JIRSS (2026)
Vol. 25, No. 2, pp 53-78
DOI: 10.22034/jirss.2026.2077431.1162

Improving location estimators for spherically symmetric models with residual under modified balanced loss functions

Mohamed Alahiane[1,2], **Lahoucine Hobbad**[2], **Ismail Arjdal**[3], **Idir Ouassou**[2]

[1]Chouaib Doukkali University, National School of Business and Management, LERSEM Laboratory, corner Avenue Ahmed Chaouki and Fez Street, BP. 122 -24000, El-Jadida, Morocco.
[2]Cadi Ayyad University, National School of Applied Sciences, Complex Systems Modeling Laboratory, Av. Abdelkrim Khattabi, BP. 575, Marrakesh, Morocco.
[3]Chouaib Doukkali University, Faculty of Sciences, Department of Mathematics, 24000, El-Jadida.

Received: 2025-11-11, Accepted: 2026-07-23, Published online: 2026-08-10

Abstract.

We consider the problem of estimating a d-dimensional parameter $\theta = (\theta_1, \ldots, \theta_d)$ when the observation is a $d + k$ dimensional vector (X, U) where $\dim X = d$ and U is a residual vector with $\dim U = k$. The distributional assumption is that (X, U) has a spherically symmetric distribution around $(\theta, 0)$. The loss functions are assumed to be modified balanced loss functions of the form: **(i)** $\omega\rho(\|\delta - \delta_0\|^2) + (1 - \omega)\rho(\|\delta - \theta\|^2)$ and **(ii)** $\ell(\omega\|\delta - \delta_0\|^2 + (1 - \omega)\|\delta - \theta\|^2)$ where δ_0 is a target estimator for θ, and where ρ and ℓ are increasing and concave functions. In the case when $d \geq 4$ and the target estimator $\delta_0(X) = X$, we establish conditions under which the estimators of the form $\delta_{\omega,g}(X, \|U\|^2) = X + a\|U\|^2(1 - \omega)g(X)$ dominate $\delta_0(X) = X$ and are minimax, where we suppose that there exists a nonpositive function $h(.)$ such that $h(X)$ is subharmonic, $E_{R,\theta}\left[R^2 h(W)\right]$ is nonincreasing with $W \sim U_{R,\theta}$, $E_\theta\left[|h(X)|\right] < \infty$ and the function $g(X)$ is weakly differentiable and also satisfies **(a)** $div(g(X)) \leq h(X)$, **(b)** $\|g(X)\|^2 + 2h(X) \leq 0$.

Keywords. Modified balanced loss, Baranchik type estimators, Dominance, Shrinkage estimation, Spherically symmetric distribution with residual.

Corresponding Author: Mohamed Alahiane (alahiane.mohamed@ucd.ac.ma).
Lahoucine Hobbad (la.hobbad@gmail.com).
Ismail Arjdal (arjdal.i537@ucd.ac.ma).
Idir Ouassou (i.ouassou@uca.ac.ma).

MSC: 62F10, 62H12, 62C20, 62C15, 62E20.

1 Introduction

Let the random observation have the form (X, U) which is spherically symmetrically distributed around a vector of the form $(\theta, 0)$ with $\dim(X) = \dim(\theta) = p$ and $\dim(U) = \dim(0) = k$, denoted $(X, U) \sim SS_{(d,k)}(\theta, 0)$, the distribution of (X, U) is invariant under orthogonal transformations in $\mathbb{R}^{d+k}$, and the location parameter θ enters the model only through the X-component, while the residual vector U carries no information about θ. Consider $(X - \theta, U)$ to be jointly spherically symmetric. The vector U plays the role of a residual component: it carries no directional information about θ and depends only on the radial structure of the model. Residual quantities arise naturally in statistical inference to separate signal from noise, quantifying uncertainty, and constructing invariant procedures. In spherical and elliptical models, they capture essential features of the data, including tail heaviness, radial dispersion, and the presence of outliers, and therefore play a key role in the construction of adaptive and robust methods such as Stein-type shrinkage estimators, M-estimators, Tyler's shape estimator, and robust multivariate tests. In high-dimensional regimes, residual norms exhibit concentration phenomena: as $d \to \infty, \|X\|^2 \approx \mathbb{E}\|X\|^2$, so the radial component becomes nearly deterministic, a property underlying the stability of ridge regularization, the behavior of principal component analysis and several effects studied in random matrix theory.

This model with a residual component was studied by Cellier et al. (1989), Brandwein and Strawderman (1991), Brandwein et al. (1993), Cellier and Fourdrinier (1995) and Fourdrinier and Strawderman (1996). These papers study classes of estimators which improve on the usual minimax estimator X. A particularly important class of estimators when a residual vector U is available is the class of robust James–Stein estimators $\delta_{JS}(X, \|U\|^2) = \left(1 - a\|U\|^2/\|X\|^2\right) X$. The estimator $\delta_{JS}(X, \|U\|^2)$ improves upon X when $0 < a < 2(d-2)/(k+2)$ under quadratic loss (see Fourdrinier, D. and Strawderman, W.E., 1996). Brandwein and Strawderman (1991) showed that the estimator $\delta_{0,g}(X, \|U\|^2)$ dominates the unbiased estimator X under the concave loss function $L(\theta, \delta) = \ell(\|\theta - \delta\|^2)$ where the concave nondecreasing function ℓ also satisfies $t^{\alpha}\ell(t)$ is nondecreasing for $0 < a < (d-2-2\alpha)/(d(k+2))$. Fourdrinier, Strawderman and Wells (2003) showed that the Baranchik-type estimator, defined as: $\delta_{a,S}(X, \|U\|^2) = \left(1 - a\|U\|^2 S(\|X\|^2/\|U\|^2)/\|X\|^2\right) X$ dominates the standard minimax estimator X under the condition that the shrinkage parameter a satisfies the inequality: $0 < a < (2(d-2))/(k+2)$. Fourdrinier and Ouassou (2000) studied the problem of estimating, under quadratic loss, the mean of a spherically symmetric distribution when its norm is supposed to be known and when a residual vector is available. Fourdrinier et al. (2003) considered the problem of estimating a d-dimensional parameter $\theta = (\theta_1, \ldots, \theta_d)$ when the observation is a $(d+k)$ dimensional vector (X, U). Two restrictions on the parameter θ are considered: first, they assume that $\theta_i \geq 0$ for $i = 1, \ldots, p$ and, secondly, they assume that only a subset of

these θ_i are nonnegative.
The Balanced Loss Functions (BLFs), introduced by Zellner (1994), are formulated to reflect two criteria, namely goodness of fit and the precision of estimation. This loss function is defined as: $\omega\|\delta - \delta_0\|^2 + (1 - \omega)\|\delta - \theta\|^2$, with the first term measuring the proximity of estimate δ to the target δ_0 and the second term measuring proximity of δ to the estimand θ, weighted respectively by ω and $1-\omega$. The special case where $\omega = 0$ corresponds to the scaled unbalanced loss function. In this case, the decisions are not influenced by the target δ_0. Giles et al. (1996), Dey et al. (1999), Chaturvedi and Shalabh (2004), Jozani et al. (2006) studied the application of balanced loss functions. Marchand and Strawderman (2020) introduced a novel class of loss functions called Modified Balanced Loss Functions (MBLFs), denoted as in **(i)** and **(ii)**, where $\rho(\cdot) \geq 0$ and $l(\cdot) \geq 0$ are increasing, concave and satisfy a completely monotone property. Notably, these MBLFs can be considered as a natural extension of Zellner's balanced loss function, where the functions $\rho(\cdot)$ and $\ell(\cdot)$ take the simple form of t for all non-negative values of t in the real numbers $\mathbb{R}_+$. Hobbad et al. (2021) generalized this class of the loss functions **(i)** and **(ii)**, requiring only that ρ and ℓ are increasing and concave functions.
Shrinkage estimation improves a given raw estimator by incorporating additional information into the estimation process. Although shrinkage estimators are generally biased, it is well established that they can achieve a smaller quadratic risk than natural estimators, most notably the maximum likelihood estimator. A shrinkage estimator typically takes the form $X + g(X)$, where the adjustment term $g(X)$ acts to pull the estimator toward a target or central value. In practice, this adjustment often results in a contraction of X, hence the term shrinkage. While the notion of shrinkage may have a negative connotation in everyday language, it is fundamentally advantageous in statistical estimation. Indeed, many widely used and theoretically optimal estimators, including Bayesian, minimax, and Gamma-minimax estimators, can be viewed as instances of shrinkage estimators.
Shrinkage estimation under balanced loss functions has been extensively studied in recent years. Karamikabir and Afshari (2018) considered shrinkage estimation of a nonnegative mean vector with unknown covariance, while Karamikabir and Afshari (2019) investigated generalized Bayes shrinkage for spherically symmetric distributions under balance-type loss. Afshari and Arashi (2021) extended SURE thresholding to spherically symmetric models with a residual vector, and Karamikabir and Afshari (2022) studied generalized Bayes estimation for multivariate normal models with unknown mean and covariance under balanced LINEX loss. More recently, Hamdaoui et al. (2022) analyzed risk ratios of shrinkage estimators under balanced loss, and Hobbad et al. (2025) established dominance of Baranchik-type estimators over the usual estimator for spherically symmetric distributions under modified balanced loss functions.
Brandwein and Strawderman (1991) established a general dominance result for shrinkage estimators of the form

$$\delta_a(X) = X + a\,\|U\|^2\, g(X),$$

under spherically symmetric distributions centered at θ, with respect to quadratic loss and its concave extensions. Their approach relies on the divergence theorem and imposes structural conditions on the shrinkage function g through the existence of an auxiliary function h such that

$$\|g(x)\|^2/2 \leq -h(x) \leq -\nabla \cdot g(x),$$

where $-h$ is superharmonic. Together with a technical radial monotonicity condition on $\mathbb{E}_\theta[R^2 h(V)]$, this inequality allows the quadratic cost of the correction term $g(X)$ to be dominated by its contraction effect after integration over spherical regions. A further restriction on the tuning parameter a, namely $0 < a \leq \{d\, \mathbb{E}_0(\|X\|^{-2})\}^{-1}$, ensures that the resulting risk reduction is global. These conditions extend the classical James-Stein estimator to a broad class of shrinkage procedures under spherical symmetry. In the presence of an unknown scale and an accompanying residual vector, the same geometric principles continue to apply, with the residual component providing a natural normalization for the shrinkage intensity. This observation motivates the developments in the next section, where dominance results are established without reliance on the technical monotonicity condition. In the presence of a residual vector, it is natural to consider shrinkage estimators of the form

$$\delta_a(X) = X + a\,\|U\|^2 g(X),$$

rather than $\delta_a(X) = X + a\, g(X)$. Under the model $(X, U) \sim SS_{d+k}((\theta, 0), I_{d+k})$, the factor $\|U\|^2$ provides an intrinsic normalization that is independent of θ, preserves the geometric structure induced by spherical symmetry, and leads to dominance conditions that are free of additional scale parameters.

