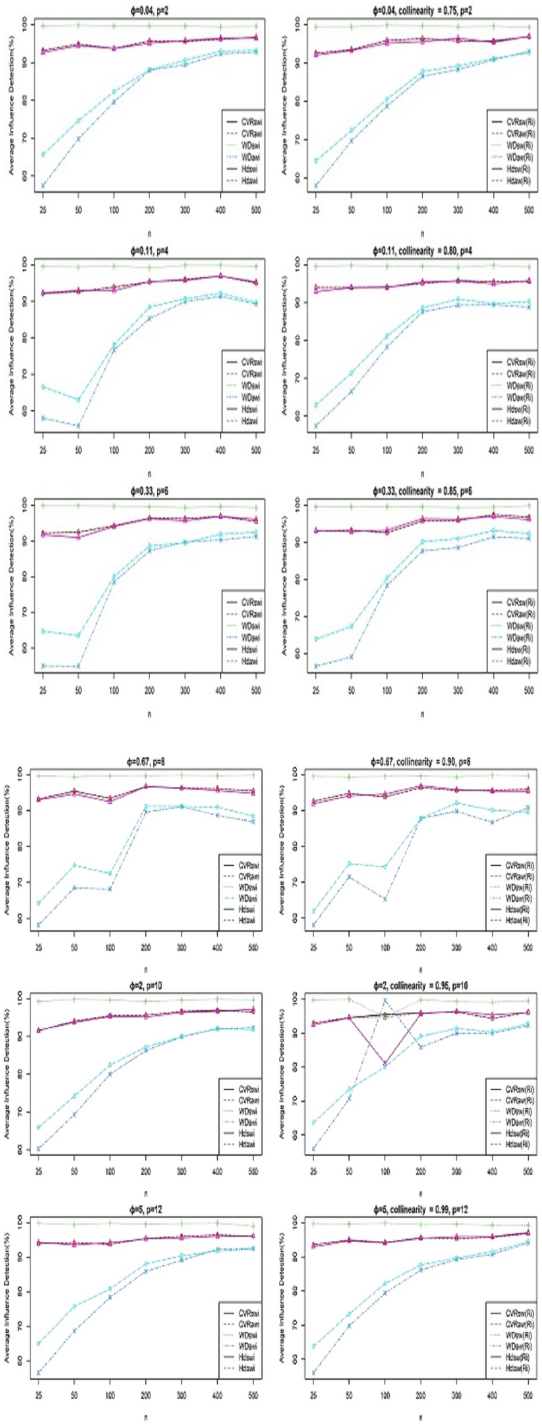


Figure 1. Performance of the G-PRM and G-PRRM for influence detection (%) using different diagnostic measures varying dispersion and multicollinearity levels

Table 3: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 0.11$, $\alpha_0^2 = 0.80$ and $p = 4$

