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Statistical inference for generalized exponential distribution under improved adaptive Type-II progressive censoring

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Abstract. This paper explores an approach to analyse the estimation of parameters of generalized exponential distribution using an improved adaptive type-II progressive censoring scheme. The point and interval estimations, utilizing two classical estimation methods maximum likelihood and maximum product of spacing are employed to estimate the unknown parameters, as well as the reliability and hazard rate functions. The approximate confidence intervals for these quantities are derived using the asymptotic normality of the maximum likelihood and maximum product of spacing methods. Bayesian estimations are explored using Markov Chain Monte Carlo (MCMC) techniques, based on two established approaches. The Bayesian estimation is examined under both symmetric and asymmetric loss functions. A Monte Carlo simulation study is conducted to compare the performance of the proposed estimates, and the effectiveness of the estimation approach is illustrated using a real dataset.

Keywords. Improved adaptive type-II progressive censoring, generalized exponential distribution, maximum likelihood estimation, maximum product spacing estimation, Bayesian estimation.

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1 Introduction

Censored sampling occurs in life-testing experiments when experimenters are unable to observe the failure times of all units. In medical and industrial applications, researchers frequently encounter censored data because they may not have enough time to observe the lifetimes of all subjects. Furthermore, subjects/items may fail due to causes unrelated to the study. Over the past few decades, several censoring methodologies have been developed for analyzing such situations. In the existing literature, two commonly employed traditional censoring schemes are type I and type II, where the experiment is terminated after a predetermined time point and number of failures, respectively. However, neither of these censoring strategies allows the experimenter to remove live units from the experiment before its termination time. A more generalized scheme, known as progressive Type II censoring, introduced by Cohen (1963), addresses this limitation and allows removal of units at various points throughout the experiment. This gives more experiment design flexibility, improved efficiency for certain research questions, and a reduction in evaluating the duration and expenses to a specific extent compared to conventional censoring schemes. The progressive type-II censoring was extensively explored by Balakrishnan and Aggarwala (2000) and detailed in the thorough review by Balakrishnan (2007). The progressive type-II censoring scheme is defined as follows. Let $X_1, \cdots, X_n$ be independent and identically distributed (i.i.d.) random lifetimes of n items and let m be the number of failures to be observed. At the time of the first failure, noted as $X_{1:m:n}, R_1$ units are randomly removed from $(n-1)$ surviving units. Similarly at the time of the second failure, noted as $X_{2:m:n}, R_2$ units from the $(n-R_1-2)$ units are randomly removed. This process continuous until, at the time of the m^{th} observed failure, the remaining $(n-m-R_1-R_2-\cdots-R_{m-1})$ units are all removed from the experiment. The enhanced reliability of products, resulting from technological advances in manufacturing, can lengthen experimental periods when employing progressive type-II censoring schemes. For this reason, the progressively Type-II hybrid censoring scheme (PT-II HCS) was proposed by Kundu and Joarder (2006). Nevertheless, in the context of this censoring scheme, the experimental duration is fixed, resulting in a randomization of the effective sample size. There are situations in which the effective sample size can reach zero, hence reducing the efficacy of statistical inference. To overcome this drawback, an adaptive censoring scheme (AT-II PCS) has been proposed by Ng et al. (2009), in which the effective number of failures m is predetermined and the test time can exceed a prefixed time T. Unfortunately, if the test units have an extreme lifespan, it is not guaranteed a satisfactory total test duration because the experiment time will be very long. Recently, to solve this problem, Yan et al. (2021) proposed an improved adaptive progressive Type-II censoring scheme (IAT-II PCS) which is a generalization of two popular censoring schemes, namely: AT-II PCS and PT-II HCS. They stated that the IAT-II PCS can effectively guarantee that the experiment stops within prespecified times such as T_1 and T_2. Recently, IAT-II PCS has been explored by a few researchers see, Nassar and Elshahhat (2024), Dutta and Kayal (2023), Dutta et al. (2024), Zhang and Yan (2024).

The procedure of IAT-II PCS can be described as follows: Suppose a random sample of n units is placed on a life-testing experiment at time zero, the effective number of failures $m(< n)$ is predetermined, and the progressive censoring scheme $R = (R_1, R_2, ..., R_m)$ with $R_i \geq 0$ is also fixed in advance, but some values of R_i may change during the experiment. Let $T_1, T_2 \in (0, \infty)$, where $T_1 < T_2$, be the two threshold time points that are determined according to the reliability information on the product of interest. Suppose $d_1(< d_2)$ and $d_2(d_2 < m)$ are the number of failures that occur before times T_1 and T_2, respectively. At the time of the first failure (say $X_{1:m:n}$), R_1 items from the experiment are randomly withdrawn from $n-1$ live items. Similarly, at the time of the second failure (say $X_{2:m:n}$), R_2 of $n - R_1 - 2$ items are randomly removed from the experiment, and so on. If $X_{m:m:n}$ occurs before time T_1, i.e. $X_{m:m:n} < T_1$ (Case-I), the experiment stops at the time of m^{th} failure with the censoring scheme $R = (R_1, R_2, \cdots, R_m)$. If $X_{d_1:m:n} < T_1 < X_{d_1+1:m:n}$ (Case-II), where $d_1 > 0$ and $d_1 + 1 < m$, the experiment stops at the time of m^{th} failure with censoring scheme $R = (R_1, R_2, \cdots, R_{d_1}, 0, \cdots, 0, R^*)$, where $R^* = n - m - \sum_{i=1}^{d_1} R_i$, then no live units will be removed from the experiment by setting $R_i = 0$ for $i = d_1 + 1, ..., m - 1$ and at the time of the m^{th} failure all remaining units are removed. On the other hand, if $T_2 < X_{m:m:n}$ (Case-III), the experiment stops at T_2 with censoring scheme $R_i = 0$ for $i = d_1 + 1, \cdots, d_2$, and at T_2 all the remaining units are removed, i.e. $R^* = n - d_2 - \sum_{i=1}^{d_1} R_i$. A diagrammatic representation of IAT-II PCS is depicted in figure 1 . The lifetimes of observed units in these three cases are provided by

Case I : $X_{1:m:n} < \cdots < X_{m:m:n}$,
Case-II : $X_{1:m:n} < \cdots < X_{d_1:m:n} < \cdots < X_{m:m:n}$,
Case-III : $X_{1:m:n} < \cdots < X_{d_1:m:n} < \cdots < X_{d_2:m:n}$.

The above cases are respectively taken place when

Case I : $X_{m:m:n} < T_1 < T_2$,
Case-II : $X_{d_1:m:n} < T_1 < X_{d_1+1:m:n} < \cdots X_{m:m:n} < T_2$,
Case-III : $X_{d_2:m:n} < T_2 < X_{d_2+1:m:n} < \cdots X_{m:m:n}$.

Suppose we have an IAT-II PC sample taken from a continuous population with cumulative distribution function (CDF) F(.) and probability density function (PDF) f(.), the joint likelihood function (LF) of the observed data can be written as follows

$$L(\Theta) = C \prod_{i=1}^{D_2} f(x_{i:m:n}) \prod_{i=1}^{D_1} [1 - F(x_i)]^{R_i} [1 - F(T^*)]^{R^*}, \tag{1.1}$$

where C is a constant which does not depend on the parameters and Θ is the vector of the unknown parameters. Let $X_i = X_{i:m:n}$ for simplicity, then from (1.1), the quantities D_1, D_2, R_i, R^* and T^* for the different cases are presented in Table 1.
Clearly, T_1 indicates a warning about the experimental time, while T_2 represents the maximum time that the experimenter can afford to continue. If the experiment reaches

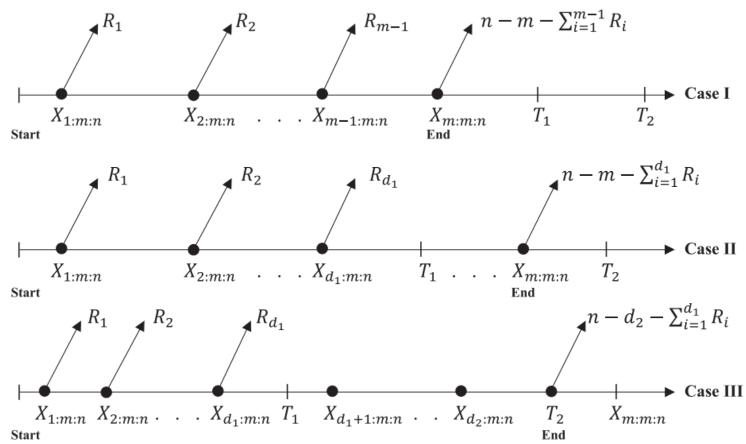


Figure 1: Graphical Illustration of Improved Adaptive Progressive Type-II Censoring Scheme Based on Three Cases

the specified time T_2 and the number of desired failures m does not occur, the life-test must be stopped at T_2. This improvement overcomes the drawback of AT-II PCS, which cannot guarantee the test time, by guaranteeing that the total test time will not exceed the time T_2.

Table 1: Various choices of D_1, D_2, R^*, T^* and $\{x_i, R_i\}$

Case	D_1	D_2	R^*	T^*	$\{X_i, R_i\}$
I	m	m	R_m	X_m	$\{(X_1, R_1), \cdots, (X_{m-1}, R_{m-1}), (X_m, R^*)\}$
II	d_1	m	$n-m-\sum_{i=1}^{d_1} R_i$	X_m	$\left\{(X_1, R_1), \cdots, (X_{d_1}, R_{d_1}), (X_{m-1}, 0), (X_m, R^*)\right\}$
III	d_1	d_2	$n-d_2-\sum_{i=1}^{d_1} R_i$	T_2	$\left\{(X_1, R_1), \cdots, (X_{d_1}, R_{d_1}), (X_{d_1+1}, 0) \cdots (X_{d_2-1}, 0), (X_{d_2}, 0)\right\}$

In this paper, we consider the statistical inference for a generalized exponential (GE) distribution under IAT-II PC. Kundu and Gupta (1999) provided the PDF and CDF of the GE distribution, which are, respectively, given by,

$$f(x; \alpha, \lambda) = \alpha\lambda \, exp(-\lambda x)[1 - exp(-\lambda x)]^{\alpha-1} \quad , x, \alpha, \lambda > 0 \tag{1.2}$$

and

$$F(x; \alpha, \lambda) = [1 - exp(-\lambda x)]^{\alpha} \quad , x, \alpha, \lambda > 0, \tag{1.3}$$

where α and λ denote the shape and scale parameters, respectively. Let $\delta(x; \lambda) = 1 - exp(-\lambda x)$; then, PDF and CDF of the GE distribution can be respectively rewritten as

$$f(x; \alpha, \lambda) = \alpha\lambda \, exp(-\lambda x)[\delta(x; \lambda)]^{\alpha-1} \quad , x, \alpha, \lambda > 0 \tag{1.4}$$

and

$$F(x; \alpha, \lambda) = [\delta(x; \lambda)]^{\alpha} \quad , x, \alpha, \lambda > 0 \ . \tag{1.5}$$

Then, the reliability (survival) and hazard functions can be written, respectively, as follows:

$$S(x;\alpha,\lambda) = 1 - [\delta(x;\lambda)]^{\alpha} \quad ,x,\alpha,\lambda > 0 \tag{1.6}$$

and

$$H(x;\alpha,\lambda) = \frac{f(x;\alpha,\lambda)}{S(x;\alpha,\lambda)} = \frac{\alpha\lambda\ exp(-\lambda x)[\delta(x;\lambda)]^{\alpha-1}}{1-[\delta(x;\lambda)]^{\alpha}}, \quad x,\alpha,\lambda > 0 \tag{1.7}$$

The GE distribution is a very flexible and favorably skewed model that is used to replace the lognormal, gamma, and Weibull distributions. Since then, GE distribution has been studied by many authors including Raqab and Ahsanullah (2001), Gupta and Kundu (2002), Gupta and Kundu (2007), Jaheen (2004), Raqab and Madi (2005), and Sarhan (2007). Kundu and Gupta (1999) mentioned that the two-parameter GE distribution could be used quite effectively in analyzing many lifetime data, particularly, in place of the two-parameter gamma and two-parameter Weibull distributions. Even though progressive Type-II censoring has been extensively investigated for different distributions, the improved adaptive Type-II progressive censoring scheme has only been applied to a small number of lifetime distributions so far. Despite its broad relevance in industrial life-testing, medical research, survival analysis, and reliability engineering, its application to the GE distribution remains unexplored. Thus, this censoring method in combination with the GE distribution is novel and fills an existing gap in the literature.

