JIRSS (2026)
Vol. 25, No. 2, pp 137-156
DOI: 10.22034/jirss.2026.2080191.1168

Evaluation of the efficiency of third-order response surface designs in the spherical region

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Received: 2025-12-07, Accepted: 2026-07-16, Published online: 2026-08-16

Abstract. Third-order response surface designs (RSDs) are essential when second-order models fail to capture higher-order curvature or complex prediction structures. Although several third-order designs have been proposed, comparative evaluations across multiple efficiency criteria remain limited. This study evaluates the efficiencies of five third-order RSDs: augmented Box-Behnken designs (ABBDs), augmented fractional Box-Behnken designs (AFBBDs), new augmented Box-Behnken designs (NABBDs), orthogonal array composite designs ($OACD_4$), and orthogonal uniform composite designs ($OUCD_4$) in the spherical region. These designs were assessed using experimental cost efficiencies (run-size and degree-of-freedom), parameter estimation efficiencies (D- and A-efficiency), and prediction accuracy efficiencies (G-, I-, T-, and per-observation G-efficiency) for $3 \leq k \leq 6$ factors using $0 \leq n_0 \leq 5$ center points. The results show that increasing the number of factors and center points raises run size, D-efficiency, and A-efficiency but reduces degree-of-freedom, G-, I-, and per-observation G-efficiencies. $OACD_4$ and $OUCD_4$ were the most economical; NABBDs and $OUCD_4$ showed the highest D- and A-efficiency values, respectively; and $OACD_4$ showed the best prediction performance based on G- and per-observation G-efficiencies. Overall, $OACD_4$ provided the best balance between experimental economy and prediction performance across most criteria.

Keywords. Third-order designs, Box-Behnken designs, Orthogonal composite designs, Prediction variance, Design efficiency.

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MSC: 62-XX, 62Rxx, 62R10.

1 Introduction

Response surface methodology (RSM) provides a structured and efficient approach for modelling and optimizing systems in which a response variable depends on several quantitative factors (Box and Wilson, 1951).The primary objective of RSM is to develop an empirical model that adequately represents the underlying process for analysis, prediction, and optimization (Khuri, 2017; Didiugwu et al., 2024). Depending on the complexity of the experimental system, RSM employs first-, second-, or higher-order response surface designs (RSDs). Second-order RSDs are widely used because of their moderate run sizes and ability to estimate curvature in many practical situations (Atkinson and Donev, 1992; Myers et al., 2009).However, several studies have shown that quadratic models may exhibit lack-of-fit when the true response surface contains higher-order nonlinearities that cannot be adequately represented by second-order terms. In such situations, third-order models become more appropriate because they accommodate cubic effects, more flexible curvature, and complex interaction structures. These models also improve prediction accuracy and support pure-error estimation and lack-of-fit testing (Mee, 2007; Arshad et al., 2012; Verma et al., 2024; Yankam and Oladugba, 2024).

Several third-order RSDs have been proposed in the literature to address these limitations. These include nested face-centered designs (Landman et al., 2007), augmented Box-Behnken designs (ABBDs) (Arshad et al., 2012), augmented fractional Box-Behnken designs (AFBBDs) (Rashid et al., 2017), new augmented Box-Behnken designs (NABBDs) (Arshad et al., 2020), third-order orthogonal array composite designs ($OACD_4$) (Zhang et al., 2018), sequential asymmetric third-order rotatable designs(Hemavathi et al., 2020), and third-order orthogonal uniform composite designs ($OUCD_4$) (Yankam and Oladugba, 2022). These designs differ substantially in terms of structure, run size, estimation precision, and prediction properties. Consequently, selecting an appropriate design requires systematic comparison across multiple efficiency criteria because practitioners often seek designs that balance experimental economy, parameter estimation precision, and prediction performance simultaneously.

Many criteria exist for selecting competing RSDs, among which D-, A-, G-, and I-optimality criteria and related efficiency measures are the most commonly used (Wangui et al., 2019; Oladugba and Yankam, 2022; Ezievuo et al., 2025; Oladugba et al., 2025). Graphical methods such as variance dispersion graphs (VDG), fraction of design space (FDS) plots, and quantile dispersion graphs (QDG) are also useful for assessing prediction variance behaviour and complementing numerical efficiency measures. Another important consideration in response surface experimentation is the number of center points, which influences process stability, pure-error estimation, and lack-of-fit testing. Draper (1982) noted that center points improve design stability and efficiency estimation, whereas Lucas (1974) and Morris (2000) cautioned that excessive center points may adversely affect design efficiency. Myers et al. (1992) and Arshad et al. (2012) recommended the use of between two and five center points in practical applications.

Despite the availability of several third-order RSDs, most existing studies have focused on individual designs using limited optimality criteria. Comprehensive comparative studies integrating experimental economy, parameter estimation efficiency, prediction performance, and graphical assessment using FDS plots remain limited, particularly in the spherical region. Furthermore, designs with strong parameter estimation properties may not necessarily exhibit superior prediction performance, thereby creating important trade-offs among competing efficiency criteria. Therefore, this study evaluates the performance of five RSDs: ABBDs, AFBBDs, NABBDs, $OACD_4$, and $OUCD_4$ for $3 \leq k \leq 6$ factors using $0 \leq n_0 \leq 5$ center points in the spherical region. The designs are compared using efficiency criteria based on experimental economy (run-size and degree-of-freedom efficiencies), parameter estimation quality (D- and A-efficiencies), and prediction performance (G-, I-, T-, and per-observation G-efficiencies), together with graphical assessment using FDS plots.

The remainder of this study is organized as follows: 2 presents the methodology, including the design structures, the third-order model, and the design efficiency criteria; 3 presents and discusses the results; and 4 provides the conclusion.

2 Materials and Methodology

This section is divided into sources of designs for comparison, third-order models, design efficiency criteria (in terms of run size, degrees of freedom, D-, A-, G-, I-, T- and per-observation G-efficiency).

2.1 Sources of designs

This section presents the five third-order RSDs considered in this study. The design points for all the compared designs are summarized in Table 6. For all designs, the axial distance α was set to $\alpha = \sqrt{k}$ to achieve a spherical experimental region following the construction procedures of Arshad et al. (2012); Rashid et al. (2017); Zhang et al. (2018); Yankam and Oladugba (2022). All analyses were performed using Design-Expert version 13 and R software.