In this work, we generalize the results of Hobbad et al. (2021) and Hobbad et al. (2025) to the case where the joint observation (X, U) follows a spherically symmetric distribution centered at $(\theta, 0)$. To establish dominance results without assuming the existence of a probability density, we exploit intrinsic properties of spherically symmetric distributions, including the radial angular decomposition, uniformity on the sphere, and analytical arguments based on superharmonicity and concavity. When a probability density exists, we additionally employ a suitable change of measure to recover and sharpen the corresponding density-based results.

This manuscript is organized as follows, in Section 2, we introduce the loss functions and estimator framework. In Section 3, we provide conditions for the constant a such that the estimators $\delta_{\omega,g}(X, \|U\|^2)$ in (2.1) dominate $\delta_0(X) = X$ under MBLFs **(i)** for spherically symmetric distributions with residual, both with and without density. In Section 4, we provide similar dominance results under MBLFs **(ii)**. In Section 5, we present a simulation study. Finally, the proofs are given in Section 6.

2 Loss functions and estimation framework

In this work, we are interested in the estimation of the unknown parameter θ under the Modified Balanced Loss Functions (MBLFs) **(i)** and **(ii)**. Our objective is to find sufficient conditions on the constant a and the function g under which the robust shrinkage estimators

$$\delta_{\omega,g}(X, \|U\|^2) = X + a(1-\omega)\|U\|^2 g(X) \tag{2.1}$$

where $\delta_0(X) = X$ is the target estimator. In this context, we assume that there exists a non-positive function $h(\cdot)$ such that $h(X)$ is subharmonic, $E_\theta\,[\|h(X)\|] < \infty$ and $E_{R,\theta}\left[R^2 h(W)\right]$ in nonincreasing where $E_{R,\theta}$ denotes the expectation with respect to the uniform distribution $U_{R;\theta}$ on the sphere $S_{R;\theta} = \{y \in R^{p+k} : \|y-\theta\| = R\}$ of radius R and centered at θ. Additionally, the function g satisfies: $g(\cdot)$ is weakly differentiable and satisfies **(a)** $div(g) \leq h$ and **(b)** $\|g\|^2 + 2h \leq 0$. These conditions ensure that the Stein-type expansion of the risk difference between the shrinkage estimator $\delta(\omega, g)$ and the natural estimator X is non-positive. The divergence term arises from Stein's identity, while the auxiliary function $h(X)$ provides a bound that controls this contribution. Intuitively, $\|g(X)\|^2$ reflects the magnitude of the shrinkage applied to X, whereas $h(X)$ offsets the divergence effect. Together, these conditions guarantee that the shrinkage direction $g(X)$ yields a systematic reduction in risk, thereby establishing the dominance of the proposed estimator.
Note that the James-Stein estimator and the Baranchik estimator can be viewed as special cases of a more general class of robust shrinkage estimators in (2.1). More specifically: the James-Stein estimator can be written in the form:

$$\delta^{JS}_{\omega,a}(X, \|U\|^2) = \left(1 - (1-\omega)a\frac{\|U\|^2}{\|X\|^2}\right)X.$$

with the specific choices $g(X) = -X/\|X\|^2$ and $h(X) = -b/\|X\|^2$, where $0 \leq b \leq d-2$. Similarly, the Baranchik type-estimators

$$\delta_{\omega,a}(X, \|U\|^2) = \left(1 - a(1-\omega)\|U\|^2\frac{S(\|X\|^2)}{\|X\|^2}\right)X$$

where $a > 0$ and the function S is twice differentiable and satisfies: $0 \leq S(\cdot) \leq 1$, $S'(\cdot) \geq 0$ and $S''(\cdot) \leq 0$, can be considered as special cases of robust shrinkage estimators with $g(X) = -\left(S(\|X\|^2)/\|X\|^2\right)X$ and $h(X) = -bS(\|X\|^2)/\|X\|^2$ where $\frac{1}{2} \leq b \leq d-2$.
We consider two broad classes of modified balanced loss functions. The first class is defined by **(i)** $L_{\omega,\delta_0,\rho}(\delta,\theta) = \omega\,\rho\left(\|\delta-\delta_0\|^2\right) + (1-\omega)\,\rho\left(\|\delta-\theta\|^2\right)$, while the second class is given by **(ii)** $L_{\omega,\delta_0,\ell}(\delta,\theta) = \ell\left(\omega\|\delta-\delta_0\|^2 + (1-\omega)\|\delta-\theta\|^2\right)$. Here δ_0 denotes a fixed benchmark estimator of θ, the weight $\omega \in [0,1)$ controls the balance between the two components, and the functions ρ and ℓ are assumed to be increasing and concave.
For loss functions of type **(i)**, we impose the regularity condition:

(H1) $\rho(0) = 0, \qquad 0 < \rho'(0) < \infty, \qquad \rho$ is concave.

This assumption encompasses a wide range of commonly used losses, including the quadratic loss, exponential-type losses such as $1-\exp(-t/\alpha)$ with $\alpha > 0$, logarithmic

losses of the form $\log(1+t)$, and fractional power losses $(1+t/\gamma)^{\beta}$ with $\gamma > 0$ and $\beta \in (0,1)$. Other admissible examples include bounded losses such as $\arctan(t)$ and $\tanh(t)$, although these functions do not satisfy complete monotonicity.
For loss functions of type **(ii)**, we assume:

$$\textbf{(H2)} \quad \ell(t) \geq 0, \qquad \ell'(t) > 0, \qquad \ell \text{ is concave.}$$

In contrast with earlier contributions, no complete monotonicity requirement is imposed on ℓ'. This relaxation allows for greater flexibility and includes all losses satisfying **(H1)**, as well as additional examples such as $L^{q/2}$-type losses with $\ell(t) = t^q$, $0 < q < 1$. The class also covers the loss functions introduced by Kubokawa et al. (2017), defined by $\ell(t) = 2Q(\sqrt{t}) - 1$, where Q is a cumulative distribution function on $\mathbb{R}$ with an even and unimodal density.

3 Main dominance results

In this section, we investigate the minimax properties of the robust shrinkage estimators (2.1) for the location vector θ, assuming that the observed vector (X,U) follows a spherically symmetric distribution with location parameter $(\theta,0)$, under the MBLFs **(i)** and **(ii)**. The results corresponding to the cases where (X,U) admits a density and where it does not admit a density are presented below. If the vector (X,U) follows the distribution $SS_{(d,k)}(\theta,0)$ and has a density with respect to the Lebesgue measure, then its density is necessarily of the form $f(\|x-\theta\|^2 + \|u\|^2)$, and the radius R has density $t \mapsto 2\pi^{p+k}\Gamma^{-1}((d+k)/2) f(t^2) t^{d+k-1}$. Throughout this work, we deal the expectations conditioned on the radius R of a spherically symmetric distribution in $\mathbb{R}^d \times \mathbb{R}^k$ centered at $(\theta,0)$, where $\theta \in \mathbb{R}^d$. These expectations reduce to integrals with respect to the uniform distribution $U_{R,\theta}$ on the sphere $S_{R,\theta} = \left\{ y = (x,u) \in \mathbb{R}^d \times \mathbb{R}^k \mid (\|x\|^2 + \|u\|^2)^{\frac{1}{2}} = R \right\}$. If $E_{R,\theta}[\psi]$ denotes the expectation of some function ψ with respect to $U_{R,\theta}$, the expectation with respect to the entire distribution is given by $E_\theta[\psi] = E[E_{R,\theta}[\psi]]$ where E is the expectation with respect to the distribution of the radius. We begin by presenting the results for distributions that admit a density.
We denote by E_θ the expectation with respect to the probability density f. After an appropriate change of measure, we write E^*_θ for the expectation taken with respect to the transformed density f^*, which is obtained from f through this change of measure. We now present the main results of the paper. We first treat the case of spherically symmetric distributions admitting a density, and then extend the results to the general setting of spherically symmetric distributions without assuming the existence of a density. Assume that $(X,U) \sim SS_{d+k}((\theta,0))$ admits a probability density of the form $f\left(\|x-\theta\|^2 + \|u\|^2\right)$. In our analysis, we employ the modified balanced loss function of type **(i)** to study the estimation of θ. The derivation of the domination and minimaxity results is based on an appropriate change-of-measure argument, which plays a central role in the proof. As a first step, we establish a technical lemma that provides the main analytical ingredient required to implement this approach and that will be repeatedly

invoked in the proof of the main theorem. The proofs of the next theorems and the associated lemmas are provided in the Section 6.

Lemma 3.1. *Let (X, U) have a spherically symmetric distribution around $(\theta, 0)$ with a density of the form $f(\|x-\theta\|^2 + \|u\|^2)$, where $\theta \in \mathbb{R}^d$ and ρ satisfies Assumption (**H1**). Then, for $d \geq 4$, we have*

$$E_\theta\left[\rho\left(-\|U\|^2 h(X)\right)\right] \leq \rho'(0) E_\theta\left[-\|U\|^2 h(X)\right] \leq \rho'(0) E_\theta\left[-\|V\|^2 h(Y)\right], \tag{3.1}$$

where $r = \|X-\theta\|$, $R^2 = \|X-\theta\|^2 + \|U\|^2$, $K = \int_{\mathbb{R}^{d+k}} \rho'((1-\omega)\|x-\theta\|^2) f(\|x-\theta\|^2 + \|u\|^2) d(x,u) = E_\theta[\rho'((1-\omega)\|X-\theta\|^2)]$ and $f^\star(x,u) = \left(\rho'((1-\omega)\|x-\theta\|^2) f(\|x-\theta\|^2 + \|u\|^2)\right)/K$.

The next theorem establishes dominance conditions for the shrinkage estimator $\delta_{\omega,g}(X, U)$ under the modified balanced loss function of type **(i)** when the observation follows a spherically symmetric distribution with density. The proof relies on a change-of-measure argument and the concavity of ρ.

Theorem 3.2. *Suppose that $(X, U) \sim SS_{(d,k)}(\theta, 0)$ with density of the form $f(\|x-\theta\|^2 + \|u\|^2)$ and $\delta_{\omega,g}(X, U)$ is defined by (2.1). Then, with respect to the loss function **(i)**, $\delta_{\omega,g}(X, U)$ has a smaller risk than δ_0 provided that*

$$0 < a \leq \left(\frac{K}{(k+2)(\omega\rho'(0) + K(1-\omega))}\right),$$

where $K = \int_{\mathbb{R}^{d+k}} \rho'((1-\omega)\|x-\theta\|^2) f(\|x-\theta\|^2 + \|u\|^2) d(x,u) = E_\theta[\rho'((1-\omega)\|X-\theta\|^2)]$,

We continue to assume that $(X, U) \sim SS_{d+k}((\theta, 0))$ admits a probability density of the form $f\left(\|x-\theta\|^2 + \|u\|^2\right)$. In this setting, we present a result establishing dominance conditions for the estimators $\delta_{\omega,g}(X, U)$ defined in (2.1) under the balanced loss functions of type **(ii)**. Prior to stating this result, we proceed with a preparatory lemma that exploits the concavity of ℓ, and which relates the difference in losses L_{ω,ℓ,δ_0} between estimates $\delta(X) = \delta_0(X) + a(1-\omega)g(X)$ and $\delta_0(X)$, to the balanced squared-error loss difference $\Delta_{\omega,\ell,\delta_0}(\theta, \delta)$.