The rest of the paper is organized as follows: Section 2 discusses point and interval estimations via the maximum likelihood approach. In Section 3, the point and interval estimation are investigated using product spacing estimation method. Bayes estimates under both squared error and LINEX loss functions are provided in Section 4. Furthermore, highest posterior density (HPD) credible intervals for the parameters are obtained. The simulation results are presented in Section 5. In section 6, a real data is considered for illustration. Finally, a concise overview of the paper is presented in Section 7.

2 Maximum Likelihood Estimation

In this section, we consider the maximum likelihood estimation method to estimate the unknown parameters of GE distribution based on IAT-II PC data. In addition to the maximum likelihood estimates (MLEs) of the parameters, the MLEs of RF and HRF and the corresponding ACIs are also obtained.

Let $X_{1:m:n},\cdots,X_{D_1:m:n},\cdots,X_{D_2:m:n}$ be an IAT-II PC-sample of size D_2 from the GE distribution with PDF and CDF given by (1.4) and (1.5), respectively, with censoring scheme R = $(R_1,...,R_{D_1},0,\cdots,0,R^*)$. Substituting (1.4), and (1.5) in (1.1), we can write the LF in the following form

$$L(\alpha,\lambda|\underline{x}) = C\prod_{i=1}^{D2}\alpha\lambda e^{-\lambda x_i}[\delta(x_i;\lambda)]^{\alpha-1}\prod_{i=1}^{D_1}[1-(\delta(x_i;\lambda))^{\alpha}]^{R_i}[1-(\delta(T^*;\lambda)^{\alpha})]^{R^*}. \tag{2.1}$$

Then, the log-likelihood function of (2.1) is as follows:

$$\ln L(\alpha, \lambda|\underline{x}) = lnC + D_2 ln\alpha + D_2 ln\lambda + (\alpha - 1)\sum_{i=1}^{D_2} ln[\delta(x_i, \lambda)] - \lambda \sum_{i=1}^{D_2} x_i$$
$$+ \sum_{i=1}^{D_1} R_i ln[1 - (\delta(x_i, \lambda))^{\alpha}] + R^* ln[1 - \delta(T^*, \lambda)^{\alpha}]. \tag{2.2}$$

To obtain the normal equations for the unknown parameters, we differentiate (2.2) partially with respect to the parameters α and λ and equate them to zero, and are given below.

$$\frac{D_2}{\alpha} + \sum_{i=1}^{D_2} ln[\delta(x_i; \lambda)] - \sum_{i=1}^{D_1} R_i \frac{[\delta(x_i; \lambda)]^{\alpha} ln[\delta(x_i; \lambda)]}{[1 - [\delta(x_i; \lambda)]^{\alpha}]} - R^* \frac{[\delta(T^*; \lambda)]^{\alpha} ln[\delta(T^*; \lambda)]}{[1 - [\delta(T^*; \lambda)]^{\alpha}]} = 0\,, \tag{2.3}$$

$$\frac{D_2}{\lambda} - \sum_{i=1}^{D_2} x_i + (\alpha - 1)\sum_{i=1}^{D_2} \frac{x_i[1 - \delta(x_i; \lambda)]}{\delta(x_i; \lambda)} - \alpha \sum_{i=1}^{D_1} R_i \frac{x_i[1 - \delta(x_i; \lambda)][\delta(x_i; \lambda)]^{\alpha-1}}{1 - [\delta(x_i; \lambda)]^{\alpha}} - \tag{2.4}$$
$$\alpha R^* \frac{T^*[1 - \delta(T^*; \lambda)][\delta(T^*; \lambda)]^{\alpha-1}}{[1 - [\delta(x_i; \lambda)]^{\alpha}]} = 0\,.$$

The system of nonlinear equations represented by (2.3) and (2.4) lacks a closed-form solution due to the intractable nature of the terms involved. Consequently, numerical methods are indispensable for obtaining the ML estimates $\hat{\alpha}$ and $\hat{\lambda}$. The Newton-Raphson method is a common numerical approach employed to find these estimates. The MLEs $\hat{\alpha}$ and $\hat{\lambda}$ can be substituted into (1.6) and (1.7) to generate the MLEs $\hat{R(t)}$ and $\hat{H(t)}$ (for any given time $t > 0$) of the RF and HRF by using the invariance property of the MLEs.

2.1 Approximate confidence intervals

The $100(1 - \tau)\%$ Approximate Confidence Intervals (ACIs) for model parameters and the reliability characteristics are obtained in this subsection. The MLEs' asymptotic normality serves as the basis for obtaining these ACIs. The asymptotic variance-covariance (VC) matrix needs to be computed in order to create such ACIs, and this can be done by inverting the Fisher information matrix (FIM). In order to do this, we first determine the second derivatives of (2.2) with regard to α and λ as

2.1.1 Confidence intervals of parameters

Based on the asymptotic normality of the relevant MLEs, the ACIs of $\alpha, \lambda, S(t)$, and $h(t)$ are determined. To obtain the ACIs, the asymptotic variance-covariance matrix (VCM) has to be constructed based on the MLEs. This VCM can be obtained from the inverse of the FIM, and the FIM, $I(\alpha, \lambda)$ can be expressed as

$$I(\alpha,\lambda) = -E\begin{bmatrix} \dfrac{\partial^2 lnL(\alpha,\lambda)}{\partial\alpha^2} & \dfrac{\partial^2 lnL(\alpha,\lambda)}{\partial\alpha\partial\alpha} \\ \dfrac{\partial^2 lnL(\alpha,\lambda)}{\partial\alpha\partial\lambda} & \dfrac{\partial^2 lnL(\alpha,\lambda)}{\partial\lambda^2} \end{bmatrix}. \tag{2.5}$$

Here,

$$\begin{aligned}\frac{\partial^2 lnL(\alpha,\lambda)}{\partial\alpha^2} =& -\frac{D_2}{\alpha^2} - \sum_{i=1}^{D_1} R_i \left\{ \frac{[\delta(x_i;\lambda)]^{\alpha} ln[\delta(x_i;\lambda)]^2}{1-[\delta(x_i;\lambda)]^{\alpha}} + \frac{[\delta(x_i;\lambda)]^{2\alpha}(ln[\delta(x_i;\lambda)])^2}{(1-[\delta(x_i;\lambda)])^{\alpha}} \right\} \\ & - R^* \left\{ \frac{[\delta(T^*;\lambda)]^{\alpha} ln[\delta(T^*;\lambda)]^2}{1-[\delta(T^*;\lambda)]^{\alpha}} + \frac{[\delta(T^*;\lambda)]^{2\alpha}(ln[\delta(T^*\lambda)]^2)}{(1-[\delta(T^*;\lambda)]^{\alpha})^2} \right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 lnL(\alpha,\lambda)}{\partial\alpha\partial\lambda} = \frac{\partial^2 lnL(\alpha,\lambda)}{\partial\lambda\partial\alpha} =& \sum_{i=1}^{D_2} \frac{x_i[1-\delta(x_i;\lambda)]}{[\delta(x_i;\lambda)]} - \sum_{i=1}^{D_1} R_i \left\{ \frac{x_i[1-\delta(x_i;\lambda)][\delta(x_i;\lambda)]^{\alpha}}{\delta(x_i;\lambda)[1-[\delta(x_i;\lambda)]^{\alpha}]} \right. \\ & + \frac{\alpha x_i(1-[\delta(x_i;\lambda)])[\delta(x_i;\lambda)]^{\alpha} ln[\delta(x_i;\lambda)]}{\delta(x_i;\lambda)([\delta(x_i;\lambda)]^{\alpha})} \\ & \left. + \frac{\alpha x_i(1-[\delta(x_i;\lambda)])[\delta(x_i;\lambda)]^{2\alpha} ln[\delta(x_i;\lambda)]}{\delta(x_i;\lambda)([\delta(x_i;\lambda)]^{\alpha})^2} \right\} \\ & - R^* \left\{ \frac{T^*[1-\delta(T^*;\lambda)][\delta(T^*;\lambda)]^{\alpha}}{\delta(T^*;\lambda)[1-[\delta(T^*;\lambda)]^{\alpha}]} \right. \\ & + \frac{\alpha T^*(1-[\delta(T^*;\lambda)])[\delta(T^*;\lambda)]^{\alpha} ln[\delta(T^*;\lambda)]}{\delta(T^*;\lambda)([\delta(T^*;\lambda)]^{\alpha})} \\ & \left. + \frac{\alpha T^*(1-[\delta(T^*;\lambda)])[\delta(T^*;\lambda)]^{2\alpha} ln[\delta(T^*;\lambda)]}{\delta(T^*;\lambda)([\delta(T^*;\lambda)]^{\alpha})^2} \right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 lnL(\alpha,\lambda)}{\partial\lambda^2} =& -\frac{D_2}{\lambda^2} - (\alpha-1)\sum_{i=1}^{D_2} \left\{ \frac{x_i^2[1-\delta(x_i;\lambda)]}{[\delta(x_i;\lambda)]} + \frac{x_i^2[1-\delta(x_i;\lambda)]^2}{[\delta(x_i;\lambda)]^2} \right\} \\ & + \sum_{i=1}^{D_1} R_i \left\{ \frac{\alpha x_i^2[1-\delta(x_i;\lambda)]^2[\delta(x_i;\lambda)]^{\alpha}}{[\delta(x_i;\lambda)]^2[1-\delta(x_i;\lambda)]^{\alpha}} - \frac{\alpha^2 x_i^2[1-\delta(x_i;\lambda)]^2[\delta(x_i;\lambda)]^{\alpha}}{[\delta(x_i;\lambda)]^2[1-\delta(x_i;\lambda)]^{\alpha}} \right. \\ & \left. \frac{\alpha x_i^2[\delta(x_i;\lambda)]^{\alpha}[1-\delta(x_i;\lambda)]}{[\delta(x_i;\lambda)](1-[\delta(x_i;\lambda)]^{\alpha}])^2} - \frac{\alpha^2 x_i^2[\delta(x_i;\lambda)]^{2\alpha}[1-\delta(x_i;\lambda)]^2}{[\delta(x_i;\lambda)]^2(1-[\delta(x_i;\lambda)]^{\alpha}])^2} \right\} \\ & + R^* \left\{ \frac{\alpha {T^*}^2[1-\delta(T^*;\lambda)]^2[\delta(T^*;\lambda)]^{\alpha}}{[\delta(T^*;\lambda)]^2[1-\delta(T^*;\lambda)]^{\alpha}} - \frac{\alpha^2 {T^*}^2[1-\delta(T^*;\lambda)]^2[\delta(T^*;\lambda)]^{\alpha}}{[\delta(T^*;\lambda)]^2[1-\delta(T^*;\lambda)]^{\alpha}} \right. \\ & \left. \frac{\alpha {T^*}^2[\delta(T^*;\lambda)]^{\alpha}[1-\delta(T^*;\lambda)]}{[\delta(T^*;\lambda)](1-[\delta(T^*;\lambda)]^{\alpha}])^2} - \frac{\alpha^2 {T^*}^2[\delta(T^*;\lambda)]^{2\alpha}[1-\delta(T^*;\lambda)]^2}{[\delta(T^*;\lambda)]^2(1-[\delta(T^*;\lambda)]^{\alpha}])^2} \right\},\end{aligned}$$