Augmented Box-Behnken Designs (ABBDs)

Arshad et al. (2012) extended second-order Box-Behnken designs to develop augmented Box-Behnken designs (ABBDs) for third-order model estimation. The ABBDs are formed by adding factorial points of levels $'a'$ denoted by $F(a)^k$ and axial points denoted by $A[\alpha]^k$ of levels $'\alpha'$ into BBD, where k is the number of factors. For some cases (e.g., for k = 6, 7, ..., 12), complementary points denoted by $B'[k]$ are also added. The factorial and axial levels are not simultaneously equal to one because ABBDs require at least four levels for third-order estimation. A typical example is given in Table 1.

Table 1: Typical Layout of ABBD for k = 3 factors with n_0 = 2

Box–Behnken portion $B[3]$			**Factorial portion** $F(\alpha)^k$			**Axial portion** $A(\alpha)^3$			**Center point** (n_0)		
-1	1	0	a	a	a	α	0	0	0	0	0
-1	-1	0	a	a	-a	$-\alpha$	0	0	0	0	0
1	-1	0	a	-a	a	0	α	0	0	0	0
1	1	0	a	-a	-a	0	$-\alpha$	0	0	0	0
-1	0	-1	-a	a	a	0	0	α	0	0	0
-1	0	1	-a	a	-a	0	0	$-\alpha$	0	0	0
1	0	-1	-a	-a	a						
1	0	1	-a	-a	-a						
0	-1	-1									
0	-1	1									
0	1	-1									
0	1	1									

Table 2: Typical Layout of AFBBD for k = 3 factors with n_0 = 2

Box–Behnken portion $B[3]$			**Factorial portion** $F(\alpha)^k$			**Axial portion** $A(\alpha)^3$			**Center point** (n_0)		
-1	-1	0	a	a	a	α	0	0	0	0	0
1	-1	0	a	a	-a	$-\alpha$	0	0	0	0	0
-1	0	-1	a	-a	a	0	α	0			
1	0	-1	a	-a	-a	0	$-\alpha$	0			
0	-1	-1	-a	a	a	0	0	α			
0	1	-1	-a	a	-a	0	0	$-\alpha$			
			-a	-a	a						
			-a	-a	-a						

Augmented Fractional Box-Behnken Designs (AFBBDs)

Rashid et al. (2017) proposed augmented fractional Box-Behnken designs (AFBBDs), which use fractional BBDs (FBBDs) of Edwards and Mee (2011) instead of the original BBD in ABBDs. AFBBDs also include factorial points $F(a)^k$, axial points $A[\alpha]^k$, and complementary points $B'[k]$, providing a smaller run size while maintaining third-order estimation capabilities. A typical example is given in Table 2.

New Augmented Box-Behnken Designs (NABBDs)

Arshad et al. (2020) introduced new augmented Box-Behnken designs (NABBDs), which are generated from balanced or partially balanced incomplete block designs (BIBDs/PBIBDs). Additional points, say $b \times t$ (b = number of blocks, t = number of factors per block), are added along with $2k$ axial points, while overlapping blocks from the

Table 3: Typical Layout of NABBD for k = 4 factors with n_o = 2

Box–Behnken portion $B[4]$				**BIBDs/PBIBDs**				**Axial portion** $A(\alpha)^3$				**Center point** (n_0)			
-1	-1	0	0	1	1	0	1	α	0	0	0	0	0	0	0
1	-1	0	0	1	1	0	-1	$-\alpha$	0	0	0	0	0	0	0
-1	1	0	0	1	-1	0	1	0	α	0	0				
1	1	0	0	1	-1	0	-1	0	$-\alpha$	0	0				
-1	0	-1	0	-1	1	0	1	0	0	α	0				
1	0	1	0	-1	1	0	-1	0	0	$-\alpha$	0				
-1	0	-1	0	-1	-1	0	1	0	0	0	α				
1	0	1	0	-1	-1	0	-1	0	0	0	$-\alpha$				
-1	0	0	-1	1	1	0	1								
1	0	0	-1	1	1	0	-1								
-1	0	0	1	1	-1	0	1								
1	0	0	1	1	-1	0	-1								
0	-1	-1	0	-1	1	0	1								
0	1	-1	0	-1	1	0	-1								
0	-1	1	0	-1	-1	0	1								
0	1	1	0	-1	-1	0	-1								
0	-1	0	-1	1	1	0	1								
0	1	0	-1	1	1	0	-1								
0	-1	0	1	1	-1	0	1								
0	1	0	1	1	-1	0	-1								
0	0	-1	-1	-1	1	0	1								
0	0	1	-1	-1	1	0	-1								
0	0	-1	1	-1	-1	0	1								
0	0	1	1	-1	-1	0	-1								
				1	1	0	1								
				1	1	0	-1								
				1	-1	0	1								
				1	-1	0	-1								
				-1	1	0	1								
				-1	1	0	-1								
				-1	-1	0	1								
				-1	-1	0	-1								

original BBD are removed. NABBDs do not exist for k = 3 factors. A typical example is given in Table 3.

Orthogonal Array Composite Designs ($OACD_4$)

Zhang et al. (2018) proposed a new class of third-order designs called orthogonal array composite designs ($OACD_4$). $OACD_4$ are formed by adding a two-level orthogonal

Table 4: A typical layout of a complete $OACD_4$ for $k = 3$ factors using with $n_o = 2$

Factorial portion			Axial portion			Center point (n_0)		
-1	1	1	$-\alpha$	$-\alpha$	$-\alpha$	0	0	0
1	-1	1	$-\alpha$	$-\alpha/3$	$-\alpha/3$	0	0	0
1	-1	-1	$-\alpha$	$\alpha/3$	$\alpha/3$			
-1	-1	-1	$-\alpha$	α	α			
1	1	1	$-\alpha/3$	$-\alpha$	$-\alpha/3$			
-1	-1	1	$-\alpha/3$	$-\alpha/3$	$-\alpha$			
1	1	-1	$-\alpha/3$	$\alpha/3$	α			
-1	1	-1	$-\alpha/3$	α	$\alpha/3$			
			$\alpha/3$	$-\alpha$	$\alpha/3$			
			$\alpha/3$	$-\alpha/3$	α			
			$\alpha/3$	$\alpha/3$	$-\alpha$			
			$\alpha/3$	α	$-\alpha/3$			
			α	$-\alpha$	α			
			α	$-\alpha/3$	$\alpha/3$			
			α	$\alpha/3$	$-\alpha/3$			
			α	α	$-\alpha$			

Table 5: A typical layout of a complete $OUCD_4$ for $k = 3$ factors using with $n_o = 2$

