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Evaluation of the System Lifetime Using a New Mixed δ-Shock Model

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Abstract. Reliability assessment of complex systems requires accurate modeling of failure mechanisms induced by random external shocks. Classical shock models often focus either on shock magnitudes or on the inter-arrival times between successive shocks, whereas many real-world systems are affected by the joint impact of these two factors. This paper introduces a mixed shock model for systems whose failure is governed by both shock magnitudes and inter-arrival times. In the proposed model, system failure occurs either when the inter-arrival time falls within the interval $[\alpha, \delta]$, where $0 \leq \alpha < \delta$, or when the magnitude of a single shock exceeds a critical threshold γ. The probability distribution of the system's stopping time is derived, and the reliability properties of the system's lifetime are investigated. Numerical examples are provided to illustrate the theoretical findings.

Keywords. Mixed δ-shock model, Inter-arrival time, Magnitude of shock, Reliability.

MSC: 62N05, 90B25, 62E15.

1 Introduction

In the context of shock modeling, two important factors that underlie failure scenarios for shock-exposed systems are the shock magnitude and the inter-arrival time between successive shocks. When the magnitude of a shock exceeds a critical threshold, it is

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called an *extreme shock*, and the shock model used to study the reliability of a system under the influence of such shocks is called the *extreme shock model*. When system failure occurs due to the inter-arrival time between two successive shocks falling below a critical threshold $\delta > 0$, the model used to study the reliability of the system is called the *δ-shock model*. In more advanced approaches, complex models may be obtained from the combination of basic models. In general, shock models that are produced by combining several shock models are called *mixed shock models*.

Many studies have been done on mixed delta shock models in the relevant literature. In the following, we only address some of the recent developments related to our works. Parvardeh and Balakrishnan (2015) introduced a mixed δ-shock model, under which the system fails when the inter-arrival time between two successive shocks is smaller than a threshold δ or the magnitude of the shock is larger than another threshold γ. Clearly, their model is a mix of the δ-shock model and the extreme shock model. Lorvand et al. (2020) discussed a mixed δ-shock model for multi-state systems by assuming a renewal process of shocks where the system fails in three specific states. Bohlooli-Zefreh et al. (2021) studied the reliability characteristics of a system under a generalized mixed δ-shock model where the system is subjected to shocks according to a stochastic process. In their model, the system fails either if the inter-arrival time between two successive shocks is less than a threshold δ and simultaneously the size of the damage is larger than another critical threshold γ_1, or if the magnitude of the damage caused by a shock exceeds a threshold γ_2 (with $\gamma_2 > \gamma_1$). Goyal et al. (2022) introduced a general δ-shock model in which the recovery time depends on both the inter-arrival time and the magnitude of the shocks. Lorvand and Nematollahi (2022) studied a mixed δ-shock model that is based on both inter-arrival times between two successive shocks and magnitudes of shocks. In their model, the system fails with a probability θ_1 when the inter-arrival time between two successive shocks falls in the interval $[\delta_1, \delta_2]$, and the system fails with a probability θ_2 when the magnitude of a shock falls in the interval $[\eta_1, \eta_2]$; also, the system fails with probability 1 as soon as the inter-arrival time between two successive shocks is less than δ_1 or a shock with magnitude greater than η_2 occurs. Doostmoradi et al. (2023) studied the reliability of a system whose efficiency is k. Under their model, the performance of the system is such that if the magnitude of the shock is greater than d or the inter-arrival time between two successive shocks is less than δ, the efficiency of the system reduces by one unit, and if k_1 is the number of shocks that have a magnitude greater than d and k_2 is the number of the inter-arrival times less than δ while $k_1 + k_2 = k$, the system loses its efficiency and fails. Roozegar et al. (2023) introduced a mixed δ-shock model, which is defined by combining δ-shock and extreme shock models for a multi-state system in which its failure occurs in two ways: first, when k inter-arrival times between two successive shocks with a magnitude larger than the critical threshold γ are in $[\delta_1, \delta_2]$ (with $\delta_1 < \delta_2$), and second, when the inter-arrival time between two successive shocks is less than threshold δ_1. Chadjiconstantinidis et al. (2023) introduced a mixed δ-shock model and investigated the reliability properties of the system under a particular change-point model when a change in the distribution of the shock magnitudes occurs upon the occurrence of a shock that is above a certain critical threshold. Chadjiconstantinidis (2024) introduced a mixed censored δ-shock model, which combines the censored δ-shock model and the classical extreme shock model. Under this model, the system fails whenever no shock occurs within a δ-length time period from the last shock, or the magnitude of a shock is greater than a critical threshold γ.