Generally, the exact expressions of (2.5) are difficult to obtain. Therefore, the approximate Fisher information matrix $I(\hat{\alpha}, \hat{\lambda})$ is obtained using the MLEs $\hat{\alpha}$ and $\hat{\lambda}$. Then the asymptotic variance-covariance matrix of $\hat{\alpha}$ and $\hat{\lambda}$ can be given by $I^{-1}(\hat{\alpha}, \hat{\lambda})$, which is expressed as

$$
\begin{aligned}
I^{-1}(\hat{\alpha}, \hat{\lambda}) &= \begin{bmatrix} -\dfrac{\partial^2 lnL(\alpha, \lambda)}{\partial \alpha^2} & -\dfrac{\partial^2 lnL(\alpha, \lambda)}{\partial \alpha \partial \alpha} \\ -\dfrac{\partial^2 lnL(\alpha, \lambda)}{\partial \alpha \partial \lambda} & -\dfrac{\partial^2 lnL(\alpha, \lambda)}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\alpha,\lambda)=(\hat{\alpha},\hat{\lambda})} \\
&= \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\lambda}) \\ cov(\hat{\lambda}, \hat{\alpha}) & var(\hat{\lambda}) \end{bmatrix}, \qquad (2.6)
\end{aligned}
$$

where $var(\hat{\alpha})$ and $var(\hat{\lambda})$ are the approximate variances of $\hat{\alpha}$ and $\hat{\lambda}$, respectively, and $cov(\hat{\alpha}, \hat{\lambda}) = cov(\hat{\lambda}, \hat{\alpha})$ is the covariance between them. According to the asymptotic normality of the MLE, it is known that $(\hat{\alpha}, \hat{\lambda}) \sim N_2[(\alpha, \lambda), I^{-1}(\hat{\alpha}, \hat{\lambda})]$, where $I^{-1}(\hat{\alpha}, \hat{\lambda})$ is given by (13). Therefore, the $100(1-\tau)$ ACIs for α and λ can be expressed, respectively, as follow

$$
\hat{\alpha} \pm z_{\tau/2} \sqrt{var(\hat{\alpha})} \text{ and } \hat{\lambda} \pm z_{\tau/2} \sqrt{var(\hat{\lambda})} ,
$$

where $z_{\tau/2}$ is the upper $(\tau/2)th$ percentile point of $N(0, 1)$.

2.1.2 ACIs of S(t) and h(t)

To construct the ACIs of R(t) and h(t), it is necessary to obtain the variances of $\hat{R(t)}$ and $\hat{h(t)}$. Then we use the delta method to approximate these variances.

Let

$$
D_R = \left(\frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial \lambda} \right)
$$

and

$$
D_h = \left(\frac{\partial h(t)}{\partial \alpha}, \frac{\partial h(t)}{\partial \lambda} \right),
$$

where

$$
\frac{\partial R(t)}{\partial \alpha} = -[\delta(x_i; \lambda)]^{\alpha} ln[\delta(x_i; \lambda)],
$$

and

$$
\frac{\partial R(t)}{\partial \lambda} = -\alpha[\delta(x_i; \lambda)]^{\alpha-1} x_i [1 - \delta(x_i; \lambda)]
$$

in which

$$\frac{\partial h(t)}{\partial \alpha} = \frac{\lambda[1-\delta(x_i;\lambda)][\delta(x_i;\lambda)]^{\alpha}}{[\delta(x_i;\lambda)][1-\delta(x_i;\lambda)^2]} + \frac{\alpha\lambda[1-\delta(x_i;\lambda)][\delta(x_i;\lambda)]^{\alpha} ln[\delta(x_i;\lambda)]}{[\delta(x_i;\lambda)][1-\delta(x_i;\lambda)^{\alpha}]} + \frac{\alpha\lambda[1-\delta(x_i;\lambda)][\delta(x_i;\lambda)]^{2\alpha} ln[\delta(x_i;\lambda)]}{[\delta(x_i;\lambda)][1-\delta(x_i;\lambda)^{\alpha}]^2}$$

and

$$\frac{\partial h(t)}{\partial \lambda} = \frac{\alpha[\delta(x_i;\lambda)]^{\alpha-1}[1-\delta(x_i;\lambda)]}{1-[\delta(x_i;\lambda)]^{\alpha}} - \frac{\alpha\lambda x_i[\delta(x_i;\lambda)]^{\alpha-1}[1-\delta(x_i;\lambda)]}{1-[\delta(x_i;\lambda)]^{\alpha}} + \frac{\alpha(\alpha-1)\lambda x_i[1-\delta(x_i;\lambda)]^2[\delta(x_i;\lambda)]^{\alpha-2}}{1-[\delta(x_i;\lambda)]^{\alpha}} + \frac{\alpha^2\lambda x_i[1-\delta(x_i;\lambda)]^2[\delta(x_i;\lambda)]^{2\alpha}}{[\delta(x_i;\lambda)]^2[1-[\delta(x_i;\lambda)]^{\alpha}]^2} .$$

Then the approximat variances of $\hat{R(t)}$ and $\hat{h(t)}$ can be obtained, respectively, as follow

$$\hat{V_R} = [D_R^T I^{-1}(\alpha,\lambda) D_R]_{(\alpha=\hat{\alpha},\lambda=\hat{\lambda})}$$

and

$$\hat{V_h} = [D_h^T I^{-1}(\alpha,\lambda) D_h]_{(\alpha=\hat{\alpha},\lambda=\hat{\lambda})} .$$

Then, the ACIs for R(t) and h(t) can be determined as follows:

$$\hat{R} \pm Z_{\tau/2}\sqrt{\hat{V_R}} \tag{2.7}$$

and

$$\hat{h} \pm Z_{\tau/2}\sqrt{\hat{V_h}} .$$

3 Product of spacing estimation

The maximum product of spacing (MPS) method was introduced independently by Cheng and Amin (1983) and Ranneby (1984) as an alternative to the method of maximum likelihood. By maximizing the product of spacing functions, the MPS estimators (MPSEs) can be obtained using the same explanation as the MLEs. Recently, Nassar and Elshahhat (2024) introduced the method of product spacing in improved adaptive type-II progressive censoring schemes for the Weibull distribution. Here, based on IAT-II PCS data, we estimate the parameters, reliability, and hazard rate functions of the GE distribution using the MPS estimation method. Additionally, the ACIs of the unknown quantities are derived using the asymptotic normality of the MPSEs.

Let $\underline{X} = (X_{1:m:n}, \cdots, X_{D_1:m:n}, ..., X_{D_2:m:n})$ be an IAT-II PC sample with a censoring scheme $R = (R_1, \cdots, R_{D_1}, 0, \cdots, 0, R^*)$, then the MPSEs are obtained by maximizing the geometric mean of the spacings, or equivalently, the product spacing (PS) function expressed as follows:

$$P(\alpha,\lambda) = C \prod_{i=1}^{D_2+1} [F(x_i) - F(x_{i-1})] \prod_{i=1}^{D_1} [1-F(x_i)]^{R_i} [1-F(T^*)]^{R^*} , \tag{3.1}$$

Using (1.4), (1.5) and (3.1), the PS function can be written as follows

$$P(\alpha,\lambda) = C\prod_{i=1}^{D_2+1}\left([\delta(x_i;\lambda)]^{\alpha}-[\delta(x_{i-1};\lambda)]^{\alpha}\right)\prod_{i=1}^{D_1}\left(1-[\delta(x_i;\lambda)]^{\alpha}]\right)^{R_i}\left(1-[\delta(T^*;\lambda)]^{\alpha}\right)^{R^*}. \tag{3.2}$$

The natural logarithm of (3.2) can be expressed as follows

$$\begin{aligned} lnP(\alpha,\lambda|\underline{x}) = &\sum_{i=1}^{D_2+1} ln\left([\delta(x_i;\lambda)]^{\alpha}-[\delta(x_{i-1};\lambda)]^{\alpha}\right)+\sum_{i=1}^{D_1} R_i ln\left(1-[\delta(x_i;\lambda)]^{\alpha}\right)+ \\ & R^* ln\left(1-[\delta(T^*;\lambda)]^{\alpha}\right). \end{aligned} \tag{3.3}$$

Differentiating (3.3) with respect to α and λ, we get the following

$$\begin{aligned} \frac{\partial lnP(\alpha,\lambda|\underline{x})}{\partial\alpha} = &\sum_{i=1}^{D_2+1}\frac{[\delta(x_i;\lambda)]^{\alpha}ln[\delta(x_i;\lambda)]-[\delta(x_{i-1};\lambda)]^{\alpha}ln[\delta(x_{i-1};\lambda)]}{[\delta(x_i;\lambda)]^{\alpha}-[\delta(x_{i-1};\lambda)]^{\alpha}} \\ &-\sum_{i=1}^{D_1} R_i\frac{[\delta(x_i;\lambda)]^{\alpha}ln[\delta(x_i;\lambda)]}{[1-\delta(x_i;\lambda)]^{\alpha}} - R^*\frac{[\delta(T^*;\lambda)]^{\alpha}ln[\delta(T^*;\lambda)]}{[1-[\delta(T^*;\lambda)]^{\alpha}]} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{\partial lnP(\alpha,\lambda|\underline{x})}{\partial\lambda} = &\sum_{i=1}^{D_2+1}\frac{\alpha x_i[\delta(x_i;\lambda)]^{\alpha-1}[1-\delta(x_i;\lambda)]-\alpha x_{i-1}[\delta(x_{i-1};\lambda)]^{\alpha-1}[1-\delta(x_{i-1};\lambda)]}{[\delta(x_i;\lambda)]^{\alpha}-[\delta(x_{i-1};\lambda)]^{\alpha}} \\ &-\sum_{i=1}^{D_1} R_i\frac{\alpha x_i[\delta(x_i;\lambda)]^{\alpha-1}[1-\delta(x_i;\lambda)]}{1-[\delta(x_i;\lambda)]^{\alpha}} - R^*\frac{\alpha T^*[\delta(T^*;\lambda)]^{\alpha-1}[1-\delta(T^*;\lambda)]}{1-[\delta(T^*;\lambda)]^{\alpha}}. \end{aligned} \tag{3.5}$$

The MPSEs of the parameters α and λ, represented by $\tilde{\alpha}$ and $\tilde{\lambda}$, can be computed by equating (3.4) and (3.5) to zero and then solving them simultaneously. Since these equations cannot be solved explicitly, we obtain $\tilde{\alpha}$ and $\tilde{\lambda}$ using any numerical method. Cheng and Traylor (1995) demonstrated that the MPSEs are consistent, similar asymptotic properties to those of MLEs, and invariance properties as well. Therefore, the MPSEs of $R(t)$ and $h(t)$ can be readily calculated using the invariance concept.