Factorial portion			Axial portion			Center point (n_0)		
-1	1	1	α	$-\alpha$	α	0	0	0
1	-1	1	α	$-\alpha/3$	$-\alpha$	0	0	0
1	-1	-1	α	$\alpha/3$	$-\alpha/3$			
-1	-1	-1	$\alpha/3$	$-\alpha/3$	$-\alpha$			
1	1	1	$\alpha/3$	$-\alpha$	$\alpha/3$			
-1	-1	1	$-\alpha/3$	$\alpha/3$	$-\alpha/3$			
1	1	-1	$-\alpha$	α	α			
-1	1	-1	$\alpha/3$	$-\alpha/3$	$\alpha/3$			
			$-\alpha$	α	α			
			$-\alpha/3$	$-\alpha/3$	α			
			$-\alpha$	$-\alpha$	$-\alpha/3$			
			$-\alpha/3$	$\alpha/3$	$-\alpha/3$			
			$\alpha/3$	$-\alpha$	$\alpha/3$			
			$-\alpha/3$	α	$-\alpha$			
			$-\alpha$	$\alpha/3$	$-\alpha$			
			α	α	$\alpha/3$			

array with levels (-1 and 1) as the factorial portion, a four-level orthogonal array with levels (α ,$-\alpha$, α/a,$-\alpha/a$; $\alpha > 0$, a = 2, 3, 4, . . .) as the axial portion, and the center points with level (0). A typical example is given in Table 4.

Table 6: Design points for all the compared designs

	Design Points (n)					
k	**OACD$_4$**	**OUCD$_4$**	**ABBDs**	**AFBBDs**	**NABBDs**	**Specific α**
3	$24 + n_0$	$24 + n_0$	$26 + n_0$	$20 + n_0$	—	$\sqrt{3}$
4	$32 + n_0$	$32 + n_0$	$48 + n_0$	$42 + n_0$	$64 + n_0$	$\sqrt{4}$
5	$32 + n_0$	$32 + n_0$	$82 + n_0$	$62 + n_0$	$140 + n_0$	$\sqrt{5}$
6	$64 + n_0$	$64 + n_0$	$172 + n_0$	$124 + n_0$	$172 + n_0$	$\sqrt{6}$

Orthogonal Uniform Composite Designs (OUCD$_4$)

Yankam and Oladugba (2022) proposed a new third-order design called orthogonal uniform composite designs (OUCD$_4$). OUCD4 are formed by adding two-level full or fractional factorial design with levels (-1 and 1) as the factorial portion denoted by n_f, four-level uniform design with levels ($\alpha, -\alpha, \alpha/a, -\alpha/a; \alpha > 0, a = 2, 3, 4, \ldots$) as the axial portion denoted by n_α, and center points denoted by n_0. A typical example is given in Table 5.
The study is restricted to the spherical region for uniformity since ABBDs, AFBBDs, and NABBDs are defined only in spherical regions.

2.2 Third-order model

All designs were evaluated using the reduced third-order response surface model:

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{iii} x_i^3 + \varepsilon$$

where $y, \beta_0, \beta_i, \beta_{ii}, \beta_{iii}$, and β_{ij} are the response, intercept, linear, quadratic, cubic, and interaction parameters, respectively, and $\varepsilon \sim N(0, \sigma^2)$. The number of parameters $p = 1 + 3k + \binom{k}{2}$.

2.3 Design Efficiency Criteria

This section discusses the design efficiency criteria used to evaluate the five third-order RSDs in terms of: experimental cost, parameter estimation, and prediction accuracy efficiencies.

Experimental Cost Efficiencies

Run-size efficiency (RS_{eff})

Run-size efficiency measures the experimental economy of a design based on the total number of experimental runs required. Designs requiring fewer runs are considered more economical and practically attractive (Box and Wilson, 1951). Since the actual run size directly reflects experimental cost, the total number of runs n was used for comparison, where smaller values indicate better efficiency. For a design with n points, run-size efficiency is defined as:

$$RS_{\text{eff}} = \min(n).$$

Degrees of freedom efficiency (DF_{eff})

Degrees of freedom efficiency evaluates how effectively a design utilizes the available degrees of freedom relative to the number of model parameters. Designs with larger residual degrees of freedom provide better support for pure-error estimation and lack-of-fit testing. It is computed as:

$$DF_{\text{eff}} = \max\left[\frac{p}{n} \times 100\right]$$

Parameter Estimation Efficiencies

D-efficiency (D_{eff})

D-efficiency assesses the precision of parameter estimation by maximizing the determinant of the information matrix $\mathbf{X}^{\mathrm{T}}\mathbf{X}$. Higher values indicate a design with greater overall information and more precise parameter estimates. It is defined as:

$$D_{\text{eff}} = \frac{100|X^T X|^{1/p}}{n}$$

A-efficiency (A_{eff})

A-efficiency minimizes the average variance of the estimated regression coefficients by minimizing the trace of the inverse information matrix. Designs with higher A-efficiency provide smaller average variances. It is computed as

$$A_{\text{eff}} = \frac{100p}{\text{trace}[N(X^T X)^{-1}]}.$$

Prediction Accuracy Efficiencies

G-efficiency (G_{eff})

G-efficiency evaluates the worst-case prediction accuracy by minimizing the maximum scaled prediction variance over the spherical region. Designs with a higher value of G_{eff} are more efficient. It is expressed as:

$$G_{\text{eff}} = \frac{p}{n \max[f^T(x)(X^T X)^{-1} f(x)]} \times 100,$$

where $f(x)$ is the model vector of form $1 \times p$, and $(X^T X)^{-1}$ is the inverse of the information matrix.

I-efficiency (I_{eff})

I-efficiency measures the average or integrated prediction variance across the region. A design with higher I-efficiency provides more uniform prediction accuracy. The I-efficiency of a design (ξ_n) is given by

$$I_{\text{eff}} = (2^k)^{-1}\text{trace}[(X^TX)^{-1}M(\xi_n)],$$

where, M is the moment of the design.

T-efficiency (T_{eff})

T-efficiency evaluates prediction performance based on the trace of the information matrix, with higher values indicating stronger overall prediction capability. It is given by

$$T_{\text{eff}} = \frac{\text{trace}(X^TX)}{np}$$

Per-Observation G-Efficiency (POG_{eff})

This measure adjusts G-efficiency by accounting for the number of design points, thereby assessing prediction efficiency per experimental run. Designs with a higher value of POG_{eff} are more efficient. It is computed as

$$POG_{\text{eff}} = \max\left[\frac{G_{\text{eff}}}{n} \times 100\right]$$

3 Discussion of Results

This study compared the efficiencies of five third-order RSDs: ABBDs, AFBBDs, NABBDs, OACD_4, and OUCD_4 for $3 \leq k \leq 5$ factors using $0 \leq n_0 \leq 5$ center points in the spherical region. The designs were evaluated using Design Expert based on experimental cost, parameter estimation, and prediction accuracy efficiencies, and the results are presented in Tables 7-10.

