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Optimal progressive censoring under non-identical lifetimes: A Weibull proportional hazards framework

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Abstract. Progressive censoring schemes have numerous applications in survival and reliability analyses. A key problem in this area is the determination of an optimal censoring plan. Existing literature addresses this problem under the assumption of independent and identically distributed lifetimes. However, no study has yet proposed an optimal plan for scenarios where lifetimes are independent but non-identically distributed. This paper proposes an approach to investigate the optimal progressive censoring plan under a proportional hazards model with a two-parameter Weibull baseline distribution for the item lifetimes. To this end, we evaluate the performance of the estimators of the parameters of interest under both the classical and Bayesian frameworks. The optimal censoring scheme is selected by minimizing the root mean squared errors of the estimators. The proposed method is illustrated using a real data set, and some concluding remarks are provided.

Keywords. Bayes estimation, Optimal censoring plan, Proportional hazard rate model, Survival function, Weibull distribution.

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1 Introduction

In medical and industrial experiments, researchers often face situations where the lifetimes of some test units cannot be fully observed. This incompleteness typically arises not from technical failure but from intentional censoring, which is employed to reduce either the duration or the financial cost of the experiment. Among the various censoring schemes available, Progressive Type-II censoring is one of the most widely adopted and flexible designs. In this scheme, unlike conventional censoring where all surviving units are removed only at the end, units may be removed at multiple intermediate time points during the experiment.

More precisely, the procedure begins by simultaneously placing n units on a lifetime test. At the moment of the first observed failure, a predetermined number R_1 of the surviving units are randomly selected and removed (censored) from the test. The process continues: at the time of the i-th observed failure ($1 \leq i \leq m \leq n$), a specified number R_i of the remaining surviving units are censored. The experiment concludes when the m-th failure is observed. Consequently, the total number of censored units equals $R_1 + R_2 + \cdots + R_m$, while exactly m units are observed to fail. The vector $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ is referred to as the progressive censoring plan, and m denotes the number of observed failures (i.e., the effective sample size). A notable special case arises when $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$. Under these conditions, no units are removed at the first $m - 1$ failures, and all remaining $n - m$ surviving units are removed only at the time of the m-th failure. In this situation, the progressive censoring plan reduces exactly to the conventional Type-II censoring plan, where only the first m failures are observed and the test terminates thereafter. For more details about the Progressive Type-II censoring one may refer to the key references Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014). The frequentist and Bayesian analyses of the parameters of lognormal distribution were studied by Singh et al. (2015) and Singh et al. (2018), respectively. Estimating the reliability characteristics of generalized Rayleigh distribution was done by Maiti and Kayal (2021). Dey et al. (2023) investigated both frequentist and Bayesian parametric estimation methods for the gamma distribution using progressively Type II censored data.

The problem of finding optimal progressive censoring plan has been studied by several authors based on different optimal criteria. Ng et al. (2004) obtained optimal progressive censoring plans for the Weibull distribution. Burkschat et al. (2006) discussed about the optimal schemes in progressive censoring. Balakrishnan et al. (2008) obtained optimal progressive censoring plan based on the Fisher information criterion. Asymptotically optimal progressive censoring plans based on Fisher information was studied by Cramer and Ensenbach (2011). Pradhan and Kundu (2013) investigated on inference and optimal censoring plans for progressively censored Birnbaum–Saunders distribution. Vishwanath and Kundu (2017) studied Bayesian inference and optimal censoring scheme under progressive censoring. Salemi et al. (2019) found A-optimal and D-optimal censoring plans in progressively Type-II right censored order statistics. Abo-Kasem et al. (2023) concentrated on optimal sampling and statistical inferences for Kumaraswamy distribution under progressive Type-II censoring schemes. Chan-

dra et al. (2024) considered optimal inference for the competing risks model when the latent failure times follow the two-parameter Kies distribution with a common shape parameter. Dhameliya et al. (2024) investigated compound optimal design for generalized exponential distribution under progressively censored data. Chandra et al. (2025) addressed Bayesian analysis and optimal life testing for new Pareto distribution under progressive censoring. Prakash et al. (2026) proposed parameter estimation and optimal plan under progressive censoring for new exponentiated transformed Weibull distribution.

All the mentioned works have been done for the case where the progressive censored samples are extracted from independent and identically distributed (IID) random variables. The basic distribution theory of progressively censored order statistics for the case of independent but non-identically distributed (INID) random variables has been developed by Balakrishnan and Cramer (2008). See, Fischer et al. (2008), Cramer and Lenz (2010) and Mao and Hu (2010) for the stochastic properties of these statistics. Rezapour et al. (2013) investigated some more properties of progressively Type-II censored order statistics in the INID case.

The primary contribution of this manuscript is the determination of optimal progressive censoring plans under the INID proportional hazards model. While the INID censoring scheme has been introduced before, its inferential aspects remain largely underdeveloped. For instance, Razmkhah and Simriz (2018) addressed the problem of point estimation as well as calculating the Fisher information under INID censoring, but did not consider the problem of optimal plan selection, either in the classical or Bayesian framework. Our work fills this gap by extending and adapting established optimality criteria originally developed for the IID setting to the INID framework. This extension is not straightforward, as it introduces significant complexities in both formulation and computation, particularly when deriving objective functions and implementing the associated algorithms.

In this paper, we propose an approach to obtain the optimal progressive censoring plan for the case of INID lifetimes. The practical importance of this paper lies in its ability to address real-world scenarios where traditional assumptions of identical lifetime distributions fail. Here are the key practical implications:

- **Applicability to heterogeneous populations:** In real life, items or subjects often have different risk profiles. For example, in medical studies, patients in a clinical trial may have different ages, genetic markers, or baseline health conditions, leading to non-identical survival times; Components in industrial reliability from different production batches, manufacturers, or operating environments (e.g., varying stress levels) will not have identical lifetime distributions; Engineering systems with mixed components (e.g., different brands or vintages) exhibit heterogeneous failure times.

- **More efficient and cost-effective testing:** Progressive censoring allows experimenters to remove surviving units at intermediate stages, saving time and resources. By optimizing the censoring plan for non-identical data, the paper enables: 1- Reduced test duration without sacrificing estimation accuracy, 2- Lower

costs by strategically deciding when to withdraw units, and 3- Better resource allocation in industries like automotive, aerospace, or electronics where lifetime testing is expensive.

- **Improved estimation accuracy:** By minimizing the root mean squared errors (RMSE) of estimators, the optimal plan yields more precise estimates of key parameters.

Proportional hazard rate model (PHRM) with baseline two-parameter Weibull distribution is considered as a widely used model for the heterogeneous populations. The PHRM has been studied by many authors in different fields. Razmkhah and Ahmadi (2011) derived nonparametric prediction intervals for future order statistics in a PHRM. Demiray and Kızılaslan (2022) studied the estimation problem of stress-strength parameter in a consecutive k-out-of-n system based on a PHRM. So, it is worthwhile to model the INID lifetimes on the basis of a PHRM. The efficiency of the estimators for the parameters of this model is considered as a criterion to obtain the optimal censoring plan. It will be shown that the optimal plan depends on different factors such as the parameter of interest and its estimator, the size of units in the test and the effective sample size, and the proportionality rate of the PHRM; the last factor clarifies that ignoring the nonidentical distribution of the lifetimes does not lead to the optimal plan.

This paper addresses the problem of determining optimal progressive censoring plans based on the performance of point estimators for the baseline two-parameter Weibull distribution within a PHRM, under both classical and Bayesian approaches. Identifying the optimal progressive censoring plan offers several concrete benefits, particularly in contexts where reliability, cost control, and data-driven decision-making are critical. By specifying exactly when to remove units from a life test, test durations can be shortened without sacrificing statistical precision. Moreover, fewer units need to be tested to failure, leading to savings in materials, labor, and equipment usage.