		G-PRM						G-PRRM					
		$\phi = 0.11$ and $\alpha_0^2 = 0.80$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
	5	92.2	92.3	99.6	58.0	66.6	92.3	94.0	94.0	99.5	57.4	62.9	92.9
	10	83.0	83.0	97.4	79.3	79.5	82.9	83.0	82.8	97.4	77.8	77.9	83.1
25	15	73.0	74.4	93.6	69.1	67.7	73.7	73.3	73.7	93.2	67.1	65.6	73.5
	20	63.0	66.1	89.2	59.5	57.3	64.9	64.1	65.4	89.0	60.0	58.9	66.0
	25	46.6	54.6	82.9	43.5	47.9	53.9	48.8	54.8	82.1	45.9	48.7	54.3
	5	92.6	92.6	99.4	55.9	63.1	93.0	94.0	94.0	99.8	66.5	71.4	93.9
	10	86.4	85.9	98.1	81.8	81.8	86.2	88.8	88.7	98.1	84.9	84.9	88.8
50	15	72.5	73.0	93.8	68.2	67.0	73.3	83.6	83.7	97.3	80.5	80.0	83.8
	20	60.4	63.5	90.8	56.3	55.6	61.8	78.3	78.6	94.3	75.8	74.7	78.9
	25	45.9	53.8	84.5	43.6	46.5	53.0	70.0	70.4	91.0	68.0	67.0	70.2
	5	93.9	93.9	99.7	76.7	78.1	92.9	94.1	94.0	99.5	78.3	81.1	94.0
	10	91.9	91.9	98.3	89.1	88.6	91.3	90.2	90.1	98.1	86.6	87.2	88.5
100	15	87.7	87.9	97.5	86.1	85.1	87.8	87.3	87.4	97.6	85.1	84.0	87.1
	20	86.0	86.3	96.3	83.2	82.7	85.5	86.0	86.0	95.6	84.2	83.6	86.1
	25	80.5	81.3	94.5	79.3	78.9	80.4	82.9	82.8	95.7	80.0	79.7	83.0
	5	95.3	95.3	99.1	85.3	88.4	95.4	95.1	95.1	99.6	87.6	88.6	95.5
	10	94.9	94.9	99.1	92.6	92.5	94.3	93.7	93.6	98.7	90.8	90.9	93.6
200	15	90.5	90.7	98.7	88.9	89.4	90.9	92.9	93.0	98.3	90.9	90.4	92.5
	20	89.8	89.7	97.9	88.5	88.2	90.2	89.8	89.6	96.9	88.9	87.6	89.1
	25	85.9	85.9	96.9	85.0	84.5	86.3	88.0	87.9	97.3	87.0	86.6	88.1
	5	96.0	96.0	99.8	89.9	90.7	95.7	95.8	95.7	99.4	89.3	90.9	95.7
	10	94.2	94.2	99.0	92.4	92.6	93.7	94.6	94.6	99.4	92.9	93.4	94.9
300	15	93.4	93.5	98.7	91.5	91.8	92.8	92.7	92.6	98.0	91.5	91.5	92.8
	20	91.6	91.5	98.2	90.9	91.1	92.6	92.3	92.7	98.5	91.6	90.8	92.3
	25	89.8	90.0	97.9	88.8	87.9	89.2	90.8	91.0	97.3	90.2	89.2	90.7
	5	96.9	96.9	99.8	91.3	92.2	96.9	95.5	95.5	99.8	89.4	89.7	94.9
	10	93.6	93.7	98.7	91.0	91.9	94.1	93.7	93.7	99.1	92.7	92.5	94.4
400	15	92.0	92.0	98.1	91.1	90.6	91.8	92.9	92.9	98.4	91.9	91.5	93.0
	20	92.1	92.0	98.2	90.7	90.6	92.4	91.0	91.2	98.6	90.0	89.9	91.4
	25	89.5	89.8	97.6	88.4	88.1	89.8	91.2	90.8	97.8	89.8	89.8	91.1
	5	95.0	95.0	99.6	89.3	89.7	95.3	95.6	95.6	99.4	88.8	90.3	95.8
	10	93.5	93.4	99.6	94.1	94.1	95.2	96.1	96.1	98.9	94.1	94.6	95.9
500	15	94.1	93.8	99.1	92.2	92.6	93.6	93.4	93.5	98.1	92.1	91.9	93.1
	20	91.2	91.4	98.0	90.6	90.2	91.2	92.4	92.4	98.4	91.5	91.1	92.1
	25	90.8	91.0	98.0	90.1	89.5	90.4	91.3	91.3	98.4	90.3	89.7	91.2

multicollinearity with regressor. It can be observed that all diagnostics measures performed nearly equally well for large sample sizes but WD_{Ri} and H_{dRi} perform better for small sample sizes. The G-PRM and G-PRRM effect diagnostics simulation findings with ridge estimations which contains two explanatory variables that CVR_{Ri} and CVR_{Ri} perform well. With the ridge estimator, leverage causes a slight discrepancy between CVR_i and CVR_{Ri}. As multicollinearity rises, the influence detection percentages of the CVR_i and CVR_{Ri} approaches somewhat increase. As the dispersion parameter drops, so does the influence detection of these procedures. Additionally, due to multicollinearity we find that the CVR_{Ri} approach performs worse than the CVR_i method. When $\phi \geq 2$, the Covariance ratio technique detections with and without the ridge estimator are about the same. Although the influence detection of the WD_i and H_{di} approaches is the same for all dispersions, Covariance ratio methods provide better diagnostics. Similar to Covariance ratio technique, multicollinearity and dispersion have an impact on

Table 4: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 0.33$, $\alpha_0^2 = 0.85$ and $p = 6$

