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New estimation for the traffic intensity parameter of a single-server queueing system with finite capacity

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Abstract. In queueing theory, system is evaluated using performance metrics such as the average number of of customers in the queue and system, and the average waiting time. One of the most important parameters for these metrics is the traffic intensity, which must be estimated. This paper proposes a new estimation method and compares it with existing approaches. We focus on the $M/M/1/K$ single-server queueing model (finite capacity) and estimate the traffic intensity using Bayesian, E-Bayesian, hierarchical Bayesian, and a new EE-Bayesian method. Beacause reducing costs and minimizing customer waiting time are central goales in queueing systems, We consider an estimator suitable if it minimizes average customer waiting time. Using Monte Carlo simulation and a real data set, We demonstrate the superiority of the proposed method over other estimators.

Keywords. Traffic intensity parameter; E-Bayesian estimation; EE-Bayesian estimation; Average customer waiting time in the queue; $M/M/1/K$ queueing model.

MSC: 62C12, 62K05.

1 Introduction

Queueing theory is one of the most important tools for modeling production and service systems. Estimating model parameters by various methods and developing new estimation techniques are necessary system efficiency, gain compeetitive advantages, and meet quality standards. Managers try to balance the costs of providing services

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with the costs associated with customer waiting. They aim to control queue lengths the customers do not become tired, upset, or abandon the queue. Indeed, customer waiting can lead to behaviors that impose significant costs on the system and both micro and macro levels.

Maximum likelihood and Bayesian estimation methods in queueing theory have received significant attention. Notable contributions include Clark (1975), who estimated the parameters of the $M/M/1$ queue under steady-state conditions using maximum likelihood; Muddapur (1972), who estimated arrival and service rates in $M/M/1$ and $M/M/\infty$; Thiruvaiyaru and Basava (1992), who proposed an empirical Bayesian estimator for $M/M/1$ and $M/M/\infty$; Chowdhury and Mukherjee (2013), who studied maximum likelihood and Bayesian estimation for $M/M/1$; Schweer and Wichelhaus (2020), who considered nonparametric estimation of service time distributions in discrete-time queues. In this paper, we introduce a new E-Bayesian estimator for the traffic intensity of the $M/M/1/K$ queue and compare its performance with existing Bayesian, E-Bayesian, and hierarchical Bayesian methods.

Many methods have been proposed for estimating distributional parameters. Among them, Bayesian approaches—based on prior distributions—are widely used. A suitable choice of priors and hyperpriors can substantially reduce posterior estimation error. Hierarchical Bayesian and E-Bayesian techniques explicitly model uncertainty in prior parameters and have been successfully applied in various settings (Lindley and Smith, 1972; Han, 1977, 2009, 2011; Jaheen and Okasha, 2011; Wang et al., 2012; Yousefzadeh, 2017; Yaghoobzadeh Shahrastani, 2019). These approaches have been used for parameters of exponential, binomial-proportion, Burr Type XII, Pascal, and Gompertz distributions, often under censored or fuzzy data schemes. Several authors have also used E-Bayes and Hierarchical Bayes estimation methods in queueing theory, some of which are mentioned below (Makhdoom and Yaghoobzadeh Shahrastani, 2023; Yaghoobzadeh Shahrastani et al., 2025; Yaghoobzadeh Shahrastani and Makhdoom, 2026).

In this paper we focus on the $M/M/1/K$ single-server queue with finite capacity and aim to estimate its traffic intensity, a key quantity affecting system performance. We propose a new estimator (denoted here as EE-Bayesian) and compare its efficiency with classical Bayesian, E-Bayesian, and hierarchical Bayesian estimators. Performance is evaluated by Monte Carlo simulation and an analysis of real data. In addition to statistical accuracy, we assess the estimators in terms of practical metrics such as customers' waiting time and system cost, which are important for managerial decisions.

The remainder of the paper is organized as follows. Section 2 presents the necessary background and notation. Sections 3 and 4 derive the E-Bayesian and hierarchical Bayesian estimators for the $M/M/1/K$ traffic intensity, respectively. Section 5 introduces the proposed EE-Bayesian estimator. Section 6 compares the estimators via Monte Carlo simulation and a real data example. Section 7 concludes the paper.

2 Basic concepts

In this section we present the preliminary concepts and definitions required for this paper. First, we introduce the $M/M/1/K$ queueing model, and then we derive the average waiting time in the customers' queue and the system cost function.

