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Lower quantile inference for constant stress accelerated life tests via p-spline modeling under type-II progressively censoring

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Abstract. The purpose of this paper is to propose a semiparametric modeling approach for constant stress accelerated life test (CSALT) under the type-II progressively censoring scheme (T-II PCS). In the proposed model, the quantile regression (QR) approach based on the asymmetric Laplace distribution (ALD) is presented, and nonparametric functions are used to describe relationships between lifetime distribution characteristics and accelerated stresses. The relationships are approximated by the B-spline method. Additionally, a penalized expectation-maximization algorithm (EM) is developed for maximum likelihood estimation of parameters of the model at the τ^{th} quantile of the lifetime distribution, using a hierarchical representation of the ALD. The proposed model is appropriate to deal with one or more accelerated stresses scenarios. To demonstrate the performance and effectiveness of the proposed model, a simulation study and a real data set analysis are conducted.

Keywords. Constant Stress Accelerated Life Test, P-spline, Lower Quantile, Asymmetric Laplace Distribution, Penalized Expectation-Maximization Algorithm.

MSC: 62N05, 62N01, 62G08.

1 Introduction

Today, manufacturers produce highly reliable products, making it challenging and time-consuming to assess reliability and conduct life testing under use conditions. To address this, accelerated life test (ALT) models have been developed to evaluate unit reliability while significantly reduc-

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ing total testing time. These tests expose units to accelerated stresses, allowing more failures to be observed in a shorter timeframe. The accelerated stresses include voltage, temperature, humidity, vibration, pressure, load, and their combinations. According to information obtained from the tests conducted at high levels of one or more accelerated stresses, an appropriate ALT model predicts the unit's lifetime under the use conditions.

Most current analyses of the ALT models focus on parametric ALT models. A parametric ALT model is based on two assumptions. First, that the distribution of failure times belongs to the same parametric distribution family across all levels of stress; and second, that there is a known parametric relationship between the accelerated stresses and the lifetime distribution characteristics.

The CSALT model is one of the more commonly used ALT models, in which the accelerated stress levels remain constant throughout the test duration.

Many authors have investigated the CSALT model. Among others, Lin et al. (2019) considered the CSALT model based on a general log-location-scale distribution, with the location parameter being a linear function of the accelerated stress levels and the fixed scale parameter. Li et al. (2020) proposed the ALT model for electronic devices that experience multiple failure mechanisms. This model employs the Weibull lifetime distribution, assuming the Arrhenius relationship between the scale parameter and the accelerated stress levels, while keeping the shape parameter constant. Zhang et al. (2021) proposed a new planning method for the CSALT model, in which the failure times follow the lognormal lifetime distribution. They assumed that the scale parameter is invariant, while the multiple accelerated stresses impact the location parameter through a second-order linear model. Abd El-Raheem (2021) considered the optimal allocation problem in the CSALT model. They assumed an extension of the exponential distribution as the lifetime distribution, in which the logarithm of the scale parameter is linearly related to the multiple accelerated stresses, while the shape parameter is unaffected by the accelerated stresses. Kumar et al. (2022) used a generalized inverse Lindley distribution in the CSALT model, such that the logarithm of the shape parameter is linked to the accelerated stress levels by a linear function and the scale parameter remains constant. Feng et al. (2024) assumed that the failure times in the CSALT model follow the exponential lifetime distribution and that the lifetime distribution parameter and the accelerated stresses have the log-linear generalized Eyring relationship. A statistical analysis of the CSALT model with the two-parameter exponential lifetime distribution was performed by Wang et al. (2024). In this model, the scale parameter has the inverse power-law relationship with the accelerated stress levels, whereas the location parameter is assumed to be constant. Feng and Tang (2025) applied the Weibull distribution to describe the lifetime of the test units, establishing the log-linear relationship between the scale parameter and the multiple accelerated stresses, and the shape parameter is fixed.

The Arrhenius relationship, the log-linear relationship, the inverse power law relationship, the Eyring relationship, and the generalized Eyring relationship are examples of commonly used parametric and empirical relationships. However, in many real situations, the true relationship between the accelerated stresses and characteristics of the lifetime distribution is uncertain and could be complicated (Escobar and Meeker, 2006). So, most of the existing ALT models that use the parametric relationships mentioned above are inappropriate in many applications. Moreover, the scale parameter of the lifetime distribution is often stress-dependent through complex mechanisms, whereas the ALT models assume a constant scale parameter across all stress levels or a linear function of the logarithm of the scale parameter with the stress levels. This major limitation of the ALT models makes them too simplified. These oversimplified models cause the lifetime estimation of the test unit to be biased.

Si and Yang (2017) proposed a semiparametric ALT model to overcome the above problems. They ignored the second assumption of the parametric ALT model. They assumed the parameters of the Weibull lifetime distribution are nonparametrically related to the accelerated stresses. They used the B-spline method to model nonparametric relationships between the accelerated stresses and the parameters of the lifetime distribution. Meanwhile, Vanegas and Paula (2017) extended log-symmetric regression models, assuming that the lifetime distribution belongs to the log-symmetric distribution class, and the location parameter and the scale parameter are semiparametric functions of explanatory variables. They utilized the P-spline approach proposed by Eilers and Marx (1996) or natural cubic splines to approximate nonparametric components of the model. It is important to note that both studies presented by Si and Yang (2017) and Vanegas and Paula (2017) considered the type-I censoring scheme. Additionally, these models may involve many spline coefficients to be estimated. Consequently, to avoid overfitting, the authors estimated the model parameters using the maximum penalized likelihood estimation (MPLE) approach.

In various practical applications, data sets can be equally well fitted by either the lognormal distribution or the Weibull distribution, particularly over the middle of the distribution. When fitted to the same data set, the lognormal distribution tends to exhibit a later lower tail compared to the corresponding Weibull distribution. That is, a low Weibull percentile is below the corresponding lognormal one (Nelson, 1990). Nevertheless, engineers and researchers often concentrate on particular lower quantiles of the lifetime distribution (Yu and Chang, 2012). Lv et al. (2015) showed that by using a model in which the shape parameter decreases with the voltage stress and increases with the temperature stress, more accurate estimates of the lifetime distribution quantiles can be achieved. A common objective of the ALT model is to estimate a specific quantile in the lower tail of the lifetime distribution under the use conditions.

The QR method, a basic and robust method for estimating the conditional quantiles of the lifetime distribution, is introduced by Chen et al. (2015). The QR method provides an exhaustive scan of the whole lifetime distribution and gives an overall evaluation of the effects of the accelerated stresses on the different lifetime distribution quantiles.

Davino et al. (2013) pointed out that the QR method is robust to the nonconstant scale parameter. Moreover, Nelson (1990) noticed that estimating the lower tail quantiles using the nonconstant scale parameter yields improved performance. Ying et al. (1995) discussed a semiparametric method for median regression, in which the median of the failure times is a linear function of the covariate. Yu and Chang (2012) employed Bayesian model averaging (BMA), which considers model uncertainty to analyze the ALT data and estimates the quantile under the use conditions. Lin et al. (2016) developed a semiparametric model with the log-linear regression function for the median of the failure times. Chen et al. (2015) considered the log-location-scale model, where both the location parameter and the scale parameter are linear functions of the transformed stress. Xue et al. (2018) studied the censored QR model in which the quantile of the failure times is linearly related to the covariate. Zhou et al. (2020) presented a Bayesian method for optimum planning in the QR field to analyze the ALT data better.

Koenker and Machado (1999) applied a skew distribution, the ALD, to estimate the QR model.

The ALD has several location-scale mixture representations. Kozumi and Kobayashi (2011) provided a mixture representation for the ALD based on the normal distribution and the exponential distribution. Zhou et al. (2016) developed a Bayesian QR method to analyze the ALT data based on scale mixtures of the normal distributions. They assumed a linear function relating the location parameter and the accelerated stress levels, while the scale parameter is fixed. Lee et al. (2023) introduced quantile forward regression model specifically designed for high-dimensional survival data. Their approach focuses on selecting variables by maximizing the ALD's likelihood function. They further determined the final model using the extended Bayesian information criterion (EBIC). This methodology allows for quantile-specific risk prediction, effectively capturing varying effects of the covariates across the outcome distribution. Bae and Lim (2025) used a nonlinear QR method based on the ALD for accelerated destructive degradation test (ADDT) data, which, by estimating parameters through the generalized EM algorithm, enables the estimation of different quantiles of the degradation distribution. Extending ideas from the QR method, Sheng and Henao (2025) proposed a parametric survival analysis method based on the ALD, which fits a single distributional model that yields closed-form expressions for the mean, median, mode, variance, and quantiles of the event-time distribution.

The selection of the ALD is motivated by its well-established connection to the QR method. Specifically, the location parameter of the ALD corresponds directly to the quantile of interest, allowing inference to focus on a specific conditional quantile rather than requiring correct specification of the entire lifetime distribution. This feature is particularly appealing in the ALT studies, where the underlying lifetime distribution is often unknown. Furthermore, as noted in the recent QR literature, constructing sufficiently flexible distributions that are directly parameterized by quantiles while simultaneously maintaining tractable inference remains challenging. Consequently, the ALD continues to serve as a practical and widely adopted foundation for likelihood-based quantile modeling and inference. In other words, the ALD, as an asymmetric distribution, allows the likelihood function to emphasize specific regions of the lifetime distribution — particularly the tails — rather than a central location, thereby enabling accurate estimation of the lower quantiles of the lifetime distribution. In practical applications, the lower quantiles are applied to determine conservative replacement or maintenance intervals, ensuring that only a small percentage of units fail before a scheduled preventive action. Additionally, they provide a direct and interpretable basis for setting warranty and reliability guarantees by controlling the probability of early failures under the use conditions. From a risk assessment perspective, the lower quantiles measure the risk associated with the early failures, which are often the most critical from safety, cost, and customer satisfaction viewpoints. Consequently, modeling and predicting the lower quantiles serve as practical, valuable tools for analyzing the ALT data.