The rest of this paper is as follows. In Section 2, the proposed model is described. The problem of finding the maximum likelihood and the Bayes estimators are studied in Section 3. The Metropolis-Hastings sampling is used to compute the Bayes estimators. A simulation study is presented in Section 4, where the sensitivity of the optimal progressive censoring plan to various model parameters is discussed. A real data set is used in Section 5 to illustrate the proposed procedure. Conclusions are stated in Section 6.

2 Model description and assumptions

Let $T_1, \ldots, T_n$ denote the lifetimes of n units placed simultaneously and independently on a lifetime test. Let us denote the probability density function (pdf), cumulative distribution function (cdf), survival function, and hazard function of T_i by $f_i(t;\boldsymbol{\theta})$, $F_i(t;\boldsymbol{\theta})$, $S_i(t;\boldsymbol{\theta})$, and $h_i(t;\boldsymbol{\theta})$, respectively, all characterized by the same parameter vector $\boldsymbol{\theta}$. For $1 \le i \le n$, assume that

$$S_i(t;\boldsymbol{\theta}) = [S(t;\boldsymbol{\theta})]^{\lambda_i}, \tag{2.1}$$

where λ_i is a known positive constant representing the proportionality rate, and $S(\cdot;\boldsymbol{\theta})$ denotes the survival function of a baseline distribution which is the same for all units. The proportionality rates $\lambda_1,\ldots,\lambda_n$ may be interpreted as the relative alteration rates of the distributions with respect to one another. In this setting, without loss of generality, we may order the distributions according to their proportionality rates and, upon dividing by the smallest rate, obtain $1=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$. The distribution corresponding to the smallest proportionality rate is then regarded as the baseline distribution (Razmkhah and Ahmadi, 2011). The relationship in (2.1) is widely recognized in the literature as the proportional hazard rate model (PHRM). It is equivalent to

$$h_i(t;\boldsymbol{\theta})=\lambda_i h(t;\boldsymbol{\theta}),$$

where $h(t;\boldsymbol{\theta})=\frac{f(t;\boldsymbol{\theta})}{S(t;\boldsymbol{\theta})}$ is the hazard rate function of the baseline distribution, and $h_i(t;\boldsymbol{\theta})=\frac{f_i(t;\boldsymbol{\theta})}{S_i(t;\boldsymbol{\theta})}$ stands for the hazard rate of T_i.

In this paper, we consider a Weibull baseline distribution with the survival function

$$S(t;\alpha,\beta)=\exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\},\quad t>0, \tag{2.2}$$

where α and β are the scale and the shape parameters, respectively. The Weibull distribution is one of the most widely used distributions in reliability and life data analysis, owing to its versatility across various fields of engineering and biology. For a complete treatment of the distribution's theory and properties, one may refer to Rinne (2009). Further, Almalki and Nadarajah (2014) provided a broad overview of modifications and generalizations on the Weibull distribution.

Let $\mathbf{R}=(R_1,\ldots,R_m)$ denote the progressive censoring plan, where R_i is a nonnegative integer representing the number of surviving units removed at the time of the i-th observed failure. The total number of units initially placed on test, denoted by n, is then given by $n=m+\sum_{i=1}^{m}R_i$, where m is the effective sample size. Furthermore, let $T^{\mathbf{R}}_{1:m:n}, T^{\mathbf{R}}_{2:m:n},\ldots,T^{\mathbf{R}}_{m:m:n}$ represent the progressively Type-II censored order statistics obtained under this plan. In contrast to the classical progressive censoring literature, which typically assumes IID lifetimes, the order statistics considered here are based on INID random variables.

The main goal of the paper is to determine the optimal progressive censoring plan $\mathbf{R}$ based on the performance of various estimators of the parameters of the baseline distribution: α, β and survival function $S(t;\alpha,\beta)$. To this end, the root mean squared error (RMSE) is employed as a criterion for comparing the performance of the estimators under different progressive censoring plans.

For a given progressive censoring plan $\mathbf{R}$ and vector of proportionality rates $\boldsymbol{\lambda}$, let us denote by $\hat{\phi}^{\mathbf{R},\boldsymbol{\lambda}}$, the estimator of any parameter ϕ. Then, the RMSE is defined as

$$\text{RMSE}=\sqrt{E\left(\left(\hat{\phi}^{\mathbf{R},\boldsymbol{\lambda}}-\phi\right)^2\right)}.$$

For given vector of proportionality rates $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$, the progressive censoring plan with the smallest RMSE is defined as the optimal $\mathbf{R}$. Clearly, the optimal plan

depends on both the parameter of interest and the type of estimator employed, namely, the maximum likelihood and Bayes estimators, which are derived in the next section.

3 Estimation based on INID progressive censored samples

Here, we first derive the likelihood function of the parameter vector $\boldsymbol{\theta} = (\alpha, \beta)$ based on the INID progressively Type-II censored order statistics. Since the units are not identically distributed, we need some indicator functions to identify which units are failed and which ones are removed during the censoring plan. Assume $\Delta_i^{(j)} = 1$ if the lifetime of the ith unit coincides with the jth failure time, while $\Delta_i^{(j)} = 0$ otherwise. Moreover, assume $H_i^{(j)} = 1$ if the ith unit is removed from the test after the jth failure time, and $H_i^{(j)} = 0$ otherwise. According to the proposed censoring plan, for each $j = 1, \ldots, m$, we get $\sum_{i=1}^{n} \Delta_i^{(j)} = 1$ and $\sum_{i=1}^{n} H_i^{(j)} = R_j$. Denoting the pdf and the survival function of T_i by $f_i(\cdot; \boldsymbol{\theta})$ and $S_i(\cdot; \boldsymbol{\theta})$, respectively, Razmkhah and Simriz (2018) showed that the likelihood function of $\boldsymbol{\theta}$ is

$$L(\boldsymbol{\theta}) = \Big(\prod_{j=1}^{m} \gamma_j\Big) \prod_{j=1}^{m} \Big\{ \prod_{i=1}^{n} [f_i(t_j; \boldsymbol{\theta})]^{\delta_i^{(j)}} [S_i(t_j; \boldsymbol{\theta})]^{\eta_i^{(j)}} \Big\}, \tag{3.1}$$

where t_j, $\delta_i^{(j)}$ and $\eta_i^{(j)}$ are the observed values of $T^{\mathbf{R}}_{j:m:n}$, $\Delta_i^{(j)}$ and $H_i^{(j)}$, respectively. Moreover, for $1 \le j \le m$, $\gamma_j = \sum_{i=j}^{m}(R_i + 1)$ stands for the number of units remaining in the experiment before the jth failure time, with convention $\gamma_{m+1} = 0$. Note that the coefficient γ_j does not depend on $\boldsymbol{\theta}$, and it is therefore omitted when making inference about $\boldsymbol{\theta}$.

3.1 Maximum likelihood estimation

Using Equation (3.1) and under the PHRM (2.1) with the baseline survival function (2.2), the log-likelihood function of the vector of $\boldsymbol{\theta} = (\alpha, \beta)$ based on the censored data $T^{\mathbf{R}}_{1:m:n}, \ldots, T^{\mathbf{R}}_{m:m:n}$ is derived as

$$\begin{aligned} \ell(\boldsymbol{\theta}) \;=\; & \sum_{j=1}^{m} \log \gamma_j + m \log\Big(\frac{\beta}{\alpha}\Big) + \sum_{j=1}^{m}\sum_{i=1}^{n} \delta_i^{(j)} \log \lambda_i \\ & + (\beta - 1) \sum_{j=1}^{m} \log\Big(\frac{t_j}{\alpha}\Big) - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i \Big(\frac{t_j}{\alpha}\Big)^{\beta} \Big(\delta_i^{(j)} + \eta_i^{(j)}\Big), \end{aligned} \tag{3.2}$$

where, for $1 \le j \le m$, t_j is the observed value of $T^{\mathbf{R}}_{j:m:n}$. Therefore, the maximum likelihood estimators (MLEs) of α and β, denoted by $\hat{\alpha}_M$ and $\hat{\beta}_M$, respectively, can be derived by solving the equations