2.1 Queueing model $M/M/1/K$

The $M/M/1/K$ model is a single-server queueing system in which interarrival times between successive customers are exponentially distributed with rate λ, and service times are exponentially distributed with rate μ. In this paper, the interarrival times $\{U_r, r \geq 1\}$ are assumed to be a sequence of independent and identically distributed (i.i.d.) random variables with density $f(u, \lambda)$, and the service times $\{V_r, r \geq 1\}$ are assumed to be an i.i.d. sequence with density $g(v, \mu)$, such that

$$f(u, \lambda) = \lambda e^{-\lambda u}, u > 0, \lambda > 0, \quad g(v, \mu) = \mu e^{-\mu v}, v > 0, \mu > 0 \tag{2.1}$$

We define $\rho = \frac{\lambda}{\mu}$ as the traffic intensity, and throughout this paper we assume $\rho \neq 1$. Managers of queueing systems pay particular attention to minimizing customer waiting time and reducing system costs; hence, minimizing waiting time is a guiding principle for queueing-system design and has a significant impact on system performance. According to Allen (1990), the distribution of the number of customers in the system (P_n), the average number of customers in the system (L), the average number of customers in the queue (L_q), and the average waiting time in the queue every customer (W_q) are given as follows:

$$\pi_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{K+1}} & n = 0, 1, \dots K \quad \rho \neq 1 \\ \frac{1}{K+1} & n = 0, 1, \dots K \quad \rho = 1 \end{cases} \tag{2.2}$$

$$L = \frac{K\rho^{K+2} - (K+1)\rho^{K+1} + \rho}{(1-\rho)(1-\rho^{K+1})} \tag{2.3}$$

$$L_q = \frac{\rho^2 - K\rho^{K+1} + (K-1)\rho^{K+2}}{(1-\rho)(1-\rho^{K+1})} \tag{2.4}$$

$$W_q = \frac{\rho^2 - K\rho^{K+1} + (K-1)\rho^{K+2}}{\lambda(1-\rho)(1-\rho^{K+1})} \tag{2.5}$$

2.2 Cost function of the queueing model system $M/M/1/K$

In any queueing system, the goal is to reduce queue length and customer waiting time while simultaneously increasing customer satisfaction. Customer time possesses significant socio-economic value, and the loss of this time is therefore regarded as a cost. Generally, in a queueing system, the expected total cost per unit of time serves as the standard metric for system evaluation, as the associated costs depend heavily on their type and nature. In this paper, the cost funct propose is defined as follows:

$$C(\rho) = C_1 L_q + C_2(L - L_q) + C_3(\lambda - \bar{\lambda}), \quad \bar{\lambda} = \lambda(1 - P_K) \tag{2.6}$$

1. $C_1 L_q$ is defined as the cost of lost customer time in the queue, which is the average total of this cost being equal to the cost of lost time for each customer in the queue (C_1) multiplied by the average number of customers in the queue (L_q).

2. $C_2(L - L_q)$ is defined as the cost of lost customer time while receiving service, which is the average total of this cost being equal to the cost of lost time for each customer receiving service (C_2) multiplied by the average number of customers receiving service ($L - L_q$).

3. $C_3(\lambda - \bar{\lambda})$ is the cost of lost customers. The system incurs this cost when it prevents a customer from entering due to capacity being full, or when a customer is deterred from entering due to observing a long queue. Therefore, by considering $\bar{\lambda}$ as the arrival rate of customers entering the system, $(\lambda - \bar{\lambda})$ is the average number of customers who did not enter the system. If the cost of losing each customer is assumed to be C_3, the total average loss the system sustains for losing customers per unit of time is $C_3(\lambda - \bar{\lambda})$.

Therefore, considering relations (2.4), (2.5) and the relation $\lambda - \bar{\lambda} = \frac{\lambda(1-\rho)\rho^K}{1-\rho^{K+1}}$, Equation (2.6) becomes the following equation

$$C(\rho) = C_1 \frac{K\rho^{K+2} - (K+1)\rho^{K+1} + \rho}{(1-\rho)(1-\rho^{K+1})} + C_2 \frac{\rho(1-\rho^K)}{1-\rho^{K+1}} + C_3 \frac{\lambda(1-\rho)\rho^K}{1-\rho^{K+1}} \tag{2.7}$$

Now, according to Han (1977), Bayesian and hierarchical Bayesian estimates are defined as follows:

Definition 2.1. Assume b_1 and b_2 are hyperparameters in the prior distribution of θ, $\pi(b_1, b_2)$ is the joint prior distribution of (b_1, b_2), and $\hat{\theta}_B(b_1, b_2)$ is the Bayesian estimator of θ. Then the E-Bayesian estimator of θ, denoted by $\hat{\theta}_{EB}$, is obtained as follows:

$$\hat{\theta}_{EB} = E[\hat{\theta}_B(b_1, b_2)] = \int_{\Lambda_1} \int_{\Lambda_2} \hat{\theta}_B(b_1, b_2)\pi(b_1, b_2)db_1 db_2, \quad b_1 \in \Lambda_1, b_2 \in \Lambda_2$$

Definition 2.2. If $\pi(\theta|\lambda)$ and $\pi^{'}(\lambda)$ are, respectively, the corresponding prior distributions for the parameter θ and the hyperparameter λ, then the hierarchical prior distribution of θ is obtained as $\pi^{''}(\theta) = \int \pi(\theta|\lambda)\pi^{'}(\lambda)d\lambda$.