Lemma 3.3. *Suppose that X has a spherically symmetric distribution. For the problem of estimating θ under the MBLFs **(ii)**, we have*

$$\Delta L_{\omega,\delta_0,\ell}(\theta, \delta) \leq (1-\omega)^2 \ell'((1-\omega)\|\delta_0 - \theta\|) \Delta L_{0,\delta_0,\ell}(\theta, \delta).$$

The next theorem provides analogous dominance results under the modified balanced loss function of type **(ii)**. The proof exploits the concavity of ℓ and uses a similar change-of-measure technique to control the risk difference.

Theorem 3.4. *Suppose that $(X, U) \sim SS_{(d,k)}(\theta, 0)$ with density of the form $f(\|x-\theta\|^2 + \|u\|^2)$ and $\delta_{\omega,g}(X, U)$ is defined by (2.1). Then, with respect to the MBLFs **(ii)**, $\delta_{\omega,g}(X, U)$ has a smaller risk than δ_0 provided the condition **1.** of Theorem 3.2 holds and $0 < a < ((k+2)(1-\omega))^{-1}$.*

The following theorem extends the dominance analysis to spherically symmetric distributions without assuming the existence of a density. The proof combines covariance inequalities, properties of superharmonic functions, and careful conditioning on the radial and angular components.

Theorem 3.5. *Suppose that* (X, U) *have a spherically symmetric distribution with location parameter* $(\theta, 0)$*. Then, with respect to the MBLFs* ***(i)***, $\delta_{\omega,g}(X, U)$ *has smaller risk than* δ_0 *provided*

1. $\|g\|^2/2 \leq -h \leq div(g)$, *where* $-h$ *is superharmonic,*
2. $d > 4, k > 2 + \frac{4d}{d-4}$,
3. $0 < a \leq \dfrac{1}{k+2} \times \dfrac{E[r^2\rho'(r^2)]}{\omega\rho'(0)E\left[r^2\right] + (1-\omega)E\left[r^2\rho'(r^2)\right]}$, *where* $r^2 = \|X - \theta\|^2$.

The following theorem establishes dominance under the loss function **(ii)** for general spherically symmetric distributions. The argument uses integration-by-parts identities specific to the residual vector and the concavity of ℓ.

Theorem 3.6. *Suppose that* (X, U) *have a spherically symmetric distribution with location parameter* $(\theta, 0)$ *and* $\delta_{\omega,g}(X)$ *is defined by (2.1). Then, with respect the loss MBLFs* ***(ii)***, $\delta_{\omega,g}(X)$ *has smaller risk than* $\delta_0(X) = X$ *provided the condition of Theorem 3.2 holds and* $0 < a < ((k+2)(1-\omega))^{-1}$.

the following last theorem strengthens the dominance condition under loss **(ii)** by imposing the additional monotonicity assumption that $t^\alpha\ell(t)$ is nondecreasing. The proof relies on covariance inequalities and careful manipulation of the conditional expectations involving the radial variable.

Theorem 3.7. *Suppose that* (X, U) *have a spherically symmetric distribution with location parameter* $(\theta, 0)$*. Assume that the concave nondecreasing function* $\ell(t)$ *also satisfies* $t^\alpha\ell(t)$ *is nondecreasing. Then, the estimator* $\delta_{\omega,g}(X)$ *given by (2.1) dominates* $\delta_0(X) = X$ *under MBLFs* ***(ii)*** *for all* θ*, with strict inequality for some* θ*, provided*

$$0 < a \leq \frac{(d-2-2\alpha)}{d(k+2)(1-\omega)}.$$

The preceding sections have established theoretical conditions under which the proposed shrinkage estimators dominate the usual estimator under modified balanced loss functions. In the next section, we turn to numerical experiments to validate these theoretical findings and illustrate the practical behavior of the estimators.

4 Numerical illustration

In this section, we present some simulations illustrating the theoretical results of the previous section. We consider Baranchik-type estimators and deal with two models:

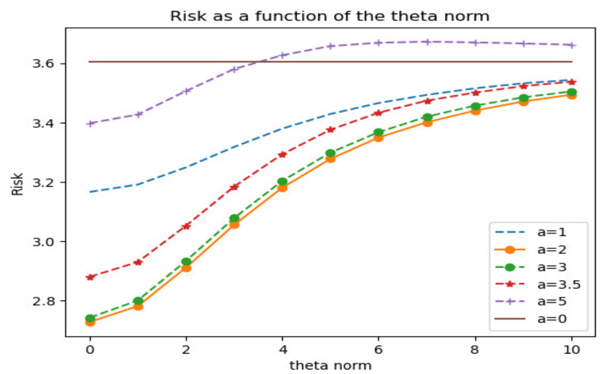


Figure 1: The risk $\mathcal{R}(\theta, \ell, \delta_{\omega,a}(X, \|U\|^2)$ as functions of the norm $\|\theta\|$.

the first with density and the second without density. The simulations are performed in Python. The Baranchik-type estimators $\delta_{\omega,a}(X, \|U\|^2) = \left(1 - a(1-\omega)\|U\|^2 S(\|X\|^2)/\|X\|^2\right) X$ dominates the estimator $\delta_0(X) = X$ if the parameter a verifies $0 < a < 2(d-2)/((k+2)(1-\omega))$. This follows from $div(g(X)) = -2aS'(\|X\|^2) - a(d-2)S(\|X\|^2)/\|X\|^2 \leq -a(d-2)S(\|X\|^2)/\|X\|^2$ and $\|g(X)\|^2 = a^2 S^2(\|X\|^2)/\|X\|^2 \leq a^2 S(\|X\|^2)/\|X\|^2$. In this simulation, we consider $\ell(t) = t^q$, $q = 1/2$, $S(t) = t/(1+t)$, $\omega = 1/3$, $d = 20$ and $k = 10$. We obtain the cutoff point $a_0 = 2(d-2)/((k+2)(1-\omega)) = 4.5$

4.1 Example 1

In this case, we consider that $(X, U) \in \mathbb{R}^{d+k}$ to be normally distributed with the identity covariance matrix. Figure 1 compares the risk of: $\delta_0(X) = X$ and the Baranchik-type estimators $\delta_{\omega,a}(X, \|U\|^2)$ under the MBLFs **(ii)** with $\ell(t) = t^{1/2}$, for various values of a. The benchmark estimator $\delta_{\omega,0}(X, \|U\|^2)$ is minimax with constant risk $\mathcal{R}(\theta, \ell, \delta_0) = E_0((1-\omega)\|X\|^2) \approx 3.605$, obtained from $\|X - \theta\|^2 \sim \chi^2_d$. Baranchik estimators dominate if the parameter a is smaller than the cutoff point $a_0 = 4.5$. This Python simulation indicates that for values of the parameter a less than a_0 ($a = 1, a = 2, a = 3, a = 3.5$). The risk curve of the Baranchik-type estimator $\delta_{\omega,a}(X, \|U\|^2)$ is lower than that of the usual estimator $\delta_{\omega,0}(X, \|U\|^2)$. However, once the value of a ($a = 5$) surpasses the bound a_0, the risk curve of the Baranchik-type estimator rises above that of the usual estimator, demonstrating the theoretical result of dominance.

4.2 Example 2

Before presenting the second simulation example, we state a fundamental characterization theorem for spherically symmetric distributions. This result provides the theoretical framework for our simulation methodology when a density is not assumed, allowing us to generate random vectors $(X, U) \sim SS_{(d,k)}(\theta, 0)$.
In this example, we examine a scenario in which a spherically symmetric vector does

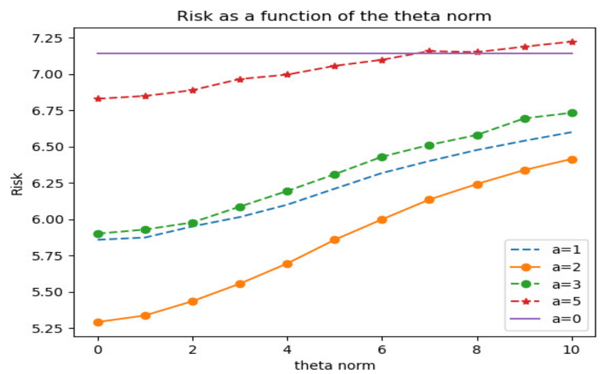
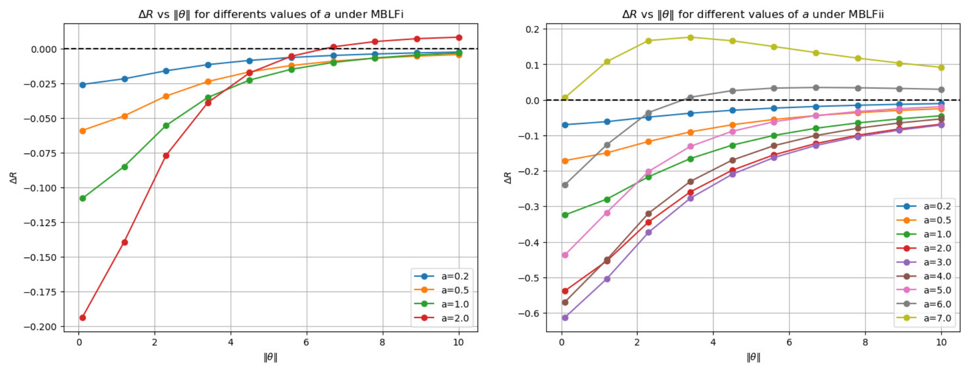


Figure 2: The risk $\mathcal{R}(\theta, \ell, \delta_{\omega,a}(X, \|U\|^2)$ as functions of the norm $\|\theta\|$.

not necessarily possess a probability density function. We begin by considering a real variable R that follows Poisson distribution with a parameter $\lambda = 10$. Additionally, we introduce a random vector $D = (D_1, D_2)$ that follows a Dirichlet distribution $Dirichlet(d/2, k/2)$. Using the normal distribution, we simulate two random vectors: W_1, uniformly distributed on the unit sphere in $\mathbb{R}^d$, and W_2, distributed on the unit sphere in $\mathbb{R}^k$. We then define the vectors X and U as follows: $X = R\sqrt{D_1}W_1 + \theta$ and $U = R\sqrt{D_2}W_2$. According Theorem 6.1, the vector (X, U) has a spherically symmetric distribution centered around $(\theta, 0)$. Figure 2 shows the variation in the risk of the estimator $\delta_{\omega,a}(X, \|U\|^2)$ as a function of the norm of θ for different values of parameter a under MBLFs **ii**, clearly showing that the estimators $\delta_{\omega,a}(X, \|U\|^2)$ dominates the usual estimator $\delta_{\omega,0}(X, \|U\|^2) = \delta_0(X) = X$ for values of a below the cutoff point $a_0 = 4.5$. This figure was generated by simulating the data using Python, and we determined that the constant risk $\mathcal{R}(\theta, \ell, \delta_0) = E_0((1-\omega)\|X\|^2) \approx 7.14$. As long as the parameter value a is below a_0 ($a = 1, a = 2, a = 3$), the risk curve of Baranchik-type estimator stays below the risk curve of the benchmark estimator $\delta_{\omega,0}(X, \|U\|^2)$. However, if a surpasses a_0 ($a = 5$), the risk curve of Baranchik-type estimator exceeds that of the benchmark estimator, thus confirming the theoretical result obtained.