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} = \frac{\beta}{\alpha}\Big(- m + \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i \big(\frac{t_j}{\alpha}\big)^{\beta} \Big(\delta_i^{(j)} + \eta_i^{(j)}\Big)\Big),$$

and

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} = \frac{m}{\beta} + \sum_{j=1}^{m} \log(\frac{t_j}{\alpha}) - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i (\frac{t_j}{\alpha})^{\beta} \log(\frac{t_j}{\alpha}) \Big(\delta_i^{(j)} + \eta_i^{(j)}\Big).$$

simultaneously. By doing some algebraic calculations, it can be shown that the MLE $\hat{\beta}_M$ satisfies

$$\frac{\sum_{k=1}^{m}\sum_{i=1}^{n} \lambda_i \Big(\Delta_i^{(k)} + H_i^{(k)}\Big) \Big(T^{\mathbf{R}}_{k:m:n}\Big)^{\hat{\beta}_M} \log\Big(T^{\mathbf{R}}_{k:m:n}\Big)}{\sum_{k=1}^{m}\sum_{i=1}^{n} \lambda_i \Big(\Delta_i^{(k)} + H_i^{(k)}\Big) \Big(T^{\mathbf{R}}_{k:m:n}\Big)^{\hat{\beta}_M}} = \frac{1}{\hat{\beta}_M} + \frac{1}{m}\sum_{k=1}^{m} \log\Big(T^{\mathbf{R}}_{k:m:n}\Big). \tag{3.3}$$

The MLE of α is then derived as

$$\hat{\alpha}_M = \left\{ \frac{1}{m} \sum_{k=1}^{m}\sum_{i=1}^{n} \lambda_i \Big(\Delta_i^{(k)} + H_i^{(k)}\Big) \Big(T^{\mathbf{R}}_{k:m:n}\Big)^{\hat{\beta}_M} \right\}^{\frac{1}{\hat{\beta}_M}}. \tag{3.4}$$

Since the MLEs of α and β don't have closed-forms, they may be obtained through numerical methods. The MLE of β is obtained numerically by solving the nonlinear equation (3.3) using the R routine *optimize*(). The MLE of α is then obtained by substituting the minimizer into (3.4).

3.2 Bayes estimation

In order to obtain the Bayes estimators, the squared error (SE) and the LINEX loss functions are used. Generally, the SE loss function for estimating any parameter θ is given by

$$L(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2,$$

where $\hat{\theta}$ is the estimate of θ. When both over- and under-estimation are equally consequential, the SE loss function is appropriate due to its symmetry. In many practical settings, however, one type of error may be more serious than the other. In such cases, the LINEX loss function is suitable, as it increases exponentially on one side of the true parameter and linearly on the other Soliman (2000). This loss function is given by

$$L(\theta, \hat{\theta}) = \exp(k(\hat{\theta} - \theta)) - k(\hat{\theta} - \theta) - 1,$$

where k is a tuning parameter such that for $k > 0$ ($k < 0$), the over-estimation (under-estimation) is penalized more heavily, whereas for the values of k close to zero, the LINEX loss function is approximately symmetric. This loss function was introduced by Varian (1975) and has been widely used in the literature.

Suppose the bivariate prior density function for α and β is $\pi(\alpha, \beta)$. By using the Bayes theorem, the posterior density function of $\boldsymbol{\theta} = (\alpha, \beta)$ is given by

$$\pi(\boldsymbol{\theta}|\boldsymbol{data}) = \frac{L(\alpha, \beta)\pi(\alpha, \beta)}{\int_0^{\infty}\int_0^{\infty} L(\alpha, \beta)\pi(\alpha, \beta) d\alpha d\beta}, \tag{3.5}$$

where $L(\boldsymbol{\theta}) \equiv L(\alpha, \beta)$ is as defined in (3.1) and

$$\boldsymbol{data} = \{t_j, \delta_i^{(j)}, \eta_i^{(j)},\ \ 1 \le j \le m,\ \ 1 \le i \le n\}$$

stands for the observed value of the INID progressively Type-II censored data. Using the posterior function, the Bayes estimates can be obtained for the common parameters α and β, or any function of them, say $\phi \equiv \phi(\alpha, \beta)$. It is well-known that under the SE loss function, the Bayes estimator of ϕ is given by

$$\begin{aligned} \hat{\phi}_S &= E\left(\phi(\alpha,\beta)|\boldsymbol{data}\right) \\ &= \int_0^\infty \int_0^\infty \phi(\alpha,\beta)\pi(\alpha,\beta|\boldsymbol{data})d\alpha d\beta. \end{aligned} \tag{3.6}$$

Moreover, under the LINEX loss function, the Bayes estimate of $\phi(\alpha, \beta)$ is obtained as

$$\begin{aligned} \hat{\phi}_L &= -\frac{1}{k}\log E\left(e^{-k\phi(\alpha,\beta)}|\boldsymbol{data}\right) \\ &= -\frac{1}{k}\log\left\{\int_0^\infty \int_0^\infty e^{-k\phi(\alpha,\beta)}\pi(\alpha,\beta|\boldsymbol{data})d\alpha d\beta\right\}. \end{aligned} \tag{3.7}$$

Since the normalizing constant in (3.5) is often intractable, the Bayes estimators under both SE and LINEX loss functions (denoted by BSE and BLINEX, respectively) are involved complicated calculations. It is sometimes possible to sample from the posterior distribution even without knowing the normalizing constant in the denominator of Equation (3.5). Toward this end, the algorithm of Metropolis-Hastings iterations is implemented in the sequel. For more details one may refer to Metropolis et al. (1953) and Hastings (1970).

Note that the parameters α and β in our model are strictly positive. Hence, to improve the efficiency of proposal distributions in the Metropolis-Hastings algorithm, we employ parameterization to unbounded domains, which simplifies the choice of proposals (e.g., Gaussian) by avoiding boundary constraints. After performing computations in the convenient parameterization, the obtained samples are transformed back to the original parameter space. Precisely, we set $\nu_1 = \log(\alpha)$ and $\nu_2 = \log(\beta)$, and assume that ν_1 and ν_2 are independent random variables with identical normal distribution with mean zero and standard deviation σ, denoted by $N(0, \sigma)$. That is, the joint prior density function of ν_1 and ν_2 is $\pi(\nu_1, \nu_2) = \pi_{0,\sigma}(\nu_1)\pi_{0,\sigma}(\nu_2)$, where $\pi_{0,\sigma}(\cdot)$ stands for the pdf of $N(0, \sigma)$ distribution. Thus, the posterior density function is given by

$$\pi(\nu_1, \nu_2|\boldsymbol{data}) \propto f(\boldsymbol{data}|e^{\nu_1}, e^{\nu_2})\pi(\nu_1, \nu_2).$$

The steps of the Metropolis-Hastings algorithm for B iterations are summarized as follows:

Step 1. Choose the initial status of the chain as $\nu_1^{(0)} = \log\hat{\alpha}_M$ and $\nu_2^{(0)} = \log\hat{\beta}_M$, where $\hat{\alpha}_M$ and $\hat{\beta}_M$ are the MLEs of α and β, respectively.