In this paper, we introduce a new mixed δ-shock model. According to the new model, the system fails if either the inter-arrival time between two successive shocks falls within the interval $[\alpha, \delta]$, where $0 \leq \alpha < \delta$, or if the magnitude of a single shock exceeds a critical threshold γ. This model is in fact a mix of the generalized δ-shock model introduced by Farhadian and Jafari (2025) and the extreme shock model. It is clear that when $\alpha = 0$, the new model reduces to the mixed δ-shock model proposed by Parvardeh and Balakrishnan (2015). The proposed model offers practical utility in real-world reliability analysis, particularly for electronic components susceptible to sparking or overheating under electrical shocks. As an illustration, consider an electronic component subjected to random electrical shocks: if the magnitude of a shock exceeds a critical threshold γ (extreme shock), it causes immediate failure via sparking, while shocks with magnitude below γ (weak shocks) contribute to cumulative thermal stress. Suppose spontaneous thermal recovery is achieved if the inter-arrival time between successive weak shocks exceeds 150 seconds. Otherwise, a cooling fan installed to regulate the temperature will operate. However, due to a technical defect, the cooling fan only activates when the inter-arrival time is less than 100 seconds. Consequently, if the inter-arrival time falls between 100 and 150 seconds, the component neither cools naturally nor activates the fan, leading to cumulative overheating and potential failure. Thus, the system's failure mechanism, defined by the critical interval $[100, 150]$ and the shock magnitude threshold γ, serves as a compelling illustration of the proposed model's applicability.

The remainder of this paper is organized as follows. Section 2 describes the new mixed δ-shock model. Section 3 presents the reliability analysis of the system's lifetime. Section 4 investigates the system's mean residual lifetime. Finally, Section 5 concludes the paper.

2 Model Description

Consider a system that starts operating at time $t = 0$ and is subjected to a sequence of random external shocks that are the only cause of the system's failure. Let X_1 be the time until the occurrence of the first shock, and X_i (for $i = 2, 3, \dots$) be the inter-arrival time between the $(i-1)$th and ith shocks. Let also Z_i represent the magnitude of the ith shock for $i = 1, 2, \dots$. We assume that the random pairs (X_i, Z_i) are independent for $i = 1, 2, \dots$, and have a common joint cumulative distribution function (CDF) $H(x, z) = 1 - \bar{H}(x, z) = \Pr(X \leq x, Z \leq z)$ with marginal CDFs $F(x) = 1 - \bar{F}(x) = \Pr(X \leq x)$ and $G(z) = 1 - \bar{G}(z) = \Pr(Z \leq z)$ of X and Z, respectively. In the model we propose here, it is assumed that the system fails when either the inter-arrival time between two successive shocks falls within the critical interval $C_{\alpha,\delta} = [\alpha, \delta]$, where $0 \leq \alpha < \delta$, or when the magnitude of a single shock exceeds a critical threshold γ. The lifetime of the system is

$$T = \sum_{i=1}^{N} X_i, \tag{2.1}$$

where the random variable N (which has support in $\mathbb{N} = \{1, 2, \dots\}$) is the stopping time of the system, and the event $\{N = n\}$ is defined as follows:

$$\begin{aligned}
\{N = n\} =& \Big\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), \\
& \Big((X_n \in C_{\alpha,\delta}, Z_n \leq \gamma) \cup (X_n \in C_{\alpha,\delta}, Z_n > \gamma)\Big)\Big\} \\
& \cup \Big\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), (X_n \notin C_{\alpha,\delta}, Z_n > \gamma)\Big\} \\
=& \Big\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), X_n \in C_{\alpha,\delta}\Big\} \\
& \cup \Big\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), (X_n \notin C_{\alpha,\delta}, Z_n > \gamma)\Big\}.
\end{aligned} \tag{2.2}$$

Hence, the probability mass function (PMF) of N is obtained as follows:

Theorem 2.1. *The PMF of the random variable N is*

$$\Pr(N = n) = \Big(\bar{F}(\delta) - \bar{H}(\delta, \gamma) + H(\alpha, \gamma)\Big)^{n-1} \Big(F(\delta) + \bar{H}(\delta, \gamma) - H(\alpha, \gamma)\Big)$$

for $n \in \mathbb{N}$.