4.3 Example 3

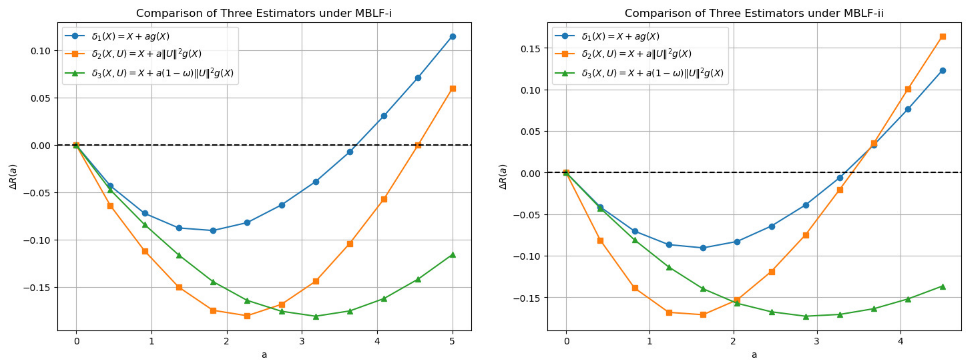
Let $(X, U) \sim SS_{d,k}(\theta, 0)$ and consider the estimation of $\theta \in \mathbb{R}^d$. We study estimators of the form $\delta_{\omega,g}(X, \|U\|^2) = X + a(1-\omega)\|U\|^2 g(X)$, and compare them with the benchmark estimator $\delta_0(X) = X$ under the modified balanced losses MBLF(i) and MBLF(ii) with $\rho(t) = \log(1 + t/2)$ and $\ell(t) = \sqrt{t}$, both increasing and concave. Let $g(X) = -X/(1 + \|X\|^2)$, the $\operatorname{div}(g(X)) = -d/(1 + \|X\|^2) + 2\|X\|^2/(1 + \|X\|^2)^2$, $\|g(X)\|^2 = \|X\|^2/(1 + \|X\|^2)^2$. We choose the auxiliary function $h(X) = -1/2\|g(X)\|^2 - \varepsilon, \quad \varepsilon > 0$, so that $\operatorname{div}(g) \leq h$ and h is superharmonic. For illustration under a non-Gaussian spherical model we use the representation $X = \theta + R\sqrt{D_1}W_1, U = R\sqrt{D_2}W_2$, where $R \sim \Gamma(2, 1.5), (D_1, D_2) \sim \operatorname{Dir}(d/2, 2/k)$, and W_1, W_2 are independent random directions uniformly distributed on $\mathbb{S}^{d-1}$ and $\mathbb{S}^{k-1}$.



(a) (b)

Figure 3: (a) The risk difference ΔR as functions of $||\theta||$ under MBLF(i). (b) The risk difference $\Delta R(a, ||\theta||)$ as functions of $||\theta||$ under MBLF(ii)

This construction yields $(X, U) \sim SS_{d,k}(\theta, 0)$ with heavier tails than the Gaussian model. Figure 3 reports the Monte Carlo estimate of the risk difference $\Delta R = R(\delta_{\omega,g}) - R(\delta_0)$ as a function of $||\theta||$ for several values of a under MBLF(i) and MBLF(ii). The experiment is conducted with $d = 20$, $k = 5$, $\omega = 0.3$, using $N = 120\,000$ replications per configuration. The benchmark estimator is $\delta_0(X) = X$, whereas $\delta_{\omega,g}(X, ||U||^2) = X + a(1-\omega)||U||^2 g(X)$, Negative values of ΔR indicate risk improvement and the horizontal line $\Delta R = 0$ separates the dominance regions. The results show that moderate shrinkage ($0.2 \le a < 2$ for MBLF(i) and $0.2 \le a < 6$ for MBLF(ii)) yields a uniform reduction in risk, particularly when $||\theta||$ is small, while larger values of a lead to over-shrinkage and loss of dominance. Figure 4 reports the Monte Carlo estimate of the risk difference $\Delta R(a) = R(\delta) - R(\delta_0)$ for three shrinkage estimators, with $\delta_0(X) = X$ as benchmark. The simulation is performed under a spherical model with residual vector with $d = 20$, $k = 5$, $\omega = 0.3$, and $N = 120\,000$ replications for each a, where a ranges over 12 equally spaced values in $[0, 5]$. The loss functions are the modified balanced loss MBLF(i) and MBLH(ii). The competing rules are $\delta_1(X) = X + ag(X)$, $\delta_2(X, U) = X + a||U||^2 g(X)$, and $\delta_3(X, U) = X + a(1-\omega)||U||^2 g(X)$, with $g(X) = -X/(1 + ||X||^2)$. Negative values of $\Delta R(a)$ indicate dominance over δ_0. The figure therefore illustrates the effect of the shrinkage level a, the residual factor $||U||^2$, and the balancing coefficient $(1-\omega)$ on the risk performance of the estimators. We investigate three shrinkage estimators in a spherically symmetric model with residual vector (X, U), whose joint radial representation allows the residual component to capture intrinsic scale information that can be exploited for adaptive estimation. The comparison is performed under two modified balanced loss functions. We compare the natural estimator $\delta_0(X) = X$ with the shrinkage rules $\delta_(X) = X + ag(X), \delta_(X, U) = X + a||U||^2 g(X), \delta_(X, U) = X + a(1-\omega)||U||^2 g(X)$, where $g(X) = -X/(1 + ||X||^2)$. Under both loss functions, the classical shrinkage estimator δ_1 yields a modest but systematic reduction in risk relative to δ_0, with empirical gains of



(a) (b)

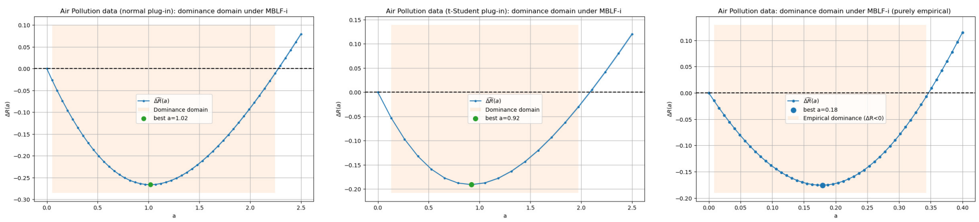
Figure 4: (a) The risk difference ΔR as function of a under MBLF(i). (b) The risk difference ΔR as function of a under MBLF(ii)

about 5%–10%. Incorporating the residual energy through δ_2 substantially strengthens this effect, increasing the maximal improvement to roughly 15%–25%, indicating that the residual component provides informative scale adaptivity. The fully adjusted estimator δ_3, which aligns the shrinkage magnitude with the curvature of the balanced loss through the factor $(1-\omega)$, achieves the strongest and most stable performance, with observed reductions of approximately 20%–30% relative to the benchmark and a wider dominance region in the tuning parameter a. Overall, the results demonstrate that the residual vector is structurally informative rather than auxiliary: combining spherical symmetry, residual-based scaling, and loss-sensitive weighting yields substantial efficiency gains and systematic dominance over the natural estimator.

4.4 Air Pollution Data

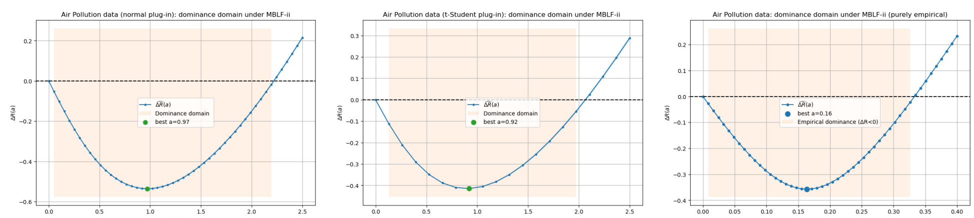
In this section, we further investigate the empirical performance of the proposed residual-based shrinkage estimator using the classical U.S. air pollution dataset analyzed in Everitt and Hothorn (2011). This dataset contains environmental and demographic measurements collected for several U.S. cities in 1981. The variables considered are the sulfur dioxide concentration in micrograms per cubic meter (SO_2), average annual temperature in degrees Fahrenheit (temp), number of manufacturing enterprises employing 20 or more workers (manu), population size according to the 1970 census in thousands (popul), average annual wind speed in miles per hour (wind), average annual precipitation in inches (precip), and the average number of days with precipitation per year (predays). These seven variables form a multivariate observation vector of dimension $d = 7$. Because the variables are measured on different physical scales, the data are first standardized before conducting the analysis.

We study the performance of the shrinkage estimator $\delta_{\omega,g}(X,U) = X + a(1-\omega)||U||^2 g(X)$, under the modified balanced loss function MBLF(i) with $\rho(t) = \log(1 + t/2)$ and MBLF(ii)



(a) (b) (c)

Figure 5: (a) The risk difference ΔR as function of a under the normal plug-in and under MBLF(i). (b) The risk difference ΔR as function of a under the Student-t plug-in and under MBLF(i). (c) The risk difference ΔR as function of a with purely empirical cross-validation procedure and under MBLF(i).



(a) (b) (c)

Figure 6: (a) The risk difference ΔR as function of a under the normal plug-in and under MBLF(ii). (b) The risk difference ΔR as function of a under the Student-t plug-in and under MBLF(ii). (c) The risk difference ΔR as function of a with purely empirical cross-validation procedure and under MBLF(ii).

with $\ell(t) = \sqrt{t}$, and the benchmark estimator $\delta_0(X) = X$ and balance parameter $\omega = 0.3$.To assess the risk behavior of the estimator, we consider three complementary approaches. First, a parametric plug-in normal approximation is used, where the mean vector and covariance matrix are estimated from the standardized data and synthetic samples are generated from a multivariate normal distribution with these estimated parameters. Second, to account for potential heavy-tailed behavior in the data, a plug-in multivariate Student-t model is fitted by estimating the degrees of freedom parameter via likelihood maximization, and simulations are performed from the resulting Student-t scale-mixture representation. Third, a purely empirical analysis is conducted using repeated K-fold cross-validation, which evaluates the predictive loss directly on the observed data without relying on a parametric model. Across these three approaches, the risk difference $\Delta R = R(delta_{\omega,g}) - R(\delta_0)$ is estimated for a range of shrinkage parameters a, and negative values indicate dominance of the shrinkage estimator over the natural estimator. The results obtained under the normal plug-in and Student-t plug-in models both exhibit a clear region of dominance where the shrinkage estimator achieves lower expected loss than the benchmark estimator. The Student-t

model generally produces slightly more conservative improvements due to the heavier tails captured by the fitted distribution. The purely empirical cross-validation analysis leads to qualitatively similar conclusions, confirming the existence of a nontrivial range of shrinkage parameters for which the proposed estimator yields a systematic reduction in expected loss. Overall, the consistency of the results across the parametric and purely empirical analyses provides strong evidence that incorporating residual-based scaling and loss-sensitive weighting leads to a robust improvement over the natural estimator in this real-data setting.