		G-PRM						G-PRRM					
		$\phi = 0.33$ and $\alpha_0^2 = 0.85$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
	5	92.2	92.2	99.9	55.0	64.7	91.7	93.1	93.1	99.6	56.6	63.9	93.2
	10	83.2	83.6	97.3	79.3	79.3	83.6	84.5	84.3	97.3	79.6	79.4	84.1
25	15	74.3	75.4	94.2	70.5	69.4	75.7	72.8	73.4	94.2	68.1	66.5	73.2
	20	60.5	63.9	90.7	57.5	58.0	62.3	61.9	63.0	90.1	58.2	58.5	63.8
	25	48.9	54.0	84.8	44.8	47.2	53.4	48.0	55.2	85.8	45.4	47.3	54.0
	5	92.5	92.5	99.9	54.9	63.5	91.0	93.3	93.3	99.7	59.1	67.3	92.9
	10	83.6	83.9	97.2	78.6	78.5	83.5	86.0	85.8	97.3	80.8	80.9	85.9
50	15	72.5	72.7	93.5	68.8	66.6	73.0	76.0	76.3	96.1	71.3	70.5	76.3
	20	60.8	64.1	90.0	56.9	57.8	63.4	60.0	64.5	89.8	57.0	56.7	63.6
	25	46.4	54.1	84.3	43.2	46.6	53.6	48.5	55.0	84.0	45.6	48.1	53.7
	5	94.3	94.3	98.8	78.5	80.0	94.1	92.6	92.6	98.7	78.4	80.4	95.3
	10	91.8	91.8	98.9	89.7	88.4	91.7	92.6	92.5	99.3	90.6	89.6	92.7
100	15	86.8	87.0	97.7	83.9	83.0	86.4	90.0	90.0	96.8	87.4	87.0	90.1
	20	86.7	87.0	97.7	85.3	84.1	86.5	84.3	84.3	96.1	82.7	82.1	84.5
	25	81.5	81.5	96.2	79.3	78.5	81.9	83.7	83.7	95.9	82.1	81.4	83.8
	5	96.4	96.4	99.7	87.3	88.7	96.3	95.8	95.8	99.6	87.7	90.2	96.4
	10	93.3	93.3	98.9	91.6	91.4	94.1	92.8	92.9	98.5	90.9	91.5	93.4
200	15	91.6	91.5	98.0	89.5	89.2	91.0	90.0	89.9	98.3	88.9	89.1	90.8
	20	88.2	88.2	97.3	86.6	86.3	88.3	89.4	89.5	97.8	88.5	88.4	89.9
	25	86.8	87.1	97.1	85.8	85.9	86.9	88.4	88.2	98.0	87.0	86.7	88.6
	5	96.3	96.3	99.4	89.7	89.5	95.7	95.9	95.9	99.4	88.6	91.0	96.3
	10	94.6	94.5	98.8	92.7	92.8	94.7	94.4	94.4	98.7	93.0	92.2	94.5
300	15	93.4	93.2	98.9	91.7	91.7	93.2	93.3	93.3	99.1	92.4	92.1	94.5
	20	92.5	92.8	98.8	92.3	91.5	92.6	90.6	90.6	98.5	89.5	89.8	91.2
	25	91.2	91.4	97.9	90.1	89.5	91.2	89.3	89.3	98.1	87.9	87.7	88.9
	5	96.9	96.9	99.7	90.3	91.9	96.9	97.5	97.5	99.5	91.5	93.2	97.0
	10	93.2	93.1	99.4	91.6	90.6	92.9	94.8	94.8	99.4	92.8	93.1	94.7
400	15	93.7	93.8	99.0	91.8	92.0	93.6	94.4	94.5	98.6	93.4	92.8	94.1
	20	91.8	92.0	98.2	90.4	90.2	91.6	91.9	92.1	99.1	90.8	90.7	92.3
	25	90.9	90.9	97.7	89.7	88.8	90.7	90.1	90.1	98.0	89.2	89.5	89.9
	5	95.5	95.5	99.4	91.3	92.5	96.1	96.8	96.8	99.9	91.1	92.4	96.2
	10	95.5	95.3	99.3	93.8	94.0	95.3	94.1	94.2	98.8	92.8	92.7	93.9
500	15	93.8	93.8	98.3	92.4	92.2	93.1	92.8	93.0	98.7	91.3	91.5	92.7
	20	94.4	94.5	98.8	93.9	93.2	94.3	92.7	92.7	98.6	91.8	92.0	92.9
	25	93.7	93.7	98.4	93.0	92.9	93.5	92.4	92.4	98.3	91.5	91.7	92.3

these two statistics both with and without the ridge estimator when it comes to identifying the influential observations.