Step 2. For $i = 1, \cdots, B$:

(i) Sample (ν_1^*, ν_2^*) from proposal distribution $q(\nu_1^*, \nu_2^* | \nu_1^{(i-1)}, \nu_2^{(i-1)})$. Here, we use independent $N(\nu_1^{(i-1)}, 0.25)$ and $N(\nu_2^{(i-1)}, 0.25)$ as proposal distributions for ν_1 and ν_2, respectively. The choice of standard deviation 0.25 ensures that the proposals are informative. Denoting the corresponding pdfs by $\pi_{\nu_1^{(i-1)},\sigma}(\cdot)$ and $\pi_{\nu_2^{(i-1)},\sigma}(\cdot)$, respectively, we have

$$q(\nu_1^*, \nu_2^* | \nu_1^{(i-1)}, \nu_2^{(i-1)}) = \pi_{\nu_1^{(i-1)},\sigma}(\nu_1^*)\pi_{\nu_2^{(i-1)},\sigma}(\nu_2^*).$$

(ii) Compute the acceptance probability

$$\omega = \min\left\{1, \frac{\pi(\nu_1^*, \nu_2^* | \boldsymbol{data}) q(\nu_1^{(i-1)}, \nu_2^{(i-1)} | \nu_1^*, \nu_2^*)}{\pi(\nu_1^{(i-1)}, \nu_2^{(i-1)} | \boldsymbol{data}) q(\nu_1^*, \nu_2^* | \nu_1^{(i-1)}, \nu_2^{(i-1)})}\right\}.$$

Since symmetric proposal distributions are chosen, the acceptance probability simplifies as

$$\omega = \min\left\{1, \frac{\pi(\nu_1^*, \nu_2^* | \boldsymbol{data})}{\pi(\nu_1^{(i-1)}, \nu_2^{(i-1)} | \boldsymbol{data})}\right\}.$$

(iii) Generate a random observation u from the uniform distribution on interval (0,1). If $u < \omega$, update the status of the chain from $(\nu_1^{(i-1)}, \nu_2^{(i-1)})$ to (ν_1^*, ν_2^*), and set $(\nu_1^{(i)}, \nu_2^{(i)}) = (\nu_1^*, \nu_2^*)$. Otherwise, set $(\nu_1^{(i)}, \nu_2^{(i)}) = (\nu_1^{(i-1)}, \nu_2^{(i-1)})$.

Step 3. Denoting the number of burn-in samples by M, the generated samples of the Markov chain Monte Carlo (MCMC) for ν_1 and ν_2 are obtained as $\{\nu_1^{(M+1)}, \cdots, \nu_1^{(B)}\}$ and $\{\nu_2^{(M+1)}, \cdots, \nu_2^{(B)}\}$, respectively.

Step 4. By transforming back, the generated samples of α and β are given by

$$\{\alpha^{(M+1)}, \ldots, \alpha^{(B)}\} = \{\exp(\nu_1^{(M+1)}), \cdots, \exp(\nu_1^{(B)})\} \tag{3.8}$$

and

$$\{\beta^{(M+1)}, \ldots, \beta^{(B)}\} = \{\exp(\nu_2^{(M+1)}), \cdots, \exp(\nu_2^{(B)})\}, \tag{3.9}$$

respectively.

Using (3.8) and (3.9), the approximated values of Bayes estimators of ϕ under the SE and the LINEX loss functions are given by

$$\hat{\phi}_S \approx \frac{1}{B-M} \sum_{i=M+1}^{B} \phi(\alpha^{(i)}, \beta^{(i)}) \tag{3.10}$$

and

$$\hat{\phi}_L \approx -\frac{1}{k}\log\left(\frac{1}{B-M}\sum_{i=M+1}^{B}\exp\left\{-k\phi(\alpha^{(i)},\beta^{(i)})\right\}\right), \tag{3.11}$$

respectively. In the next sections, we focus on three special cases of $\phi = \alpha$, $\phi = \beta$ and $\phi = S(t;\alpha,\beta)$.

4 Simulation study

In this section, we conduct an MCMC simulation study to analyze the sensitivity of the optimal plan to various model parameters. To demonstrate the effect of heterogeneous distributions of the units in the test, we assume that the survival function of the lifetime of the ith unit obeys the PHRM in (2.1) with proportionality rate $\lambda_i = (\lambda_0)^{i-1}$ for $i = 1,\ldots,n$, where λ_0 may be interpreted as a baseline proportionality rate. Moreover, the values $\lambda_0 = 1, 1.1, 1.3$ are taken into account. The case $\lambda_0 = 1$ corresponds to the IID setting; however, for $\lambda_0 > 1$, the lifetime of the ith unit is stochastically smaller than that of the $(i+1)$th unit. On the other hand, larger values of λ_0 result in greater heterogeneity in the distributions.

To study the effect of varying the prior hyperparameters, we consider two distinct prior distributions for ν_1 and ν_2: $N(0, 0.25)$ and $N(0, 5)$. Both priors center the parameters α and β around 1, which serves as a natural baseline. However, the prior $N(0, 0.25)$ assigns low probability to extreme values; for example, less than 1% probability that $\alpha > e^{0.59} \approx 1.80$ or $\beta > 1.80$. This distribution is therefore informative. In contrast, the alternative prior $N(0, 5)$ is more diffuse, assigning substantial probability to extreme values; for instance, more than 20% probability that $\alpha > e^{4.2} \approx 66.68$ or $\beta > 66.68$. Thus, this alternative represents a noninformative prior. Furthermore, to obtain the Bayes estimators, we set the number of Metropolis-Hastings iterations and the number of burn-in samples to $B = 5\times 10^4$ and $M = 5000$, respectively.

For $n = 30$ and $m = 15, 23$, four different censoring plans are considered to cover the cases that the survival units are removed from the test at the initial, middle or last failure times, or even when the removals are uniformly distributed throughout the failure process. For simplicity in notation, the letters I, M, L and U are used to denote the initial, middle, last and uniform removal patterns (RP), respectively, which are presented in Table 1. In this table, the notation $\{a\}^b$ indicates b repetitions of a. For example, $(\{0\}^4) = (0, 0, 0, 0)$. Note that the effective sample size m is chosen to include either 50% or approximately 75% of the total sample size n.

For each censoring plan in Table 1, a simulation with 1000 repetitions is performed, and the RMSE values of the MLE and the Bayes estimators (under the SE and LINEX loss functions) for the parameters α, β and $S(1;\alpha,\beta)$ are presented in Tables 2, 3 and 4, respectively.

From Tables 2-4, it is observed that:

Table 1: Different censoring plans for $n = 30$ and $m = 15, 23$.

m	Removal part (RP)	Censoring plan (**R**)
15	I	$(15, \{0\}^{14})$
	M	$(\{0\}^{7}, 15, \{0\}^{7})$
	L	$(\{0\}^{14}, 15)$
	U	$(\{1\}^{15})$
23	I	$(7, \{0\}^{22})$
	M	$(\{0\}^{11}, 7, \{0\}^{11})$
	L	$(\{0\}^{22}, 7)$
	U	$(1, 0, 0, \{1, 0, 0, 0\}^{4}, 1, 0, 0, 1)$

Table 2: Values of RMSE of $\hat{\alpha}$ for $n = 30$ and $m = 15, 23$.