Figure 5 displays the results of the real-data analysis based on the U.S. air pollution dataset under the modified balanced loss MBLF(i). The figure compares the three approaches considered in this study: the plug-in normal approximation, the plug-in multivariate Student-t approximation, and the purely empirical cross-validation procedure. Figure 6 reports the corresponding results obtained under the alternative loss function MBLF(ii).

5 Conclusion

This work focuses on estimating a multivariate location parameter under modified balanced loss functions within the class of spherically symmetric models with a residual vector. Using the geometric invariance induced by spherical symmetry, analytical arguments based on concavity, superharmonicity, divergence identities and an appropriate change-of-measure technique, we derive general conditions guaranteeing dominance and minimaxity for a large class of scale-invariant shrinkage estimators, both for spherically symmetric distributions with and without densities. The proposed results extend the framework of Hobbad et al. (2021, 2025) to the two classes of modified balanced loss functions. The resulting framework unifies and generalizes several existing results in the literature on shrinkage estimation, while providing a flexible theoretical basis for balanced loss estimation in multivariate problems. Numerical investigations corroborate the theoretical results and highlight the practical efficiency of the proposed estimators.

Beyond the theoretical developments, the framework of spherically symmetric models with residual structure has wide applicability. It arises naturally in robust multivariate analysis, where heavy-tailed and elliptical distributions are modeled through radial components; in high-dimensional inference, where shrinkage improves stability and mitigates overfitting; in invariant decision theory, where orthogonal symmetry simplifies risk evaluation; and in Bayesian and generalized Bayes procedures, where balanced and modified balanced loss functions provide flexible decision criteria. Applications also extend to signal processing, image reconstruction, machine learning regularization methods, and risk modeling in finance, where radial shrinkage mechanisms enhance robustness and predictive performance.

6 Proofs and Technical Lemmas

This section presents the proofs of the main theoretical results established in the paper. The presentation is organized into two parts. First, in Subsection 6.1, we provide the detailed proofs of Lemmas 3.1-3.3 and Theorems 3.2–3.7, which state sufficient conditions for the dominance of the proposed shrinkage estimators under modified balanced loss functions, both in the presence and in the absence of a density. Second, in Subsection 6.2, we collect and prove several technical lemmas that are essential for the subsequent derivations. These lemmas exploit the spherical symmetry of the model, concavity properties of the loss functions, and integration-by-parts identities.

6.1 Proofs of Main Results

We now provide the detailed proofs of Lemmas 3.1-3.3 and Theorems 3.2–3.7. Each proof leverages the technical lemmas established in the following subsection, together with the assumptions on the loss functions and the spherical symmetry of the model.

Proof of Lemma 3.1.
The first inequality of (6.10) follows from $\rho(t) \leq \rho(0) + \rho'(0)t = \rho'(0)t$ which holds because ρ satisfies hypothesis **H1**. Let $r^2 = \|X-\theta\|^2, R^2 = \|X-\theta\|^2 + \|U\|^2$ and note that the conditional distribution of X given r and R is $SS_d(\theta)$. Then, the random variable $\beta = \|X-\theta\|^2/(\|X-\theta\|^2 + \|U\|^2)$ has a $Beta(d/2, k/2)$ distribution and independent of $R^2 = \|X-\theta\|^2 + \|U\|^2$. Let E^*_θ denote the expectation with respect to the density f^*. We have

$$
\begin{aligned}
E_\theta\left[-\|U\|^2 h(X)\right] &= E_{R,r}\left[E_\theta\left(-\|U\|^2 h(X) \mid r, R\right)\right] \\
&= E_{R,r}\left[E_\theta\left(-(R^2 - r^2) h(X) \mid r, R\right)\right] \\
&= E_{R,r}\left[E_\theta\left(-(R^2 - \beta R^2) h(X) \mid R, \beta\right)\right] \text{ (note that } \beta R^2 = r^2) \\
&= E_{R,\beta}\left[\frac{\rho'(\beta R^2)}{K} E_\theta\left(-(1-\beta) R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right) \frac{K}{\rho'(\beta R^2)}\right] \\
&= E^*_{R,\beta}\left[E^*_\theta\left(-(1-\beta) R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right) \frac{K}{\rho'(\beta R^2)}\right].
\end{aligned}
$$

For fixed R, from Lemma 6.3 the functions $\beta \longrightarrow E_\theta\left[-(1-\beta)R^2 h(X)/\|X-\theta\| = \beta R^2\right]$ is decreasing and $\beta \longrightarrow \left(\rho'(\beta R^2)\right)^{-1}$ is increasing. Using the covariance inequality we get

$$
\begin{aligned}
E_\theta\left[-\|U\|^2 h(X)\right] &\leq E^*_{R,\beta}\left[E^*_\theta\left(-(1-\beta) R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\right] \times E^*_{R,\beta}\left[\frac{K}{\rho'(\beta R^2)}\right] \\
&= E^*_{R,\beta}\left[E^*_\theta\left(-\|U\|^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\right] = E^*_\theta\left[-\|U\|^2 h(X)\right].
\end{aligned}
$$

Hence we obtai the desired result. ■

Proof of Lemma 3.3.
Since $\ell(\cdot)$ is a concave function, for any points α and β, we have $\ell(\alpha+\beta)-\ell(a) \leq \beta\ell'(\alpha)$. Thus, for $\alpha = L_{\omega,\delta_0}(\theta,\delta_0)$ and $\alpha+\beta = L_{\omega,\delta_0}(\theta,\delta)$, we get

$$\Delta L_{\omega,\delta_0,\ell}(\theta,\delta) = L_{\omega,\ell,\delta_0}(\theta,\delta) - L_{\omega,\ell,\delta_0}(\theta,\delta_0) \leq (1-\omega)^2\ell'((1-\omega)\|\delta_0-\theta\|^2)\Delta L_{0,\delta_0}(\theta,\delta).$$

■

Proof of Theorem 3.2.
As in the proof of Theorem 3.2, the risk difference $\Delta\mathcal{R}(\omega,\delta_0,\rho)/(a(1-\omega)^2)$ is bounded by

$$\begin{aligned} & E_\theta\Big\{\omega\rho'(0)a^2(1-\omega)^2\|U\|^4\|g(X)\|^2 + (1-\omega)K\times\frac{\rho'(\|X-\theta\|^2)}{K}\times \\ & \Big[2a(1-\omega)\|U\|^2 g(X).(X-\theta) + a^2(1-\omega)^2\|g(X)\|^2\|U\|^4\Big]\Big\} \qquad (6.1) \\ = \; & E_\theta\Big[a\omega\rho'(0)\|U\|^4\|g(X)\|^2\Big] + KE^\star_\theta\Big[2\|U\|^2 g(X).(X-\theta)\Big] + a(1-\omega)KE^\star_\theta\Big[\|g(X)\|^2\|U\|^4\Big]. \end{aligned}$$

In the second expectation of the right-hand side of the last equality, the function idepends on θ. We can eliminate this dependence using Lemma 6.5 given by Cellier and Fourdrinier (1995).
For a weakly differentiable function g and by setting $q = 1$ in (6.11), we obtain

$$E_\theta((1-\omega)\|x-\theta\|^2)\Big[\|U\|^2 g(X)'(X-\theta)\Big] = \frac{1}{k+2}E_\theta((1-\omega)\|x-\theta\|^2)\Big[\|U\|^4 div(g(X))\Big].$$

Then the quantity in (6.1) is bounded by

$$a\omega\rho'(0)^2E_\theta\Big[\|U\|^4\|g(X)\|^2\Big] + KE^\star_\theta\Big[\Big(\frac{2\|U\|^4}{k+2}div(g(X)) + a(1-\omega)\|g(X)\|^2\|U\|^4\Big)\Big].$$

Since the function g satisfies condition **1.** of Theorem 3.2, then the risk difference $\Delta\mathcal{R}(\omega,\delta_0,\rho)/(a(1-\omega)^2)$ is bounded by

$$E^\star_\theta\Big[-2\|U\|^4 h(X)\Big(a(\omega\rho'(0)+K(1-\omega)) - \frac{K}{(k+2)}\Big)\Big],$$

hence the desired result. ■

Proof of Theorem 3.4.
Let E^*_θ denote the expectation with respect to the density f^*, where

$$f^*(\|x-\theta\|^2+\|u\|^2) = L^{-1}\ell'((1-\omega)\|x-\theta\|^2)f(\|x-\theta\|^2+\|u\|^2)$$

and $L = \int_{\mathbb{R}^{d+k}}\ell'((1-\omega)\|x-\theta\|^2)f(\|x-\theta\|^2+\|u\|^2)d(x,u) = E_\theta[\ell'((1-\omega)\|X-\theta\|^2)]$.
Using the same approach as Brandwein and Strawderman (1980) and Marchand and

Strawderman (2020) we calculate the difference between the risks of two estimators $\delta_0(X) = X$ and $\delta_{\omega,g}(X)$ with respect to the MBLFs **(ii)**. Using Lemmas 6.6, 6.5 and the conditions **1.** of Theorem 3.2 respectively, we obtain

$$
\begin{aligned}
\Delta\mathcal{R}(\theta) &= E_\theta\left[\Delta_{\theta,\delta_0,\ell}(\theta,\delta_{\omega,g})\right] \\
&\leq E_\theta\left[(1-\omega)^2\ell'((1-\omega)\|\delta_0-\theta\|^2)\Delta_0(\theta,\delta_{\omega,g})\right] \\
&= (1-\omega)^2E_\theta\left[\ell'((1-\omega)\|X-\theta\|^2)(\|\delta_{\omega,g}(X)-\theta\|^2-\|X-\theta\|^2)\right] \\
&= (1-\omega)^2E_\theta\left[\ell'((1-\omega)\|X-\theta\|^2)\left(\|X+a(1-\omega)\|U\|^2g(X)-\theta\|^2-\|X-\theta\|^2\right)\right] \\
&= a(1-\omega)^3L\left\{E^\star_\theta\left[a(1-\omega)\|U\|^4\|g(X)\|^2\right]+2E^\star_\theta\left[\|U\|^2g(X).(X-\theta)\right]\right\} \\
&= a(1-\omega)^3\ LE^\star_\theta\left[a(1-\omega)\|U\|^4\|g(X)\|^2+\frac{2\|U\|^4}{k+2}div(g(X))\right] \\
&\leq a(1-\omega)^3LE^\star_\theta\left[-2a(1-\omega)\|U\|^4h(X)+\frac{2\|U\|^4}{k+2}h(X)\right] \\
&= a(1-\omega)^3LE^\star_\theta\left[-2\|U\|^4h(X)(a(1-\omega))-\frac{1}{k+2})\right].
\end{aligned}
$$