Proof. By using the definition of N in Eq. (2.2), we have

$$\begin{aligned}
\Pr(N = n) =& \Big(\Pr(X \notin C_{\alpha,\delta}, Z \leq \gamma)\Big)^{n-1} \Pr(X \in C_{\alpha,\delta}) \\
& + \Big(\Pr(X \notin C_{\alpha,\delta}, Z \leq \gamma)\Big)^{n-1} \Pr(X \notin C_{\alpha,\delta}, Z > \gamma) \\
=& \Big(\Pr\big((X < \alpha \cup X > \delta) \cap Z \leq \gamma\big)\Big)^{n-1} \\
& \times \Big(\Pr(\alpha \leq X \leq \delta) + \Pr\big((X < \alpha \cup X > \delta) \cap Z > \gamma\big)\Big) \\
=& \Big(\Pr\big((X < \alpha \cap Z \leq \gamma) \cup (X > \delta \cap Z \leq \gamma)\big)\Big)^{n-1} \\
& \times \Big(\Pr(\alpha \leq X \leq \delta) + \Pr\big((X < \alpha \cap Z > \gamma) \cup (X > \delta \cap Z > \gamma)\big)\Big) \\
=& \Big(\Pr(X < \alpha \cap Z \leq \gamma) + \Pr(X > \delta \cap Z \leq \gamma)\Big)^{n-1} \\
& \times \Big(\Pr(\alpha \leq X \leq \delta) + \Pr(X < \alpha \cap Z > \gamma) + \Pr(X > \delta \cap Z > \gamma)\Big).
\end{aligned} \tag{2.3}$$

Besides, we have $\Pr(X < \alpha \cap Z \leq \gamma) = H(\alpha, \gamma)$, $\Pr(X > \delta \cap Z \leq \gamma) = \bar{F}(\delta) - \bar{H}(\delta, \gamma)$, $\Pr(\alpha \leq X \leq \delta) = F(\delta) - F(\alpha)$, $\Pr(X < \alpha \cap Z > \gamma) = F(\alpha) - H(\alpha, \gamma)$, and $\Pr(X > \delta \cap Z > \gamma) = \bar{H}(\delta, \gamma)$. By substituting these in Eq. (2.3), the desired result is obtained. □

It is clear from Theorem 2.1 that the random variable N has a geometric distribution with parameter $F(\delta) + \bar{H}(\delta,\gamma) - H(\alpha,\gamma)$. Thus, the expected value and variance of N are obtained as follows, respectively:

$$E(N) = \frac{1}{F(\delta) - \bar{H}(\delta,\gamma) - H(\alpha,\gamma)}, \tag{2.4}$$

and

$$Var(N) = \frac{1 - \Big(F(\delta) + \bar{H}(\delta,\gamma) - H(\alpha,\gamma)\Big)}{\Big(F(\delta) + \bar{H}(\delta,\gamma) - H(\alpha,\gamma)\Big)^2} = \frac{\bar{F}(\delta) - \bar{H}(\delta,\gamma) + H(\alpha,\gamma)}{\Big(F(\delta) + \bar{H}(\delta,\gamma) - H(\alpha,\gamma)\Big)^2}.$$

Remark 2.2. The stated shock model in this section is a generalization of the mixed δ-shock model studied by Parvardeh and Balakrishnan (2015). More precisely, in the particular case where $\alpha = 0$, we have $\{N = n\} = \{(X_1 \notin [0,\delta], Z_1 \le \gamma), (X_2 \notin [0,\delta], Z_2 \le \gamma), \ldots, (X_{n-1} \notin [0,\delta], Z_{n-1} \le \gamma), X_n \in [0,\delta]\} \cup \{(X_1 \notin [0,\delta], Z_1 \le \gamma), (X_2 \notin [0,\delta], Z_2 \le \gamma), \ldots, (X_{n-1} \notin [0,\delta], Z_{n-1} \le \gamma), (X_n \notin [0,\delta], Z_n > \gamma)\} = \{(X_1 > \delta, Z_1 \le \gamma), (X_2 > \delta, Z_2 \le \gamma), \ldots, (X_{n-1} > \delta, Z_{n-1} \le \gamma), X_n \le \delta\} \cup \{(X_1 > \delta, Z_1 \le \gamma), (X_2 > \delta, Z_2 \le \gamma), \ldots, (X_{n-1} > \delta, Z_{n-1} \le \gamma), (X_n > \delta, Z_n > \gamma)\}$. Thus, the model reduces to the mixed δ-shock model investigated by Parvardeh and Balakrishnan (2015).