				Bayes with $\sigma = 0.25$: Informative prior					Bayes with $\sigma = 5$: Noninformative prior				
					BLINEX					BLINEX			
					k					k			
m	λ_0	RP	MLE	BSE	-1	-0.5	0.5	1	BSE	-1	-0.5	0.5	1
15	1	I	0.2563	0.4219	0.3998	0.4110	0.4327	0.4432	0.2714	0.2917	0.2782	0.2672	0.2550
		M	0.2569	0.3265	0.3069	0.3168	0.3361	0.3455	0.2750	0.5693	0.2998	0.2649	0.2584
		L	0.2876	0.2902	0.2702	0.2803	0.3000	0.3098	0.3475	3.4321	2.1590	0.3145	0.2946
		U	0.2585	0.3209	0.3005	0.3108	0.3309	0.3406	0.2777	0.9644	0.5511	0.2645	0.2563
	1.1	I	0.3112	0.2619	0.2468	0.2543	0.2695	0.2771	0.3752	1.8504	0.9187	0.3421	0.3224
		M	0.3760	0.2638	0.2481	0.2559	0.2719	0.2800	0.5058	7.1903	5.1596	0.4275	0.3895
		L	0.5754	0.2584	0.2391	0.2485	0.2687	0.2749	0.9627	17.4197	50.462	0.6323	0.5307
		U	0.4336	0.2632	0.2460	0.2545	0.2721	0.2810	0.5487	10.0703	7.2319	0.4447	0.4007
	1.3	I	0.4479	0.3715	0.3506	0.3612	0.3815	0.3913	0.5533	4.8022	3.2402	0.4701	0.4301
		M	0.5038	0.4090	0.3837	0.3966	0.4209	0.4324	0.6970	12.5350	10.1082	0.5471	0.4927
		L	1.1868	0.5382	0.4976	0.5185	0.5567	0.5742	2.4247	56.4879	53.9040	0.9743	0.7598
		U	0.6688	0.4800	0.4488	0.4648	0.4944	0.5082	1.0036	33.9743	31.3105	0.6607	0.5760
23	1	I	0.2162	0.3551	0.3374	0.3463	0.3638	0.3725	0.2113	0.2189	0.2145	0.2090	0.2077
		M	0.2152	0.3126	0.2964	0.3045	0.3206	0.3286	0.2152	0.2248	0.2194	0.2122	0.2101
		L	0.2033	0.2916	0.2749	0.2832	0.2999	0.3081	0.2091	0.2241	0.2156	0.2041	0.2003
		U	0.2118	0.3137	0.2968	0.3052	0.3221	0.3304	0.2014	0.2118	0.2059	0.1982	0.1959
	1.1	I	0.2444	0.2016	0.1930	0.1971	0.2062	0.2111	0.2694	0.3839	0.2910	0.2553	0.2449
		M	0.2687	0.2086	0.1997	0.2040	0.2135	0.2186	0.2963	0.4101	0.3226	0.2797	0.2681
		L	0.3196	0.1969	0.1899	0.1930	0.2014	0.2063	0.3805	1.3719	0.5704	0.3420	0.3177
		U	0.2806	0.2163	0.2064	0.2112	0.2216	0.2271	0.3160	0.7590	0.3920	0.2937	0.2790
	1.3	I	0.3509	0.2994	0.2848	0.2921	0.3067	0.3140	0.3953	0.7118	0.4594	0.3650	0.3451
		M	0.3628	0.3112	0.2940	0.3026	0.3197	0.3281	0.4274	1.0949	0.5563	0.3908	0.3696
		L	0.5013	0.3760	0.3514	0.3639	0.3879	0.3993	0.6319	4.5293	2.9434	0.5283	0.4777
		U	0.4257	0.3529	0.3341	0.3435	0.3620	0.3710	0.5033	1.6949	0.8277	0.4530	0.4248

- The optimal progressive censoring plan, defined as the one with the smallest RMSE, depends on several factors, including the parameter of interest, the proportionality rates, the effective sample size m, the type of estimator (frequentist or Bayesian), and the hyperparameters of the prior distribution. Notably, for fixed hyperparameters of the prior distribution, the optimal plan remains the same across all Bayes estimators considered, including those under the SE and LINEX loss functions with various values of tuning parameter k. The resulting optimal plans for the different configurations are reported in Table 5.

- For $\lambda_0 = 1.3$, which corresponds to a high degree of heterogeneity in the PHRM distributions, the censoring plan $RP = I$ is always optimal. In other words, when the PHRM distributions are strongly heterogeneous, removing the surviving units at the first failure time is optimal.

- Depending on the censoring plan and the parameter of interest, the RMSE of the

Table 3: Values of RMSE of $\hat{\beta}$ for $n = 30$ and $m = 15, 23$.

				Bayes with $\sigma = 0.25$: Informative prior					Bayes with $\sigma = 5$: Noninformative prior				
					BLINEX					BLINEX			
					k					k			
m	λ_0	RP	MLE	BSE	-1	-0.5	0.5	1	BSE	-1	-0.5	0.5	1
15	1	I	0.4311	0.5235	0.4971	0.5104	0.5364	0.5490	0.4115	0.4647	0.4354	0.3927	0.3788
		M	0.5080	0.4635	0.4339	0.4487	0.4781	0.4927	0.4391	0.5005	0.4677	0.4149	0.3953
		L	0.6474	0.5231	0.4864	0.5050	0.5407	0.5578	0.5766	0.7233	0.6413	0.5270	0.4909
		U	0.5501	0.5007	0.4685	0.4847	0.5164	0.5317	0.4932	0.5799	0.5329	0.4609	0.4358
	1.1	I	0.3792	0.2704	0.258	0.2639	0.2773	0.2846	0.3850	0.4277	0.4048	0.3680	0.3541
		M	0.4116	0.2034	0.2020	0.2021	0.2059	0.2093	0.3796	0.4167	0.3969	0.3650	0.3531
		L	0.6116	0.1812	0.1841	0.1814	0.1832	0.1870	0.5200	0.6390	0.5730	0.4788	0.4485
		U	0.4978	0.1980	0.1967	0.1965	0.2010	0.2053	0.4429	0.5107	0.4743	0.4166	0.3953
	1.3	I	0.2286	0.1618	0.1693	0.1654	0.1584	0.1552	0.2159	0.2238	0.2196	0.2128	0.2103
		M	0.2314	0.1740	0.1827	0.1783	0.1699	0.1659	0.2254	0.2354	0.2301	0.2212	0.2175
		L	0.4696	0.2366	0.2534	0.2449	0.2286	0.2208	0.3943	0.4434	0.4170	0.3752	0.3597
		U	0.3152	0.2086	0.2215	0.2150	0.2024	0.1964	0.2933	0.3158	0.3040	0.2838	0.2754
23	1	I	0.3840	0.4727	0.4489	0.4609	0.4844	0.4959	0.3369	0.3696	0.3519	0.3247	0.3151
		M	0.3862	0.4567	0.4320	0.4443	0.4690	0.4811	0.3605	0.3945	0.3762	0.3472	0.3366
		L	0.4539	0.4845	0.4558	0.4702	0.4985	0.5123	0.4074	0.4589	0.4309	0.3885	0.3739
		U	0.3887	0.4842	0.4586	0.4714	0.4967	0.5091	0.3538	0.3908	0.3707	0.3400	0.3294
	1.1	I	0.3133	0.2281	0.2195	0.2235	0.2332	0.2387	0.2872	0.3072	0.2964	0.2795	0.2734
		M	0.3232	0.2062	0.2014	0.2034	0.2097	0.2138	0.3047	0.3280	0.3156	0.2952	0.2872
		L	0.3865	0.2162	0.2103	0.2127	0.2207	0.2259	0.3648	0.4022	0.3820	0.3507	0.3396
		U	0.3540	0.2092	0.2041	0.2062	0.2130	0.2175	0.3176	0.3438	0.3298	0.3074	0.2992
	1.3	I	0.1703	0.1369	0.1452	0.1425	0.1365	0.1312	0.1733	0.1775	0.1753	0.1716	0.1703
		M	0.1748	0.1398	0.1457	0.1427	0.1377	0.1343	0.1765	0.1817	0.1790	0.1743	0.1724
		L	0.2341	0.1765	0.1865	0.1814	0.1717	0.1670	0.2211	0.2306	0.2256	0.2173	0.2141
		U	0.2102	0.1642	0.1717	0.1679	0.1606	0.1571	0.2006	0.2076	0.2039	0.1977	0.1951

Bayes estimator under an informative prior may be either smaller or larger than that under a noninformative prior.

- The RMSE of a given estimator under heterogeneous distributions may be smaller or larger than that under the IID setting. For given m and censoring plan, the RMSE of the MLE of α increases with λ_0, whereas that of β decreases.

Remark 4.1. In the frequentist point of view, when the estimation problem of more than one parameter is of interest, the determinant and the trace of the covariance matrix of the estimators, which are called the generalized variance (GV) and the total variance (TV), respectively, may be used to determine the optimal censoring plan. They correspond to the D-optimality and A-optimality criteria which are commonly used in optimal design of experiments. For more details, see Atkinson et al. (2007). Accordingly, we perform a simulation study with 10^4 repetitions of an MCMC algorithm to obtain approximated values of the GV and TV of the MLEs of α and β, based on the censoring plans presented in Table 1 and different values of the proportionality rate λ_0. The results are reported in Table 6. In the simulation algorithm, the censored data are generated from the PHRM with a baseline Weibull distribution with parameters $\alpha = \beta = 2$. From Table 6, it is observed that, for all values of λ_0 and m, removing the surviving units at the initial and middle failure times (i.e., RP = I and M) yield nearly identical GV values, which are also the smallest among the remaining plans. Thus, based on the GV criterion, either RP = I or RP = M may be recommended for practical implementation. On the other hand, based on the TV criterion, the optimal progressively Type-II censoring plan is RP = I for $\lambda_0 = 1$, whereas the optimal plan is RP = M for $\lambda_0 > 1$.