The result follows immediately from the hypothesis $-h(\cdot) > 0$. ■

Proof of Theorem 3.5.
First we derive an expression for the risk difference:

$$
\begin{aligned}
\Delta\mathcal{R}(\omega,\delta_0,\rho) &= \mathcal{R}(\omega,\rho,\delta)-\mathcal{R}(\theta,\rho,\delta_0) = E_\theta\left[L_{\omega,\rho,\delta_0}(\theta,\delta)-L_{\omega,\rho,\delta_0}(\theta,\delta_0)\right] \\
&= E_\theta\left[\omega\rho(\|\delta(X)-\delta_0(X)\|^2)+(1-\omega)\rho(\|\delta(X)-\theta\|^2)-(1-\omega)\rho(\|\delta_0(X)-\theta\|^2)\right] \\
&= E_\theta\Big[\omega\rho(\|X+a(1-\omega)\|U\|^2g(X)-X\|^2)+ \\
&\qquad (1-\omega)\rho(\|X+a(1-\omega)\|U\|^2g(X)-\theta\|^2)(1-\omega)\rho(\|X-\theta\|^2)\Big] \\
&= E_\theta\left[\omega\rho(\|a(1-\omega)\|U\|^2g(X)\|^2)+(1-\omega)\left\{\rho(\|X-\theta+a(1-\omega)\|U\|^2g(X)\|^2)-\ \rho(\|X-\theta\|^2)\right\}\right] \\
&\leq E_\theta\Big[\omega\rho'(0)a^2(1-\omega)^2\|U\|^4\|g(X)\|^2+(1-\omega)\rho'(\|X-\theta\|^2)\times \\
&\qquad \left(2a(1-\omega)\|U\|^2g(X).(X-\theta)+a^2(1-\omega)^2\|g(X)\|^2\|U\|^4\right)\Big].
\end{aligned}
$$

The last inequality follows from the fact that $\rho(t) \leq \rho(0) + \rho'(0)t = \rho'(0)t$, which holds since ρ is concave with $\rho(0) = 0$. Then, the risk difference $(\Delta\mathcal{R}(\omega,\delta_0,\rho))/a(1-\omega)^2$ is bounded by:

$$
E_\theta\left[a\omega\rho'(0)\|U\|^4\|g(X)\|^2+2\rho'(\|X-\theta\|^2)\|U\|^2g(X).(X-\theta)+a(1-\omega)\rho'(\|X-\theta\|^2)\|g(X)\|^2\|U\|^4\right]. \tag{6.2}
$$

Using Lemma 3.3, we obtain

$$\begin{aligned} E_{R,\theta}\left[\rho'(R^2-\|U\|^2)\|U\|^2 g(X).(X-\theta)\right] &\leq E_{R,\theta}\left[\rho'(R^2-\|U\|^2)\frac{\|U\|^2\frac{k}{2}+1}{(k+2)(\|U\|^2)\frac{k}{2}-1}div(g(X))\right] \\ &= E_{R,\theta}\left[\rho'(R^2-\|U\|^2)\frac{\|U\|^4}{k+2}div(g(X))\right]. \end{aligned}$$

Hence the risk difference in (6.2) is bounded by

$$\begin{aligned} & E_\theta\left[a\omega\rho'(0)\|U\|^4\|g(X)\|^2+2\rho'(R^2-\|U\|^2)\frac{\|U\|^4}{k+2}div(g(X))+a(1-\omega)\rho'(\|X-\theta\|^2)\|g(X)\|^2\|U\|^4\right] \\ = \;& E_\theta\left[a\left(\omega\rho'(0)+(1-\omega)\rho'(\|X-\theta\|^2)\right)\|g(X)\|^2\|U\|^4+\frac{2}{k+2}\rho'(R^2-\|U\|^2)\|U\|^4 div(g(X))\right]. \quad (6.3) \end{aligned}$$

Furthermore, suppose that the function $g(.)$ is weakly differentiable and also satisfies $div(g(X)) \leq h(X) \leq -1/2\|g(X)\|^2$. Then the expectation in equation (6.3) is bounded by

$$E\left[\left(a\omega\rho'(0)+a(1-\omega)\rho'(\|X-\theta\|^2)-\frac{\rho'(\|X-\theta\|^2)}{(k+2)}\right)(-2\|U\|^4h(X))\right]. \quad (6.4)$$

Conditioning with respect to the to radius $r=\||X-\theta\|$ and $R^2=\||X-\theta\|^2+\||U\|^2$, the expectation in (6.4) becomes

$$\begin{aligned} & E\left\{\left(a\omega\rho'(0)+a(1-\omega)\rho'(r^2)-\frac{\rho'(r^2)}{k+2}\right)(R^2-r^2)^2E_{R,\theta}\left[-2h(X)\||X-\theta\|=r,R\right]\right\} \\ = \;& E\left\{E\left[\left(a\omega\rho'(0)+a(1-\omega)\rho'(\beta R^2)\right)(R^2-\beta R^2)^2E_{R,\theta}\left[-2h(X)\||X-\theta\|^2=\beta R^2\right]|R\right]\right\} \\ & -\frac{1}{k+2}E\left\{E\left[\rho'(\beta R^2)(R^2-\beta R^2)^2E_{R,\theta}\left[-2h(X)\||X-\theta\|^2=\beta R^2\right]|R\right]\right\} \\ = \;& E\left\{E\left[R^2\left(a\omega\rho'(0)+a(1-\omega)\rho'(\beta R^2)\right)\frac{(1-\beta)^2}{\beta}E_\theta\left[-2\beta R^2h(X)\||X-\theta\|^2=\beta R^2\right]|R\right]\right\} \\ & -\frac{1}{k+2}E\left\{E\left[\beta R^2\left(-\rho'(\beta R^2)\right)\frac{(1-\beta)^2}{\beta}R^2E_{R,\theta}\left[-2h(X)\||X-\theta\|^2=\beta R^2\right]|R\right]\right\}. \quad (6.5) \end{aligned}$$

We note that the functions $(1-\beta)^2/\beta$ is decreasing and $\left(a\omega\rho'(0)+a(1-\omega)\rho'(\beta R^2)\right)(1-\beta)^2\beta^{-1}$ is nondecreasing in β, since ρ is concave. Also, for fixed R, $E_\theta\left[-2\beta R^2h(X)/\|X-\theta\|=\beta R^2\right]$ is nonnegative and increasing in β. Using the covariance inequality and the fact that β

and R are independent, the first expectation in (6.5) is bounded by

$$\begin{aligned}
& E\left[R^2\left(a\omega\rho'(0)+\rho'(\beta R^2)a(1-\omega)\right)\frac{(1-\beta)^2}{\beta}E_{R,\theta}\left[-2\beta R^2h(X)/\|X-\theta\|^2=\beta R^2\right]\mid R\right] \qquad (6.6)\\
\leq\ & E\left[R^2\left(a\omega\rho'(0)+\rho'(\beta R^2)a(1-\omega)\right)\frac{(1-\beta)^2}{\beta}\right]\times E\left[E_{R,\theta}\left[-2\beta R^2h(X)\mid \|X-\theta\|^2=\beta R^2\right]/R\right]\\
\leq\ & E\left[\beta R^2\left(a\omega\rho'(0)+\rho'(\beta R^2)a(1-\omega)\right)\right]\times E\left[\frac{(1-\beta)^2}{\beta^2}\right]\times E\left[E_{R,\theta}\left[-2\beta R^2h(X)\mid \|X-\theta\|^2=\beta R^2\right]\mid R\right]\\
=\ & \left(a\omega\rho'(0)E\left[r^2\right]+a(1-\omega)E\left[r^2\rho'(r^2)\right]\right)\times E\left[\frac{(1-\beta)^2}{\beta^2}\right]\times E\left[E_{R,\theta}\left[-2\beta R^2h(X)/\|X-\theta\|^2=\beta R^2\right]|R\right].
\end{aligned}$$

Similarly, we know from Lemma 6.3 that, for fixed R, $E_\theta\left[-2h(X)/\|X-\theta\|=\beta R^2\right]$ is nonnegative and decreasing in β and $-\beta R^2\rho'(\beta R^2)(1-\beta)^2$ is decreasing in β. It then follows from the covariance inequality and the independence of β and R that the expectation in the second term of (6.5) is bounded by

$$\begin{aligned}
& \frac{1}{k+2}E\left[\beta R^2\left(-\rho'(\beta R^2)\right)\frac{(1-\beta)^2}{\beta}R^2E_\theta\left[-2h(X)\|\|X-\theta\|^2=\beta R^2\right]|R\right]\\
\leq\ & \frac{1}{k+2}E\left[\beta R^2\left(-\rho'(\beta R^2)\right)\right]\times E\left[\frac{(1-\beta)^2}{\beta}R^2E_{R,\theta}\left[-2h(X)\|\|X-\theta\|^2=\beta R^2\right]|R\right]\\
=\ & \frac{1}{k+2}E\left[\beta R^2\left(-\rho'(\beta R^2)\right)\right]\times E\left[\frac{(1-\beta)^2}{\beta}R^2E_{R,\theta}\left[-2h(X)\|\|X-\theta\|^2=\beta R^2\right]|R\right]\\
\leq\ & \frac{1}{k+2}E\left[\beta R^2\left(-\rho'(\beta R^2)\right)\right]\times E\left[\frac{(1-\beta)^2}{\beta^2}\right]\times E\left[E_{R,\theta}\left[-2\beta R^2h(X)\|\|X-\theta\|^2=\beta R^2\right]|R\right]\\
=\ & -\frac{1}{k+2}E\left[r^2\rho'(r^2)\right]\times E\left[\frac{(1-\beta)^2}{\beta^2}\right]\times E\left[E_{R,\theta}\left[-2\beta R^2h(X)\|\|X-\theta\|^2=\beta R^2\right]|R\right]. \qquad (6.7)
\end{aligned}$$

Combining equations (6.5), (6.6) and (6.7), we obtain an upper bound on the risk difference:

$$\begin{aligned}
\frac{\Delta\mathcal{R}(\theta)}{a(1-\omega)^2} \leq\ & \left(a\omega\rho'(0)E\left[r^2\right]+a(1-\omega)E\left[r^2\rho'(r^2)\right]-\frac{1}{k+2}E\left[r^2\rho'(r^2)\right]\right)\times\\
& E\left\{E\left[E_\theta\left[-2\beta R^2h(X)/\|X-\theta\|^2=\beta R^2\right]|R\right]\right\}.
\end{aligned}$$

Thus, a necessary condition for the risk difference to be negative is

$$a\omega\rho'(0)E\left[r^2\right]+a(1-\omega)E\left[r^2\rho'(r^2)\right]-\frac{1}{k+2}E\left[r^2\rho'(r^2)\right]\leq 0,$$

because $-h(\cdot)$ is positive. Hence, we obtain the desired result. ■

Proof of Theorem 3.6.
In the same way as in the proof of Theorem 3, using the fact that the function $\ell(.)$ is concave, we show that the risk difference $\Delta\mathcal{R}(\theta)$ is bounded by

$$\begin{aligned}
& a(1-\omega)^3 E_\theta\left[\ell'((1-\omega)\|X-\theta\|^2)\left(2\|U\|^2 g(X)'(X-\theta)+a(1-\omega)\|U\|^4\|g(X)\|^2\right)\right] \\
= \;& a(1-\omega)^3 E_\theta\left[\ell'((1-\omega)(R^2-\|U\|^2))\left(2\|U\|^2 g(X)'(X-\theta)+a(1-\omega)\|U\|^4\|g(X)\|^2\right)\right] \\
= \;& a(1-\omega)^3\left(E_\theta\left[2\|U\|^2\ell'((1-\omega)(R^2-\|U\|^2))g(X)'(X-\theta)\right]\right. \qquad (6.8)\\
& \left.+E_\theta\left[\ell'((1-\omega)(R^2-\|U\|^2))a(1-\omega)(\|U\|^2)^2\|g(X)\|^2\right]\right).
\end{aligned}$$