3.1 Approximate confidence intervals

Here, we determine the $100\,(1-\tau)\,\%$ ACIs for $\alpha, \lambda, R(t)$, and $h(t)$.

3.1.1 Confidence intervals of parameters

Similar to the MLEs, the ACIs are determined using the asymptotic normality of the MPSEs. To obtain the approximate asymptotic VC matrix for the MPSEs, we need the second order derivatives (3.3) with respect to α and λ as follows:

$$\frac{\partial^2 lnP(\alpha,\lambda|\underline{x})}{\partial\alpha^2} = \sum_{i=1}^{D_2}\left\{\frac{\left\{s_i^{\alpha}-[\delta(x_{i-1};\lambda]^{\alpha}\right\}\left\{s_i^{\alpha}(lns_i)^2-s_{i-1}^{\alpha}(ln\delta(x_{i-1};\lambda))^2\right\}}{\left\{s_i^{\alpha}-s_{i-1}^{\alpha}\right\}^2}\right\}$$
$$-\sum_{i=1}^{D_1}R_i\left\{\frac{s_i^{\alpha}[lns_i]^2}{[1-s_i^{\alpha}]}+\frac{s_i^{\alpha}[lns_i]^2}{[1-s_i^{\alpha}]^2}\right\}-\sum_{i=1}^{D_1}R^*\left\{\frac{s_T^{\alpha}[lns_T]^2}{[1-s_T^{\alpha}]}+\frac{s_T^{\alpha}[lns_T]^2}{[1-s_T^{\alpha}]^2}\right\},$$

$$\frac{\partial^2 lnP(\alpha,\lambda|\underline{x})}{\partial\lambda^2} = \sum_{i=1}^{D_2}\left\{\frac{x_i^2\alpha(\alpha-1)s_i^{(\alpha-2)}(1-s_i)^2}{(s_i^{\alpha}-s_{i-1}^{\alpha})}+\frac{x_is_i^{\alpha-1}(1-s_i)}{(s_i^{\alpha}-s_{i-1}^{\alpha})}-\frac{\alpha^2x_is_i^{(2\alpha-2)}(1-s_i)^2}{(s_i^{\alpha}-s_{i-1}^{\alpha})^2}\right.$$
$$\left.+\frac{\alpha^2x_is_i^{\alpha-1}s_{i-1}^{\alpha-1}x_{i-1}(1-s_i)(1-s_{i-1})}{(s_i^{\alpha}-s_{i-1}^{\alpha})^2}\right\}-\sum_{i=1}^{D_1}R_i\left\{\frac{x_i^2\alpha s_i^{\alpha-1}(1-s_i)}{1-s_i^{\alpha}}\right.$$
$$\left.+\frac{\alpha(\alpha-1)x_i^2s_i^{\alpha-2}(1-s_i)^2}{(1-s_i^{\alpha})^2}+\frac{x_is_i^{\alpha-1}(1-s_i)}{(1-s_i^{\alpha})^2}+\frac{\alpha^2x_i^2s_i^{2\alpha-2}(1-s_i)^2}{(1-s_i^{\alpha})^2}\right\}$$
$$-R^*\left\{\frac{T^{*^2}\alpha s_T^{\alpha-1}(1-s_T)}{1-s_T^{\alpha}}+\frac{\alpha(\alpha-1)T^{*^2}s_T^{\alpha-2}(1-s_T)^2}{(1-s_T^{\alpha})^2}+\frac{T^*s_T^{\alpha-1}(1-s_T)}{(1-s_T^{\alpha})^2}\right.$$
$$\left.+\frac{\alpha^2T^{*^2}s_T^{2\alpha-2}(1-s_T)^2}{(1-s_T^{\alpha})^2}\right\}$$

and

$$\frac{\partial^2 lnP(\alpha,\lambda|\underline{x})}{\partial\alpha\partial\lambda} = \sum_{i=1}^{D_2}\left\{\frac{s_i^{\alpha-1}x_i(1-s_i)}{s_i^{\alpha}-s_{i-1}^{\alpha}}+\frac{\alpha x_is_i^{\alpha-1}(1-s_i)lns_i}{s_i^{\alpha}-s_{i-1}^{\alpha}}-\frac{x_{i-1}s_{i-1}^{\alpha-1}(1-s_{i-1})}{s_i^{\alpha}-s_{i-1}^{\alpha}}\right.$$
$$-\frac{(s_i^{\alpha}lns_i-s_{i-1}^{\alpha-1}lns_{i-1})(\alpha s_i^{\alpha-1}x_i(1-s_i))-\alpha s_{i-1}^{\alpha-1}x_i(1-s_{i-1})}{(s_i^{\alpha}-s_{i-1}^{\alpha})^2}$$
$$\left.-\frac{\alpha x_{i-1}s_{i-1}^{\alpha-1}(1-s_{i-1})lns_{i-1}}{s_i^{\alpha}-s_{i-1}^{\alpha}}\right\}-\sum_{i=1}^{D_1}R_i\left\{\frac{s_i^{\alpha-1}x_i(1-s_i)}{1-s_i^{\alpha}}+\frac{\alpha x_is_i^{\alpha-1}(1-s_i)lns_i}{1-s_i^{\alpha}}\right.$$
$$\left.+\frac{\alpha x_is_i^{2\alpha-1}lns_i(1-s_i)}{(1-s_i^{\alpha})^2}\right\}-R^*\left\{\frac{s_T^{\alpha-1}T^*(1-s_T)}{1-s_T^{\alpha}}+\frac{\alpha T^*s_T^{\alpha-1}(1-s_T)lns_T}{1-s_T^{\alpha}}\right.$$
$$\left.+\frac{\alpha T^*s_T^{2\alpha-1}lns_T(1-s_T)}{(1-s_T^{\alpha})^2}\right\},$$

where, $s_i = \delta(x_i; \lambda)$ and $s_T = \delta(T^*; \lambda)$.
From the above expressions, the approximate assymptotic VC matrix can be obtained as follows

$$\begin{aligned} I^{-1}(\tilde{\alpha}, \tilde{\lambda}) &= \begin{bmatrix} -\dfrac{\partial^2 lnP(\alpha,\lambda)}{\partial\alpha^2} & -\dfrac{\partial^2 lnP(\alpha,\lambda)}{\partial\alpha\partial\alpha} \\ -\dfrac{\partial^2 lnP(\alpha,\lambda)}{\partial\alpha\partial\lambda} & -\dfrac{\partial^2 lnP(\alpha,\lambda)}{\partial\lambda^2} \end{bmatrix}^{-1}_{(\alpha,\lambda)=(\tilde{\alpha},\tilde{\lambda})} \\ &= \begin{bmatrix} var(\tilde{\alpha}) & cov(\tilde{\alpha},\tilde{\lambda}) \\ cov(\tilde{\lambda},\tilde{\alpha}) & var(\tilde{\lambda}) \end{bmatrix} \end{aligned} \tag{3.6}$$

Now the $100(1-\tau)$ ACIs for α and λ can be obtained, respectively, as follows

$$\tilde{\alpha} \pm z_{\tau/2}\sqrt{var(\tilde{\alpha})} \text{ and } \tilde{\lambda} \pm z_{\tau/2}\sqrt{var(\tilde{\lambda})} .$$

3.1.2 ACIs of S(t) and h(t)

Applying the delta method, we can obtain the approximated variances as follow

$$\tilde{V_R} = [D_R^T I^{-1}(\alpha,\lambda) D_R]_{(\alpha=\tilde{\alpha},\lambda=\tilde{\lambda})}$$

and

$$\tilde{V_h} = [D_h^T I^{-1}(\alpha,\lambda) D_h]_{(\alpha=\tilde{\alpha},\lambda=\tilde{\lambda})} .$$

Hence the ACIs of R(t) and h(t) can be obtained, respectively, as follow

$$\tilde{R} \pm Z_{\tau/2}\sqrt{\tilde{V_R}} \tag{3.7}$$

and

$$\tilde{h} \pm Z_{\tau/2}\sqrt{\tilde{V_h}} .$$

4 Bayesian Estimation

The main focus of this section is to estimate α and λ, as well as $S(t)$ and $h(t)$, of the GE distribution using Bayesian approaches. In this study, we obtain Bayesian estimates (BEs) using both the likelihood function (LF) and the product spacing (PS) function, thereby providing a comparison between the two approaches. The Bayesian estimation utilizes both symmetric and asymmetric loss functions, including the squared error loss (SEL) function and LINEX loss (LL) function.

The parameters α and λ are assumed to follow independent gamma prior distributions, $G(a, b)$ and $G(c, d)$, respectively, with PDFs given by

$$\pi_1(\alpha; a, b) = \frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha} \quad ; \alpha, a, b > 0 \tag{4.1}$$

and

$$\pi_2(\lambda; c, d) = \frac{d^c}{\Gamma c}\lambda^{c-1}e^{-d\lambda} \quad ; \lambda, c, d > 0\ . \tag{4.2}$$

The joint prior distribution of α and λ can be written in the following form.

$$\pi(\alpha, \lambda) \propto \alpha^{a-1}\lambda^{c-1}e^{-b\alpha+d\lambda} \quad ; a, b, c, d > 0\ . \tag{4.3}$$

4.1 Bayes estimators via LF-based

Based on the likelihood function in (1.1), the joint posterior distribution of α and λ is given by the following expression

$$\pi^*(\alpha, \lambda|\ \underline{x}) = \frac{L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)d\alpha d\lambda}\ .$$

Then the Bayes estimates of any function ψ of α and λ under SEL and LL functions can be respectively obtained as follows.

$$\hat{\psi}_{SE} = \frac{\int_0^\infty \int_0^\infty \psi(\alpha, \lambda)L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)d\alpha d\lambda}, \tag{4.4}$$

and

$$\hat{\psi}_{LL} = -\left(\frac{1}{p}\right)\frac{\int_0^\infty \int_0^\infty e^{-p\psi(\alpha,\lambda)}L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\underline{x})\pi(\alpha, \lambda)d\alpha d\lambda}, \quad p \neq 0\ . \tag{4.5}$$

Obviously, (4.4) and (4.5) cannot get an explicit solution. Hence, the Markov chain Monte Carlo (MCMC) method is employed to generate samples from the posterior density function, facilitating the computation of Bayesian estimation for the unknown parameters.