Table 4: Values of RMSE of $\hat{S}(1;\alpha,\beta)$ for $n = 30$ and $m = 15, 23$.

				Bayes with $\sigma = 0.25$: Informative prior					Bayes with $\sigma = 5$: Noninformative prior				
					BLINEX					BLINEX			
					k					k			
m	λ_0	RP	MLE	BSE	-1	-0.5	0.5	1	BSE	-1	-0.5	0.5	1
15	1	I	0.0769	0.1842	0.1815	0.1828	0.1856	0.1870	0.0810	0.0796	0.0803	0.0817	0.0825
		M	0.0702	0.1481	0.1459	0.1470	0.1492	0.1503	0.0680	0.0671	0.0676	0.0684	0.0689
		L	0.0673	0.1464	0.1442	0.1453	0.1474	0.1485	0.0650	0.0644	0.0647	0.0653	0.0656
		U	0.0703	0.1520	0.1497	0.1508	0.1531	0.1542	0.0678	0.0670	0.0674	0.0682	0.0687
	1.1	I	0.0631	0.0968	0.0955	0.0961	0.0974	0.0981	0.0559	0.0552	0.0555	0.0563	0.0568
		M	0.0699	0.0842	0.0829	0.0835	0.0848	0.0854	0.0657	0.0644	0.0650	0.0664	0.0671
		L	0.1041	0.0821	0.0807	0.0814	0.0827	0.0834	0.0869	0.0836	0.0852	0.0886	0.0903
		U	0.0828	0.0842	0.0829	0.0836	0.0848	0.0855	0.0718	0.0701	0.0709	0.0728	0.0738
	1.3	I	0.1051	0.1073	0.1051	0.1062	0.1085	0.1096	0.0921	0.0892	0.0907	0.0936	0.0951
		M	0.1243	0.1202	0.1172	0.1187	0.1218	0.1233	0.1220	0.1168	0.1194	0.1246	0.1272
		L	0.2808	0.1732	0.1668	0.1700	0.1766	0.1800	0.2083	0.1929	0.2005	0.2162	0.2242
		U	0.1883	0.1479	0.1435	0.1457	0.1501	0.1524	0.1622	0.1534	0.1578	0.1667	0.1711
23	1	I	0.0689	0.1564	0.1541	0.1553	0.1576	0.1588	0.0638	0.0630	0.0634	0.0642	0.0646
		M	0.0630	0.1414	0.1393	0.1403	0.1424	0.1435	0.0626	0.0619	0.0622	0.0629	0.0632
		L	0.0628	0.1401	0.1380	0.1391	0.1412	0.1423	0.0608	0.0603	0.0605	0.0610	0.0613
		U	0.0633	0.1459	0.1438	0.1448	0.1470	0.1481	0.0593	0.0587	0.0590	0.0597	0.0600
	1.1	I	0.0458	0.0694	0.0686	0.069	0.0698	0.0703	0.0422	0.0419	0.042	0.0424	0.0425
		M	0.0461	0.0656	0.0648	0.0652	0.066	0.0663	0.046	0.0456	0.0458	0.0462	0.0464
		L	0.0479	0.0625	0.0617	0.0621	0.0629	0.0632	0.0456	0.045	0.0453	0.0459	0.0462
		U	0.0505	0.0693	0.0685	0.0689	0.0697	0.0701	0.0443	0.0438	0.0440	0.0445	0.0448
	1.3	I	0.0694	0.0785	0.0772	0.0779	0.0792	0.0799	0.064	0.0627	0.0634	0.0647	0.0654
		M	0.0730	0.0822	0.0806	0.0814	0.083	0.0837	0.0735	0.0716	0.0725	0.0744	0.0754
		L	0.1101	0.1055	0.1030	0.1042	0.1068	0.1081	0.0996	0.0957	0.0976	0.1017	0.1037
		U	0.0951	0.0967	0.0948	0.0957	0.0977	0.0987	0.0912	0.0886	0.0899	0.0925	0.0938

Table 5: Optimal plan for $n = 30$ and $m = 15, 23$ in estimating different parameters.

				Bayes estimators	
Parameter	m	λ_0	MLE	$\sigma = 0.25$	$\sigma = 5$
α	15	1	I	L	I
		1.1	I	L	I
		1.3	I	I	I
	23	1	L	L	U
		1.1	I	L	I
		1.3	I	I	I
β	15	1	M	M	I
		1.1	I	L	M
		1.3	I	I	I
	23	1	I	I	I
		1.1	I	M	I
		1.3	I	I	I
$S(1;\alpha,\beta)$	15	1	L	L	L
		1.1	I	L	I
		1.3	I	I	I
	23	1	L	L	U
		1.1	I	L	I
		1.3	I	I	I

Table 6: Simulated values of GV and TV for $n = 30$ and $m = 15, 23$.

		$\lambda_0 = 1$		$\lambda_0 = 1.1$		$\lambda_0 = 1.3$	
m	RP	GV	TV	GV	TV	GV	TV
15	I	0.0088	0.2012	0.0065	0.1935	0.0024	0.2158
	M	0.0086	0.1965	0.0064	0.2250	0.0025	0.2944
	L	0.0146	0.3005	0.0131	0.4831	0.0069	0.9661
	U	0.0107	0.2300	0.0097	0.3341	0.0043	0.5304
23	I	0.0043	0.1457	0.0031	0.1319	0.0011	0.1407
	M	0.0043	0.1420	0.0030	0.1406	0.0011	0.1647
	L	0.0056	0.1725	0.0045	0.2055	0.0018	0.2857
	U	0.0047	0.1528	0.0038	0.1683	0.0014	0.2071

5 Application to a real data set

To investigate the performance of the proposed procedure, we use a subset of the data on the *between-failures times* (BFT) of the air conditioning system of jet airplanes, originally analyzed by Proschan (1963). Specifically, five jet airplanes numbered 7907, 7910, 7912, 7914, and 7917 are selected. To examine whether the distributions of these data conform to the PHRM in (2.1) with a baseline two-parameter Weibull distribution, we verify if the BFTs for the i-th airplane follow the survival function

$$S_i(t;\alpha,\beta) = e^{-\lambda_i(\frac{t}{\alpha})^{\beta}}, \quad t > 0. \tag{5.1}$$

Note that if T_i follows (5.1), then the transformed variable $Y_i = (T_i/\alpha)^{\beta}$ follows an exponential distribution with rate $\lambda_i = 1/\mu_i$, where μ_i denotes the sample mean of Y_i. Setting $\alpha = 180$ and $\beta = 0.8$, the inverse sample means of $Y_1, \ldots, Y_5$ for the selected airplanes are computed as $\lambda_1 = 2.0007$, $\lambda_2 = 1.4473$, $\lambda_3 = 2.6674$, $\lambda_4 = 2.3363$, and $\lambda_5 = 1.6650$, where these quantities are assumed to be the known proportionality rates of the PHRM described in Section 2.

Applying the Kolmogorov–Smirnov test for exponentiality to $Y_1, \ldots, Y_5$ with rates $\lambda_1, \ldots, \lambda_5$, based on the complete data, yields distance statistic values of 0.23971, 0.20556, 0.10312, 0.15738, and 0.18223 for the first through fifth airplanes, respectively. The corresponding p-values are 0.8099, 0.5288, 0.9464, 0.5925, and 0.7571. These results confirm that Weibull distributions with proportionality rates $\lambda_1, \ldots, \lambda_5$ and common parameters $\alpha = 180$ and $\beta = 0.8$ provide good fits to the BFTs of the air conditioning systems for the selected jet airplanes. Based on these values, the baseline survival probabilities at times $t = 20$ and $t = 100$ are $S(20;\alpha,\beta) = 0.8416$ and $S(100;\alpha,\beta) = 0.5353$, respectively. In the sequel, we evaluate the performance of the proposed censoring scheme in estimating these parameters.