3.1 Experimental Cost Efficiencies

The comparison of run-size efficiency (RS_{eff}) showed that OACD_4 and OUCD_4 consistently required the smallest number of runs for $4 \leq k \leq 6$,confirming their economical structure. In contrast, NABBDs required the largest run sizes for all factor levels; reflecting their construction based on BIBDs/PBIBDs, which introduces substantial additional design points. For $k = 3$, AFBBDs provided the smallest run size, whereas ABBDs produced the largest.

Degree-of-freedom efficiency (DF_{eff}) further confirmed this trend: OACD_4 and OUCD_4 yielded the highest DF_{eff} -values for $4 \leq k \leq 6$,while NABBDs had the lowest across all the corresponding settings. For k = 3, AFBBDs and ABBDs recorded the highest and lowest DF_{eff} -value, respectively. These results highlight OACD_4 and OUCD_4 as the most economical designs because they offer greater flexibility for pure-error estimation and lack-of-fit testing.

3.2 Parameter Estimation Efficiencies

The D-efficiency values indicated that OACD_4 had the lowest DF_{eff} -values for all factors across all center-point levels. In contrast, NABBDs exhibited the highest D-efficiency for $4 \leq k \leq 6$, while ABBDs had the largest DF_{eff} -values for $k = 3$. This demonstrates that NABBDs, despite their larger run size, provide the greatest precision in estimating model parameters for most factor levels.

A-efficiency results revealed an opposite pattern: NABBDs recorded the lowest A_{eff} -values for all factor levels except $k = 3$, where AFBBDs had the smallest values. OUCD_4 consistently had the highest A_{eff} -values across factor levels, with the exception of $k = 5$, where ABBDs slightly outperformed. This positions OUCD_4 as the design with the best average parameter variance performance.

Overall, NABBDs and OUCD_4 emerged as the most efficient designs for D-efficiency and A-efficiency, respectively.

3.3 Prediction Accuracy Efficiencies

The G-efficiency result showed that for $4 \leq k \leq 6$, OACD_4 had the highest G_{eff} -values, indicating superior worst-case prediction performance across the spherical region. For $k = 3$, AFBBDs attained the highest G_{eff} -values, while NABBDs had the lowest G_{eff} -values for all factor-level combination.

For I-efficiency, NABBDs had the highest I_{eff} -values across all factor settings except for $k = 3$, where AFBBDs were superior. Conversely, OACD_4 generally recorded the lowest I_{eff} -values for most factor settings.

T-efficiency results followed a similar pattern: NABBDs recorded the highest T_{eff} -values across factors, with AFBBDs leading only for $k = 3$. OUCD_4 consistently yielded the smallest T_{eff} -values except for $k = 5$, where OACD_4 recorded the lowest.

Per-observation G-efficiency further emphasized the superior predictive economy of OACD_4 as it consistently had the highest POG_{eff} -values for $4 \leq k \leq 6$, while AFBBDs led for $k = 3$. NABBDs, owing to their large design size, had the lowest POG_{eff} -values throughout.

Overall, the prediction efficiency results highlight OACD_4 as the most effective design for G- and per-observation G-efficiencies, whereas NABBDs showed superior I- and T-efficiency performance, and OUCD_4 performs competitively depending on the prediction metric.

In general, the results demonstrate clear trade-offs in design performance. OACD_4 and OUCD_4 are the most economical designs in terms of run size and degrees of freedom. NABBDs offer the strongest D- and T-efficiency performance but are limited by greater run sizes and weaker G- and POG-efficiency. OUCD_4 performs best in terms of A-efficiency, while $OACD_4$ dominates in G- and per-observation G-efficiency. AFBBDs show notable advantages for $k = 3$, particularly in predictive performance.

3.4 Composite Efficiency Index (CEI)

To obtain an overall ranking of the competing third-order RSDs, a composite efficiency index (CEI) was introduced. The CEI combines experimental economy, parameter estimation, and prediction efficiency into a single measure, thereby avoiding conflicting conclusions based on individual criteria (Mohanty, 2016).In this study, the efficiency measures were classified into benefit-type and cost-type criteria. Benefit-type criteria (DF_{eff}, D_{eff}, A_{eff}, G_{eff}, I_{eff}, T_{eff}, and POG_{eff}) are measures for which larger values indicate better performance, while the cost-type criterion (RS_{eff}) prefers smaller run sizes due to improved experimental economy. The efficiency measures were normalized using min-max transformation:

$$S_{ij} = \frac{X_{ij} - \min(X_j)}{\max(X_j) - \min(X_j)}$$

for benefit-type criteria, and

$$S_{ij} = \frac{\max(X_j) - X_{ij}}{\max(X_j) - \min(X_j)}$$

for the cost-type criterion, where S_{ij} denotes the normalized score of the i^{th} design under the j^{th} criterion, X_{ij} represents the original value of the j^{th} criterion for the i^{th} design, while $\min X_j$ and $\max X_j$ denote the minimum and maximum values of the j^{th} criterion across all competing designs, respectively. The overall composite index for each design is then obtained as the average of the normalized efficiency scores:

$$CEI_i = \frac{1}{m} \sum_{j=1}^{m} S_{ij}$$

where m is the number of efficiency criteria considered. Higher CEI values indicate better overall design performance. The CEI was computed across all factor levels $3 \le k \le 6$ and center point $0 \le n_0 \le 5$, and the results are presented in Table 11.

The results in Table 11 show that $OACD_4$ consistently achieved the highest CEI across all factor levels, indicating superior overall performance when experimental economy, parameter estimation, and prediction efficiency are jointly considered. $OUCD_4$ ranked second, demonstrating stable and competitive performance across the evaluation criteria. AFBBDs showed moderate overall performance, while ABBDs and NABBDs exhibited comparatively lower composite efficiencies due to trade-offs among run size, estimation precision, and prediction capability. Overall, the CEI results identify $OACD_4$ as the most suitable general-purpose design among the competing third-order response surface designs.