By using the likelihood function in (2.1) and joint prior distribution in (4.3), the joint posterior distribution of α and λ is given by

$$\begin{aligned}\pi^*(\alpha, \lambda|\ \underline{x}) \propto \alpha^{D_2+a-1} exp\left(-b\alpha - d\lambda\right) exp\left(-\lambda\sum_{i=1}^{D_2} x_i\right) exp\left((\alpha-1)\sum_{i=1}^{D_2} ln\delta(x_i; \lambda)\right)\\ exp\left(\sum_{i=1}^{D_1} R_i ln[1-[\delta(x_i;\lambda)]^\alpha] + R^* ln[1-[\delta(x_i;\lambda)]^\alpha]\right)\ .\end{aligned} \tag{4.6}$$

From equation (4.6), the conditional distributions for α and λ can be obtained as

$$\pi^*(\alpha|\lambda;\underline{x}) \propto \alpha^{D_2+a-1} exp\,(-b\alpha)\, exp\left\{(\alpha-1)\sum_{i=1}^{D_2} ln[\delta(x_i;\lambda)]\right\}$$
$$exp\left\{\sum_{i=1}^{D_1} R_i ln(1-[\delta(x_i;\lambda)]^{\alpha}) + R^* ln(1-[\delta(x_i;\lambda)]^{\alpha})\right\} \tag{4.7}$$

and

$$\pi^*(\lambda|\alpha,\underline{x}) \propto \lambda^{D_2+c-1} exp\left(-\lambda\sum_{i=1}^{D_2}\right) exp\left\{(\alpha-1)\sum_{i=1}^{D_2} ln[\delta(x_i;\lambda)]\right\}$$
$$exp\left\{\sum_{i=1}^{D_1} R_i ln(1-[\delta(x_i;\lambda)]^{\alpha})) + R^* ln(1-[\delta(x_i;\lambda)]^{\alpha})\right\}. \tag{4.8}$$

Moreover, the conditional posterior distributions of α and λ in (4.7) and (4.8) cannot be reduced analytically to well-known distributions and therefore it is not possible to generate sample directly by standard methods, but the plots of them show that they are similar to normal distribution. So, to generate sample from these distributions, we use the M–H algorithm within the Gibbs Sampling scheme with normal proposal distribution. For that we use the following algorithm:

1. Start with the initial values $(\alpha^{(0)}, \lambda^{(0)})$.
2. Set $i = 1$.
3. Generate $\alpha^{(i)}$ from $\pi^*(\alpha|\lambda^{(i-1)};\underline{x})$.
4. Generate $\lambda^{(i)}$ from $\pi^*(\lambda|\alpha^{(i)};\underline{x})$.
5. Set $i = i + 1$.
6. Repeat steps 3–6 N times.
7. The Bayes estimates $\hat{\psi}_{SB}$ and $\hat{\psi}_{LB}$ of ψ under SEL and LL functions, respectively, are obtained as

$$\hat{\psi}_{SB} = \frac{1}{N-M}\sum_{j=M+1}^{N} \hat{\psi}^{(j)}, \tag{4.9}$$

and

$$\hat{\psi}_{LB} = -\frac{1}{p}\left(\frac{1}{N-M}\sum_{j=M+1}^{N} e^{-p\hat{\psi}^{(j)}}\right), p \neq 0 \tag{4.10}$$

where ψ stands for α and λ.

8. Substituting $\alpha^{(i)}$ and $\lambda^{(i)}$ into (1.6) and (1.7) to compute $S^{(1)}(t), S^{(2)}(t), \cdots, S^{(N)}(t)$ and $h^{(1)}(t), h^{(2)}(t), \cdots, h^{(N)}(t)$

9. An approximate Bayesian estimate of a function $g(t)$ which could be $S(t)$ and $h(t)$ under both the SEL and LL in a unified form as follows:

$$\hat{g}(t) = \begin{cases} \dfrac{1}{N-M}\sum\limits_{i=M+1}^{N} g^{(i)}(t), & \text{under SEL,} \\ -\dfrac{1}{p}\log\left(\dfrac{1}{N-M}\sum\limits_{i=M+1}^{N}\exp\left(-pg^{(i)}(t)\right)\right), & \text{under LL.} \end{cases}$$

4.2 Bayesian Estimation via PS-based

In this section, we consider the Bayesian estimation procedure for the parameters α and λ, using PS. The concept was adapted from an idea by Coolen and Newby (1994). Recently, many authors investigated Bayesian estimations of some life time distributions utilizing the PS function; see, for example, (Nassar and Elshahhat, 2024; Basu et al., 2019, 2017). Here, we obtain the BEs of $\alpha, \lambda, S(t)$ and $h(t)$ by using the PS function as an alternative to the LF. Instead of deriving the joint posterior distribution based on the LF, the joint posterior distribution is derived using the PS function in (3.1). Then, the joint posterior distribution of α and λ is as follows,

$$\pi^{*}_{P}(\alpha, \lambda|\,\underline{x}) = \frac{P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)}{\int_0^\infty\int_0^\infty P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)d\alpha d\lambda}\,.$$

Combining (3.1) and (4.3), we get

$$\pi^{*}_{P}(\alpha,\lambda|\underline{x}) = \alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}exp\left[\sum_{i=1}^{D_2+1} ln[(\delta(x_i;\lambda))^{\alpha} - (\delta(x_{i-1};\lambda))^{\alpha}]\right] \tag{4.11}$$

$$exp\left[\sum_{i=1}^{D_1} R_i ln[1-(\delta(x_i;\lambda))^{\alpha}]\right] + R^{*}ln[1-(\delta(T^{*};\lambda))^{\alpha}]\,. \tag{4.12}$$

Similar to the usual Bayesian estimation method, the Bayes estimates under SELF and LLF can be obtained through the following formula.

$$\hat{\psi}_{SE_p}(\alpha,\lambda) = \frac{\int_0^\infty\int_0^\infty \psi(\alpha,\lambda)P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty\int_0^\infty P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)d\alpha d\lambda}, \tag{4.13}$$

and

$$\hat{\psi}_{LL_p}(\alpha,\lambda) = -\left(\frac{1}{p}\right)\frac{\int_0^\infty\int_0^\infty e^{-p\psi(\alpha,\lambda)}P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty\int_0^\infty P(\alpha,\lambda|\underline{x})\pi(\alpha,\lambda)d\alpha d\lambda}, \quad p\neq 0\,. \tag{4.14}$$

Again, (4.13) and (4.14) cannot be obtained analytically, therefore, the MCMC technique can be implemented to obtain the BEs in this case. The conditional posterior distributions of α and λ can be expressed, respectively, as

$$\pi_p^*(\alpha|\lambda,\underline{x}) \propto \alpha^{a-1}e^{-b\alpha}exp\left[\sum_{i=1}^{D_2+1} ln[(\delta(x_i;\lambda)^{\alpha}) - (\delta(x_{i-1};\lambda)^{\alpha})]\right]$$
$$exp\left[\sum_{i=1}^{D_1} R_i ln[1-(\delta(x_i;\lambda))^{\alpha}] + R^* ln[1-(\delta(T^*;\lambda))^{\alpha}]\right] \tag{4.15}$$

and

$$\pi_p^*(\lambda|\alpha,\underline{x}) \propto \lambda^{c-1}e^{-d\lambda}exp\left[\sum_{i=1}^{D_2+1} ln[(\delta(x_i;\lambda)^{\alpha}) - (\delta(x_{i-1};\lambda)^{\alpha})]\right]$$
$$exp\left[\sum_{i=1}^{D_1} R_i ln[1-(\delta(x_i;\lambda))^{\alpha}] + R^* ln[1-(\delta(T^*;\lambda))^{\alpha}]\right]. \tag{4.16}$$

It is observed that the conditional posterior distributions in (4.15) and (4.16) do not match any well-known distributions. Then, the M-H sampling technique is considered to generate samples for α and λ, then find the estimates similar to the method described in section 4.1.

4.3 HPD credible intervals

To construct HPD credible interval estimates, we propose to utilise the method offered by Chen and Shao (1999). First, arrange the MCMC samples of $\alpha, \lambda, R(t)$ and $h(t)$ after taking first M iterations as burn-in period $(\alpha^{(M+1)}, \cdots, \alpha^{(N)}), (\lambda^{(M+1)}, \cdots, \lambda^{(N)}), (R^{(M+1)}, \cdots, R^{(N)})$ and $(h^{(M+1)}, \cdots, h^{(N)})$. Applying the method proposed by Chen and Shao (1999), the $100(1-\tau)\%$ two-sided HPD credible intervals for α and λ, $R(t)$ or $h(t)$ (say ψ) are given by

$$\left[\psi^{(j^*)}, \psi^{(j^*+(1-\tau)(N-M))}\right], \tag{4.17}$$

where $j^* = M+1, M+2, \cdots, N$ is chosen such that

$$\psi^{(j^*+(1-\tau)(N-M))} - \psi^{(j^*)} = \min_{1\le j\le(N-M)}(\psi^{(j+(1-\tau)(N-M))} - \psi^{(j)}). \tag{4.18}$$

where $[x]$ denotes the largest integer less than or equal to x. Using generated MCMC samples based on both LF and PS functions, this method may be used to derive the HPD credible intervals.

5 Simulation Study

An extensive Monte Carlo simulation analysis is conducted to assess the behavior of the theoretical results produced in the preceding sections. The performance of the point estimates is compared based on the mean squared error (MSE) and interval estimates is compared based on the average length (AL) of the intervals and the coverage probability (CP). In the classical paradigm, we computed the ML estimates and the 95% approximate confidence intervals for $\alpha, \lambda, S(t = 0.4)$ and $H(t = 0.4)$. To obtain Bayes estimates, the M-H algorithm has been used to generate MCMC samples. We take $p = 1$ to evaluate Bayes estimators under the LINEX loss function L_2. Hyper-parameters are selected in a manner that the expectation of the prior distribution are equal to the true values of the parameters. Thus we take the values for the hyper parameters as $(a, b, c, d) = (3.5, 2, 1.5, 2)$. For generating IAPC-TII samples, we use the following algorithm developed by Yan et al. (2021).
Step 1. Set the parameter values of α and λ
Step 2. Carried out an ordinary progressive Type-II censored sample using the algorithm described by Balakrishnan and Sandhu (1995) as:

1. Generate $\omega_1, \omega_2, ..., \omega_m$ from uniform distribution.
2. For a specific values of n, m and $R_i, i = 1, 2, ..., m$, set $v_i = w_i^{(i+\sum_{j=m-i+1}^{m} R_j)^{-1}}, i = 1, 2, \cdots, m$.
3. Set $U_i = 1 - v_m v_{m-1} \cdots v_{m-i+1}$ for $i = 1, 2, ..., m$. Then, $U_i, i = 1, 2, \cdots, m$, is a progressive Type-II censored sample of size m from $U(0, 1)$ distribution.
4. Generate progressive Type-II censored sample from GE(α, λ) by using $X_i = F^{-1}(U_i; \alpha, \lambda), i = 1, 2, \cdots, m$, where F is the CDF given in (1.3).

Step 3. Determine d_1 at given time T_1, and discard the remaining sample $X_i, i = d_1 + 2, ..., m$.
Step 4. Generate the first $m - d_1 - 1$ order statistics from a truncated distribution $f(x)/[1 - F(x_{d_1+1})]$ with sample size $n - d_1 - 1 - \sum_{i=1}^{d_1} R_i$ as $X_{d_1+2}, \cdots, X_m$.
Step 5. The sample data under an IAPCS-TII will include one of the following scenarios, such as:

1. If $X_m < T_1$, the experiment stops at X_m with failure times $X_i, i = 1, 2, \cdots, m$, and the remaining units $n - m - \sum_{i=1}^{m-1} R_i$ are removed from the test, that is Case-I.
2. If $T_1 < X_m < T_2$, the experiment stops at X_m with failure times $X_i, i = 1, 2, ..., m$, and the remaining units $n - m - \sum_{i=1}^{d_1} R_i$ are removed from the test, that is Case-II.
3. If $T_1 < T_2 < X_m$, the experiment stops at T_2 with failure times $X_i, i = 1, 2, \cdots, d_2$, and the remaining units $n - d_2 - \sum_{i=1}^{d_1} R_i$ are removed from the test, that is Case-III.