In this paper, we propose a semiparametric CSALT (SPCSALT) model under the T-II PCS. In the SPCSALT model, we employ the location-scale mixture representation of the ALD, as presented in Kozumi and Kobayashi (2011), and use the B-spline approach to approximate the nonparametric functions that describe the relationships between the accelerated stresses and the location parameter and the scale parameter of the ALD.

The existing semiparametric ALT literature, for example, the semiparametric models presented by Si and Yang (2017) and Vanegas and Paula (2017), offer valuable flexibility in relating the accelerated stresses to the lifetime distribution characteristics. Similar developments have also been conducted more broadly in the survival analysis literature. Although the standard Cox proportional hazards model (PHM) restricts the covariate effects to be linear on the log-hazard scale, this restriction has been extended by representing the covariate effects nonparametrically through the B-spline approach (Molinari et al., 2001); however, such survival models have critical structural limitations in representing diverse quantile behaviors. In these models, the nonparametric functions that relate the covariates to the characteristics of the survival distribution are estimated once from the full likelihood function, rather than separately at each quantile level. Although these models do not explicitly model quantiles, any quantile of interest can in principle be recovered from the fitted model through its survival (reliability) function, since it is simply the value at which the survival function equals $1 - \tau$. Since this recovery only locates a point on the survival function whose covariate-dependent components are already fixed, the resulting covariate-quantile relationship is forced to follow the same rigid structure across all quantile levels, regardless of τ. This limitation is particularly consequential for the lower quantiles, which characterize early failures and are of central interest in reliability and warranty analysis. The mechanism driving early failures may respond to the accelerated stresses in a manner that is not adequately captured by a structure estimated from the entire lifetime distribution, since the stress-dependent components in these models are fixed once and not re-estimated specifically for the lower quantile region. The proposed model addresses this gap by directly and independently estimating the relationship between the accelerated stresses and the lifetime distribution at each quantile τ, particularly enabling a more flexible and accurate characterization of the lower quantiles, which are often of particular interest in reliability applications.

We estimate the unknown parameters in the SPCSALT model by the penalized EM algorithm. Finally, we illustrate the performance of the proposed model by a simulation study and by analyzing a real-life data set. To the best of our knowledge, this is the first work to apply this concept to the CSALT analysis.

The rest of the paper is organized as follows: In Section 2, the SPCSALT model is outlined, and the penalized EM algorithm is employed to estimate parameters. The performance of the proposed model is evaluated on simulated and real-world data sets in Section 3. In the last section, a conclusion is provided.

2 The SPCSALT model

In this section, we provide the basic settings and assumptions for the SPCSALT model and also obtain the estimates of the model's unknown parameters based on the T-II PCS.

2.1 Type-II progressively censoring scheme

Using a censoring scheme is common in reliability experiments. The two basic censoring schemes are type-I and type-II censoring schemes. One of the drawbacks of these censoring schemes is that test units can only be removed at the end of the test. The T-II PCS was introduced in reliability and life testing to overcome this drawback. It allows the test units to be removed at different points during the test.

Suppose there are q accelerated stresses marked with $\boldsymbol{s} = (s_1, \ldots, s_q)^\top$ and J combined stress levels in the CSALT model. Let n units be put on the test at time 0 and allocated to J groups. The sample size for each group is denoted as $n_1, \ldots, n_J$, respectively, so that $\sum_{j=1}^{J} n_j = n$. When the j^{th} group is set under the j^{th} combined stress level, only m_j ($m_j < n_j$) failures are observed, for $j = 1, \ldots, J$. The following is the procedure for performing the T-II PCS under the CSALT model: under the j^{th} combined stress level, when the first failure, T_{j1}, is observed, $R_{j,1}$ surviving units are randomly removed from the remaining $(n_j - 1)$ units. Similarly, at the time of the second failure, T_{j2}, $R_{j,2}$ surviving units are randomly withdrawn from the remaining $(n_j - R_{j,1} - 2)$ units. The process is ongoing, at the time of m_j^{th} failure, T_{jm_j}, all the remaining $R_{j,m_j} = n_j - m_j - \sum_{\ell=1}^{m_j-1} R_{j,\ell}$ units are taken out and the test is terminated. So, $\boldsymbol{T}_j = (T_{j1}, \ldots, T_{jm_j})$ is vector of failure times at the j^{th} combined stress level and $\boldsymbol{T} = (\boldsymbol{T}_1, \ldots, \boldsymbol{T}_J)$ is vector of all failure times. The n_j, m_j, and $R_{j,1}, \ldots, R_{j,m_j}$ for $j = 1, \ldots, J$ are predetermined values.

The T-II PCS can be considered as an incomplete data problem. The vector of all censored failure times is defined by $\boldsymbol{T}^\star = (\boldsymbol{T}_1^\star, \ldots, \boldsymbol{T}_J^\star)$, where $\boldsymbol{T}_j^\star = (\boldsymbol{T}_{j1}^\star, \ldots, \boldsymbol{T}_{jm_j}^\star)$ is the vector of censored failure times at the j^{th} combined stress level. Each $\boldsymbol{T}_{j\ell}^\star$ is a $1 \times R_{j,\ell}$ vector with $\boldsymbol{T}_{j\ell}^\star = (T_{j\ell 1}^\star, \ldots, T_{j\ell R_{j,\ell}}^\star)$ for $\ell = 1, \ldots, m_j$, $j = 1, \ldots, J$, and they are not observable. The censored failure time vector $\boldsymbol{T}^\star$ can be thought of as missing data.

The type-II censoring situation and the complete sample situation are special cases of this censoring scheme. For more details on the T-II PCS, the reader may refer to Balakrishnan and Aggarwala (2000) and Balakrishnan (2007).

2.2 Preliminaries and the model description

Under the mentioned experiment in 2.1, the following basic assumptions are made:

1. The unit lifetime in the SPCSALT model follows the ALD at each accelerated stress level.
2. Increasing the stress levels reduces the location parameter and the scale parameter.
3. The multiple stresses in the ALT model with multiple accelerated stresses have additive effects on the unit lifetime.

The probability density function (PDF) of the ALD is

$$f(t \mid \vartheta(\boldsymbol{s}), \zeta(\boldsymbol{s})) = \frac{\tau(1-\tau)}{\zeta(\boldsymbol{s})} \exp\left\{-\frac{\rho_\tau(t - \vartheta(\boldsymbol{s}))}{\zeta(\boldsymbol{s})}\right\}, \tag{2.1}$$

where $\vartheta(\boldsymbol{s})$ and $\zeta(\boldsymbol{s})$ are the location parameter and the scale parameter that get impacted by the accelerated stresses, respectively. $\tau \in (0, 1)$ is the quantile parameter. $\vartheta(\boldsymbol{s})$ is the τ^{th} quantile. $\rho_\tau(\cdot)$ is the loss function that satisfies $\rho_\tau(u) = \tau\, u$ if $u > 0$ and $\rho_\tau(u) = (\tau - 1)\, u$ if $u \leqslant 0$.

Moreover, the mean and the standard deviation (SD) of the ALD, which depend on the accelerated stresses, are formulated as

$$\mu(\boldsymbol{s}) = \vartheta(\boldsymbol{s}) + \frac{1 - 2\tau}{\tau(1-\tau)}\zeta(\boldsymbol{s}), \tag{2.2}$$

and

$$\sigma(\boldsymbol{s}) = \zeta(\boldsymbol{s}) \sqrt{\frac{1 - 2\,\tau + 2\,\tau^2}{\tau^2\,(1-\tau)^2}}, \tag{2.3}$$

respectively.

Additive models offer significant advantages as they extend the capabilities of linear regression by allowing for the assessment of the effects of each variable separately. Additive models are widely used in reliability literature. Anderson (1991) proposed the ALT model in which there is a linearly additive relationship between the mean lifetime and the multiple accelerated stresses. Meeker et al. (1998) assumed that the location parameter of the glass capacitor lifetime distribution is an additively linear function of the two accelerated stresses, while the scale parameter is unaffected by the accelerated stresses. Elsayed and Zhang (2007) developed optimal multiple-stress ALT plans using the PHM. Zhu and Elsayed (2013) presented an approach for designing ALT plans with multiple stresses, considering the Weibull distribution as the lifetime distribution, and assuming the logarithm of the scale parameter is affected by a vector of stresses through a linear additive function.