Table 7: Three factor comparative efficiencies of ABBDs, AFBBDs, $OACD_4$ and $OUCD_4$

k	p	n_0	Designs	RS_{eff}	DF_{eff}	D_{eff}	A_{eff}	G_{eff}	I_{eff}	T_{eff}	POG_{eff}
			ABBD	26	50.00	12.86	89.29	48.28	1.02	14.56	185.69
		0	AFBBD	20	65.00	12.23	59.17	26.27	1.50	21.97	131.35
			$OACD_4$	24	54.17	4.51	122.18	58.80	0.60	10.64	244.58
			$OUCD_4$	24	54.17	6.27	150.29	44.06	0.56	8.65	183.58
			ABBD	27	48.15	12.65	108.42	57.72	0.83	11.99	213.78
		1	AFBBD	21	61.90	11.69	90.22	71.84	0.98	14.41	342.10
			$OACD_4$	25	52.00	4.61	123.22	57.73	0.57	10.55	230.92
			$OUCD_4$	25	52.00	6.42	152.58	43.59	0.54	8.52	174.36
			ABBD	28	46.43	12.71	116.59	55.88	0.76	11.15	199.57
3	13	2	AFBBD	22	59.09	11.76	99.31	67.62	0.89	13.09	307.36
			$OACD_4$	26	50.00	4.73	123.93	56.33	0.56	10.49	216.65
			$OUCD_4$	26	50.00	6.59	154.21	42.11	0.53	8.43	161.96
			ABBD	29	68.42	12.87	121.04	54.06	0.73	10.74	186.41
		3	AFBBD	23	56.52	11.97	103.67	65.00	0.86	12.54	282.61
			$OACD_4$	27	48.15	4.85	124.40	55.03	0.54	10.45	203.81
			$OUCD_4$	27	48.15	6.77	155.32	41.67	0.53	8.37	154.33
			ABBD	30	43.33	13.09	123.93	52.32	0.72	10.49	174.40
		4	AFBBD	24	54.17	12.24	106.21	63.19	0.83	12.24	263.29
			$OACD_4$	28	46.43	4.97	124.64	53.36	0.54	10.43	190.57
			$OUCD_4$	28	46.43	6.95	156.25	40.63	0.52	8.32	145.11
			ABBD	31	41.94	13.33	125.97	50.67	0.70	10.32	163.45
		5	AFBBD	25	52.00	12.56	107.79	60.24	0.82	12.06	240.96
			$OACD_4$	29	44.83	5.10	125.00	51.75	0.53	10.40	178.45
			$OUCD_4$	29	44.83	7.14	157.00	39.59	0.52	8.28	136.52

3.5 Fraction of Design Space (FDS) Plot Comparison

The fraction of design space (FDS) plot introduced by Zahran et al. (2003) displays the distribution of the scaled prediction variance (SPV) across the entire experimental region. A large number of points are sampled uniformly from the design space, and the corresponding SPV values are computed and sorted. The sorted SPV values are then plotted against the cumulative fraction of the design space, where the x-axis and y-axis represent the proportion of the design space and the SPV values, respectively. Lower SPV at any given fraction indicates better prediction precision, and a design with a lower, flatter FDS curve is preferred because it maintains small prediction variance over a larger portion of the design space. Although variance dispersion graphs (VDGs) are commonly used for assessing prediction variance properties of RSDs, FDS plots were adopted in this study because they provide a more comprehensive visualization of the distribution of scaled prediction variance across the entire experimental region. FDS plots are particularly useful for comparing overall prediction behaviour and identifying designs with superior worst-case and average prediction performance. The FDS plots

Table 8: Four factor comparative efficiencies of ABBDs, AFBBDs, NABBDs, $OACD_4$ and $OUCD_4$

k	p	n_0	Designs	RS_{eff}	DF_{eff}	D_{eff}	A_{eff}	G_{eff}	I_{eff}	T_{eff}	POG_{eff}
			ABBD	48	39.58	20.76	97.38	47.03	1.20	13.35	97.98
			AFBBD	42	45.24	19.48	131.94	52.54	1.28	14.40	125.10
		0	NABBD	64	29.69	31.70	82.50	21.89	2.38	23.03	34.20
			$OACD_4$	32	59.38	4.80	135.71	59.71	0.71	14.00	186.59
			$OUCD_4$	32	59.38	8.04	171.95	37.53	0.83	11.05	117.28
			ABBD	49	38.78	20.98	99.16	46.15	1.17	13.11	94.18
			AFBBD	43	44.19	19.69	135.62	51.40	1.24	14.01	119.53
		1	NABBD	65	29.23	31.77	84.52	21.71	2.29	22.48	33.40
			$OACD_4$	33	57.58	4.92	135.91	58.02	0.70	13.98	175.82
			$OUCD_4$	33	57.58	8.22	173.04	34.75	0.82	10.98	105.30
4	19		ABBD	50	38.00	21.23	100.46	45.28	1.14	12.94	90.56
			AFBBD	44	43.18	19.94	138.18	50.28	1.21	13.75	114.27
		2	NABBD	66	28.79	31.92	85.86	21.60	2.23	22.13	32.73
			$OACD_4$	34	55.88	5.04	136.10	56.57	0.70	13.96	166.38
			$OUCD_4$	34	55.88	8.40	173.83	34.61	0.82	10.93	101.79
			ABBD	51	37.25	21.50	148.32	44.43	1.13	12.81	87.12
			AFBBD	45	42.22	20.22	139.91	49.19	1.18	13.58	109.31
		3	NABBD	67	28.36	32.12	86.84	21.38	2.19	21.88	31.91
			$OACD_4$	35	54.29	5.16	136.20	55.63	0.69	13.95	158.94
			$OUCD_4$	35	54.29	8.59	174.47	33.44	0.81	10.89	95.54
			ABBD	52	36.54	21.78	149.49	43.61	1.11	12.71	83.87
			AFBBD	46	41.30	20.51	141.26	48.15	1.17	13.45	104.67
		4	NABBD	68	27.94	32.36	87.56	21.16	2.16	21.70	31.12
			$OACD_4$	36	52.78	5.28	136.30	54.11	0.69	13.94	150.31
			$OUCD_4$	36	52.78	8.79	174.95	34.09	0.81	10.86	94.69
			ABBD	53	35.85	22.08	150.43	42.81	1.10	12.63	80.77
			AFBBD	47	40.43	20.82	142.32	47.14	1.16	13.35	100.30
		5	NABBD	69	27.54	32.62	88.09	20.89	2.14	21.57	30.28
			$OACD_4$	37	51.35	5.40	136.40	52.95	0.69	13.93	143.11
			$OUCD_4$	37	51.35	8.98	175.28	32.30	0.81	10.84	87.30

for $k = 4$ using $0 \leq n_0 \leq 5$ center points are represented in Figures 1-6. Figures 1-6 show the distribution of SPV for the competing designs across the experimental region. At $n_0 = 0$, NABBDs exhibited the largest SPV values, indicating poorer prediction precision, whereas $OUCD_4$ showed the smallest SPV values over a large portion of the design space. However, at higher fractions of the design space (approximately 0.9 and above), $OACD_4$ and AFBBDs produced smaller SPV values. As the number of center points increased from 1 to 5, NABBDs consistently maintained the highest SPV values, while ABBDs and AFBBDs remained intermediate. $OUCD_4$ generally showed better