Similarly, the performance WD_{Ri} and H_{dRi} are the same with the lower dispersion. However, H_{dRi} performs better than WD_{Ri} for larger dispersion. However, because to multicollinearity these diagnostics function exactly the same in the G-PRM situation producing false positives. Additionally, we find that when multicollinearity increases influence diagnostics work better. The performance influence diagnostics are also impacted by the number of explanatory variables. The number of explanatory factors has a direct impact on how well Covariance ratio and Hadi's measure with ridge estimates perform. Similarly, as the number of explanatory variables increases, so does the performance of WD_{Ri} and H_{dRi} influence diagnostic approaches. Figure **??** displays the simulation results (for $n = 25, 50, 100, 200, 300, 400$ and 500) graphically. Because these residuals and diagnostic measures produce different results, we plot Covariance

Table 5: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 0.67$, $\alpha_0^2 = 0.90$ and $p = 8$

		G-PRM						G-PRRM					
		$\phi = 0.67$ and $\alpha_0^2 = 0.90$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
	5	93.2	93.2	99.7	58.2	64.2	93.0	92.6	92.6	99.5	58.1	61.9	91.9
	10	83.0	82.6	97.2	77.1	76.9	82.3	86.1	85.6	97.7	81.7	81.8	85.8
25	15	72.5	73.2	93.6	68.7	67.4	73.2	75.7	76.3	94.3	72.0	70.6	76.1
	20	62.9	66.7	90.6	59.3	58.6	65.7	63.9	66.1	90.9	59.9	59.3	65.6
	25	49.7	56.9	86.1	46.4	50.2	54.7	47.8	53.3	82.4	45.4	48.0	52.5
	5	95.4	95.5	99.4	68.5	74.8	94.6	94.8	94.8	99.4	71.5	75.2	94.1
	10	86.7	86.8	97.7	82.8	82.4	87.0	89.7	89.8	98.2	85.9	85.9	89.9
50	15	82.3	82.4	96.3	79.3	78.3	82.4	83.7	83.8	97.5	80.6	79.8	84.0
	20	76.5	77.2	95.1	74.7	72.4	76.9	79.5	80.2	94.6	76.6	76.1	80.1
	25	71.7	72.1	93.4	68.5	67.7	72.1	73.9	75.1	92.9	71.3	70.5	74.4
	5	93.4	93.4	99.7	68.2	72.5	92.5	93.8	93.8	99.5	65.3	74.3	94.6
	10	89.3	89.2	98.1	84.8	84.9	89.3	88.5	88.3	98.7	84.4	84.7	89.0
100	15	83.6	83.5	97.1	80.5	80.0	84.2	80.8	81.6	95.7	78.7	77.7	81.3
	20	78.3	79.4	95.2	76.0	75.3	78.7	75.9	76.3	94.1	73.7	72.4	75.9
	25	70.9	72.7	92.9	69.1	68.9	71.6	71.8	71.5	91.4	69.2	67.3	72.0
	5	96.7	96.6	99.7	89.6	91.2	96.7	96.4	96.4	99.6	87.8	87.8	96.9
	10	92.1	92.3	98.8	90.6	90.4	93.1	93.4	93.6	99.3	91.1	91.1	93.6
200	15	91.6	91.6	98.3	90.4	90.0	91.7	92.2	92.3	98.4	91.0	90.7	92.5
	20	89.9	90.1	98.0	87.7	87.8	89.5	89.3	89.6	97.4	87.5	87.7	88.6
	25	89.4	89.2	97.8	87.7	87.5	89.3	89.2	89.3	98.3	88.3	87.4	89.3
	5	96.3	96.3	99.8	91.1	91.2	96.2	95.6	95.6	99.4	89.8	92.1	95.9
	10	94.7	94.6	99.1	92.2	92.8	94.2	93.7	93.7	99.0	92.4	92.1	94.2
300	15	91.8	91.6	98.5	90.9	91.1	92.2	92.8	92.8	98.7	91.5	91.8	92.8
	20	90.4	90.2	97.9	89.1	88.9	90.8	89.3	89.5	97.9	87.2	87.2	88.7
	25	90.2	90.2	97.6	88.7	88.2	90.3	88.4	88.6	97.6	87.6	87.1	89.0
	5	96.1	96.1	99.7	88.7	91.0	93.6	95.7	95.7	99.8	86.7	90.1	95.4
	10	94.7	94.7	98.8	93.4	92.6	94.6	95.0	95.1	99.1	94.9	93.9	95.7
400	15	94.1	93.9	98.7	92.5	92.2	94.4	92.0	91.8	98.1	90.5	90.1	91.8
	20	91.1	91.1	97.8	90.3	90.1	91.3	92.5	92.5	98.4	91.7	90.8	92.0
	25	89.5	89.9	98.1	89.0	88.9	89.8	90.2	90.0	97.4	89.0	88.0	90.6
	5	95.5	95.5	99.8	86.9	88.4	94.8	95.9	95.9	99.7	90.8	89.5	95.4
	10	94.5	94.4	99.0	93.0	92.3	94.1	96.1	96.1	99.4	94.4	94.0	95.8
500	15	93.2	93.2	98.6	91.6	91.9	93.5	93.5	93.7	98.8	92.1	91.8	93.1
	20	91.8	91.8	98.3	90.3	90.6	91.2	91.4	91.5	98.6	90.5	90.2	91.9
	25	91.2	91.2	98.0	90.1	90.0	91.4	90.3	90.5	98.3	89.0	88.1	90.3