In the proposed SPCSALT model, we apply an additive model to determine the overall effects of the several accelerated stresses on the τ^{th} quantile and the scale parameter at each combined stress level $\boldsymbol{s}_j = (s_{1j}, \ldots, s_{qj})^\top$ for $j = 1, \ldots, J$, and formulate $\vartheta(\boldsymbol{s})$ and $\zeta(\boldsymbol{s})$ nonparametrically and additively as

$$g_1(\vartheta(\boldsymbol{s}_j)) = g_1(\vartheta_j) = a_1 + \sum_{i=1}^{q} f_{1i}(s_{ij}), \tag{2.4}$$

and

$$g_2(\zeta(\boldsymbol{s}_j)) = g_2(\zeta_j) = a_2 + \sum_{i=1}^{q} f_{2i}(s_{ij}), \tag{2.5}$$

respectively. In (2.4) and (2.5), we choose monotonic link functions $g_k(x) = \log(x)$ for $k = 1, 2$, to certify that both $\vartheta(\boldsymbol{s})$ and $\zeta(\boldsymbol{s})$ are nonnegative, $f_{1i}(s_{ij})$ and $f_{2i}(s_{ij})$ are continuous, smooth, and unknown functions in which the effectiveness of the i^{th} accelerated stress on the τ^{th} quantile and the scale parameter at each combined stress level $\boldsymbol{s}_j$ is nonparametrically modeled by them, for $j = 1, \ldots, J$, $i = 1, \ldots, q$, respectively, and a_1 and a_2 are two constants.

We use the B-spline method to approximate the smooth functions $f_{1i}(s_{ij})$ and $f_{2i}(s_{ij})$ as

$$f_{1i}(s_{ij}) = \sum_{p=1}^{L_i} B_p^{(M)}(s_{ij})\,\omega_{1ip} = \boldsymbol{B}^{(M)}(s_{ij})\,\boldsymbol{\omega}_{1i}, \tag{2.6}$$

and

$$f_{2i}(s_{ij}) = \sum_{p=1}^{L_i} B_p^{(M)}(s_{ij})\,\omega_{2ip} = \boldsymbol{B}^{(M)}(s_{ij})\,\boldsymbol{\omega}_{2i}, \tag{2.7}$$

respectively, where $B_p^{(M)}(s_{ij})$ for $j = 1, \ldots, J$, $i = 1, \ldots, q$, and $p = 1, \ldots, L_i$, is a M^{th} order B-spline basis function, and $\boldsymbol{B}^{(M)}(s_{ij}) = (B_1^{(M)}(s_{ij}), \ldots, B_{L_i}^{(M)}(s_{ij}))$. ω_{1ip} and ω_{2ip} are unknown coefficients of the B-spline basis functions, and $\boldsymbol{\omega}_{ki} = (\omega_{ki1}, \ldots, \omega_{kiL_i})^\top$ for $i = 1, \ldots, q$, $k = 1, 2$. L_i is the number of basis functions for the i^{th} accelerated stress. To make it easier to use,

we select $L_i = L$ for $i = 1, \dots, q$.
Based on (2.6) and (2.7), (2.4) and (2.5) can be expressed as

$$g_1(\vartheta_j) = a_1 + \sum_{i=1}^{q} \boldsymbol{B}^{(M)}(s_{ij})\, \boldsymbol{\omega}_{1i}, \tag{2.8}$$

and

$$g_2(\zeta_j) = a_2 + \sum_{i=1}^{q} \boldsymbol{B}^{(M)}(s_{ij})\, \boldsymbol{\omega}_{2i}, \tag{2.9}$$

respectively. According to De Boor (1978), every B-spline basis function is nonnegative. So, based on the second assumption, a sufficient condition for the τ^{th} quantile and the scale parameter to be decreasing monotonously is that $\triangle^1 \omega_{kip} = \omega_{kip} - \omega_{ki(p-1)} < 0$ for $p = 1, \dots, L$, $i = 1, \dots, q$, and $k = 1, 2$. In this paper, we impose this constraint by reparametrizing the constrained coefficients, $\boldsymbol{\omega}_{ki}$, as $\boldsymbol{\omega}_{ki} = \boldsymbol{\Sigma}\tilde{\boldsymbol{\beta}}_{ki}$, where $\boldsymbol{\beta}_{ki} = (\beta_{ki1}, \dots, \beta_{kiL})^\top$, $\tilde{\boldsymbol{\beta}}_{ki} = (\beta_{ki1}, \exp(\beta_{ki2}), \dots, \exp(\beta_{kiL}))^\top$, and $\Sigma_{ru} = 0$ if $r < u$; $\Sigma_{ru} = 1$ if $u = 1$, $r \geq 1$; $\Sigma_{ru} = -1$ if $u \geq 2$, $r \geq u$.
To ensure the model is identifiable, the intercept parameter of the monotone smooth functions f_{ki} is zero, i.e., $\omega_{ki1} = \beta_{ki1} = 0$ for $i = 1, \dots, q$, $k = 1, 2$. Hence, we can rewrite $\boldsymbol{\beta}_{ki}$ and $\tilde{\boldsymbol{\beta}}_{ki}$ as $\boldsymbol{\beta}_{ki} = (\beta_{ki2}, \dots, \beta_{kiL})^\top$ and $\tilde{\boldsymbol{\beta}}_{ki} = (\exp(\beta_{ki2}), \dots, \exp(\beta_{kiL}))^\top$, respectively, remove the first column of $\boldsymbol{B}^{(M)}(s_{ij})$, i.e., $\boldsymbol{B}^{(M)}(s_{ij}) = (B_2^{(M)}(s_{ij}), \dots, B_L^{(M)}(s_{ij}))$, and delete the first row and column of the $\boldsymbol{\Sigma}$ matrix. So, $\boldsymbol{\beta}_{ki}$ is the unknown unconstrained parameter vector, $\boldsymbol{\beta}_k = (\boldsymbol{\beta}_{k1}^\top, \dots, \boldsymbol{\beta}_{kq}^\top)^\top$, $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top)^\top$, $\tilde{\boldsymbol{\beta}}_k = (\tilde{\boldsymbol{\beta}}_{k1}^\top, \dots, \tilde{\boldsymbol{\beta}}_{kq}^\top)^\top$, and $\tilde{\boldsymbol{\beta}} = (\tilde{\boldsymbol{\beta}}_1^\top, \tilde{\boldsymbol{\beta}}_2^\top)^\top$. Thus for $k = 1, 2$, $\boldsymbol{\theta}_k = (a_k, \boldsymbol{\beta}_k^\top)^\top$, $\tilde{\boldsymbol{\theta}}_k = (a_k, \tilde{\boldsymbol{\beta}}_k^\top)^\top$, $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\theta}}_1^\top, \tilde{\boldsymbol{\theta}}_2^\top)^\top$, and the parameter vector of the model is $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$.
By definition $\boldsymbol{Z}_{ij} = \boldsymbol{Z}(s_{ij}) = \boldsymbol{B}^{(M)}(s_{ij})\boldsymbol{\Sigma} \in \mathbb{R}^{L-1}$, such that $\boldsymbol{Z}(s_{ij}) = (Z_1(s_{ij}), \dots, Z_{L-1}(s_{ij}))$, the j^{th} row of the design matrix is defined as $\boldsymbol{Z}_j = \boldsymbol{Z}(\boldsymbol{s}_j) = \{1, \boldsymbol{Z}(s_{1j}), \dots, \boldsymbol{Z}(s_{qj})\}$ for $j = 1, \dots, J$, $i = 1, \dots, q$.
Furthermore, to ensure that the full additive model is uniquely identifiable, a sum-to-zero (centering) constraint is imposed on each monotone smooth function by centering the corresponding columns of the design matrix $\boldsymbol{Z}$ following the reparameterization process. This constraint eliminates the confounding between the global intercept (a_k) and arbitrary constant shifts in the individual monotone smooth functions. As a result, the decomposition of the additive predictor into the intercept and the monotone smooth functions is unique. Since centering the corresponding columns of $\boldsymbol{Z}$ only changes each smooth function by a constant, it does not affect the monotonicity established by the reparameterization process.

2.3 Parameter estimation of the SPCSALT model

Within the SPCSALT framework that involves more than one accelerated stress, we refer to the ℓ^{th} observed failure time at the j^{th} combined stress level of the T-II PCS by the term $t_{j\ell}$ for $\ell = 1, \dots, m_j$, $j = 1 \dots, J$. Thus, $\boldsymbol{t} = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_J)$ is the vector of all observed failure times, such that $\boldsymbol{t}_j = \left(t_{j1}, \dots, t_{jm_j}\right)$ for $j = 1, \dots, J$.
In this paper, we propose an estimation procedure based on the penalized EM algorithm. Following Kozumi and Kobayashi (2011), we employ the location-scale mixture representation of

the ALD and rewrite our model as

$$t_{j\ell} \overset{d}{=} \vartheta_j + \phi_1\, z_{j\ell} + \phi_2\, \xi_{j\ell}\, \sqrt{\zeta_j\, z_{j\ell}}, \tag{2.10}$$

where $\phi_1 = \dfrac{1-2\,\tau}{\tau\,(1-\tau)}$, $\phi_2 = \sqrt{\dfrac{2}{\tau\,(1-\tau)}}$, $z_{j\ell} \sim \mathrm{Exp}\left(\dfrac{1}{\zeta_j}\right)$, $\xi_{j\ell} \sim \mathrm{N}\,(0,1)$, and $z_{j\ell}$ is independent of $\xi_{j\ell}$ for $\ell = 1,\ldots,m_j$, $j = 1,\ldots,J$. In (2.10), we regard $z_{j\ell}$ like $\boldsymbol{T}^{\star}_{j\ell}$ as missing data and denote all of them at the j^{th} combined stress level as $\boldsymbol{z}_j = (z_{j1},\ldots,z_{jm_j})$ for $j = 1,\ldots,J$, and $\boldsymbol{z} = (\boldsymbol{z}_1,\ldots,\boldsymbol{z}_J)$. Given $z_{j\ell}$ for $\ell = 1,\ldots,m_j$, $j = 1,\ldots,J$, we have a hierarchical model as follows

$$\begin{aligned} t_{j\ell}\,\big|\,z_{j\ell} &\sim \mathrm{N}\left(\vartheta_j + \phi_1\, z_{j\ell}\,,\ \phi_2^2\,\zeta_j\, z_{j\ell}\right)\\ z_{j\ell} &\sim \mathrm{Exp}\left(\frac{1}{\zeta_j}\right). \end{aligned}$$