To implement an INID progressively Type-II censored sample of size $n = 30$, we extract six BFTs from each jet airplane, as reported in Table 7. We restrict our attention to the aforementioned data and exclude the rest from the analysis. Thus, we encounter a sample of 30 BFTs in which the distribution of every six consecutive observations differs from that of the subsequent six. That is, for $i = 1, \ldots, 5$, the data $T_{6(i-1)+1}, \ldots, T_{6i}$ follow the survival function in (5.1). Hence, the proposed PHRM is applicable.

Table 7: Summary descriptions of Boeing 720 jet aircraft data.

Aircraft No.	7907	7910	7912	7914	7917
i	1	2	3	4	5
T_{6i-5}	194	74	23	50	484
T_{6i-4}	15	57	261	44	18
T_{6i-3}	41	48	87	102	100
T_{6i-2}	29	502	7	72	7
T_{6i-2}	33	12	120	22	98
T_{6i}	181	70	14	39	5
λ_i	2.0007	1.4473	2.6674	2.3363	1.6650

To illustrate the implementation of the proposed point estimators for the parameters of the baseline Weibull distribution, we first extract four distinct INID progressively

Table 8: Progressive censored samples extracted from the data in Table 7 for various censoring plans with $m = 15$.

RP	Censored samples
I	5 , 7 , 14 , 18, 23, 33 , 48, 50, 57 , 62, 70 , 72 ,100, 261, 484
M	5, 7 , 7 , 12, 14, 15, 18 22 , 23 , 44, 50, 70 , 74, 181, 484
L	5, 7 , 7 ,12 ,14, 15, 18, 22, 23, 29, 33, 39, 41, 44, 48
U	5, 7 , 7, 12 , 14, 15 , 22, 29 , 33, 41, 48, 57 , 70, 74, 194

Type-II censored samples of effective size $m = 15$, corresponding to RP = I, M, L, U, as described in Table 1, from the data presented in Table 7. The resulting censored data are reported in Table 8. Using these data, we estimate the parameters α, β, $S(20;\alpha,\beta)$, and $S(100;\alpha,\beta)$ via various estimators, including the MLE, BSE, and BLINEX. The MLEs of β and α are numerically obtained using equations (3.3) and (3.4), respectively. For the Bayes estimation, we adopt prior distributions $N(\log(180),\sigma)$ and $N(\log(0.8),\sigma)$ for the transformed parameters $\nu_1 = \log\alpha$ and $\nu_2 = \log\beta$, respectively, with $\sigma = 0.25, 5$. The Metropolis-Hastings algorithm is then run with $B = 5 \times 10^4$ iterations and a burn-in period of $M = 5000$ samples to generate observations from the posterior densities. The estimates are presented in Table 9.

Comparing the estimated parameter values presented in Table 9 with the true values obtained from the complete sample (i.e., $\alpha = 180$, $\beta = 0.8$, $S(20;\alpha,\beta) = 0.8416$, and $S(100;\alpha,\beta) = 0.5353$), the following observations can be made:

- For α, the MLE under the censoring plan RP = I is closer to the true value than those under the other censoring plans.
- All Bayes estimates of α (BSE and BLINEX with various k, under both informative and noninformative priors) perform best under RP = U.
- For β and $S(20;\alpha,\beta)$, both the MLE and all Bayes estimates are closest to the true values under RP = M.
- The MLE of $S(100;\alpha,\beta)$ under RP = U is closest to the true value under RP = U.
- Under the informative prior ($\sigma = 0.25$), all Bayes estimates of $S(100;\alpha,\beta)$ are closest to the true value under RP = L; under the noninformative prior ($\sigma = 5$), however, RP = U yields the closest estimates.
- As expected, all Bayes estimates of the given parameters under the informative prior are closer to the true values than those obtained under the noninformative prior.

It should be noted that each of the samples presented in Table 8 represents only one possible realization under a given censoring plan. Consequently, the estimates would differ if another sample had been drawn.

Convergence of the Metropolis-Hastings algorithm was assessed using standard MCMC diagnostics including three standard diagnostic procedures. Trace plots were examined for each parameter to verify adequate mixing and stationarity, with particular attention to the behavior of the proposal distribution. Autocorrelation plots were used

Table 9: Parameter estimates based on various censoring plans for $n = 30$ and $m = 15$.

					Bayes estimation			
					BLINEX			
Parameter	RP	MLE	σ	BSE	$k = -1$	$k = -0.5$	$k = 0.5$	$k = 1$
α	I	179.7396	0.25	188.6536	385.6062	375.9281	108.7425	99.4137
			5	211.2053	1596.7499	1585.9302	90.5073	81.0481
	M	174.8006	0.25	186.8934	409.3324	398.9872	104.8871	96.1275
			5	216.0871	1930.0068	1919.1873	87.9526	78.2900
	L	106.9343	0.25	172.4972	389.4740	378.6575	94.7889	85.5507
			5	139.9801	2415.1906	2404.3708	68.3750	61.3019
	U	160.5181	0.25	184.4772	367.0459	356.9237	112.3610	104.2302
			5	197.8135	1950.9517	1940.1319	88.4306	80.1316
β	I	0.9573	0.25	0.9051	0.9128	0.9089	0.9014	0.8977
			5	0.9246	0.9367	0.9306	0.9187	0.9129
	M	0.9391	0.25	0.8913	0.8983	0.8948	0.8878	0.8844
			5	0.8950	0.9064	0.9007	0.8892	0.8836
	L	1.2919	0.25	0.9496	0.9602	0.9548	0.9444	0.9394
			5	1.2136	1.2605	1.2366	1.1916	1.1703
	U	1.0267	0.25	0.9275	0.9363	0.9319	0.9231	0.9188
			5	0.9792	0.9978	0.9885	0.9702	0.9612
$S(20;\alpha,\beta)$	I	0.8850	0.25	0.8703	0.8709	0.8706	0.8700	0.8696
			5	0.8790	0.8798	0.8794	0.8786	0.8781
	M	0.8776	0.25	0.8658	0.8663	0.8661	0.8655	0.8652
			5	0.8723	0.8730	0.8726	0.8719	0.8715
	L	0.8917	0.25	0.8709	0.8714	0.8712	0.8707	0.8704
			5	0.8847	0.8854	0.8851	0.8844	0.8841
	U	0.8888	0.25	0.8734	0.8739	0.8736	0.8731	0.8728
			5	0.8823	0.8830	0.8827	0.8820	0.8816
$S(100;\alpha,\beta)$	I	0.5653	0.25	0.5610	0.5626	0.5618	0.5602	0.5594
			5	0.5790	0.5824	0.5807	0.5772	0.5755
	M	0.5533	0.25	0.5547	0.5562	0.5555	0.5539	0.5532
			5	0.5715	0.5749	0.5732	0.5699	0.5682
	L	0.3997	0.25	0.5375	0.5393	0.5384	0.5367	0.5358
			5	0.4359	0.4442	0.4401	0.4317	0.4275
	U	0.5406	0.25	0.5570	0.5585	0.5578	0.5562	0.5555
			5	0.5576	0.5613	0.5594	0.5558	0.5539

to evaluate the efficiency of the random-walk proposals; slow decay prompted adjustments to the proposal variance to achieve an optimal acceptance rate of approximately 23-44%, depending on the parameter dimension. Finally, the Gelman-Rubin shrink factor ($\hat{R}$), calculated from two parallel chains with overdispersed starting values, was monitored across iterations, with $\hat{R} < 1.05$ for all parameters indicating successful convergence to the posterior distribution. For more details see Gelman et al. (2013). Figures 1 and 2 present the Gelman, Trace and Autocorrelation plots for the generated observations of α and β from posterior distribution obtained under progressive censoring plan $RP = M$. These plots are drawn based on $B = 5 \times 10^4$ iterations of the Metropolis-Hastings algorithm, following the removal of $M = 5000$ burn-in samples. From these figures, it is clear that the convergence diagnostics indicate that the MCMC algorithm successfully converged for both α and β. The blue solid line in Gelman plot represents the median shrink factor, while the red dashed line represents the upper