Ratio (CVR_i), Welch Distance (WD_i), Hadi's Measure (H_{di}), Covariance Ratio in Ridge (CVR_{Ri}), Welch Distance in Ridge (WD_{Ri}) and Hadi's Measure in Ridge (H_{dRi}) for a graphical comparison using the standardized and adjusted working residuals. For comparison, we consider one of the forms of all the G-PRM and G-PRRM. Furthermore, the influence analysis with different diagnostic measures in G-PRM and G-PRRM affected by the increase in sample size. It is noteworthy that the rate at which influential observations are identified is significantly influenced by sample size. As sample size increases, the percentage increases as well, and at very large sample sizes, all influential observations are discovered. The behavior of every diagnostic measure is essentially the same. Analyzing the results makes it clear that multicollinearity with regressor and ϕ (levels of dispersion) affect the detection rate. It can be seen that while Welch Distance in Ridge (WD_{Ri}) and Hadi's Measure in Ridge (H_{dRi}) perform better for small sample numbers, all diagnostic measures work almost equally well for big sample sizes.

Table 6: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 2$, $\alpha_0^2 = 0.95$ and $p = 10$

		G-PRM						G-PRRM					
		$\phi = 2$ and $\alpha_0^2 = 0.95$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
	5	91.5	91.5	99.2	60.2	65.9	91.6	93.0	93.0	99.8	55.9	63.6	92.5
	10	83.4	83.9	96.8	78.5	78.2	83.5	85.2	85.0	97.8	79.7	80.1	85.2
25	15	77.1	76.6	95.0	72.4	70.2	77.5	75.8	75.7	95.3	70.2	69.8	75.9
	20	62.0	65.9	90.8	58.0	58.7	65.0	63.9	66.3	90.3	59.4	59.5	66.3
	25	47.9	54.6	84.5	44.9	48.5	55.0	48.6	56.5	82.6	45.9	49.4	56.9
	5	94.0	94.0	99.8	69.3	74.2	93.7	94.6	94.6	100.0	70.7	73.5	94.5
	10	89.3	89.2	98.1	84.9	85.5	89.4	89.3	89.2	98.5	85.0	85.3	89.3
50	15	84.4	84.1	96.6	79.9	78.1	84.1	81.5	81.5	96.4	78.9	77.4	81.8
	20	78.6	79.5	94.3	75.5	75.3	79.1	79.8	79.9	95.8	76.3	75.3	79.9
	25	72.3	74.0	92.8	70.1	68.5	73.1	70.1	72.1	91.3	68.0	66.9	71.1
	5	93.5	93.5	99.7	80.0	82.4	95.2	94.7	94.7	99.7	80.1	81.1	91.1
	10	91.7	91.8	98.7	88.7	88.7	91.8	91.7	92.5	98.4	88.8	88.6	88.6
100	15	86.7	86.8	97.4	84.3	83.8	86.9	86.7	88.8	97.4	86.8	86.3	86.3
	20	84.0	84.2	95.6	81.5	80.9	83.5	84.0	83.6	95.6	81.9	81.9	81.9
	25	82.6	83.3	96.0	81.0	79.6	82.7	82.6	81.8	94.3	79.1	79.0	79.0
	5	95.6	95.6	99.2	86.2	87.2	95.1	96.0	96.0	99.9	85.9	89.1	95.7
	10	93.7	93.8	98.1	92.3	91.9	93.9	93.0	93.0	99.2	91.2	90.6	92.5
200	15	90.9	90.9	97.8	89.2	89.0	90.9	91.7	91.6	98.0	90.4	89.9	92.0
	20	88.7	89.0	97.7	87.8	87.2	89.3	90.8	90.3	98.2	89.1	89.3	91.2
	25	90.9	90.7	97.5	89.5	88.6	90.8	88.1	88.1	97.4	87.4	86.8	88.6
	5	96.6	96.6	99.6	89.9	89.9	96.3	96.3	96.3	99.3	90.0	91.4	96.5
	10	95.0	95.0	99.6	93.0	93.0	94.4	94.0	94.0	99.3	92.5	92.6	94.4
300	15	93.2	93.1	98.6	91.3	91.0	92.4	92.5	92.5	98.2	91.0	90.8	92.4
	20	91.3	91.5	97.7	90.0	89.8	91.3	92.2	92.4	98.1	91.7	91.2	92.7
	25	89.9	90.3	97.3	88.5	88.1	89.2	90.8	90.7	97.8	89.4	89.5	90.4
	5	96.9	96.8	99.8	92.0	92.1	96.6	94.4	94.4	99.2	90.0	90.4	95.4
	10	93.2	93.2	99.0	93.7	93.1	94.5	94.0	94.0	99.0	92.9	93.2	94.5
400	15	93.3	93.3	98.3	91.9	92.1	93.0	93.6	93.4	98.9	92.1	92.8	93.7
	20	92.8	93.0	98.9	91.9	90.8	92.7	91.3	91.4	98.2	90.6	89.8	91.4
	25	91.2	91.4	98.2	90.1	90.2	91.0	91.8	91.8	97.4	91.3	91.1	92.2
	5	96.5	96.6	99.7	92.3	91.8	97.2	96.2	96.2	99.6	92.3	92.8	96.1
	10	95.0	94.9	99.2	94.4	94.0	95.4	94.6	94.5	98.9	93.2	93.5	94.6
500	15	92.9	94.1	99.0	92.6	92.6	93.5	95.2	95.2	99.3	94.3	94.1	95.0
	20	93.4	93.3	98.8	93.0	92.3	93.6	93.5	93.5	98.7	92.7	93.2	94.0
	25	93.7	93.6	98.6	93.0	92.4	93.5	91.5	91.7	98.5	90.5	90.7	91.6