Hence, given $\boldsymbol{z}$, the likelihood function based on the observed failure times $\boldsymbol{t}$ and the T-II PCS is

$$L(\boldsymbol{\theta}\,|\,\boldsymbol{t},\boldsymbol{z}) = \prod_{j=1}^{J} C_j \prod_{\ell=1}^{m_j} f\left(t_{j\ell}\,\big|\,z_{j\ell},\boldsymbol{\theta}\right)\left[1 - F\left(t_{j\ell}\,\big|\,z_{j\ell},\boldsymbol{\theta}\right)\right]^{R_{j,\ell}}, \tag{2.11}$$

where $C_j = n_j\,(n_j - 1 - R_{j,1})\ldots(n_j - m_j + 1 - \sum_{\ell=1}^{m_j-1} R_{j,\ell})$.
The combination of $(\boldsymbol{t},\boldsymbol{T}^{\star},\boldsymbol{z})$ makes the complete data set. The likelihood function based on the complete data is

$$\begin{aligned} L\left(\boldsymbol{\theta}\,\big|\,\boldsymbol{t},\boldsymbol{T}^{\star},\boldsymbol{z}\right) &= \prod_{j=1}^{J} C_j \prod_{\ell=1}^{m_j}\left[f\left(t_{j\ell}\,\big|\,z_{j\ell},\boldsymbol{\theta}\right)\times f(z_{j\ell})\times\prod_{r=1}^{R_{j,\ell}} f\left(T^{\star}_{j\ell r}\,\big|\,z_{j\ell},\boldsymbol{\theta},T^{\star}_{j\ell r} > t_{j\ell}\right)\right]\\ &= \prod_{j=1}^{J} C_j \prod_{\ell=1}^{m_j}\Bigg[\frac{1}{\sqrt{2\,\pi\,\phi_2^2\,\zeta_j\,z_{j\ell}}}\exp\left\{-\frac{\left(t_{j\ell}-\vartheta_j-\phi_1\,z_{j\ell}\right)^2}{2\,\phi_2^2\,\zeta_j\,z_{j\ell}}\right\}\times\frac{1}{\zeta_j}\exp\left\{-\frac{z_{j\ell}}{\zeta_j}\right\}\times\\ &\quad \prod_{r=1}^{R_{j,\ell}}\frac{1}{\sqrt{2\,\pi\,\phi_2^2\,\zeta_j\,z_{j\ell}}}\exp\left\{-\frac{\left(T^{\star}_{j\ell r}-\vartheta_j-\phi_1\,z_{j\ell}\right)^2}{2\,\phi_2^2\,\zeta_j\,z_{j\ell}}\right\}\Bigg]. \end{aligned} \tag{2.12}$$

The log-likelihood function based on the complete data $(\boldsymbol{t},\boldsymbol{T}^{\star},\boldsymbol{z})$ with ignoring the additive constants is

$$\begin{aligned} \mathcal{L}\left(\boldsymbol{\theta}\,\big|\,\boldsymbol{t},\boldsymbol{T}^{\star},\boldsymbol{z}\right) &= \sum_{j=1}^{J}\sum_{\ell=1}^{m_j}\Bigg[-\left(\frac{3+R_{j,\ell}}{2}\right)\log(\zeta_j) - \left(\frac{1+R_{j,\ell}}{2}\right)\log z_{j\ell} - \frac{z_{j\ell}}{\zeta_j} -\\ &\quad \frac{1}{2\,\phi_2^2\,\zeta_j}\left\{\frac{\left(t_{j\ell}-\vartheta_j-\phi_1\,z_{j\ell}\right)^2}{z_{j\ell}} + \sum_{r=1}^{R_{j,\ell}}\frac{\left(T^{\star}_{j\ell r}-\vartheta_j-\phi_1\,z_{j\ell}\right)^2}{z_{j\ell}}\right\}\Bigg]. \end{aligned} \tag{2.13}$$

Due to the initial value $\boldsymbol{\theta}^{(0)}$ of the unknown parameter vector, the estimate of the parameter vector can be obtained based on the penalized EM algorithm through two steps: the E-step and the M-step. Details of the two steps are provided in Appendices A and B.

In the proposed model, a set of smoothing parameters $\Lambda = \{\lambda_{ki}, i = 1, \ldots, q, k = 1, 2\}$ need to be specified. In this paper, we minimize the Akaike information criterion (AIC) for selecting Λ as

$$\mathrm{AIC}(\Lambda) = -\frac{2}{n}\mathcal{L}\left(\hat{\boldsymbol{\theta}}_\Lambda \mid \boldsymbol{t}\right) + \frac{2}{n}\mathrm{df}_\Lambda, \tag{2.14}$$

where n denotes the size of the sample, $\mathcal{L}(\hat{\boldsymbol{\theta}}_\Lambda \mid \boldsymbol{t})$ is the estimated pseudo log-likelihood function value dependent on Λ. df_Λ is the effective degrees of freedom of the SPCSALT model, which depends on Λ.

It is important to note that the intricacy of the estimated model can be quantified by computing the effective degrees of freedom of the model. According to Hastie and Tibshirani (1990), we can approximate the effective degrees of freedom of the proposed model as

$$\mathrm{df}_\Lambda = \sum_{k=1}^{2}\sum_{i=1}^{q}\mathrm{trace}(\mathbf{S}_{ki}) + 2, \tag{2.15}$$

where $\mathbf{S}_{ki}$ is a smoothing matrix for the efficacy of the i^{th} accelerated stress on the τ^{th} quantile or the scale parameter. Details on how to compute the smoothing matrix can be found in Hastie and Tibshirani (1990).

Predicting the lifetime distribution at different stress conditions is an important objective of the ALT model. Using the estimated $\hat{\boldsymbol{\theta}}$, the τ^{th} quantile and the scale parameter are estimated at each combined stress level $\boldsymbol{s}_j$ for $j = 0, 1, \ldots, J$, and for each $\tau \in (0, 1)$, respectively, as outlined below. The vector $\boldsymbol{s}_0 = (s_{10}, \ldots, s_{q0})^\top$ illustrates the corresponding use conditions.

$$\hat{\vartheta}(\boldsymbol{s}_j) = g_1^{-1}\left(\hat{a}_1 + \sum_{i=1}^{q}\hat{f}_{1i}(s_{ij})\right) = g_1^{-1}\left(\hat{a}_1 + \sum_{i=1}^{q}\sum_{p=1}^{L}\boldsymbol{B}^{(M)}(s_{ij})\,\hat{\boldsymbol{\omega}}_{1i}\right), \tag{2.16}$$

and

$$\hat{\zeta}(\boldsymbol{s}_j) = g_2^{-1}\left(\hat{a}_2 + \sum_{i=1}^{q}\hat{f}_{2i}(s_{ij})\right) = g_2^{-1}\left(\hat{a}_2 + \sum_{i=1}^{q}\sum_{p=1}^{L}\boldsymbol{B}^{(M)}(s_{ij})\,\hat{\boldsymbol{\omega}}_{2i}\right), \tag{2.17}$$

where $g_k^{-1}(x)$ is equal $\exp(x)$ and indicates the reverse functions of $g_k(x)$.

3 Numerical examples

In this section, we present the results of numerical studies to illustrate the performance of the proposed model. Since a particular quantile in the lower tail of the lifetime distribution at the use conditions is often the focus of interest to engineers and researchers, we consider $\tau = 0.15, 0.25$, and 0.35. These quantiles enable a comprehensive evaluation of the proposed model's ability to capture early failure behavior under the use conditions. To demonstrate the robustness of the proposed model, we generate the failure times from the generalized exponential (GE) distribution and the Weibull distribution under the T-II PCS and fit the SPCSALT model to the generated failure times in the simulation study.

For comparative purposes, we implement the parametric CSALT (PCSALT) model under the GE lifetime distribution and the ALD-based parametric QR model for the CSALT data set, which we refer to as PCSALT-GE and PCSALT-ALD in the real-world data set analysis, respectively.

3.1 Simulation study

To demonstrate how well the suggested model, SPCSALT, performs, we report the findings of a simulation study in this subsection.

The proposed model can be used in situations where there is one or more accelerated stresses. In particular, we consider the model with two accelerated stresses, temperature (indicated as s_1 = temp) and voltage (indicated as s_2 = volt), to describe how the combination of temperature and voltage affects the failure of the test unit. Following Escobar and Meeker (2006), the mean lifetime of the test unit with temperature and voltage accelerated stresses is measured using an empirical relationship as follows

$$\mu\,(\text{temp, volt}) = \frac{1}{R\,(\text{temp, volt})}\left(\frac{\text{volt}}{\delta_0}\right)^{\gamma_1}, \tag{3.1}$$

where γ_1 and δ_0 are particular parameters of the test unit, and $R\,(\text{temp, volt})$ is computed as

$$R\,(\text{temp, volt}) = \gamma_0 \exp\left\{\frac{-E_a}{k_1 \times \text{temp}}\right\} \exp\left\{\gamma_2 \log(\text{volt}) + \frac{\gamma_3 \log(\text{volt})}{k_1 \times \text{temp}}\right\}, \tag{3.2}$$

where k_1, the ideal gas constant, is equal to 8.414 $J\,mol^{-1}K^{-1}$, γ_0, γ_2, and γ_3 are characteristics of particular physical or chemical processes.