Using Lemma 6.7, for the function $m(t)=\ell'((1-\omega)(R^2-t))t$ and the corresponding function $M(t)=-\frac{1}{2}\int_0^t m(t)t^{\frac{k-2}{2}}dt$. By integrating by parts, we have

$$\begin{aligned}
M(t) &= -\frac{1}{2}\int_0^t \ell'((1-\omega)(R^2-t))t^{\frac{k}{2}}dt \\
&= -\ell'((1-\omega)(R^2-t))\frac{t^{\frac{k+2}{2}}}{k+2}-(1-\omega)\int_0^t \ell''((1-\omega)(R^2-t))\frac{t^{\frac{k+2}{2}}}{k+2}dt \\
&\geq -\ell'((1-\omega)(R^2-t))\frac{t^{\frac{k+2}{2}}}{k+2},
\end{aligned}$$

since the function $\ell(\cdot)$ is concave (i.e $\ell''(t)\leq 0$ for any $t\geq 0$). Conditioning with respect to the radius R, the first expectation in (6.8) becomes

$$\begin{aligned}
& E_{R,\theta}\left[\ell'((1-\omega)(R^2-\|U\|^2))\|U\|^2 g(X).(X-\theta)\right] \\
= \;& E_{R,\theta}\left[m\left(\|U\|^2\right)g(X).(X-\theta)\right] \\
= \;& -E_{R,\theta}\left[\frac{M(\|U\|^2)}{(\|U\|^2)^{\frac{k}{2}-1}}div(g(X))\right] \\
\leq \;& E_{R,\theta}\left[\ell'((1-\omega)(R^2-\|U\|^2))\frac{\|U\|^{k+2}}{(k+2)\|U\|^{k-2}}div(g(X))\right] \\
= \;& E_{R,\theta}\left[\ell'((1-\omega)(R^2-\|U\|^2))\frac{\|U\|^4}{k+2}div(g(X))\right] \qquad (6.9)
\end{aligned}$$

Combining condition **1.** of Theorem 3.2 and equations (6.8) and (6.9), an upper bound on the risk difference $\Delta\mathcal{R}(\theta)/(a(1-\omega)^3)$ is given by

$$\begin{aligned}
= \;& E_\theta\left[2\ell'((1-\omega)(R^2-\|U\|^2))\frac{\|U\|^4}{k+2}div(g(X))+\ell'((1-\omega)(R^2-\|U\|^2))\|U\|^4(1-\omega)\|g(X)\|^2)\right] \\
\leq \;& E_\theta\left[2\ell'((1-\omega)(R^2-\|U\|^2))\|U\|^4\frac{1}{k+2}h(X)+\ell'((1-\omega)(R^2-\|U\|^2))\|U\|^4 a(1-\omega)(-2h(X))\right] \\
= \;& E_\theta\left[-2\ell'((1-\omega)(R^2-\|U\|^2))\|U\|^4 h(X)\left(a(1-\omega)-\frac{1}{k+2}\right)\right].
\end{aligned}$$

The result follows immediately from the assumption that both functions $\ell'(\cdot)$ and $-h(\cdot)$ are positive. ∎

Proof of Theorem 3.7.
Let $r^2 = \|X-\theta\|^2$ and $R^2 = \|X-\theta\|^2 + \|U\|^2$. From the equation (6.8) and Lemma 6.8 we have

$$\begin{aligned}
\frac{\Delta\mathcal{R}(\theta)}{(1-\omega)^3} &\leq E_\theta\left[\ell'((1-\omega)\|X-\theta\|^2)\left(\|X+a(1-\omega)\|U\|^2 g(X)-\theta\|^2-\|X-\theta\|^2\right)\right] \\
&= E\left[E\left[\ell'((1-\omega)r^2)\left(2a^2(1-\omega)\|U\|^2/r^2-2a\|U\|^2/d\right)E_\theta\left[-r^2h(W)\mid \|X-\theta\|^2=r^2\right]\mid R,r\right]\right] \\
&\leq E\left[E\left[\ell'((1-\omega)\beta R^2)\left(2a^2(1-\omega)\frac{((1-\beta)R^2)^2}{\beta R^2}-\frac{2a(1-\beta)R^2}{d}\right)\right.\right. \\
&\quad \left.\left.\times E_\theta\left[-\beta R^2 h(W)\mid \|X-\theta\|^2=\beta R^2\right]\mid R\right]\right] \\
&= 2aE\left[E\left[R^2(\beta R^2)^\alpha \ell'((1-\omega)\beta R^2)\frac{1-\beta}{(\beta R^2)^\alpha}\right.\right. \\
&\quad \left.\left.\times\left(\frac{a(1-\omega)(1-\beta)}{\beta}-\frac{1}{d}\right)E_\theta\left[-\beta R^2 h(W)\mid \|X-\theta\|^2=\beta R^2\right]|R\right]\right] \\
&= 2aE\left[E\left[\left\{\frac{1-\beta}{\beta^\alpha}\left(\frac{a(1-\omega)(1-\beta)}{\beta}-\frac{1}{d}\right)\right\}\left\{R^{2-2\alpha}(\beta R^2)^\alpha\ell'((1-\omega)\beta R^2)\right.\right.\right. \\
&\quad \left.\left.\left.\times E\left[-\beta R^2h(W)\mid \|X-\theta\|^2=\beta R^2\right]|R\right]\right\}\right].
\end{aligned}$$

Next, for fixed R, both function $\beta \to (1-\beta)\beta^{-\alpha}\left(a(1-\omega)(1-\beta)\beta^{-1}-d^{-1}\right)$ and $\beta \to (\beta R^2)^\alpha\ell'((1-\omega)\beta R^2)E\left[-\beta R^2h(W)\mid \|X-\theta\|^2=\beta R^2\right]$ are nonnegative and nondecreasing in β. Hence it follows from the covariance inequality that the previous expectation is less than or equal to

$$\begin{aligned}
&2aE\left[E\left(E_{R,\theta}\left[-\beta R^2h(W)\mid \|X-\theta\|^2=\beta R^2\right]R^{2-2\alpha}(\beta R^2)^\alpha\ell'((1-\omega)\beta R^2)\mid R\right)\right] \\
&\qquad\qquad \times E\left[a(1-\omega)\beta^{-\alpha}(1-\beta)^2/\beta-\frac{\beta^{-\alpha}(1-\beta)}{d}.\right]
\end{aligned}$$

Since the first expectation in this term is nonnegative, it suffices that the second expectation is negative. But this is equivalent to:

$$0 < a \leq \frac{1}{d(1-\omega)}E[(\beta^{-\alpha}(1-\beta))]/E[(\beta^{-\alpha}(1-\beta)^2)/\beta] = (d-2-2\alpha)/(1-\omega)(k+2)d.$$

∎

6.2 Technical Lemmas

This subsection gathers auxiliary results that are repeatedly used in the main proofs. The lemmas cover properties of superharmonic functions, conditional expectations under spherical symmetry, and integration identities specific to the residual-augmented framework.

Theorem 6.1. *If the n-random vector X has a spherically symmetric distribution around* 0*, partition X into m parts* $X = \begin{pmatrix} X^{(1)'} \\ \vdots \\ X^{(m)'} \end{pmatrix}$*, each with* $n_1, \cdots, n_m$ *components respectively. Then* $\begin{pmatrix} X^{(1)'} \\ \vdots \\ X^{(m)'} \end{pmatrix}$*, has the same distribution as* $\begin{pmatrix} rd_1U_1' \\ \vdots \\ rd_mU_m' \end{pmatrix}$*, where r has the same distribution as* $\|X\|$*,* $d_i \geq 0, i = 1, \cdots, m$ *;* $(d_1^2, \cdots, d_m^2)$ *follows the Dirichlet distribution* $D_m(n_1/2, \cdots, n_m/2)$*;* $U_1, \cdots, U_m$ *and* $(d_1^2, \cdots, d_m^2)$ *are independent, and* U_j *follows the uniform distribution on the unit sphere of dimension* n_j*,* $j = 1, \cdots, m$*.*

Proof. See Fang et al. (1990) page: 43-44.

Lemma 6.2. *If h is a superharmonic function on* $\mathbb{R}^p$ *and Z is a random variable with a uniform distribution on the sphere centered at the origin and of radius* τ*, then, for any* $\theta \in \mathbb{R}^p$*, the expectation* $E_\theta\left[h(\theta + Z)\right]$ *is a non-increasing function of* τ*.*

Proof. See Du Plessis (1970), Theorem 2.30 page 54.

Lemma 6.3. *Giving X with a spherically symmetric distribution around* θ *and* $\beta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *a non-decreasing function. Then,* $E\left[R^2 \dfrac{\beta(\|X\|^2)}{\|X\|^2} / \|X - \theta\| = R\right]$ *is a non-decreasing function of* $R > 0$*.*

Proof. See Brandwein and Strawderman (1979) and Brandwein and Strawderman (1980), pages 394-395 within the proof of their Theorem 3.3.1.

Lemma 6.4. *Let* (X, U) *have a spherically symmetric distribution around* $(\theta, 0)$ *with the density of the form* $f(\|x - \theta|\|^2 + \|u\|^2)$*, with* $\theta \in \mathbb{R}^d$ *and* ρ *satisfy Assumption (**H1**). Then: For* $d \geq 4$*, we have*

$$E_\theta\left[\rho\left(-\|U\|^2 h(X)\right)\right] \leq \rho'(0) E_\theta\left[-\|U\|^2 h(X)\right] \leq \rho'(0) E_\theta\left[-\|V\|^2 h(Y)\right], \tag{6.10}$$

where $r = \|X - \theta\|$*,* $R^2 = \|X - \theta\|^2 + \|U\|^2$*,* $K = \int_{\mathbb{R}^{d+k}} \rho'((1-\omega)\|x - \theta\|^2) f(\|x - \theta\|^2 + \|u\|^2) d(x, u) = E_\theta[\rho'((1-\omega)\|X - \theta\|^2)]$ *and* $f^\star(x, u) = \left(\rho'((1-\omega)\|x - \theta\|^2) f(\|x - \theta\|^2 + \|u\|^2)\right)/K$*.*

Proof. The first inequality of (6.10) follows from $\rho(t) \leq \rho(0) + \rho'(0)t = \rho'(0)t$, which is true because ρ satisfies hypothesis **H1**. Let $r^2 = \|X - \theta\|^2, R^2 = \|X - \theta\|^2 + \|U\|^2$ and note that the conditional distribution of X given r and R is $SS_d(\theta)$. Then the random variable $\beta = \|X - \theta\|^2 / (\|X - \theta\|^2 + \|U\|^2)$ has a $Beta(d/2, k/2)$ distribution, independent of $R^2 = \|X-\theta\|^2 + \|U\|^2$. Let E^*_θ denote the expectation with respect to the density f^*. We have

$$\begin{aligned}
E_\theta\left[-\|U\|^2 h(X)\right] &= E_{R,r}\left[E_\theta\left(-\|U\|^2 h(X) \mid r, R\right)\right] \\
&= E_{R,r}\left[E_\theta\left(-(R^2-r^2)h(X) \mid r, R\right)\right] \\
&= E_{R,r}\left[E_\theta\left(-(R^2-\beta R^2)h(X) \mid R, \beta\right)\right] \text{ (note that } \beta R^2 = r^2) \\
&= E_{R,\beta}\left[\frac{\rho'(\beta R^2)}{K} E_\theta\left(-(1-\beta)R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\frac{K}{\rho'(\beta R^2)}\right] \\
&= E^*_{R,\beta}\left[E^*_\theta\left(-(1-\beta)R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\frac{K}{\rho'(\beta R^2)}\right].
\end{aligned}$$