3 Reliability Analysis

3.1 Reliability function, mean lifetime, and Laplace transform

The reliability function of a system subjected to random shocks under the model described in Section 2 is given below.

Theorem 3.1. *For a system under the model described in Section 2, we have*

$$\begin{aligned}
\Pr(T > t) =& \Big(F(\delta) - F(\max\{\alpha, t\})\Big) I_{[0,\delta)}(t) \\
&+ \sum_{n=2}^{\infty} \Big(H(\alpha,\gamma) + \bar{F}(\delta) - \bar{H}(\delta,\gamma)\Big)^{n-1} \int_0^{\infty} \Pr(S^*_{n-1} > t - x) dF(x) \\
&+ \Big(\bar{H}(t,\gamma) - \bar{H}(\alpha,\gamma)\Big) I_{[0,\alpha)}(t) + \bar{H}(\max\{\delta, t\}, \gamma) \\
&- \sum_{n=2}^{\infty} \Big(H(\alpha,\gamma) + \bar{F}(\delta) - \bar{H}(\delta,\gamma)\Big)^{n} \Pr(S^*_n > t),
\end{aligned}$$

where S^*_n is the sum of n i.i.d. random variables having the following CDF,

$$F^*(x) = \begin{cases} \frac{H(x,\gamma)}{H(\alpha,\gamma)}, & \text{if } x < \alpha, \\ \frac{H(x,\gamma) - H(\delta,\gamma)}{G(\gamma) - H(\delta,\gamma)}, & \text{if } x > \delta. \end{cases}$$

Proof. We have $\Pr(T > t) = \Pr\left(\sum_{i=1}^{N} X_i > t\right)$, where the random variable N is defined in Eq. (2.2). Applying the definition of N, we arrive at the following:

$$\begin{aligned}\Pr(T > t) = &\sum_{n=1}^{\infty} \Pr\left(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), X_n \in C_{\alpha,\delta}\right)\\ &+ \sum_{n=1}^{\infty} \Pr\left(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma),\right.\\ &\qquad\qquad \left.(X_n \notin C_{\alpha,\delta}, Z_n > \gamma)\right). \end{aligned} \tag{3.1}$$

We first evaluate the first summation in the above equation. According to the assumptions of the model, the random pairs (X_i, Z_i)'s (for $i = 1, 2 \ldots$) are independent. So, for $n = 1$, the term within the first summation in Eq. (3.1) is

$$\Pr(X_1 > t, X_1 \in C_{\alpha,\delta}) = \Big(F(\delta) - F(\max\{\alpha, t\})\Big) I_{[0,\delta)}(t),$$

and for $n \geq 2$, the probability term within the first summation in Eq. (3.1) is equal to (since $\sum_{i=1}^{n} X_i = X_n + \sum_{i=1}^{n-1} X_i$, and X_n is independent of $\sum_{i=1}^{n-1} X_i$),

$$\begin{aligned}&\int_{C_{\alpha,\delta}} \Pr\left(\sum_{i=1}^{n-1} X_i > t - x, (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma)\right) dF(x)\\ &= \Pr\Big((X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma)\Big)\\ &\quad \times \int_{C_{\alpha,\delta}} \Pr\left(\sum_{i=1}^{n-1} X_i > t - x \middle| (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma)\right) dF(x)\\ &= \Big(\Pr(X \notin C_{\alpha,\delta}, Z \leq \gamma)\Big)^{n-1}\\ &\quad \times \int_{C_{\alpha,\delta}} \Pr\left(\sum_{i=1}^{n-1} X_i > t - x \middle| (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma)\right) dF(x).\end{aligned}$$