Table 9: Five factor comparative efficiencies of ABBDs, AFBBDs, NABBDs, $OACD_4$ and $OUCD_4$

k	p	n_0	Designs	RS_{eff}	DF_{eff}	D_{eff}	A_{eff}	G_{eff}	I_{eff}	T_{eff}	POG_{eff}
			ABBD	82	31.71	31.81	173.22	37.46	1.47	15.01	45.68
			AFBBD	62	41.94	26.79	156.63	47.25	1.64	16.60	72.21
		0	NABBD	140	18.57	54.19	75.28	10.17	3.72	34.54	7.26
			$OACD_4$	32	81.25	4.11	138.22	56.91	0.98	18.81	177.84
			$OUCD_4$	32	81.25	11.48	85.58	16.77	2.59	30.38	52.41
			ABBD	83	31.33	32.11	173.80	37.02	1.46	14.96	44.60
			AFBBD	63	41.27	27.06	158.34	46.51	1.60	16.42	73.83
		1	NABBD	141	18.44	54.38	75.47	10.13	3.70	34.45	7.18
			$OACD_4$	33	78.79	4.19	138.96	56.79	0.95	18.71	172.09
			$OUCD_4$	33	78.79	11.48	85.58	17.11	2.59	30.38	47.76
5	26		ABBD	84	30.95	32.40	174.26	36.59	1.45	14.92	43.56
			AFBBD	64	40.63	27.34	159.80	45.78	1.58	16.27	71.53
		2	NABBD	142	18.31	54.59	75.65	10.03	3.68	34.37	7.06
			$OACD_4$	34	76.47	4.28	139.48	56.80	0.93	18.64	167.06
			$OUCD_4$	34	76.47	11.75	86.01	17.00	2.57	30.23	46.15
			ABBD	85	30.59	32.70	174.73	36.17	1.45	14.88	42.55
			AFBBD	65	40.00	27.64	160.79	45.08	1.56	16.17	69.35
		3	NABBD	143	18.18	54.81	75.78	9.97	3.67	34.31	6.99
			$OACD_4$	35	74.29	4.37	139.78	55.76	0.91	18.60	160.06
			$OUCD_4$	35	74.29	12.32	86.58	16.71	2.56	30.03	42.74
			ABBD	86	30.23	33.01	175.08	35.75	1.44	14.85	41.57
			AFBBD	66	39.39	27.95	161.69	44.40	1.55	16.08	67.27
		4	NABBD	144	18.06	55.04	75.91	9.90	3.65	34.25	6.88
			$OACD_4$	36	72.22	4.47	140.01	55.15	0.90	18.57	152.67
			$OUCD_4$	36	72.22	12.62	86.75	14.54	2.56	29.97	41.22
			ABBD	87	29.89	33.32	175.44	35.35	1.43	14.82	40.63
			AFBBD	67	38.81	28.27	162.40	43.74	1.54	16.01	65.28
		5	NABBD	145	17.93	55.28	76.02	9.87	3.64	34.20	6.79
			$OACD_4$	37	70.27	4.57	140.16	54.57	0.89	18.55	146
			$OUCD_4$	37	70.27	12.92	86.90	14.36	2.55	29.92	38.68

prediction precision at lower fractions of the design space, whereas $OACD_4$ became superior at higher fractions. Overall, the FDS plots confirm the strong prediction capability of $OACD_4$ and $OUCD_4$ across the spherical region.

Table 10: Six factor comparative efficiencies of ABBDs, AFBBDs, NABBDs, $OACD_4$ and $OUCD_4$

k	p	n_0	Designs	RS_{eff}	DF_{eff}	D_{eff}	A_{eff}	G_{eff}	I_{eff}	T_{eff}	POG_{eff}
			ABBD	172	19.77	48.67	237.93	24.30	1.50	14.29	14.13
			AFBBD	124	27.42	40.56	210.79	31.63	1.71	16.13	25.51
		0	NABBD	172	19.77	79.71	128.93	23.72	3.39	26.37	13.80
			$OACD_4$	64	53.13	3.75	309.65	75.10	0.57	10.98	117.34
			$OUCD_4$	64	53.13	9.05	323.19	62.10	0.95	10.52	97.03
			ABBD	173	19.65	48.88	238.43	24.16	1.49	14.26	13.97
			AFBBD	125	27.20	40.77	211.97	31.38	1.69	16.04	25.10
		1	NABBD	173	19.65	79.94	129.97	23.76	3.31	26.16	13.73
			$OACD_4$	65	52.31	3.79	310.50	75.85	0.56	10.95	116.69
			$OUCD_4$	65	52.31	9.17	323.50	65.34	0.94	10.51	100.52
6	34		ABBD	174	19.54	49.08	238.76	24.03	1.48	14.24	13.81
			AFBBD	126	26.98	40.99	212.90	31.13	1.67	15.97	24.71
		2	NABBD	174	19.54	80.18	130.82	23.77	3.24	25.99	13.66
			$OACD_4$	66	51.52	3.84	311.07	76.93	0.55	10.93	116.56
			$OUCD_4$	66	51.52	9.30	323.81	60.94	0.94	10.50	92.33
			ABBD	175	19.43	49.30	239.27	23.90	1.47	14.21	13.66
			AFBBD	127	26.77	41.21	213.70	30.89	1.66	15.91	24.32
		3	NABBD	175	19.43	80.44	131.58	23.76	3.18	25.84	13.58
			$OACD_4$	67	50.75	3.88	311.64	78.16	0.55	10.91	116.66
			$OUCD_4$	67	50.75	9.43	324.12	60.28	0.94	10.49	89.97
			ABBD	176	19.32	49.51	239.61	23.76	1.47	14.19	13.50
			AFBBD	128	26.56	41.44	214.38	30.65	1.64	15.86	23.95
		4	NABBD	176	19.32	80.71	132.19	23.73	3.13	25.72	13.48
			$OACD_4$	68	50.00	3.93	311.93	77.02	0.54	10.90	113.26
			$OUCD_4$	68	50.00	9.55	324.43	59.18	0.94	10.48	87.03
			ABBD	177	19.21	49.73	239.94	23.63	1.46	14.17	13.35
			AFBBD	129	26.36	41.68	215.05	30.42	1.63	15.81	23.58
		5	NABBD	177	19.21	80.99	132.71	23.69	3.08	25.62	13.38
			$OACD_4$	69	49.28	3.97	312.21	75.71	0.54	10.89	109.72
			$OUCD_4$	69	49.28	9.68	324.74	61.82	0.94	10.47	89.59