The population's mean and variance are often correlated in real-world situations. Taylor (1961) suggested an empirical variance-mean relationship described by a power function. The power function is widely used to investigate the relationship between the population variance and the population mean. So, we suppose the SD and the mean of the test unit lifetime have a power law relationship as

$$\sigma\,(\text{temp, volt}) = k_2 \times \mu^b\,(\text{temp, volt}), \tag{3.3}$$

where k_2 and b are positive coefficients. We determine the model parameters as $\delta_0 = \gamma_1 = 1$, $E_a = 10000\ J\,mol^{-1}$, $\gamma_0 = 10^{-4}\,h^{-1}$, $\gamma_2 = 2$, $\gamma_3 = 1000$, $k_2 = 0.1$, and $b = 1.2$.

We take into consideration five stress levels for s_1: 340, 355, 370, 385, and 400 (K), and five stress levels for s_2: 120, 140, 160, 180, and 200 (V). Therefore, there are $5 \times 5 = 25$ combined levels of s_1 and s_2. The sample size, n_j, and the number of observed failures, m_j, for each combined stress level j for $j = 1, \ldots, J$, are determined as $n_j = 14 - 2\times\left\lfloor \frac{j-1}{5} \right\rfloor$ and $m_j = 11 - 2\times\left\lfloor \frac{j-1}{5} \right\rfloor$, where $\lfloor . \rfloor$ denotes the floor function.

Three distinct censoring schemes, CS1, CS2, and CS3, are represented in Table 1. The conditions for use are $s_{10} = 320$ (K) and $s_{20} = 100$ (V).

According to the algorithm presented in Balakrishnan and Sandhu (1995) and the specified relationships (3.1)-(3.3), we simulate type-II progressively censored samples from the GE distribution and the Weibull distribution. By generating the failure times from the lifetime distributions outside the ALD family, it is possible to evaluate the performance of the proposed method under

Table 1: Prefixed T-II PCS, $R_{j,\ell}$ for $\ell = 1, \dots, m_j$, $j = 1, \dots, J$.

Scheme	$(R_{j,1}, \dots, R_{j,m_j})$
CS1	$(\mathbf{0}_{(m_j-1)}, n_j - m_j)$
CS2	$(n_j - m_j, \mathbf{0}_{(m_j-1)})$
CS3	$(n_j - m_j - 1, \mathbf{0}_{(m_j-2)}, n_j - m_j - 1)$

model misspecification, which is critical in practical reliability applications. Specifically, the order and number of B-spline basis functions are $M = 4$ corresponding to the cubic B-spline, and $L = 12$, respectively.
For each combination of the τ^{th} quantile and the censoring scheme, the SPCSALT model parameters are estimated using the proposed penalized EM algorithm.
Tables 2 and 3 provide the true and predicted values of the τ^{th} quantile of the GE distribution and the Weibull distribution under the use conditions, for each censoring scheme, corresponding to the quantiles $\tau = 0.15, 0.25$, and 0.35, based on 1000 replications, respectively.
Additionally, the corresponding relative bias (RB) criterion (in parentheses) has been calculated for each combination of the τ^{th} quantile and the censoring scheme.
Under the use conditions, if the predicted parameter in the d^{th} repeat for $d = 1, \dots, 1000$, is denoted as $\hat{\gamma}^{(d)}$, and the true value of the parameter as γ, the RB criterion is defined as follows

$$\text{RB} = \frac{1}{1000} \sum_{d=1}^{1000} \frac{\left| \hat{\gamma}^{(d)} - \gamma \right|}{\gamma}.$$

Table 2: True and predicted values of the τ^{th} quantile of the GE lifetime distribution at the use conditions for three different censoring schemes and $\tau = 0.15, 0.25$, and 0.35, based on 1000 replications.

		$\hat{\vartheta}(s_0)$		
τ	$\vartheta(s_0)$	CS1	CS2	CS3
0.15	489.1723	471.2552	478.6195	475.9080
		(0.0288)	(0.0157)	(0.0218)
0.25	556.8473	545.3041	541.319	539.4293
		(0.0204)	(0.0259)	(0.0276)
0.35	617.6606	607.1654	610.7861	612.1358
		(0.0229)	(0.0152)	(0.0105)

It can be seen in Tables 2 and 3 that the predicted τ^{th} quantile of the lifetime distributions by the proposed model is near the true τ^{th} quantile of the lifetime distributions for the three different censoring schemes at the use conditions for each $\tau = 0.15, 0.25$, and 0.35. Additionally, these predictions exhibit a small RB. Although the failure times have been generated from the lifetime distributions except for the ALD, these results demonstrate that the proposed model can accurately predict the lower quantiles of the lifetime distribution under the use conditions. Therefore, the SPCSALT model is robust under model misspecification. So, the proposed model

Table 3: True and predicted values of the τ^{th} quantile of the Weibull lifetime distribution at the use conditions for three different censoring schemes and $\tau = 0.15, 0.25$, and 0.35, based on 1000 replications.

		$\hat{\vartheta}(s_0)$		
τ	$\vartheta(s_0)$	CS1	CS2	CS3
0.15	453.6240	444.3584	456.7558	458.2489
		(0.0211)	(0.0070)	(0.0175)
0.25	553.0522	543.7513	561.4527	541.8870
		(0.0158)	(0.0217)	(0.0142)
0.35	636.2370	627.9047	620.1904	629.1347
		(0.0164)	(0.0285)	(0.0136)

provides an effective and robust framework for establishing warranty policies and assessing the risks of early failures associated with the lower tail of the lifetime distribution.

3.2 Real data analysis

The objective of this subsection is to compare the performance of the SPCSALT model with the PCSALT-GE model and the PCSALT-ALD model using a real-life data set to illustrate the advantages of the proposed model.

In the PCSALT-GE model, the following basic assumptions at each accelerated stress level s_j are considered for $j = 0, 1, \ldots, J$:

1. The unit lifetime follows the GE(α, η_j) distribution such that α is the shape parameter and η_j is the scale parameter.

2. The scale parameter, η_j, is related to the accelerated stress level s_j in the log-linear manner as follows, and the shape parameter, α, is constant.

$$\log(\eta_j) = d_1 + d_2\,\phi_j, \tag{3.4}$$

where d_1 and d_2 are unknown parameters and $\phi_j = \phi(s_j)$ is an increasing function of s_j, where $\phi(s_j)$ is defined as $\phi(s_j) = \log(s_j), -\dfrac{1}{s_j}$, and s_j for the inverse power law, the Arrhenius, and the exponential models, respectively. We choose the widely used exponential model. The parameter η_j for $j = 0, 1, \ldots, J$, can be expressed based on the link function (3.4) as follows

$$\eta_j = \eta_0 \exp\left\{d_2(s_j - s_0)\right\} = \eta_0\, \delta^{h_j}, \tag{3.5}$$

where $\delta = \exp\{d_2(s_1 - s_0)\} = \dfrac{\eta_1}{\eta_0} > 1$, η_0 is the scale parameter of the GE distribution at the use conditions and $h_j = \dfrac{s_j - s_0}{s_1 - s_0}$ for $j = 1, \ldots, J$ is the transformed accelerated stress level satisfying $h_J > h_{J-1} > \ldots > h_1 = 1$.

Within the PCSALT-ALD model, both the location parameter, ν_j, and the scale parameter, ξ_j, have the log-linear relationship with each accelerated stress level s_j for $j = 0, 1, \ldots, J$ as

$$\log(\nu_j) = b_1 + b_2 s_j, \tag{3.6}$$

and

$$\log(\xi_j) = c_1 + c_2 s_j, \tag{3.7}$$

respectively, where b_1, b_2, c_1, and c_2 are unknown parameters.
We take into account the real CSALT data set detailed in Hirose (1993). The failure times (measured in hours) of 29 solid electrical insulations exposed to voltage stress at 3, 4, 10, 15, and 20 Kv/mm levels are included in this data set. The number of units being tested, n_j, the number of observed failure times, m_j, the censoring scheme, R_j, and the observed failure times at each accelerated stress level s_j for $j = 1, \ldots, J$ are presented in Table 4. The use condition is set to $s_0 = 2\,kV/mm$.

Table 4: Number of units, n_j, number of observed failure times, m_j, censoring scheme, R_j, and observed failure times at each accelerated stress level s_j for $j = 1, \ldots, J$ for the real-life data set.