For fixed R, from Lemma 6.3 the functions $\beta \longrightarrow E_\theta\left[-(1-\beta)R^2 h(X)/\|X-\theta\| = \beta R^2\right]$ is decreasing and
$\beta \longrightarrow (\rho'(\beta R^2))^{-1}$ is increasing. Using the covariance inequality we get

$$\begin{aligned}
E_\theta\left[-\|U\|^2 h(X)\right] &\leq E^*_{R,\beta}\left[E^*_\theta\left(-(1-\beta)R^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\right] \times E^*_{R,\beta}\left[\frac{K}{\rho'(\beta R^2)}\right] \\
&= E^*_{R,\beta}\left[E^*_\theta\left(-\|U\|^2 h(X) \mid \|X-\theta\|^2 = \beta R^2\right)\right] = E^*_\theta\left[-\|U\|^2 h(X)\right].
\end{aligned}$$

Hence, we obtain the result. ■

Lemma 6.5. *For every weakly differentiable function $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, for every integer q and for every $\theta \in \mathbb{R}^d$, we have*

$$E_\theta\left[(\|X\|^2)^q g(X)'(X-\theta)\right] = \frac{1}{k+2q} E_\theta\left[(\|U\|^2)^{q+1} div(g(X))\right], \tag{6.11}$$

provided these expectations exist.

Proof. See Fourdrinier and Strawderman (1996).

Lemma 6.6. *Let (X,U) have a spherically symmetric distribution with location parameter $(\theta,0)$. For the problem of estimating θ under MBLFs **(ii)** with ℓ satisfying the condition **H2**, we have $\Delta_{\omega,\ell,\delta_0}(\theta,\delta) \leq (1-\omega)^2 \ell'((1-\omega)\|\delta_0-\theta\|)\Delta_0(\theta,\delta)$.*

Proof. The proof uses the fact that $\ell(a+b)-\ell(a) \leq b\ell'(a)$ since ℓ is concave, with $a = L_{\omega,\ell,\delta_0}(\theta,\delta_0)$ and $a+b = L_{\omega,\ell,\delta_0}(\theta,\delta_0+(1-\omega)g)$. This yields

$$\begin{aligned}
\Delta_{\omega,\ell,\delta_0}(\theta,\delta) &= L_{\omega,\ell,\delta_0}(\theta,\delta_0+(1-\omega)g) - L_{\omega,\ell,\delta_0}(\theta,\delta_0) \\
&\leq (1-\omega)^2 \ell'((1-\omega)\|\delta_0-\theta\|)\Delta_0(\theta,\delta).
\end{aligned}$$

■

Lemma 6.7. *For every weakly differentiable function $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, for every real-valued function m, then, for any $\theta \in \mathbb{R}^d$, we have*

$$E_{R,\theta}\left[m(\|U\|^2)g(X)'(X-\theta)\right] = -E_{R,\theta}\left[\frac{M(\|U\|^2)}{(\|U\|^2)^{\frac{k}{2}-1}}div(g(X))\right],$$

provided these expectations exist. Where M is the indefinite integral, vanishing at 0, of the function $t \longrightarrow -2^{-1}h(t)t^{\frac{k}{2}-1}$.

Proof. See Cellier and Fourdrinier (1995).

Lemma 6.8. *Let $X \sim SS_d(\theta)$, for $d \geq 4$ and let $g(X)$ map $\mathbb{R}^d$ into $\mathbb{R}^d$ be weakly differentiable, and such that*

1. $\|g(X)\|^2/2 \leq -h(X) \leq -div(g(X))$,
2. $-h(X)$ *is superharmonic and* $E_\theta\left[r^2h(W) \mid r\right]$ *is a nondecreasing function of r,*

where W has a uniform distribution on the sphere of radius r centered at θ. Then

$$E_\theta\left[\|X+ag(X)-\theta\|^2 - \|X-\theta\|^2\right] \leq E_\theta\left[\left(-2a^2/r^2 + 2a\frac{1}{d}\right)E_{r,\theta}[r^2h(W) \mid \|X-\theta\|^2 = r^2\right].$$

Proof. We have

$$\begin{aligned} E_\theta\left[\|X+ag(X)-\theta\|^2 - \|X-\theta\|^2\right] &= E_\theta\left[2a^2\|g(X)\|^2/2 + 2a(X-\theta)^\top g(X)\right] \\ &\leq E_\theta\left[-2a^2h(X)\right] + 2a\left[\frac{1}{d}\int_{\mathbb{R}^+} r^2 \int_{B_{r,\theta}} \nabla g(x)dV_{r,\theta}(x)d\rho(r)\right] \\ &\leq E_\theta\left[-2a^2h(X)\right] + 2a\left[\frac{1}{d}\int_{\mathbb{R}^+} r^2 \int_{B_{r,\theta}} h(x)dV_{r,\theta}(x)d\rho(r)\right]. \end{aligned}$$

By subharmonicity of h we have

$$\int_{B_{r,\theta}} h(x)dV_{r,\theta}(x)d\rho(r) \leq \int_{S_{r,\theta}} h(x)dU_{r,\theta}(x)d\rho(r).$$

Hence,

$$E_\theta\left[\|X+ag(X)-\theta\|^2 - \|X-\theta\|^2\right] \leq E_\theta\left[\left(-2a^2/r^2 + 2a\frac{1}{d}\right)E_{r,\theta}\left[r^2h(W) \mid \|X-\theta\|^2 = r^2\right]\right].$$

■

Lemma 6.9. *If (Z, U) has a spherically symmetric distribution about $(0, 0)$ when $\dim Z = d$ and $\dim U = k$, then the distribution of $Z'Z/\left(Z'Z + \|U\|^2\right)$ is Beta$\left(\frac{d}{2}, \frac{k}{2}\right)$ and is independent of $Z'Z + \|U\|^2$.*

Proof. See Muirhead (1982) page: 38-39.

acknowledgemen: The authors thank the Editor and the Reviewers for their very helpful and constructive comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

Afshari M, Arashi M. New Wavelet Sure Thresholdings of Elliptical distributions under the balance loss. Statistica Sinica. 2021;31(31):1829–1852.

Brandwein, Ralescu, Strawderman. Shrinkage estimators of the location parameter for certain spherically symmetric distributions. Annals of the Institute of Statistical Mathematics. 1993;45(3):551–565.

Brandwein A, Strawderman W. Minimax estimation of mean of spherically symmetric distributions under general quadratic loss. Journal of Multivariate Analysis. 1979;9(2):579–588.

Brandwein A, Strawderman W. Minimax estimation of location parameters for spherically symmetric distributions with concave loss. Annals of Statistics. 1980;8(2):279–284.

Brandwein A, Strawderman W. Generalizations of James-Stein Estimators Under Spherical Symmetry. Annals of Statistics. 1991;19(3):639.

Cellier D, Fourdrinier D. Shrinkage estimators under spherical symmetry for the general linear model. Journal of Multivariate Analysis. 1995;52(2):338–351.

Cellier D, Fourdrinier D, Robert C. Robust shrinkage estimators of the location parameter for elliptically symmetric distributions. Journal of Multivariate Analysis. 1989;138(12):29–52.

Chaturvedi A, Shalabh. Risk and Pitman closeness properties of feasible generalized double k-class estimators in linear regression models with non-spherical disturbances under balanced loss function. Journal of Multivariate Analysis. 2004;90(2):229–256.

Dey D, Ghosh M, Strawderman WE. On estimation with balanced loss functions. Statistics Probability Letters. 1999;45(2):97–101.

Du Plessis N. An introduction to potential theory. Wiley-Interscience, New York; 1970.

Everitt B, Hothorn T. An Introduction to Applied Multivariate Analysis with R. Springer; 2011.

Fang KT, Kotz S, Ng KW. Symmetrie Multivariate and Related Distributions. Springer Science + Business Media, B.V; 1990.

Fourdrinier D, Ouassou I. Estimation of the mean of a spherically symmetric distribution with constraints on the norm. The Canadian Journal of Statistics. 2000;28(2):399–415.

Fourdrinier D, Ouassou I, Strawderman WE. Estimation of a parameter vector when some components are restricted. Journal of Multivariate Analysis. 2003;86(1):14–27.

Fourdrinier D, Strawderman WE. A paradox concerning shrinkage estimators : should a known scale parameter be replaced by an estimated value in the shrinkage factor ? Journal of Multivariate Analysis. 1996;59(2):109–140.

Giles JA, Giles DEA, K O. The exact risks of some pre-test and Stein-type regression estimates under balanced loss. Communications in Statistics - Theory and Methods. 1996;25(12):2901–2924.

Hamdaoui A, Almutiry TM W, Benkhaled A. Comparison of Risk Ratios of Shrinkage Estimators in High Dimensions. Mathematics. 2022;10(52).

Hobbad L, Alahiane M, Ouassou I, Rachdi M. Shrinkage estimation for spherically symmetric distributions under modified balanced loss functions. Journal of the Iranian Statistical Society. 2025;23(02):151–177.

Hobbad L, Marchand E, Ouassou I. On shrinkage estimation of a spherically symmetric distribution for balanced loss functions. Journal of Multivariate Analysis. 2021;186.

Jozani J, M E Marchand, Parsian A. On estimation with weighted balanced-type loss function. Statistics & Probability Letters. 2006;76(8):773–780.

Karamikabir H, Afshari M. Generalized Bayesian shrinkage and wavelet estimation of location parameter for spherical distribution under balance-type loss: Minimaxity and admissibility. Journal of Multivariate Analysis. 2019;177(104583).

Karamikabir H, Afshari M. Wavelet Shrinkage Generalized Bayes Estimation for Multivariate Normal Distribution Mean Vectors with unknown Covariance Matrix under Balanced-LINEX Loss. Revista Colombiana de Estadística - Theoretical Statistics. 2022;45(104583):107–123.

Karamikabir H, Afshari M, Arashi M. Shrinkage estimation of non-negative mean vector with unknown covariance under balance loss. Journal of Inequalities and Applications. 2018;331(331).

Marchand, Strawderman. On shrinkage estimation for balanced loss functions. Journal of Multivariate Analysis. 2020;175.

Muirhead RJ. Aspects of Multivariate Statistical Theory. Wiley, New York; 1982.

Zellner A. Bayesian and non Bayesian estimation using balanced loss functions. in: Gupta, S.S., Berger, J.O. (Eds.), Statistical Decision Theory and Related Topics. Springer, New York, pp. 371-390; 1994.