Table 11: Composite Efficiency Index (CEI) and Ranking of Competing Designs

Design	**CEI (k = 3)**	**CEI (k = 4)**	**CEI (k = 5)**	**CEI (k = 6)**	**Overall CEI**	**Rank**
ABBD	0.48	0.42	0.40	0.38	0.42	4.5
AFBBD	0.55	0.47	0.45	0.44	0.48	3
NABBD	–	0.44	0.41	0.40	0.42	4.5
$OACD_4$	0.76	0.77	0.75	0.73	0.75	1
$OUCD_4$	0.70	0.69	0.67	0.66	0.68	2

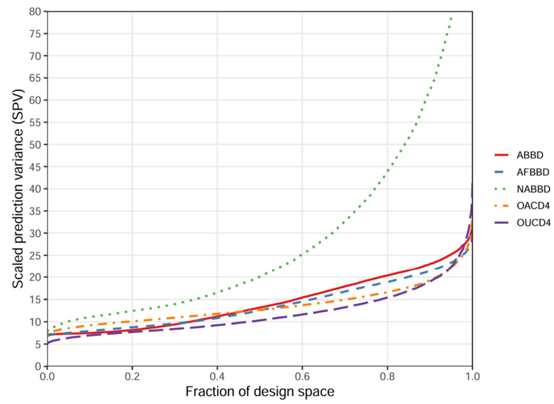

Figure 1: FDS plots of the designs for $k = 4, n_0 = 0$ on a spherical region.

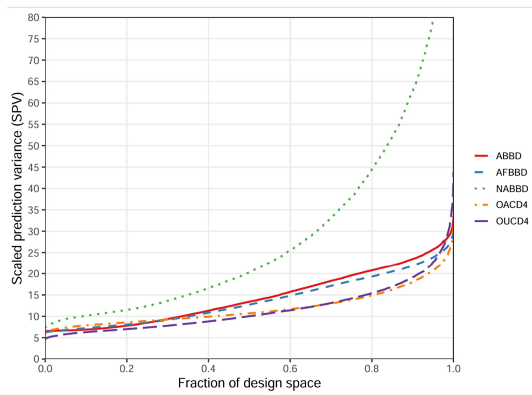

Figure 2: FDS plots of the designs for $k = 4, n_0 = 1$ on a spherical region.

4 Conclusion

This study evaluated the efficiencies of five third-order RSDs: ABBDs, AFBBDs, NABBDs, $OACD_4$, and $OUCD_4$ for $3 \leq k \leq 6$ factors using $0 \leq n_0 \leq 5$ center points in the spherical region. The designs were compared using experimental cost, parameter estimation, and prediction accuracy efficiency criteria, along with FDS plots. The results showed that $OACD_4$ and $OUCD_4$ were the most economical designs because they required smaller run sizes and provided higher degrees of freedom. NABBDs achieved the highest D-efficiency, whereas $OUCD_4$ showed superior A-efficiency performance. In terms of prediction accuracy, $OACD_4$ performed best based on G- and per-observation G-efficiency, while NABBDs showed superior I- and T-efficiency values. The FDS plots further confirmed the strong prediction performance of $OACD_4$ and $OUCD_4$. The CEI results showed that $OACD_4$ consistently achieved the highest overall ranking across all factor levels, indicating the best balance among experimental economy, parameter estimation, and prediction performance. Overall, $OACD_4$ emerged as the most efficient design across most of the considered criteria.

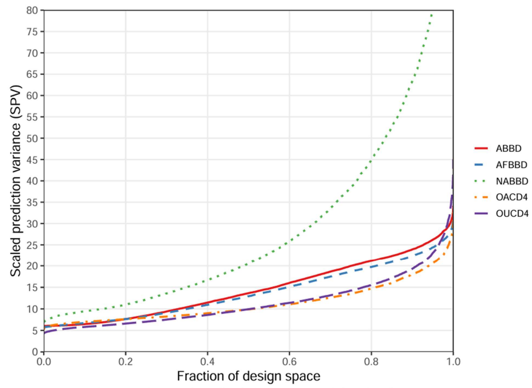

Figure 3: FDS plots of the designs for $k = 4, n_0 = 2$ on a spherical region.

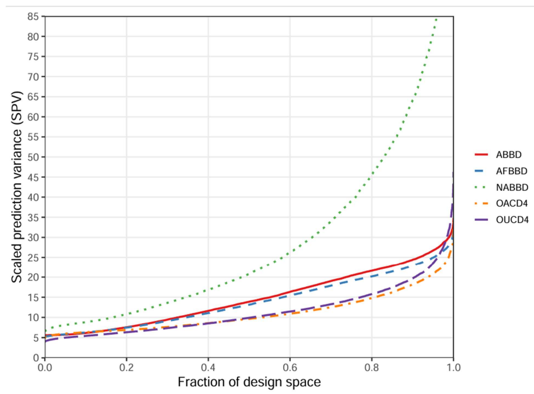

Figure 4: FDS plots of the designs for $k = 4, n_0 = 3$ on a spherical region.

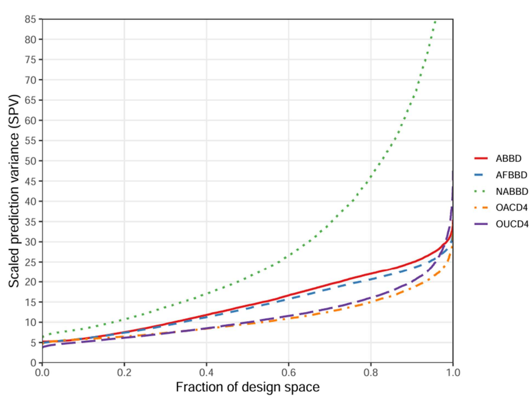

Figure 5: FDS plots of the designs for $k = 4, n_0 = 4$ on a spherical region.