Stress level	Voltage (per unit length (Kv/mm))	n	m	R	Observed failure times
$j = 1$	3	6	5	(0, 0, 1, 0, 0)	3487, 3580, 7884, 9894, 19260
$j = 2$	4	6	5	(0, 0, 1, 0, 0)	397.4, 445.6, 592.3, 707, 1642
$j = 3$	10	6	5	(1, 0, 0, 0, 0)	25.2, 44.4, 46.4, 58.1, 92.2
$j = 4$	15	5	4	(0, 0, 1, 0)	9.9, 11.9, 12.4, 20.1
$j = 5$	20	6	3	(0,3,0)	6.8, 7.4, 11.7

We employ the modified Kolmogorov-Smirnov goodness-of-fit test for the type-II progressively censored data, which was proposed by Pakyari and Balakrishnan (2012), to assess the adequacy of the ALD for each $\tau = 0.15$, 0.25, and 0.35, and of the GE lifetime distribution. The test statistic values and their corresponding P-values for each accelerated stress level are computed and presented in Table 5. These results indicate that there is no statistically significant lack of fit at the 5% significance level, supporting the suitability of the ALD for each $\tau = 0.15$, 0.25, and 0.35, and of the GE lifetime distribution for fitting lifetime data under the considered T-II PCS.
We use the SPCSALT, PCSALT-GE, and PCSALT-ALD models to analyze the real-life data set and select the order of the B-spline basis functions as $M = 3$, indicating the quadratic B-spline. The basis dimension is set to $L = 12$, as this choice yields the minimum BIC value for each quantile level considered.
The estimated parameters in the PCSALT-GE model are $\hat{\boldsymbol{\theta}}_{\text{PCSALT-GE}} = \left\{\hat{\alpha}, \hat{\eta}_0, \hat{\delta}\right\} = \{0.9316, 0.0001, 1.4398\}$. For $\tau = 0.15, 0.25$, and 0.35, the estimated parameters obtained from the PCSALT-ALD model are shown in Table 6.
Table 7 presents the sample and estimated τ^{th} quantile by the SPCSALT, PCSALT-GE, and PCSALT-ALD models for $\tau = 0.15, 0.25$, and 0.35, as well as the corresponding relative error (RE) criterion (in parentheses) for each accelerated stress level.

Table 5: Values of the modified Kolmogorov-Smirnov test statistic and the corresponding P-values at each accelerated stress level for the ALD for $\tau = 0.15, 0.25$, and 0.35, and for the GE distribution.

	Stress	3	4	10	15	20
$\tau = 0.15$	Test statistic	0.2910	0.3316	0.3340	0.4124	0.5888
	P-value	0.8080	0.4260	0.4301	0.2510	0.3280
$\tau = 0.25$	Test statistic	0.3008	0.3334	0.2750	0.4220	0.5984
	P-value	0.7150	0.4110	0.9620	0.1870	0.2502
$\tau = 0.35$	Test statistic	0.3030	0.3403	0.2812	0.4278	0.6050
	P-value	0.7580	0.3350	0.9260	0.1300	0.2130
GE	Test statistic	0.3136	0.3480	0.3924	0.4431	0.6478
	P-value	0.6560	0.3590	0.3360	0.1630	0.2510

Table 6: Estimated parameters by the PCSALT-ALD model for $\tau = 0.15, 0.25$, and 0.35.

	$\hat{b}_1$	$\hat{b}_2$	$\hat{c}_1$	$\hat{c}_2$
$\tau = 0.15$	8.1109	-0.4317	6.5267	-0.3996
$\tau = 0.25$	8.4771	-0.4683	6.9975	-0.3934
$\tau = 0.35$	8.7057	-0.4181	7.3892	-0.4002

Let $\hat{\gamma}_j$ represents the estimated value and γ_j denotes the true value of a parameter at the j^{th} stress level for $j = 1, \ldots, J$, then the RE_j criterion is defined as

$$\text{RE}_j = \frac{\left|\hat{\gamma}_j - \gamma_j\right|}{\gamma_j}.$$

It is concluded from Table 7 that for $\tau = 0.15, 0.25$, and 0.35, the proposed model estimates the τ^{th} quantile in close agreement with the corresponding sample quantiles. In contrast, the estimates obtained from the PCSAL-GE and PCSALT-ALD models exhibit significant deviations from the sample τ^{th} quantile. The proposed model outperforms the traditional parametric models and effectively captures the lower tail behavior of the lifetime distribution, which is crucial in the ALT applications.

In addition, the BIC values for the SPCSALT model at $\tau = 0.15, 0.25$, and 0.35 are 287.6275, 289.8542, and 292.6901, respectively, all of which are substantially smaller than the corresponding BIC value of 318.8761 obtained from the PCSALT-GE model, as well as smaller than the corresponding BIC values obtained from the PCSALT-ALD model at the same quantile levels, 373.1357, 376.2671, and 376.9014, respectively. These results indicate that the proposed model achieves a better balance between goodness of fit and model complexity than both competing models; thus, it provides a more accurate and reliable representation of lifetime data.

Prediction of the τ^{th} quantile and the scale parameter under the use condition is another way to compare the performance of the proposed SPCSALT model and the traditional PCSALT-GE and PCSALT-ALD models. At the use condition, the SPCSALT model predicts the τ^{th} quantile for $\tau = 0.15, 0.25$, and 0.35 as 53734.2400 h, 54240.2700 h, and 152964.8000 h, respectively. However, the corresponding predictions obtained from the PCSALT-GE model are 678.1942 h,

Table 7: Sample and estimated values of the τ^{th} quantile for $\tau = 0.15, 0.25$, and 0.35, as well as the corresponding RE criterion (in parentheses) at each accelerated stress level for the real-life data set.

	Stress	3	4	10	15	20
$\tau = 0.15$	Sample value	3556.7500	433.5500	39.6000	11.1000	7.2500
	SPCSALT	3580.0590	445.6080	35.8672	9.8997	6.8001
		(0.0065)	**(0.0278)**	**(0.0942)**	**(0.1081)**	**(0.0620)**
	PCSALT-GE	971.1951	674.5347	75.7166	12.2371	1.9777
		(0.7258)	(0.5822)	(1.0620)	(0.1330)	(0.7166)
	PCSALT-ALD	912.0283	592.2305	44.4002	5.1262	0.5918
		(0.7425)	(0.3891)	(0.2091)	(0.5253)	(0.9152)
$\tau = 0.25$	Sample value	4656.0000	482.2750	44.4250	11.9000	7.5500
	SPCSALT	4385.3600	462.2093	44.4001	12.3350	7.3477
		(0.0581)	**(0.0416)**	**(0.0005)**	**(0.0365)**	**(0.0268)**
	PCSALT-GE	1777.5581	1234.5868	138.5825	22.3974	3.6198
		(0.5034)	(1.7706)	(2.1212)	(0.9646)	(0.4901)
	PCSALT-ALD	1178.4555	737.7251	42.3016	4.2686	0.4104
		(0.6708)	(0.6555)	(0.0472)	(0.6255)	(0.9422)
$\tau = 0.35$	Sample value	6808.0000	555.6250	44.4750	12.1000	7.8500
	SPCSALT	6520.2021	592.3105	44.4210	12.3970	7.4001
		(0.0422)	**(0.0660)**	**(0.0012)**	**(0.0245)**	**(0.0573)**
	PCSALT-GE	2719.5465	1889.12	212.1423	34.2920	5.5391
		(0.4870)	(2.7461)	(3.6934)	(1.8756)	(0.2328)
	PCSALT-ALD	1721.8998	1133.4356	92.2001	11.3940	1.4080
		(0.6752)	(1.2476)	(1.0398)	(0.0445)	(0.8050)

1051.0014 h, and 1872.4080 h, respectively, and the corresponding predictions obtained from the PCSALT-ALD model are 1404.5130 h, 1882.4860 h, and 2615.8870 h, respectively. It's obvious that the parametric models underestimate the τ^{th} quantile for all τ.

Moreover, the SPCSALT model predicts the scale parameter at the use condition for $\tau = 0.15, 0.25$, and 0.35 as 36026.5800 h, 54737.0400 h, and 99376.3400 h, respectively.

The significant difference between the three models arises because the lower quantiles increase at a much faster rate than what is suggested by the log-linear relationship as the stress level decreases. Therefore, the PCSALT-GE and PCSALT-ALD models fail to capture this true relationship, leading to a substantial underestimation of the lifetime quantiles. In contrast, by applying a flexible semiparametric structure, the proposed SPCSALT model can accommodate this nonlinear behavior, resulting in more realistic extrapolation and superior predictive performance under the use condition.

4 Conclusion

In most existing parametric CSALT models, there are parametric relationships between the lifetime distribution characteristics and the accelerated stresses. However, in many real-world conditions, the true relationships are complex and unknown. Therefore, using existing models can lead to errors in data analysis and bias in estimating the unit lifetime under the use conditions.

The lower quantiles are often the focus of interest to engineers and researchers, as they provide a straightforward and interpretable framework for risk evaluation related to early failures and warranty and reliability guarantees. The ALD is an effective method for analyzing ALT data because it offers a robust framework for examining this data and accurately estimating lower lifetime quantiles. This is especially useful when the underlying lifetime distribution is unknown, as it allows for direct likelihood-based inference on quantiles of interest.

In this study, we presented the SPCSALT model based on the location-scale mixture representation of the ALD and type-II progressively censored data. In the SPCSALT model, we relaxed the second aforesaid parametric assumption and applied the B-spline approach to approximate the nonparametric relationships between the accelerated stresses and the location parameter and the scale parameter. We estimated the parameter vector of the proposed model via the penalized EM algorithm.

The proposed model is appropriate to deal with one or more accelerated stress scenarios. In the multiple accelerated stresses case, we used an additive modeling method.

Also, to compare the proposed model with the PCSALT models, we analyzed a real-world case study. The result illustrated that the proposed model provides a better fit than the traditional parametric models.

Appendix A

This section offers details on the E-step and the M-step of the proposed EM algorithm. $\boldsymbol{W} = (\boldsymbol{T}^\star, \boldsymbol{z})$ is the incomplete data.