To conduct the simulation, several combinations of n, m, T_1, and T_2 have been taken into account with various censoring schemes. Two distinct possibilities of n and m are utilized for two time thresholds $(T_1, T_2) = (0.8, 2)$ and $(1.5, 3.0)$, where $n = (40, 50)$ and $m = (20, 30)$. Three different censoring schemes were employed in our study, specifically:

Scheme 1 : $R_m = n - m, \quad R_i = 0$, for $i \neq m$.
Scheme 2 : $R_1 = n - m, \quad R_i = 0$, for $i \neq 1$.
Scheme 3 : $R_{(m+1)/2} = n - m, \quad R_i = 0$, for $i \neq (m+1)/2$; if m is odd $\quad R_{(m)/2} = n - m, \quad R_i = 0$, for $i \neq (m)/2$; if m is even.

The MLE and MPSE results of $\alpha, \lambda, S(t)$ and $H(t)$ are given in Table 2, the two-side 95% ACI and HPD credible intervals of α and λ are constructed in Table 5. Table 3 presents the MSE of Bayesian estimation by MCMC method under SEL function and LL function for α and λ. Similarly, the MSE of Bayesian estimation by MCMC method under squared error loss function and LINEX loss function for S(t) and h(t) are presented in Table 4. Based on the above tables, we can infer the following conclusions.

1. Based on the results of point estimation for $\alpha, \lambda, S(t)$ and $h(t)$, it can be observed that when the sample size n is held constant, the MSE decreases with an increase in m. Besides, for a fixed value of m, the MSE decreases when n increases from 40 to 50.

2. The 95% asymptotic intervals and HPD credible intervals are presented in Table 5.5. The ACI method demonstrates competitive performance with the HPD method in terms of interval length. Based on the presented values, it can be concluded that ACI and HPD credible intervals using the PS function outperform the LF function for both time points, (0.8, 2) and (1.5, 3). For the parameter α, HPD credible intervals outperform ACI, while for λ, ACI outperforms HPD for both time points. Additionally, all three intervals exhibit satisfactory coverage probabilities. Furthermore, as n increases, the average interval lengths decrease across all cases.

3. As evident from Table 2, the MPSE exhibits superior performance compared to MLE for estimating parameters $\alpha, \lambda, S(t)$ and $h(t)$ in terms of smaller MSE.

4. Furthermore, analysis of the data presented in Tables 3 and 4 reveal that the MCMC method employing PS exhibits superior performance in comparison to MCMC using LF across both time points.

5. Additionally, in the context of Bayesian estimation for α and λ, LF exhibits superior performance at the time point $(T_1, T_2) = (0.8, 2)$, while PS exhibits better performance at $(T_1, T_2) = (1.5, 3)$. Similarly, for S(t) and h(t), LF shows superior performance at $(T_1, T_2) = (0.8, 2)$, whereas PS outperforms it at $(T_1, T_2) = (1.5, 3)$. Thus, LF is favored with early stopping, while PS performs better with longer observation.

6. There is no significant difference in the performance of MSE based on the SEL and LL functions, for all parameters $\alpha, \lambda, S(t)$ and $h(t)$, for both time points.

7. We employed heatmaps (see Figures 2 to 7) to facilitate an intuitive comparison of the MSE performance of MLE and Bayesian methods for parameters α, $\lambda, S(t)$, and $h(t)$ across various censoring schemes. Color assignments were based on the magnitude of the MSE values, with smaller values represented in white and larger values in red.

8. From the heatmaps of MSE for α and λ based on SEL and LL functions using LF and PS functions, we can infer that there is no significant colour variation observed between SEL and LL functions of both α and λ. Also, for S(t) and h(t), the SEL and LL functions are nearly identical.

9. Across all heatmaps, it is evident that estimates derived from the PS function outperform those from the LF method, in terms of MSE.

Table 2: The MSEs of the MLE and MPSE estimates of $\alpha, \lambda, S(0.4)$, and $H(0.4)$ are presented for various values of n and m, given $\alpha = 1.75$, $\lambda = 0.75$, and $(T_1, T_2) = (0.8, 2)$ and $(1.5, 3)$.

				α		λ		$S(t)$		$h(t)$	
(T_1, T_2)	n	m	CS	MLE	MPSE	MLE	MPSE	MLE	MPSE	MLE	MPSE
(0.8, 2)	40	20	I	0.2600	0.1993	0.0450	0.0278	0.0052	0.0030	0.0358	0.0104
	40	20	II	0.3495	0.2258	0.0560	0.0299	0.0049	0.0019	0.0395	0.0062
	40	20	III	0.3551	0.2399	0.1096	0.0313	0.0051	0.0017	0.0419	0.0095
	40	30	I	0.2269	0.1807	0.0314	0.0179	0.0041	0.0023	0.0321	0.0074
	40	30	II	0.3171	0.1909	0.0384	0.0216	0.0042	0.0017	0.0335	0.0056
	40	30	III	0.3413	0.1952	0.0870	0.0209	0.0044	0.0018	0.0352	0.0076
	50	30	I	0.2697	0.1517	0.0301	0.0158	0.0033	0.0025	0.0266	0.0067
	50	30	II	0.2154	0.1755	0.0325	0.0252	0.0019	0.0016	0.0383	0.0056
	50	30	III	0.3253	0.1668	0.1026	0.0186	0.0038	0.0013	0.0303	0.0058
(1.5, 3)	40	20	I	0.2525	0.1960	0.0410	0.0213	0.0043	0.0027	0.0307	0.0087
	40	20	II	0.2751	0.1903	0.0641	0.0218	0.0046	0.0009	0.0315	0.0081
	40	20	III	0.2932	0.1928	0.0639	0.0209	0.0049	0.0019	0.0306	0.0078
	40	30	I	0.2074	0.1497	0.0211	0.0124	0.0035	0.0025	0.0230	0.0064
	40	30	II	0.2198	0.1688	0.0342	0.0132	0.0039	0.0150	0.0262	0.0063
	40	30	III	0.2222	0.1723	0.0368	0.0133	0.0037	0.0018	0.0259	0.0055
	50	30	I	0.1827	0.1347	0.0275	0.0119	0.0031	0.0021	0.0219	0.0069
	50	30	II	0.1921	0.1736	0.0274	0.0249	0.0019	0.0011	0.0380	0.0032
	50	30	III	0.2308	0.1430	0.0271	0.0136	0.0034	0.0012	0.0231	0.0034

Table 3: The MSEs of the Bayesian estimates of α and λ under SE and LINEX loss functions with different values of n and m, when $\alpha = 1.75$, $\lambda = 0.75$, $(T_1, T_2) = (0.8, 2)$ and $(1.5, 3)$ based on LF and PS.

			LF				***PS***			
n	m	CS	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LL}$	$\hat{\lambda}_{SEL}$	$\hat{\lambda}_{LL}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LL}$	$\hat{\lambda}_{SEL}$	$\hat{\lambda}_{LL}$
					$(T_1 = 0.8, T_2 = 2)$					
40	20	I	0.00663	0.00676	0.00680	0.00678	0.00425	0.00426	0.00427	0.00430
40	20	II	0.00741	0.00754	0.00677	0.00678	0.00438	0.00439	0.00418	0.00420
40	20	III	0.00665	0.00653	0.00807	0.00798	0.00361	0.00362	0.00429	0.00433
40	30	I	0.00518	0.00527	0.00651	0.00660	0.00345	0.00347	0.00316	0.00320
40	30	II	0.00718	0.00720	0.00628	0.00637	0.00362	0.00377	0.00392	0.00398
40	30	III	0.00639	0.00649	0.00681	0.00667	0.00345	0.00343	0.00314	0.00337
50	30	I	0.00562	0.00555	0.00614	0.00615	0.00319	0.00348	0.00299	0.00302
50	30	II	0.00554	0.00540	0.00633	0.00623	0.00343	0.00315	0.00282	0.00282
50	30	III	0.00568	0.00571	0.00618	0.00620	0.00307	0.00341	0.00310	0.00321
					$(T_1 = 1.5, T_2 = 3)$					
40	20	I	0.01018	0.01021	0.00831	0.00835	0.00357	0.00358	0.00350	0.00350
40	20	II	0.01106	0.01133	0.01110	0.01131	0.00382	0.00382	0.00355	0.00314
40	20	III	0.01055	0.01078	0.00971	0.00985	0.00329	0.00329	0.00339	0.00352
40	30	I	0.00840	0.00846	0.00761	0.00763	0.00338	0.00366	0.00299	0.00300
40	30	II	0.01015	0.01047	0.00978	0.00985	0.00356	0.00355	0.00333	0.00271
40	30	III	0.01032	0.01022	0.00872	0.00889	0.00309	0.00319	0.00316	0.00322
50	30	I	0.00791	0.00791	0.00677	0.00681	0.00285	0.00283	0.00233	0.00233
50	30	II	0.00996	0.00988	0.00971	0.00980	0.00317	0.00318	0.00267	0.00236
50	30	III	0.00790	0.00793	0.00827	0.00832	0.00311	0.00313	0.00237	0.00336

Table 4: The MSEs of the Bayesian estimates of $S(0.4)$ and $h(0.4)$ under SE and LINEX loss functions with different values of n and m, when $\alpha = 1.75$, $\lambda = 0.75$, $(T_1, T_2) = (0.8, 2)$ and $(1.5, 3)$ based on LF and PS.

			LF				***PS***			
n	m	CS	$S(t)_{SEL}$	$S(t)_{LL}$	$h(t)_{SEL}$	$h(t)_{LL}$	$S(t)_{SEL}$	$S(t)_{LL}$	$h(t)_{SEL}$	$h(t)_{LL}$
					$(T_1 = 0.8, T_2 = 2)$					
40	20	I	0.00138	0.00127	0.00346	0.00349	0.00070	0.00072	0.00185	0.00187
40	20	II	0.00143	0.00139	0.00346	0.00344	0.00074	0.00074	0.00169	0.00171
40	20	III	0.00150	0.00150	0.00380	0.00379	0.00068	0.00067	0.00169	0.00162
40	30	I	0.00124	0.00125	0.00309	0.00312	0.00062	0.00063	0.00162	0.00164
40	30	II	0.00110	0.00138	0.00330	0.00333	0.00071	0.00072	0.00153	0.00152
40	30	III	0.00140	0.00143	0.00302	0.00321	0.00065	0.00065	0.00162	0.00169
50	30	I	0.00124	0.00118	0.00254	0.00254	0.00060	0.00061	0.00155	0.00155
50	30	II	0.00137	0.00138	0.00251	0.00253	0.00059	0.00060	0.00133	0.00134
50	30	III	0.00110	0.00112	0.00301	0.00292	0.00058	0.00058	0.00146	0.00145
					$(T_1 = 1.5, T_2 = 3)$					
40	20	I	0.00216	0.00216	0.00364	0.00364	0.00066	0.00066	0.00178	0.00178
40	20	II	0.00235	0.00242	0.00497	0.00502	0.00067	0.00068	0.00141	0.00141
40	20	III	0.00214	0.00215	0.00464	0.00473	0.00066	0.00066	0.00165	0.00164
40	30	I	0.00195	0.00197	0.00336	0.00336	0.00062	0.00062	0.00157	0.00158
40	30	II	0.00230	0.00235	0.00422	0.00427	0.00064	0.00064	0.00125	0.00125
40	30	III	0.00197	0.00197	0.00429	0.00302	0.00062	0.00063	0.00147	0.00148
50	30	I	0.00183	0.00182	0.00298	0.00301	0.00055	0.00055	0.00129	0.00128
50	30	II	0.00176	0.00180	0.00391	0.00396	0.00052	0.00053	0.00129	0.00129
50	30	III	0.00187	0.00190	0.00300	0.00434	0.00056	0.00056	0.00109	0.00109

Table 5: Average length (AL) and coverage probability (CP) of 95% asymptotic confidence interval and HPD credible interval for parameters, when $\alpha = 1.75, \lambda = 0.75$ at two time points $(T_1, T_2) = (0.8, 2)$ and $(T_1, T_2) = (0.8, 2)$.