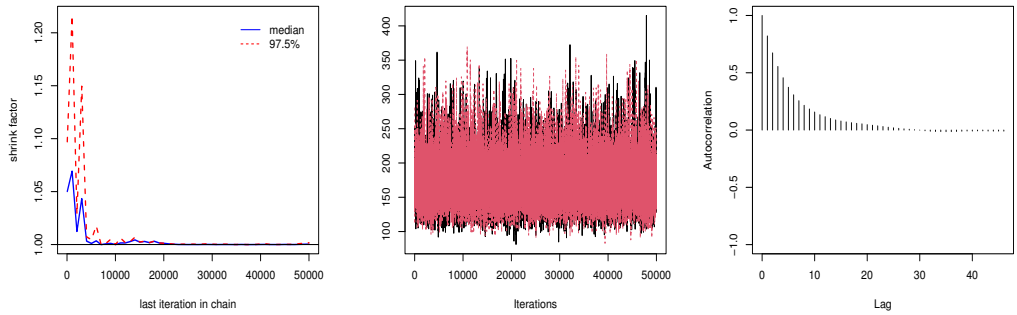


Figure 1: The Gelman (left), Trace (middle), and Autocorrelation (right) plots for generated samples from posterior distributions of α.

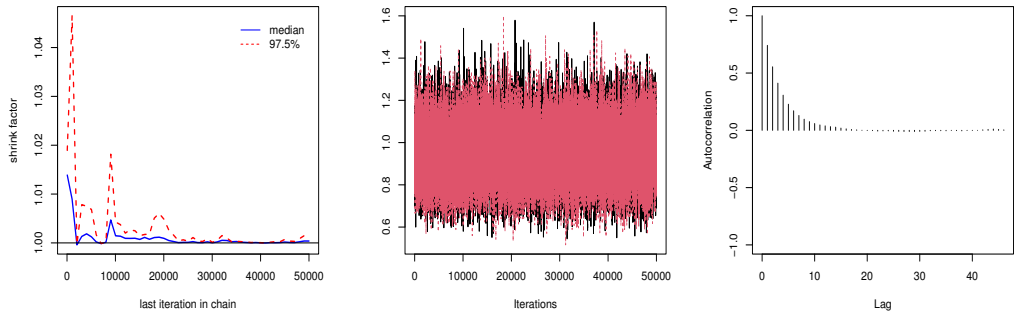


Figure 2: The Gelman (left), Trace (middle), and Autocorrelation (right) plots for generated samples from posterior distributions of β.

97.5% quantile. At the beginning of the simulation, both curves are noticeably greater than 1, indicating that the chains have not yet converged. As the number of iterations increases, both curves rapidly approach 1.0 and remain essentially equal to 1 after approximately 5000-10000 iterations. So, the Gelman diagnostics show that the potential scale reduction factors ($\hat{R}$)) rapidly approached 1.0 and remained stable throughout the remainder of the simulation, confirming convergence of the multiple chains. The trace plots exhibit stationary behavior with no discernible trends or abrupt shifts, indicating that the chains adequately explored the posterior distributions. Although the autocorrelation plots reveal dependence between successive samples, the autocorrelations decay steadily toward zero with increasing lag, demonstrating satisfactory mixing. The decay is slightly faster for β than for α, resulting in a larger effective sample size (ESS = 14688.340) compared with α (ESS = 9583.043). Both ESS values are sufficiently large, indicating that the posterior samples contain a substantial number of effectively independent draws and provide reliable estimation of the posterior distributions. Overall, the diagnostic results confirm that the MCMC algorithm produced well-converged,

well-mixed, and statistically efficient posterior samples for both model parameters.

6 Conclusions

In this paper, the optimal progressive censoring plan was investigated in the case that the lifetimes were independent but not identically distributed random variables. In fact, it was assumed that the lifetimes are coming from a PHRM with two-parameter Weibull distribution as the baseline population. The parameters α and β and the survival function $S(\cdot;\alpha,\beta)$ were focused to be estimated. The MLE and Bayes estimation of these parameters were derived. To obtain the Bayes estimators, the SE and LINEX loss functions were used, and independent normal prior distributions were considered for $\log(\alpha)$ and $\log(\beta)$. To investigate the effect of the prior, two different standard deviations were considered: $\sigma = 0.25$, corresponding to an informative prior, and $\sigma = 5$, corresponding to a noninformative prior. The Metropolis-Hastings algorithm was used to get the Bayes estimates due to complexity of the posterior distribution. The minimum RMSE of each estimator was considered as a criterion to find the optimal scheme. From the simulation results, it was deduced that the optimal scheme depends on several factors, including the parameter of interest, the proportionality rates, the effective sample size m, the type of estimator (frequentist or Bayesian), and the hyperparameters of the prior distribution. So, before implementing progressive censoring plan, the effective factors should be determined to obtain the optimal censoring scheme. Finally, the proposed model was applied to a real data set, and the performance of the various censoring plans was investigated.

The procedure proposed in this paper may be extended to the following cases:

- In this paper, a PHRM with baseline Weibull distribution was considered. Other baseline distributions in a PHRM or even other models describing the INID lifetimes may be studied.

- The optimal censoring plan may be investigated in other fields of statistical topics other than the estimation problem of unknown parameters. For example, the optimal censoring plan with maximum Fisher information and Entropy might be studied in future.

- The paper addressed the case of known proportionality rates and determined them under the assumption that there exists a hierarchical relationship among the distributions, an assumption that is common in the literature on proportional hazard rate models. In practice, these rates are typically derived based on prior experience and expert knowledge. In situations where the rates are unknown, they would need to be estimated based on a sufficient sample from each distribution. In such cases, the model would need to be extended to accommodate a data matrix of lifetimes consisting of entries $T_{i,1},\ldots,T_{i,m_i}$, for $i = 1,\ldots,n$, where m_i denotes the sample size from the ith distribution.

- The framework developed in this paper under the known proportionality rates assumption can be extended to accommodate covariate information. Consider p covariates $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{ip})'$ associated with the lifetime T_i, for $1 \leq i \leq n$. The proportionality rate λ_i may then be expressed as $\lambda_i = \exp\{-\boldsymbol{Y}_i'\boldsymbol{\beta}\}$, with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ denoting the regression coefficients. In this extended setting, the optimal censoring plan could be determined by minimizing a criterion based on the performance of the MLE of $\boldsymbol{\beta}$. This generalization represents a valuable avenue for future investigation.

- The assumption of constant proportionality rates may also be extended to time-varying rates by allowing $\lambda_i(t)$ to follow a specific dynamic form. For instance, one could consider a piecewise-constant structure, where the rate changes at predetermined change points, or a parametric function such as $\lambda_i(t) = \lambda_i t^{\gamma_i}$, which accommodates monotonic trends over time. Alternatively, a more flexible specification such as $\lambda_i(t) = \lambda_i \exp\{\alpha_i t\}$ could be adopted to capture exponential growth or decay in the hazard ratio. In such settings, the optimal censoring plan would need to be derived under the time-varying framework, and the performance of the estimators would depend not only on the baseline parameters but also on the dynamics governing the proportionality rates. This generalization represents a meaningful extension of the current work and is left for future investigation.

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Data and Code Availability

The numerical results reported in Tables 2–4, 6, and 9, along with the convergence plots in Figures 1 and 2, were generated using R version 4.4.0. The code used to generate these results is available at the following URL: https://fumdrive.um.ac.ir/index.php/s/gG7CcXGzgPRdbyf.

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