E-step. Given the $(d-1)^{\text{th}}$ iteration value of the unknown parameter vector, $\boldsymbol{\theta}^{(d-1)}$, the Q function of the d^{th} iteration is given by

$$
\begin{aligned}
Q\left(\boldsymbol{\theta} \,\middle|\, \boldsymbol{\theta}^{(d-1)}\right) &= \mathrm{E}_{\boldsymbol{W} \mid \boldsymbol{t}, \boldsymbol{\theta}^{(d-1)}}\left[\mathcal{L}\left(\boldsymbol{\theta} \,\middle|\, \boldsymbol{t}, \boldsymbol{T}^\star, \boldsymbol{z}\right)\right] \\
&= \sum_{j=1}^{J} \sum_{\ell=1}^{m_j} \Bigg[-\left(\frac{3+R_{j,\ell}}{2}\right) \log \zeta_j - \left(\frac{1+R_{j,\ell}}{2}\right) \mathrm{E}\left(\log z_{j\ell} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - \frac{1}{\zeta_j}\ \mathrm{E}\left(z_{j\ell} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - \\
&\quad \frac{1}{2\,\phi_2^2\,\zeta_j}\Bigg\{ \mathrm{E}\left(\frac{\left(t_{j\ell}-\vartheta_j\right)^2}{z_{j\ell}} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - 2\,\phi_1 \left(t_{j\ell}-\vartheta_j\right) + \phi_1^2\ \mathrm{E}\left(z_{j\ell} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \\
&\quad \sum_{r=1}^{R_{j,\ell}} \Bigg[\mathrm{E}\left(\frac{T_{j\ell r}^{\star 2}}{z_{j\ell}} \,\middle|\, T_{j\ell r}^\star > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \vartheta_j^2\ \mathrm{E}\left(\frac{1}{z_{j\ell}} \,\middle|\, T_{j\ell r}^\star > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \phi_1^2\ \mathrm{E}\left(z_{j\ell} \,\middle|\, T_{j\ell r}^\star > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) -
\end{aligned}
$$

$$
\begin{aligned}
& 2\,\vartheta_j\ \mathrm{E}\left(\frac{T^{\star}_{j\ell r}}{z_{j\ell}}\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - 2\,\phi_1\ \mathrm{E}\left(T^{\star}_{j\ell r}\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + 2\,\phi_1\,\vartheta_j\Bigg]\Bigg\}\Bigg] \\
&= \sum_{j=1}^{J}\sum_{\ell=1}^{m_j}\Bigg[-\left(\frac{3+R_{j,\ell}}{2}\right)\log\zeta_j - \left(\frac{1+R_{j,\ell}}{2}\right)\mathrm{E}\left(\log z_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - \frac{1}{\zeta_j}\ \mathrm{E}\left(z_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - \\
& \frac{1}{2\,\phi_2^2\,\zeta_j}\Bigg\{\left(t_{j\ell}-\vartheta_j\right)^2\mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - 2\,\phi_1(t_{j\ell}-\vartheta_j) + \phi_1^2\ \mathrm{E}\left(z_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \\
& \sum_{r=1}^{R_{j,\ell}}\Bigg[\mathrm{E}\left(T^{\star 2}_{j\ell r}\ \mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \vartheta_j^2\ \mathrm{E}\left(\mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + \\
& \phi_1^2\ \mathrm{E}\left(\mathrm{E}\left(z_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - 2\,\vartheta_j\ \mathrm{E}\left(T^{\star}_{j\ell r}\,\mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) - \\
& 2\,\phi_1\ \mathrm{E}\left(T^{\star}_{j\ell r}\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) + 2\,\phi_1\,\vartheta_j\Bigg]\Bigg\}\Bigg] \\
&= \sum_{j=1}^{J}\sum_{\ell=1}^{m_j}\Bigg[-\left(\frac{3+R_{j,\ell}}{2}\right)\log\zeta_j - \left(\frac{1+R_{j,\ell}}{2}\right)\Delta 1_{j\ell} - \frac{1}{\zeta_j}\,\Delta 2_{j\ell} - \\
& \frac{1}{2\,\phi_2^2\,\zeta_j}\Bigg\{\left(t_{j\ell}-\vartheta_j\right)^2\Delta 3_{j\ell} - 2\,\phi_1\left(t_{j\ell}-\vartheta j\right) + \phi_1^2\,\Delta 2_{j\ell} + \\
& \sum_{r=1}^{R_{j,\ell}}\Bigg[\Delta 4^{\star}_{j\ell} + \vartheta_j^2\,\Delta 3^{\star}_{j\ell} + \phi_1^2\,\Delta 2^{\star}_{j\ell} - 2\,\vartheta_j\,\Delta 5^{\star}_{j\ell} - 2\,\phi_1\,\Delta 6^{\star}_{j\ell} + 2\,\phi_1\,\vartheta_j\Bigg]\Bigg\}\Bigg], \qquad \text{(A.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta 1_{j\ell} &= \mathrm{E}\left(\log z_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 2_{j\ell} &= \mathrm{E}\left(z_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 3_{j\ell} &= \mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 2^{\star}_{j\ell} &= \mathrm{E}\left(\mathrm{E}\left(z_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 3^{\star}_{j\ell} &= \mathrm{E}\left(\mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 4^{\star}_{j\ell} &= \mathrm{E}\left(T^{\star 2}_{j\ell r}\ \mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 5^{\star}_{j\ell} &= \mathrm{E}\left(T^{\star}_{j\ell r}\ \mathrm{E}\left(z^{-1}_{j\ell}\,\middle|\,T^{\star}_{j\ell r}\right)\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right), \\
\Delta 6^{\star}_{j\ell} &= \mathrm{E}\left(T^{\star}_{j\ell r}\,\middle|\,T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right)\cdot
\end{aligned}
$$

To obtain $\Delta 1_{j\ell}$, $\Delta 2_{j\ell}$, and $\Delta 3_{j\ell}$ for $\ell = 1, \dots, m_j$, $j = 1, \dots, J$ firstly, we obtain the conditional pdf $f\left(z_{j\ell}\,\middle|\,t_{j\ell}\right)$. It has been noted that the joint pdf of $t_{j\ell}$ and $z_{j\ell}$ is

$$
f\left(t_{j\ell}, z_{j\ell}\,\middle|\,\boldsymbol{\theta}\right) = \frac{1}{\sqrt{2\pi\,\phi_2^2\,\zeta_j\,z_{j\ell}}}\ \exp\left\{-\frac{\left(t_{j\ell}-\vartheta_j-\phi_1\,z_{j\ell}\right)^2}{2\,\phi_2^2\,\zeta_j\,z_{j\ell}}\right\}\times\frac{1}{\zeta_j}\exp\left\{-\frac{z_{j\ell}}{\zeta_j}\right\}.
$$

Then, we have

$$\begin{aligned} f\left(z_{j\ell} \,\middle|\, t_{j\ell}\right) &\propto f\left(t_{j\ell}, z_{j\ell} \,\middle|\, \boldsymbol{\theta}\right) \\ &\propto \frac{1}{\sqrt{z_{j\ell}}} \exp\left\{-\frac{\left(t_{j\ell} - \vartheta_j - \phi_1 \, z_{j\ell}\right)^2}{2\,\phi_2^2\,\zeta_j\, z_{j\ell}} - \frac{z_{j\ell}}{\zeta_j}\right\} \\ &\propto \frac{1}{\sqrt{z_{j\ell}}} \exp\left\{-\frac{1}{2}\left[\left(\frac{\left(t_{j\ell} - \vartheta_j\right)^2}{\phi_2^2\,\zeta_j}\right) z_{j\ell}^{-1} + \left(\frac{\phi_1^2 + 2\,\phi_2^2}{\phi_2^2\,\zeta_j}\right) z_{j\ell}\right]\right\}. \end{aligned}$$

Thus, the marginal conditional pdf of $z_{j\ell}$ is

$$\begin{aligned} f\left(z_{j\ell} \,\middle|\, t_{j\ell}\right) &\propto \frac{1}{\sqrt{z_{j\ell}}} \exp\left\{-\frac{1}{2}\left[\left(\frac{\left(t_{j\ell} - \vartheta_j\right)^2}{\phi_2^2\,\zeta_j}\right) z_{j\ell}^{-1} + \left(\frac{\phi_1^2 + 2\,\phi_2^2}{\phi_2^2\,\zeta_j}\right) z_{j\ell}\right]\right\} \\ &\sim \mathrm{GIG}\left(\frac{1}{2}, \frac{\left(t_{j\ell} - \vartheta_j\right)^2}{\phi_2^2\,\zeta_j}, \frac{\phi_1^2 + 2\,\phi_2^2}{\phi_2^2\,\zeta_j}\right), \end{aligned}$$

where $\mathrm{GIG}(\kappa, \chi, \varphi)$ is the generalized inverse Gaussian distribution with pdf as

$$Y \sim f_Y(y) = \frac{\left(\frac{\varphi}{\chi}\right)^{\frac{\kappa}{2}}}{2\,K_\kappa\left(\sqrt{\chi\,\varphi}\right)}\, y^{\kappa-1} \exp\left\{-\frac{1}{2}\left(\chi\, y^{-1} + \varphi\, y\right)\right\}, \qquad y > 0,$$

where K_κ denotes the modified Bessel function of the third kind with index κ. $\kappa \in \mathbb{R}$ is the shape parameter, χ and φ are the scale parameters. The parameters must satisfy the following conditions: $\chi > 0, \varphi \geq 0$ when $\kappa < 0$; $\chi > 0, \varphi > 0$ when $\kappa = 0$; and $\chi \geq 0, \varphi > 0$ when $\kappa > 0$. If $\chi > 0$ and $\varphi > 0$, then for each $\gamma \in \mathbb{R}$, we have

$$\mathrm{E}\left(Y^\gamma\right) = \left(\frac{\chi}{\varphi}\right)^{\frac{\gamma}{2}} \frac{K_{\kappa+\gamma}\left(\sqrt{\chi\,\varphi}\right)}{K_\kappa\left(\sqrt{\chi\,\varphi}\right)}.$$