6.2 Application: Reaction rate data

The reaction rate data set is taken from Huet et al. (2004) and Saleem and Akbar (2024, 2025). The main objective of this data set is to determine the reaction rate of the catalytic isomerization of n-pentane to iso-pentane based on partial pressure of different independent variables (factors). These explanatory variables are used to speed up the reaction rate. This data set consists of $n = 24$ experimental data values with one dependent variable, i.e. reaction rate (Y) and $p = 3$ explanatory variables, i.e. partial pressure of hydrogen (X_1), partial pressure of n-pentane (X_2) and partial pressure of iso-pentane (X_3).

So, the appropriate regression model to determine the reaction rate ($\hat{\mu}$) based on these three explanatory variables such as $p = 3$ explanatory variables, i.e. partial pressure of hydrogen (x_1), partial pressure of n-pentane (x_2) and partial pressure of iso-pentane (x_3) is the G-PR model.

Table 7: Detection rate (%) of influential points (IP) using SWR and AWR in G-PRM and G-PRRM varying diagnostics measures, $\phi = 5$, $\alpha_0^2 = 0.99$ and $p = 12$

		G-PRM						G-PRRM					
		$\phi = 5$ and $\alpha_0^2 = 0.99$											
n	IP	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}	CVR_{SWR}	CVR_{AWR}	WD_{SWR}	WD_{AWR}	H_{SWR}	H_{AWR}
	5	93.9	93.9	99.8	56.5	65.0	94.3	93.6	93.6	99.8	56.0	63.7	93.0
	10	84.6	84.2	97.8	79.2	79.3	84.2	84.4	84.7	97.3	77.5	77.5	84.8
25	15	75.1	76.2	94.6	71.5	69.9	75.9	76.5	77.0	94.4	73.3	71.1	76.9
	20	63.1	66.8	89.9	59.8	59.3	65.6	61.9	65.0	88.9	58.3	57.7	63.9
	25	47.8	56.3	84.2	43.8	48.7	55.2	46.3	53.6	82.4	43.6	46.6	52.9
	5	94.2	94.2	99.4	68.6	75.7	93.5	95.0	95.1	99.7	69.8	73.3	94.7
	10	87.8	87.7	97.2	84.9	84.7	87.5	88.2	88.3	97.9	84.0	84.0	88.2
50	15	83.0	83.1	96.4	80.4	79.4	83.1	83.9	84.3	96.3	84.1	80.0	84.3
	20	78.1	78.7	95.3	76.4	74.8	78.2	79.1	79.4	95.5	76.6	75.5	79.3
	25	70.0	70.8	91.9	67.9	67.1	70.5	75.2	75.3	93.1	71.6	70.9	76.0
	5	93.7	93.7	99.8	78.4	80.5	94.2	94.3	94.3	99.9	79.4	82.3	94.1