Future studies may consider extending similar comparative evaluations to other third-order response surface designs applicable in irregular or constrained experimen-

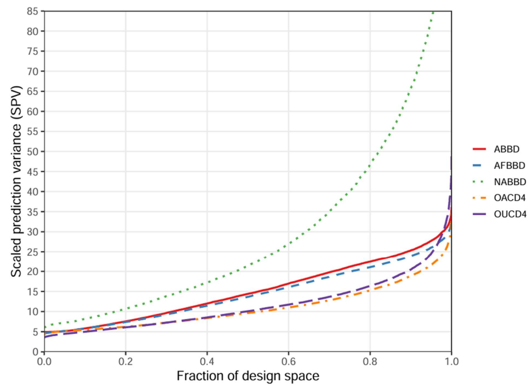

Figure 6: FDS plots of the designs for $k = 4, n_0 = 5$ on a spherical region.

tal regions.

References

Arshad HM, Ahmad T, Akhtar M. Some sequential third order response surface designs. Communications in Statistics-Simulation and Computation. 2020;49(7):1872–1885. https://doi.org/10.1080/03610918.2018.1517454.

Arshad HM, Akhtar M, Gilmour SG. Augmented Box-Behnken designs for fitting third- order response surfaces. Communications in Statistics-Theory and Methods. 2012;41(23):4225–4239. https://doi.org/10.1080/03610926.2011.583344.

Atkinson AC, Donev AN. Optimum experimental designs. 1st ed. New York, NY: Oxford University Press; 1992.

Box GEP, Wilson KB. On the experimental attainment of optimum conditions. Journal of the Royal Statistical Society, Series B. 1951;13(1):1–45. https://doi.org/10.1111/j.2517-6161.1951.tb00067.x.

Didiugwu AF, Oladugba AV, Ude OI. Effects of error in factor levels on orthogonal composite designs for second-order models. Communications in Statistics-Theory and Methods. 2024;p. 1–19. https://doi.org/10.1080/03610926.2024.2342943.

Draper NR. Center points in second-order response surface designs. Technometrics. 1982;24(2):127–133. https://doi.org/10.1080/00401706.1982.10487737.

Edwards DJ, Mee RW. Fractional Box–Behnken designs for one-step response surface methodology. Journal of Quality Technology. 2011;43(4):288–306. https://doi.org/10.1080/00224065.2011.11917866.

Ezievuo CS, Oladugba AV, Babatunde OT. Evaluation of orthogonal composite designs for second-order model in presence of missing observation. Quality and Reliability Engineering International. 2025;41(1):204–234. https://doi.org/10.1002/qre.3687.

Hemavathi A, Varghese E, Shekhar S, Jaggi S. Sequential asymmetric third-order rotatable designs (SATORDs). Journal of Applied Statistics. 2020;49(6):1364–1381. https://doi.org/10.1080/02664763.2020.1856191.

Khuri AI. A general overview of response surface methodology. Biometrics & Biostatistics International Journal. 2017;5(3):87–93. https://doi.org/10.15406/bbij.2017.05.00121.

Landman D, Simpson J, Mariani R, Ortiz F, Britcher C. Hybrid design for aircraft wind-tunnel testing using response methodologies. Journal of Aircraft. 2007;44(4):1214–1221. https://doi.org/10.2514/1.27211.

Lucas JM. Optimum composite designs. Technometrics. 1974;16(4):561–567. https://doi.org/10.1080/00401706.1974.10489237.

Mee RW. Optimal three-level designs for response surfaces in spherical experimental regions. Journal of Quality Technology. 2007;39(4):340–354. https://doi.org/10.1080/00224065.2007.11917698.

Mohanty PC. A panel data analysis of relationship between migration and development. Journal of Infrastructure Development. 2016;8(2):177–192. https://doi.org/10.1177/0974930616668738.

Morris MD. A class of three-level experimental designs for response surface modeling. Technometrics. 2000;42(2):111–121. https://doi.org/10.1080/00401706.2000.10485994.

Myers RH, Montgomery DC, Anderson-Cook CM. Response surface methodology: Process and product optimization using designed experiments. 3rd ed. New York, NY: John Wiley & Sons; 2009.

Myers RH, Vining GG, Giovannitti-Jensen A, Myers SL. Variance dispersion properties of second-order response surface designs. Journal of Quality Technology. 1992;24(1):1–11. https://doi.org/10.1080/00224065.1992.12015243.

Oladugba AV, Ude IO, Didiugwu AF, Nnanna LI. Evaluation of second-order response designs with errors in factors levels in the cuboidal region. Journal of the Iranian Statistical Society. 2025;24(1):1–20. https://doi.org/10.22034/jirss.2025.2021358.1050.

Oladugba AV, Yankam BM. Prediction variance properties of third-order response surface designs in the hypersphere. Mathematics and Statistics. 2022;10(1):61–73. https://doi.org/10.13189/ms.2022.100107.

Rashid F, Akram M, Akbar A, Javed A. Some new augmented fractional Box-Behnken designs. Communications in Statistics-Theory and Methods. 2017;46(4):2007–2012. https://doi.org/10.1080/03610926.2015.1091566.

Verma A, Jaggi S, Bhowmik A, Varghese E, Varghese C, Datta A, et al. Sequential third order response surface design from an application perspective. Communications in Statistics-Theory and Methods. 2024;19(2):301–306. https://doi.org/10.5958/2231-6744.2024.00030.7.

Wangui PM, Ayubu A, Argwings O. Selection of second order models' design using D-, A-, E-, T-optimality criteria. Asian Journal of Probability and Statistics. 2019;5(2):1–15. https://doi.org/10.9734/ajpas/2019/v5i230132.

Yankam BM, Oladugba AV. Orthogonal uniform composite designs for the third-order models. Brazilian Journal of Probability and Statistics. 2022;36(4):777–791. https://doi.org/10.1214/22-BJPS1573.

Yankam BM, Oladugba AV. Robustness of augmented third-order response surfaces designs to missing observation. Journal of the Iranian Statistical Society. 2024;23(1):33–49. https://doi.org/10.22052/jiss.2024.256860.1062.

Zahran A, Anderson-Cook CM, Myers RH. Fraction of design space to assess prediction capability of response surface designs. Journal of Quality Technology. 2003;35(4):377–386. https://doi.org/10.1080/00224065.2003.11980237.

Zhang X, Qi Z, Zhou Y, Yang F. Orthogonal-array composite design for the third- order models. Communications in Statistics-Theory and Methods. 2018;47(14):3488–3507. https://doi.org/10.1080/03610926.2017.1365315.