For every integer $k \geq 1$, we have

$$\begin{aligned}&\Pr\Big(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k \Big| (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_k \notin C_{\alpha,\delta}, Z_k \leq \gamma)\Big)\\ &= \prod_{i=1}^{k} \frac{\Pr(X \leq x_i \cap X \notin C_{\alpha,\delta}, Z \leq \gamma)}{\Pr(X \notin C_{\alpha,\delta}, Z \leq \gamma)}\\ &= \prod_{i=1}^{k} \frac{\Pr(X \leq x_i \cap X \notin C_{\alpha,\delta}, Z \leq \gamma)}{\Pr(X \notin C_{\alpha,\delta}, Z \leq \gamma)}.\end{aligned}$$

So, we can conclude that

$$\left(\sum_{i=1}^{k} X_i \middle| (X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_k \notin C_{\alpha,\delta}, Z_k \leq \gamma)\right) \stackrel{d}{=} S_k^*,$$

where the symbol $\overset{d}{=}$ means that the both sides of the equation are equal in distribution, and S_k^* is the sum of k i.i.d. random variables having the following CDF:

$$F^*(x) = \frac{\Pr(X \le x \cap X \notin C_{\alpha,\delta}, Z \le \gamma)}{\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma)} = \begin{cases} \frac{H(x,\gamma)}{H(\alpha,\gamma)}, & \text{if } x < \alpha, \\ \frac{H(x,\gamma)-H(\delta,\gamma)}{G(\gamma)-H(\delta,\gamma)}, & \text{if } x > \delta. \end{cases}$$

Hence,

$$\Pr\Big(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \le \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \le \gamma), X_n \in C_{\alpha,\delta}\Big)$$
$$= \begin{cases} \Big(F(\delta) - F(\max\{\alpha, t\})\Big) I_{[0,\delta)}(t), & \text{if } n = 1, \\ \Big(\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma)\Big)^{n-1} \int_{C_{\alpha,\delta}} \Pr(S_{n-1}^* > t - x) dF(x), & \text{if } n \ge 2, \end{cases}$$

where $\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma) = H(\alpha,\gamma) + \bar{F}(\delta) - \bar{H}(\delta,\gamma)$.

For the probability term within the second summation in Eq. (3.1), similarly we have for $n = 1$,

$$\begin{aligned}\Pr(X_1 > t, X_1 \notin C_{\alpha,\delta}, Z_1 > \gamma) &= \Pr(t < X_1 < \alpha, Z_1 > \gamma) I_{[0,\alpha)}(t) + \Pr(X_1 > \max\{\delta, t\}, Z_1 > \gamma) \\ &= \Big(\bar{H}(t,\gamma) - \bar{H}(\alpha,\gamma)\Big) I_{[0,\alpha)}(t) + \bar{H}(\max\{\delta, t\}, \gamma),\end{aligned}$$

and for $n \ge 2$,

$$\Pr\Big(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \le \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \le \gamma), (X_n \notin C_{\alpha,\delta}, Z_n > \gamma)\Big)$$
$$= \Pr\Big(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \le \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \le \gamma), X_n \notin C_{\alpha,\delta}\Big)$$
$$- \Pr\Big(\sum_{i=1}^{n} X_i > t, (X_1 \notin C_{\alpha,\delta}, Z_1 \le \gamma), \dots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \le \gamma), (X_n \notin C_{\alpha,\delta}, Z_n \le \gamma)\Big)$$
$$= \Big(\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma)\Big)^{n-1} \int_{C_{\alpha,\delta}^c} \Pr(S_{n-1}^* > t - x) dF(x) - \Big(\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma)\Big)^{n} \Pr(S_n^* > t),$$

where $C_{\alpha,\delta}^c = [\alpha, \delta]^c = (0, \alpha) \cup (\delta, \infty)$ and $\Pr(X \notin C_{\alpha,\delta}, Z \le \gamma) = H(\alpha,\gamma) + \bar{F}(\delta) - \bar{H}(\delta,\gamma)$.

The theorem is proved. □

Next, we obtain an explicit formula for the mean lifetime of the system, which defines the system's mean time to failure (MTTF). To prove the next theorem, we need the following lemma.