	10	91.3	91.1	98.8	88.4	88.4	91.4	90.6	90.6	98.4	87.9	88.5	90.6
100	15	88.7	88.4	97.6	85.6	85.6	89.2	88.7	88.7	98.5	86.6	86.1	89.2
	20	83.4	83.4	96.5	83.1	82.4	83.3	87.2	87.1	96.2	85.1	84.1	87.3
	25	79.9	80.7	95.2	78.8	77.6	80.1	81.0	81.4	96.1	78.7	78.0	81.0
	5	95.4	95.4	99.5	85.9	88.1	95.3	95.6	95.6	99.5	86.2	87.7	95.5
	10	94.4	94.5	99.0	92.1	93.0	95.0	92.7	92.7	99.4	90.0	90.4	91.9
200	15	93.0	93.1	98.0	91.5	91.2	93.1	92.8	92.6	98.6	91.0	91.0	92.7
	20	89.8	89.7	97.3	88.1	88.1	89.6	89.8	89.6	97.7	87.8	88.1	89.7
	25	89.0	89.1	97.7	87.7	86.9	88.9	88.4	88.2	97.0	87.2	87.3	88.3
	5	96.0	96.0	99.7	89.1	90.4	95.5	95.4	95.4	99.6	89.3	89.7	96.0
	10	94.1	94.2	99.0	93.2	92.6	94.3	93.7	93.7	98.6	91.8	92.5	93.9
300	15	92.4	92.5	98.3	91.3	91.0	92.4	92.4	92.3	98.8	91.2	90.5	92.5
	20	91.3	91.2	98.6	90.7	91.0	91.9	91.3	91.0	97.5	90.1	90.6	91.6
	25	90.2	90.5	96.7	88.6	89.2	90.3	91.0	91.3	98.1	89.8	89.8	91.2
	5	96.4	96.4	99.8	92.3	91.7	96.0	95.8	95.8	99.4	90.8	91.6	96.0
	10	93.9	93.8	98.7	92.3	92.1	93.3	95.5	95.4	99.4	93.3	93.2	94.8
400	15	94.7	94.7	99.0	93.8	94.0	94.6	93.7	93.8	99.2	92.3	92.1	93.4
	20	93.0	93.0	98.9	92.3	92.2	93.6	92.6	92.6	98.6	91.5	91.8	92.3
	25	91.3	91.3	98.6	90.6	89.7	91.2	91.6	91.9	98.4	90.8	90.4	91.4
	5	96.1	96.1	99.0	92.5	92.4	96.1	96.9	96.9	99.4	94.1	94.5	97.2
	10	96.1	96.2	99.2	94.6	94.5	95.7	95.4	95.3	99.5	94.2	94.2	95.5
500	15	93.4	93.5	98.3	92.4	92.2	93.5	93.5	93.6	98.7	92.7	92.6	94.1
	20	92.4	92.3	98.5	91.2	91.2	92.5	94.9	94.8	98.9	94.3	94.1	95.0
	25	92.9	92.8	98.7	92.3	92.1	92.8	93.4	93.4	98.2	92.5	92.5	93.6

The fitted G-PRM for inverse link function using real data is given by.

$$\hat{\mu}_i = (0.12[0.092, S] + 0.35x_1[0.0007, S] - 0.08x_2[0.0002, S] + 0.25x_3[0.0002, N])^{-1} \tag{6.2}$$

where the square brackets contain the standard errors of the estimated parameters. The letter N represents the non-significance and S represents the significance of the regression coefficients. This data set is well fitted to the Gamma-Pareto distribution as tested by few tests of goodness of fit and the results are given in Table 8.