Therefore, we can acquire

$$\Delta 1_{j\ell} = \mathrm{E}\left(\log z_{j\ell} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) = \frac{\mathrm{d}\,\mathrm{E}\left(z_{j\ell}^{\gamma} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right)}{\mathrm{d}\gamma}\Big|_{\gamma=0},$$

$$\Delta 2_{j\ell} = \mathrm{E}\left(z_{j\ell} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) = \frac{\left|\, t_{j\ell} - \vartheta_j^{(d-1)} \right|}{\sqrt{\phi_1^2 + 2\,\phi_2^2}} \cdot \frac{K_{\frac{1}{2}+1}\left(\frac{\left|\, t_{j\ell} - \vartheta_j^{(d-1)} \right| \cdot \sqrt{\phi_1^2 + 2\,\phi_2^2}}{\phi_2^2\,\zeta_j^{(d-1)}}\right)}{K_{\frac{1}{2}}\left(\frac{\left|\, t_{j\ell} - \vartheta_j^{(d-1)} \right| \cdot \sqrt{\phi_1^2 + 2\,\phi_2^2}}{\phi_2^2\,\zeta_j^{(d-1)}}\right)}$$

$$= \frac{\phi_2^2\, \zeta_j^{(d-1)}}{\phi_1^2 + 2\,\phi_2^2} + \frac{\left| t_{j\ell} - \vartheta_j^{(d-1)} \right|}{\sqrt{\phi_1^2 + 2\,\phi_2^2}},$$

$$\Delta 3_{j\ell} = \mathrm{E}\left(z_{j\ell}^{-1} \,\middle|\, t_{j\ell}, \boldsymbol{\theta}^{(d-1)} \right) = \frac{\sqrt{\phi_1^2 + 2\,\phi_2^2}}{\left| t_{j\ell} - \vartheta_j^{(d-1)} \right|}.$$

The $\Delta 2^\star_{j\ell}$, $\Delta 3^\star_{j\ell}$, $\Delta 4^\star_{j\ell}$, $\Delta 5^\star_{j\ell}$, and $\Delta 6^\star_{j\ell}$ can be obtained by generating $T^\star_{j\ell r}$ for $r = 1, \ldots, R_{j,\ell}$, $\ell = 1, \ldots, m_j$, and $j = 1, \ldots, J$, from the truncated ALD (see the Appendix B).

When the number of basis functions in the B-spline method and the number of accelerated stresses are large, the number of model parameters can also become large. In such cases, estimating these parameters may become computationally unstable, leading to potential overfitting. To avoid the overfitting problem in the approximation of f_{ki}, quadratic penalty term given by $-\frac{1}{2}\,\lambda_{ki}\,\boldsymbol{\beta}_{ki}^\top\,\boldsymbol{\Omega}_{ki}\,\boldsymbol{\beta}_{ki}$ is used for each smooth functions f_{ki} for $i = 1, \ldots, q$, $k = 1, 2$. The $\lambda_{ki} \geq 0$ is a smoothing parameter controlling the amount of smoothness of f_{ki}. The $\boldsymbol{\Omega}_{ki}$ is a known, nonnegative definite, and symmetric penalty matrix. Thus, the penalized pseudo log-likelihood function is

$$\begin{aligned} Q_P\left(\boldsymbol{\theta} \,\middle|\, \boldsymbol{\theta}^{(d-1)}\right) = \sum_{j=1}^{J} \sum_{\ell=1}^{m_j} \Bigg[&-\left(\frac{3 + R_{j,\ell}}{2}\right) \log \zeta_j - \left(\frac{1 + R_{j,\ell}}{2}\right) \Delta 1_{j\ell} - \frac{1}{\zeta_j}\,\Delta 2_{j\ell} - \\ &\frac{1}{2\,\phi_2^2\,\zeta_j} \Bigg\{ \left(t_{j\ell} - \vartheta_j\right)^2 \Delta 3_{j\ell} - 2\,\phi_1 \left(t_{j\ell} - \vartheta j\right) + \phi_1^2\,\Delta 2_{j\ell} + \\ &\sum_{r=1}^{R_{j,\ell}} \left[\Delta 4^\star_{j\ell} + \vartheta_j^2\,\Delta 3^\star_{j\ell} + \phi_1^2\,\Delta 2^\star_{j\ell} - 2\,\vartheta_j\,\Delta 5^\star_{j\ell} - 2\,\phi_1\,\Delta 6^\star_{j\ell} + 2\,\phi_1\,\vartheta_j \right] \Bigg\} \Bigg] - \\ &\frac{1}{2} \sum_{k=1}^{2} \sum_{i=1}^{q} \lambda_{ki}\,\boldsymbol{\beta}_{ki}^\top\,\boldsymbol{\Omega}_{ki}\,\boldsymbol{\beta}_{ki}. \end{aligned} \tag{A.2}$$

M-step. If at the d^{th} iteration, the estimate of $\boldsymbol{\theta}$ is $\boldsymbol{\theta}^{(d)}$, then $\boldsymbol{\theta}^{(d+1)}$ can be obtained by maximizing the penalized pseudo log-likelihood function in (A.2) with respect to $\boldsymbol{\theta}$. The maximization of (A.2) can be obtained by the Newton-Raphson algorithm. Thus, the so-called Newton step (N-step) takes the role of the M-step.

The nested optimization process is in the following manner: an inner loop uses a constant Λ to optimize the parameter vector $\boldsymbol{\theta}$ through the penalized EM algorithm. The algorithm iterations are continued until the maximum relative change between two successive parameter vector estimates satisfies

$$\left\| \frac{\boldsymbol{\theta}^{(d+1)} - \boldsymbol{\theta}^{(d)}}{\boldsymbol{\theta}^{(d+1)}} \right\|_\infty < 10^{-6}.$$

Then, Λ is updated in an outer loop based on the AIC minimization using the `optim()` function in R, with a relative convergence tolerance of 10^{-5} and a maximum of 1000 iterations. After convergence of Λ optimization, the penalized EM algorithm is performed using $\hat{\Lambda}$ to obtain the final parameter vector estimate.

Appendix B

In this section, we obtain $\Delta 4^{\star}_{j\ell}$. The $\Delta 2^{\star}_{j\ell}$, $\Delta 3^{\star}_{j\ell}$, $\Delta 5^{\star}_{j\ell}$, $\Delta 6^{\star}_{j\ell}$ for $\ell = 1, \ldots, m_j$, $j = 1, \ldots, J$, can be acquired in a similar way.

$$
\begin{aligned}
\Delta 4^{\star}_{j\ell} &= \mathrm{E}\left(T^{\star 2}_{j\ell r}\, \mathrm{E}\left(z^{-1}_{j\ell} \,\middle|\, T^{\star}_{j\ell r}\right) \,\middle|\, T^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) \\
&= \int \int t^{\star 2}_{j\ell r}\, z^{-1}_{j\ell}\, f\left(z_{j\ell} \,\middle|\, t^{\star}_{j\ell r}\right) f\left(t^{\star}_{j\ell r} \,\middle|\, t^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) dz_{j\ell}\, dt^{\star}_{j\ell r} \\
&= \int t^{\star 2}_{j\ell r} \left[\int z^{-1}_{j\ell}\, f\left(z_{j\ell} \,\middle|\, t^{\star}_{j\ell r}\right) dz_{j\ell}\right] f\left(t^{\star}_{j\ell r} \,\middle|\, t^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(d-1)}\right) dt^{\star}_{j\ell r} \qquad \text{(B.1)} \\
&= \int t^{\star 2}_{j\ell r} \left[\frac{\sqrt{\phi_1^2 + 2\,\phi_2^2}}{\left|t^{\star}_{j\ell r} - \vartheta^{(d-1)}_j\right|}\right] f\left(t^{\star}_{j\ell r} \,\middle|\, t^{\star}_{j\ell r} > t_{j\ell}, \boldsymbol{\theta}^{(u-1)}\right) dt^{\star}_{j\ell r} \\
&= \mathrm{E}\left[\frac{T^{\star 2}_{j\ell r}\,\sqrt{\phi_1^2 + 2\,\phi_2^2}}{\left|T^{\star}_{j\ell r} - \vartheta^{(d-1)}_j\right|} \,\middle|\, T^{\star}_{j\ell r} > t_{j\ell}\right], \qquad \text{(B.2)}
\end{aligned}
$$

where the equality in the fourth line holds because, if $t^{\star}_{j\ell r}$ were available, it would be a realization of the ALD $\left(\vartheta_j, \zeta_j, \tau\right)$ and, consequently, the inner integral in (B.1) would be equal to the $\Delta 2_{j\ell}$. Finally, the expectation in (B.2) can be approximated by generating 20000 realizations of $T^{\star}_{j\ell r}$ from truncated ALD $\left(\vartheta_j, \zeta_j, \tau; (t_{j\ell}\,,\ +\infty)\right)$.

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