Theorem 3.2. *The MTTF of a system under the model described in Section 2 is given by*

$$MTTF = \frac{E(X)}{F(\delta) + \bar{H}(\delta,\gamma) - H(\alpha,\gamma)}. \tag{3.2}$$

Proof. The random variable N is a stopping time with respect to filtration $\mathcal{F}_n = \sigma\{(X_1, Z_1), (X_2, Z_2), \ldots, (X_n, Z_n)\}$, that is, the event $\{N = n\}$ is independent of $(X_{n+1}, Z_{n+1}), (X_{n+2}, Z_{n+2}), \ldots$, for $n \geq 1$. Hence, by using the well-known Wald's identity, we can write

$$MTTF = E(T) = E\left(\sum_{i=1}^{N} X_i\right) = E(X_1)E(N). \tag{3.3}$$

Now, by using (2.4) (for $E(N)$) in Eq. (3.3), the desired result is immediately obtained. □

We present now the explicit solution of the problem in terms of the corresponding Laplace Transform (LT), which is convenient for practical applications (after inversion).

Theorem 3.3. *Under the model described in Section 2, the LT of the system's lifetime is*

$$L(s) = \frac{E\left(e^{-sX} I_{\{X \in C_{\alpha,\delta}\}}\right) + E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z > \gamma\}}\right)}{1 - E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z \leq \gamma\}}\right)}.$$

Proof. By using the properties of conditional expectation, we have

$$\begin{aligned} L(s) = E\left(e^{-sT}\right) = E\left(E\left(e^{-s\sum_{i=1}^{N} X_i} \middle| N\right)\right) &= \sum_{n=1}^{\infty} E\left(e^{-s\sum_{i=1}^{N} X_i} \middle| N = n\right) P(N = n) \\ &= \sum_{n=1}^{\infty} E\left(e^{-s\sum_{i=1}^{n} X_i} I_{\{N=n\}}\right). \end{aligned}$$

Applying the definition of N (see Eq. (2.2)), we obtain

$$\begin{aligned} L(s) &= \sum_{n=1}^{\infty} E\left(e^{-s\sum_{i=1}^{n} X_i} I_{\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), X_n \in C_{\alpha,\delta}\}}\right) \\ &\quad + \sum_{n=1}^{\infty} E\left(e^{-s\sum_{i=1}^{n} X_i} I_{\{(X_1 \notin C_{\alpha,\delta}, Z_1 \leq \gamma), \ldots, (X_{n-1} \notin C_{\alpha,\delta}, Z_{n-1} \leq \gamma), (X_n \notin C_{\alpha,\delta}, Z_n > \gamma)\}}\right) \\ &= \sum_{n=1}^{\infty} \left(E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z \leq \gamma\}}\right)\right)^{n-1} E\left(e^{-sX} I_{\{X \in C_{\alpha,\delta}\}}\right) \\ &\quad + \sum_{n=1}^{\infty} \left(E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z \leq \gamma\}}\right)\right)^{n-1} E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z > \gamma\}}\right). \end{aligned}$$

Hence (by geometric series),

$$L(s) = \frac{E\left(e^{-sX} I_{\{X \in C_{\alpha,\delta}\}}\right)}{1 - E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z \leq \gamma\}}\right)} + \frac{E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z > \gamma\}}\right)}{1 - E\left(e^{-sX} I_{\{X \notin C_{\alpha,\delta}, Z \leq \gamma\}}\right)}.$$

This completes the proof. □

3.2 Numerical Experiments

Suppose the inter-arrival times $X_1, X_2, \ldots$ are i.i.d. exponential random variables with mean $E(X) = 5$, and the shocks magnitudes $Z_1, Z_2, \ldots$ are i.i.d. exponential random variables with mean $E(Z) = 2.5$. Using Theorem 3.1, the reliability function of the system's lifetime T is computed for $t = 4$ and different values of thresholds α, δ, and γ. Using Theorem 3.2, the MTTF of the system is also calculated. All calculations are performed using the R (version 4.4.2) program. As shown in Table 1, the reliability function decreases as the critical interval $C_{\alpha,\delta} = [\alpha, \delta]$ widens. This is consistent with the model's logic, since a wider critical area increases the chance of inter-arrival times falling within the failure-inducing interval, thereby reducing the overall system reliability. It is also observed that the values of reliability functions are increasing as γ increases. This is also reasonable because when γ is considered large, it is equivalent to increasing the resistance of the system against the magnitude of the shocks. Furthermore, the MTTF of the system decreases as the critical interval $C_{\alpha,\delta} = [\alpha, \delta]$ widens, and also MTTF increases with increasing γ.