In Table 11, numerical evaluation of G-PRM and G-PRRM with and without influential observations is presented. For G-PRM, higher values of MSE and R^2 are observed when influential observations are found in the data. But the reduction in MSE and R^2 is observed when influential observations are removed from the data. On the other hand, when G-PRRM is applied without influential observations, small MSE and very large R^2 is observed which leads

Table 8: Distribution goodness of fit test for reaction rate data

Goodness of fit test		Probability Distribution						
		Gamma	Pareto	Gamma-Pareto	Weibull	Weibull-Pareto	Normal	Normal-Pareto
Anderson-Darling (AD)	Statistic	0.2519	3.1872	0.3299	0.2043	0.9912	1.2462	0.8055
	P-value	0.7538	0.5881	**0.8103**	0.6288	0.2033	0.0027	0.0390
Cramer-von Mises (CVM)	Statistic	0.0423	0.4570	0.2797	0.0521	0.2392	0.2127	0.1865
	P-value	0.8259	0.0067	**0.6922**	0.4772	0.3143	0.0033	0.0051
Pearson chi-square (PCS)	Statistic	2.0000	14.8800	24.774	6.0000	9.6522	10.667	17.995
	P-value	0.8491	0.0033	**0.9605**	0.3062	0.2071	0.0584	0.0949

Gamma-Pareto Distribution (GPD).

Table 9: Influence observations in the G-PRM and the G-PRRM

Procedure	G-PRM ($K = 0$)	Procedure	G-PRRM ($K = 3.74$)
CVR_{SWR}	6, 10	CVR_{SWR}	5, 14
CVR_{AWR}	6	CVR_{AWR}	8, 11
WD_{SWR}	5, 10, 21	WD_{SWR}	11, 21, 22
WD_{AWR}	6, 11	WD_{AWR}	5, 11, 12
H_{SWR}	11, 12, 21, 22	H_{SWR}	8, 10, 21, 22
H_{AWR}	6, 21, 22	H_{AWR}	5, 18, 22

Table 10: Absolute change (%) in the estimates after deleting the influential observation

Influential observations	G-PRM ($K = 0$)				G-PRRM ($K = 3.74$)			
	β_0	β_1	β_2	β_3	β_0	β_1	β_2	β_3
5	4.5579	9.4577	3.7947	8.9448	9.3745	7.8353	3.7847	7.3783
6	7.8245	3.7283	1.2379	1.8939	2.8958	8.9882	9.8287	9.3690
8	7.8478	8.9234	1.2988	1.2893	9.2388	1.3689	7.8478	8.2389
10	2.9889	3.4787	3.1003	1.2838	1.4888	1.8230	2.3048	1.2893
11	2.3478	8.2347	3.2898	1.2992	1.4984	1.4937	1.3497	7.4579
12	8.7878	1.9389	1.8318	1.9018	1.3089	8.8989	1.8838	1.9489
14	1.9982	2.9489	1.4884	1.0489	1.9484	4.8489	1.0488	1.9888
18	9.5989	1.8049	2.4009	2.8553	3.0859	2.8582	2.8979	2.5899
21	2.7899	2.0048	9.8589	39.891	49.842	1.8348	1.8284	1.3482
22	17.737	1.8374	1.1234	18.184	1.9834	24.824	1.8189	1.9410

Table 11: Influential observations' effect in G-PRM and G-PRRM

With influential observations (G-PRM)		Without influential observations (G-PRM)		With influential observations (G-PRRM)		Without influential observations (G-PRRM)	
MSE	R^2	MSE	R^2	MSE	R^2	MSE	R^2
20.342	0.780	5.236	0.575	14.244	0.501	6.736	0.825

to the supremacy of the G-PRRM over G-PRM. The data fits the Gamma-Pareto distribution quite well, according to Table 8. For the aforementioned data, the Gamma-Pareto regression influence diagnostics are therefore applied. In Table 10, absolute change (%) in the estimates

after deleting the influential observation is presented. In order to compare diagnostic measures with and without the ridge, we now determine the true impact of these diagnosed observations on the G-PRM and G-PRRM estimations. In Table 9, influential observations in the G-PRM and G-PRRM when $K = 0$ and $K = 3.74$ are presented.

7 Conclusion

The behavior and comparison of standardized and adjusted working residuals for the G-PRM and G-PRRM with various diagnostic measures is examined in this paper. A simulation study and a real data set are used to evaluate standardized and adjusted working residuals comparisons with different diagnostic measures and the results produced significant findings. First, different diagnostic measures for G-PRM and G-PRRM can be useful in deciding which residual to use in relation to the study's purpose. The class of Hadi's Measure in Ridge is the best option if the researcher's objective is to identify the influential observation. Large sample sizes increase the Hadi's Measure in Ridge (H_{AWR}). Both a simulation study and a real data analysis show that the dispersion parameter has an impact on the detection of influential observations.

Funding. No funding resource is available for this study.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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