$(T_1 = 0.8, T_2 = 2)$																
	ACI_LF				ACI_Ps				HPD_LF				HPD_PS			
	$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$	
CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
I	2.5355	0.94	0.9610	0.98	2.0232	0.86	0.9615	0.92	1.2973	0.80	1.1089	0.79	1.1286	0.91	0.9537	0.93
II	2.5168	0.98	0.9616	0.95	2.1584	0.91	0.9218	0.91	1.3874	0.73	1.0897	0.82	1.0848	0.90	0.9493	0.98
III	1.8223	0.77	0.8162	0.57	1.6169	0.57	0.7950	0.52	1.2823	0.76	1.1301	0.82	1.5143	0.76	1.1142	0.92
I	2.2082	0.95	0.8181	0.96	1.8619	0.85	0.8064	0.93	1.2922	0.62	1.0317	0.75	1.1487	0.85	0.8661	0.92
II	2.1823	0.99	0.7401	0.99	1.7316	0.87	0.7096	0.94	1.3124	0.78	1.0794	0.80	1.0826	0.90	0.7553	0.96
III	1.7352	0.88	0.6863	0.89	1.4847	0.64	0.6534	0.63	1.2649	0.79	1.0713	0.79	1.2589	0.59	1.1092	0.89
I	2.1170	0.98	0.8159	0.98	1.8070	0.88	0.7970	0.93	1.2889	0.76	0.9530	0.76	1.0390	0.83	0.8103	0.92
II	1.9700	0.96	0.7328	0.92	1.6394	0.88	0.6828	0.90	1.2239	0.75	1.0394	0.76	1.0652	0.80	0.7269	0.91
III	1.4508	0.62	0.6158	0.42	1.2487	0.43	0.5736	0.28	1.2647	0.70	1.0537	0.76	1.0994	0.73	0.9557	0.83

$(T_1 = 1.5, T_2 = 3)$																
	ACI_LF				ACI_Ps				HPD_LF				HPD_PS			
	$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$		$(\hat{\alpha})$		$(\hat{\lambda})$	
CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
I	2.1678	0.94	0.8407	0.98	1.9787	0.86	0.8401	0.90	1.5414	0.63	1.0969	0.82	1.3832	0.74	0.9971	0.85
II	2.1576	0.95	0.7908	0.90	1.9096	0.87	0.7407	0.73	1.4486	0.66	1.3116	0.73	1.3186	0.72	1.0477	0.95
III	2.3262	0.99	0.9542	0.95	2.0420	0.90	0.9263	0.95	1.5513	0.68	1.1968	0.10	1.3996	0.09	0.9936	0.88
I	2.0842	0.95	0.7000	0.99	1.7331	0.88	0.6905	0.96	1.2254	0.78	1.0259	0.70	1.3109	0.78	0.9285	0.89
II	1.9961	0.99	0.7031	0.98	1.7718	0.89	0.6687	0.93	1.1358	0.87	1.0725	0.80	1.2797	0.75	1.0369	0.88
III	1.8897	0.95	0.7325	0.92	1.8019	0.93	0.7265	0.92	1.4340	0.51	1.1350	0.64	1.2290	0.77	0.9955	0.84
I	1.9908	0.93	0.6964	0.98	1.7094	0.87	0.6885	0.93	1.1743	0.83	1.0109	0.79	1.2641	0.65	0.9391	0.80
II	1.9993	0.97	0.6757	0.97	1.6672	0.91	0.6747	0.93	1.0531	0.87	1.0160	0.80	1.2535	0.75	1.0025	0.89
III	1.8401	0.97	0.7141	0.98	1.6964	0.91	0.7205	0.94	1.3551	0.70	0.9521	0.83	1.2596	0.78	0.9424	0.82

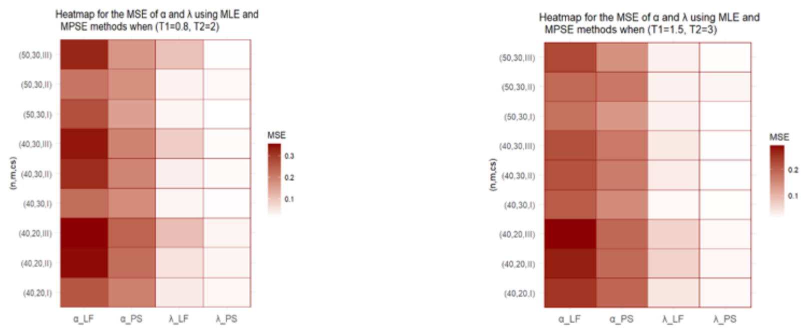


Figure 2: **The heatmaps of MSE for α and λ, based on MLE and MPSE methods, are presented for (T1, T2) values of (0.8, 2) and (1.3, 5).**

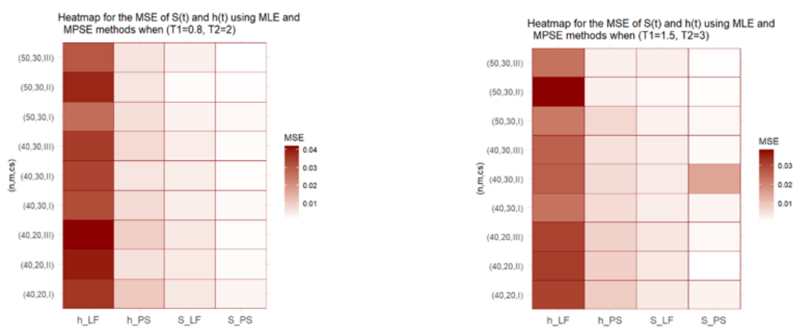


Figure 3: **The heatmaps of MSE for S(t) and h(t), based on MLE and MPSE methods, are presented for (T1, T2) values of (0.8, 2) and (1.3, 5).**

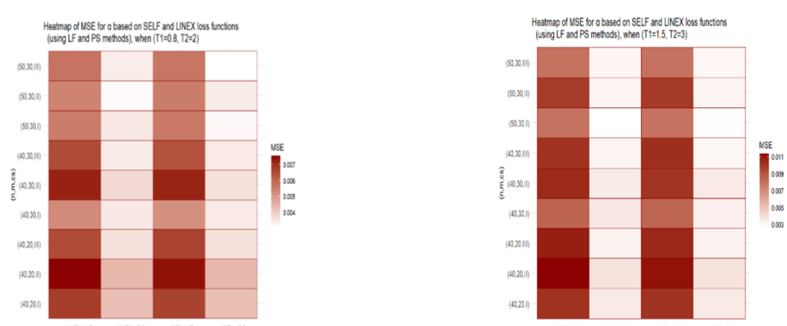


Figure 4: **The heatmaps of MSE for α, based on SELF and LINEX loss functions, using LF and PS functions, are presented for (T1, T2) values of (0.8, 2) and (1.5, 3).**

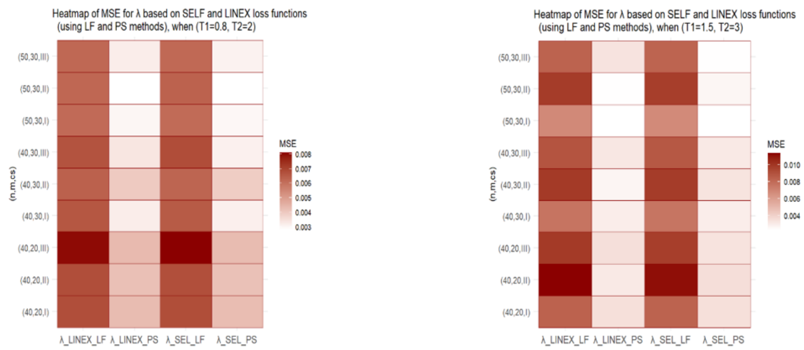


Figure 5: **The heatmaps of MSE for λ, based on SELF and LINEX loss functions, using LF and PS functions, are presented for (T1, T2) values of (0.8, 2) and (1.5, 3).**

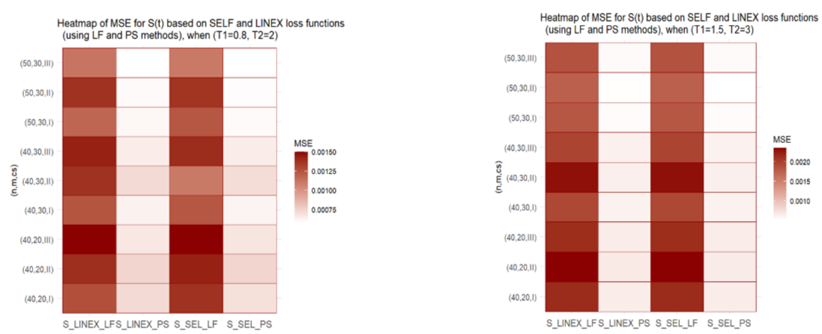


Figure 6: **The heatmaps of MSE for S(t), based on SELF and LINEX loss functions, using LF and PS functions, are presented for (T1, T2) values of (0.8, 2) and (1.5, 3).**

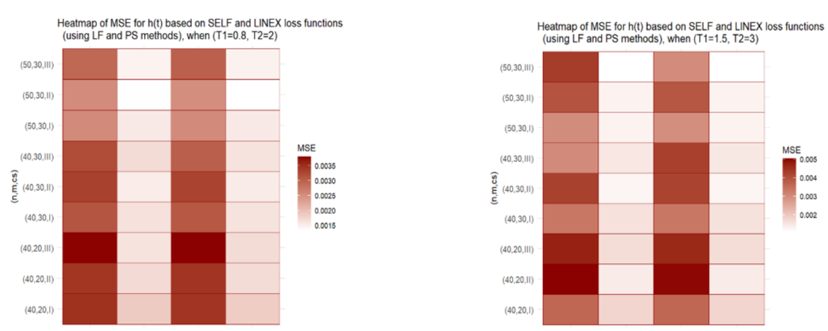


Figure 7: **The heatmaps of MSE for h(t), based on SELF and LINEX loss functions, using LF and PS functions, are presented for (T1, T2) values of (0.8, 2) and (1.5, 3).**

6 Illustrations using real data

To show the adaptability of methodologies proposed to a real phenomenon, an engineering application using a real-life data set is analyzed. Here, we shall use a real-life data set originally reported by Murthy et al. (2004). This data set shows the time between failures for 30 items of repairable mechanical equipment (RME) and is given below.