Table 1: The MTTF and reliability function (at $t = 4$) of the system for some different values of α, δ, and γ.

γ	$[\alpha,\delta]$	MTTF	$\Pr(T > 4)$	$[\alpha,\delta]$	MTTF	$\Pr(T > 4)$	$[\alpha,\delta]$	MTTF	$\Pr(T > 4)$
5	[1,8]	7.478	0.543	[1,10]	6.817	0.537	[1,12]	6.581	0.536
	[3,8]	11.436	0.807	[3,10]	9.989	0.803	[3,12]	9.529	0.801
	[5,8]	18.109	0.898	[5,10]	14.887	0.891	[5,12]	13.276	0.889
10	[1,8]	8.097	0.560	[1,10]	7.235	0.559	[1,12]	6.776	0.563
	[3,8]	13.663	0.883	[3,10]	11.629	0.881	[3,12]	10.757	0.878
	[5,8]	27.818	0.991	[5,10]	20.064	0.987	[5,12]	16.875	0.983
15	[1,8]	8.112	0.564	[1,10]	7.588	0.559	[1,12]	6.786	0.567
	[3,8]	14.207	0.884	[3,10]	11.923	0.881	[3,12]	10.921	0.880
	[5,8]	30.263	0.998	[5,10]	21.240	0.997	[5,12]	18.047	0.990

4 Mean residual lifetime function of the system

In this section, we obtain the mean residual lifetime (MRL) of the system. This function represents the status of the system when the system is working at time t, and we can evaluate the system by this function. The MRL is defined as follows:

$$m(t) = E\left(T - t | T > t\right).$$

To calculate the above equation, we need the following probability:

$$\Pr(T > s | T > t) = \frac{\Pr(T > s)}{\Pr(T > t)}, \qquad for\, s > t.$$

Hence,

$$m(t) = E\left(T - t|T > t\right) = \int_0^\infty \Pr(T > t + x|T > t)dx = \frac{\int_0^\infty \Pr(T > t + x)dx}{\Pr(T > t)}.$$

In Table 2, we present numerical results on the MRL when the inter-arrival times and the magnitude of shocks hold under the assumptions of Subsection 3.2. From Table 2, it is observed that as the critical interval $C_{\alpha,\delta} = [\alpha, \delta]$ shrinks, the MRL values increase. Also, the MRL values are increasing as the threshold γ increases. These are reasonable because any factor that increases the stability of the system also leads to an increase in MRL. Here, both the shrinkage of the critical region and the increase of the threshold γ lead to the system being more stable.

Table 2: The MRL of the system for t=4 and some different values of α, δ, and γ.

γ	$[\alpha,\delta]$	$m(4)$	$[\alpha,\delta]$	$m(4)$	$[\alpha,\delta]$	$m(4)$
5	[1,8]	11.865	[1,10]	10.739	[1,12]	10.195
	[3,8]	13.713	[3,10]	12.042	[3,12]	11.013
	[5,8]	19.780	[5,10]	16.577	[5,12]	14.669
10	[1,8]	12.056	[1,10]	11.147	[1,12]	10.236
	[3,8]	15.205	[3,10]	12.953	[3,12]	11.673
	[5,8]	28.065	[5,10]	20.822	[5,12]	17.461
15	[1,8]	12.495	[1,10]	11.237	[1,12]	10.361
	[3,8]	15.546	[3,10]	13.106	[3,12]	11.745
	[5,8]	29.700	[5,10]	21.583	[5,12]	18.086

5 Conclusions

We introduced a new mixed δ-shock model, under which the system fails either when the inter-arrival time between two successive shocks falls within the critical interval $[\alpha, \delta]$, where $0 \leq \alpha < \delta$, or when the magnitude of a single shock exceeds a critical threshold γ. Based on the proposed model, several lifetime and reliability properties of the system were investigated. Numerical results were also presented to illustrate the theoretical findings.

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