0.11, 0.30, 0.40, 0.45, 0.59, 0.63, 0.70, 0.71, 0.74, 0.77, 0.94, 1.06, 1.17, 1.23, 1.23, 1.24, 1.43, 1.46, 1.49, 1.74, 1.82, 1.86, 1.97, 2.23, 2.37, 2.46, 2.63, 3.46, 4.36, 4.73.
Recently, Elshahhat et al. (2023) reanalyzed this RME data set for AT-II PCS following weighted-exponential distribution. Before proceeding with the analysis, it is important to determine whether the proposed model adequately fits the data. We use Kolmogorov Smirnov test for checking the goodness of fit. The value for the Kolmogorov Smirnov test statistic is 0.06505 and the corresponding P-value is 0.9996. In light of the high P-value, we fail to reject the null hypothesis indicating that the data is derived from the GE distribution. Then we consider the estimation of two parameters under IAT-II PC data. By using the original dataset we have generated random sets of IAT-II PC samples for different censoring schemes and are given in Table 7. The different combinations for m and CS are given in Table 6. We have obtained the MLEs and Bayes estimates of $\alpha, \lambda, S(0.4)$ and $H(0.4)$, which are given in Table 8. For the Bayesian estimation we used the same values for the hyper parameters as those used in the simulation study.

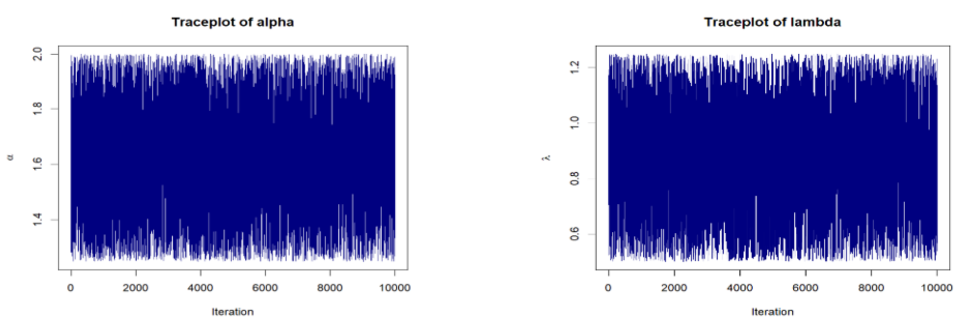
Table 6: IAT-II PC with n = 30

Sl No.	m	CS	Censoring number
1	5	I	R=(0,0,0,0,0,25)
2	5	II	R=(4,5,9,5,2)
3	5	III	R=(5,5,5,5,5)
4	10	IV	R=(0,0,0,0,0,0,0,0,0,20)
5	10	V	R=(2,4,1,2,3,1,1,1,2,3)
6	10	VI	R=(2,2,2,2,2,2,2,2,2,2)

Based on the RME data and censoring scheme VI from Table 7, traceplots based on 10,000 chain values of α and λ under LF and PS functions are plotted in Figures 8 to 11 to evaluate the convergence of the MCMC procedure. All plots for α and λ under both LF and PS functions exhibit consistent fluctuations around a stable mean, indicating that the chains have achieved convergence.

Table 7: Improved adaptive Type-II progressive censored samples generated from the data under different censoring schemes.

(T_1, T_2)	m	CS	Data	d_1	d_2	R^*	T^*
(0.8,2.0)	5	I	0.11,0.30,0.40,0.45,0.59	5	5	25	0.59
		II	0.11,0.59,1.17,1.23,1.86	2	5	16	1.86
		III	0.11,0.63,1.74,1.97	2	4	16	2.0
	10	IV	0.11,0.30,0.40,0.45,0.59,0.63,0.70,0.71,0.74,0.77	10	10	20	0.77
		V	0.11,0.63,0.74,0.94,1.23,1.24,1.46,1.74,1.82,1.97	3	10	13	1.97
		VI	0.11,0.63,0.71,0.77,1.06,1.24,1.43,1.97	4	7	12	2.0
(1.5,3.0)	5	I	0.11,0.30,0.40,0.45,0.59	5	5	25	0.59
		II	0.11,0.71,1.23,1.74,1.86	3	5	7	1.86
		III	0.11,0.70,1.06,1.82,1.97	3	5	10	1.97
	10	IV	0.11,0.30,0.40,0.45,0.59,0.63,0.70,0.71,0.74,0.77	10	10	20	0.77
		V	0.11,0.45,0.63,0.77,1.23,1.46,1.74,1.86,1.97,2.63	6	10	7	2.63
		VI	0.11,0.45,0.70,0.71,0.94,1.06,1.49,2.23,2.63	7	9	7	3.0



(a) (b)

Figure 8: The trace plots of $(a): \alpha$, and $(b): \lambda$ using the LF function from RME data when $(T_1, T_2) = (0.8, 2)$

Table 8: Estimates derived using various estimation procedures for the dataset using censoring scheme VI, when $(T_1, T_2) = (1.5, 3.0)$

Methods	α	λ	S(0.4)	h(0.4)
MLE	1.8519	0.8196	0.5449	0.8133
MPSE	1.7599	0.7898	0.9058	0.3532
SEL (LF-Based)	1.7388	0.7593	0.5902	0.3623
LINEX (LF-Based)	1.738	0.7584	0.5908	0.3622
SEL (PS-Based)	1.7428	0.7736	0.5835	0.3703
LINEX (PS-Based)	1.7421	0.7729	0.584	0.3701

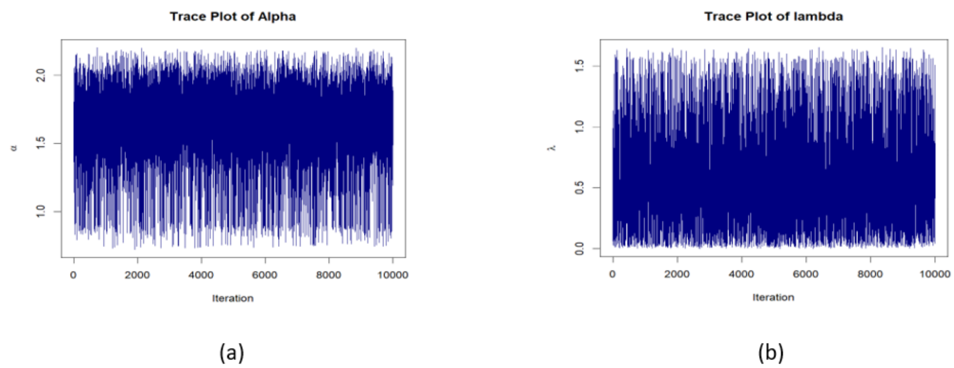


Figure 9: The trace plots of (a) : α, and (b) : λ using the PS function from RME data when $(T_1, T_2) = (0.8, 2)$

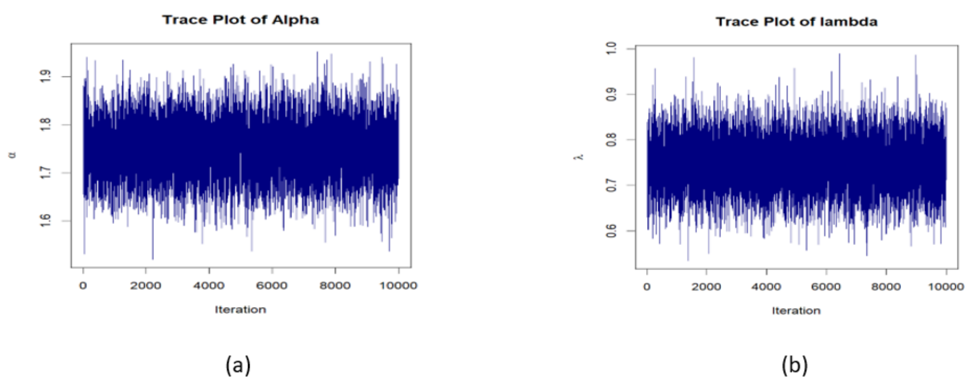


Figure 10: The trace plots of (a) : α, and (b) : λ using the LF function from RME data when $(T_1, T_2) = (1.5, 3)$

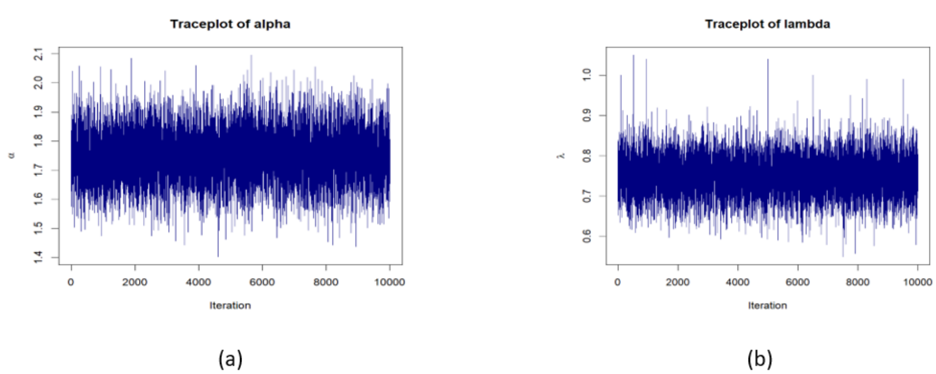


Figure 11: The trace plots of (a) : α, and (b) : λ using the PS function from RME data when $(T_1, T_2) = (1.5, 3)$

7 Conclusions

In this paper, we focused on the estimation problem of the unknown parameters, reliability, and hazard rate functions based on IAT-II PCS samples from the Generalized exponential distribution. The maximum likelihood and maximum product of spacing as classical estimation methods were employed for this purpose. To obtain the Bayes estimates, the MCMC technique was employed. Under the assumption of a gamma prior distribution, Bayesian estimates based on both LF and PS function were obtained under the squared error loss function and LINEX loss function. From the simulation study, we noticed that the mean-squared error decreases when n and m increase. Point estimations based on MLE and MPSE, demonstrates that MPSE outperforms MLE in terms of MSE. Also, the MCMC method using PS demonstrates superior performance compared to MCMC using LF, in terms of least bias. Average interval length using the PS function outperforms the LF function for both time points (0.8, 2) and (1.5, 3). For the parameter α, HPD credible intervals outperform ACI, while for λ, ACI outperforms HPD for both time points. A real dataset was also employed to demonstrate the estimation procedures developed in this paper. Although the proposed MLE and MPS methods demonstrated satisfactory performance, several limitations should be acknowledged. The inferential procedures depend on asymptotic normality, which may be unreliable for small or heavily censored samples. These methods further assume that lifetime data follow the generalized exponential distribution; thus, model misspecification may reduce estimation accuracy. Additionally, the absence of closed-form solutions for MLE, MPS, and Bayesian approaches necessitates numerical optimization, increasing computational complexity and the risk of convergence issues under the improved adaptive Type-II progressive censoring scheme. Since this study is limited to this specific censoring scheme, future research should extend the framework to alternative censoring mechanisms and compare them with other estimation methods, including least squares, weighted least squares, percentile estimation, method of moments, probability-weighted moments, and shrinkage estimation.

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