Now, the loss function is introduced next. Norstrom (1996) defined a class of loss functions in the following form.

$$L(\delta, \theta) = w(\theta)\frac{(\delta - \theta)^2}{\delta^i}, \quad 0 \le i \le 2, w(\theta) > 0$$

This loss function is scale-invariant in three cases: $i = 0, w(\theta) = \frac{1}{\theta^2}$; $i = 1, w(\theta) = \frac{1}{\theta}$; and $i = 2, w(\theta) = 1$. In the second case, i.e., $i = 1, w(\theta) = \frac{1}{\theta}$, this loss function is transformed into an asymmetric loss function as $L(\delta, \theta) = \frac{\delta}{\theta} + \frac{\theta}{\delta} - 2$. Also, the Bayesian estimator of θ under this loss function is obtained as follows:

$$\hat{\theta}_B = \sqrt{\frac{E(\theta|data)}{E(\theta^{-1}|data)}} \tag{2.8}$$

3 *E*-Bayesian estimation

In this section, λ and μ are treated as independent parameters with the following prior distributions. Moreover, $\mathbf{x} = \{U_1, \cdots, U_n, V_1, \cdots, V_n\}$, $T_1 = \sum_{i=1}^{n} U_i$, and $T_2 = \sum_{i=1}^{n} V_i$ are also considered.

$$\pi_1(\lambda|a_1, b_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda^{a_1-1} e^{-b_1\lambda}, \lambda > 0, a_1 > 0, b_1 > 0, \tag{3.1}$$

$$\pi_2(\mu|a_2, b_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \mu^{a_2-1} e^{-b_2\mu}, \mu > 0, a_2 > 0, b_2 > 0 \tag{3.2}$$

Theorem 3.1. *The Bayesian estimator of the traffic intensity parameter for the M/M/1/K queueing model is:*

$$\hat{\rho}_B(b_1, b_2) = \sqrt{\frac{(a_1 + n)(a_1 + n - 1)}{(a_2 + n)(a_2 + n - 1)} \frac{b_2 + T_2}{b_1 + T_1}}$$

Proof. Given the observation vector $\mathbf{X}$ and the distributions from Equation (2.1), the likelihood function is obtained as follows:

$$L(\lambda, \mu) = (\lambda\mu)^n e^{-\lambda T_1 - \mu T_2} \tag{3.3}$$

By using Equations (3.1), (3.2), and (3.3), the posterior distribution of λ and μ conditional on $\mathbf{X}$ is obtained as follows:

$$\pi(\lambda, \mu|\mathbf{X}) = \frac{(b_1 + T_1)^{a_1+n}(b_2 + T_2)^{a_2+n}}{\Gamma(a_1 + n)\Gamma(a_2 + n)} \lambda^{a_1+n-1} \mu^{a_2+n-1} e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}$$

Therefore, the Bayesian estimator for the traffic intensity parameter, ρ, is obtained using Equation (2.8) as follows:

$$\hat{\rho}_B(b_1, b_2) = \sqrt{\frac{E(\rho|\mathbf{X})}{E(\rho^{-1}|\mathbf{X})}} = \sqrt{\frac{\int_0^\infty \int_0^\infty \left(\frac{\lambda}{\mu}\right) \pi(\lambda, \mu|\mathbf{X}) d\lambda d\mu}{\int_0^\infty \int_0^\infty \left(\frac{\mu}{\lambda}\right) \pi(\lambda, \mu|\mathbf{X}) d\lambda d\mu}} = \sqrt{\frac{(a_1 + n)(a_1 + n - 1)}{(a_2 + n)(a_2 + n - 1)} \frac{b_2 + T_2}{b_1 + T_1}}$$

□

Next, the E-Bayesian estimator for the parameter ρ is obtained. According to Han (1977) in calculating the E-Bayesian estimator, a_1 and b_1 in Equation (3.1) are chosen such that $\pi_1(\lambda|a_1, b_1)$ is decreasing with respect to λ. Therefore, based on the relationship

$$\frac{d\pi_1(\lambda, \mu|a_1, b_1)}{d\lambda} = \frac{b_1^{a_1}\lambda^{a_1-2}e^{-b_1\lambda}}{\Gamma(a_1)}[(a_1-1)-b_1\lambda]$$

it must be that $b_1 > 0$ and $0 < a_1 \leq 1$. Berger (1985) showed that increasing b_1 causes the efficiency of the Bayesian estimator for λ to decrease. Therefore, the hyperparameter b_1 must be bounded above and restricted to $0 < b_1 < c_1$, where c_1 is a constant. Consequently, Han (1977) treated b_1 as a random variable with a continuous uniform distribution on $(0, c_1)$, i.e. $\pi_1(b_1) = \frac{1}{c_1}$ for $0 < b_1 < c_1$. Also, given $0 < a_1 \leq 1$, we assume $a_1 = 1$ without loss of generality, so Equation (3.1) becomesnto $\pi_1(\lambda|b_1) = b_1 e^{-b_1\lambda}, \lambda > 0, b_1 > 0$. By similar reasoning regarding a_2 and b_2 in Equation (3.2), the distribution of b_2, $\pi_2(b_2)$, iis assumed uniform on $(0, c_2)$, where c_2 is a constant. By selecting $a_2 = 1$, Equation (3.2) is transformed into Equation $\pi_2(\mu|b_2) = b_2 e^{-b_2\mu}.\mu > 0, b_2 > 0$.

Theorem 3.2. *The E-Bayesian estimator of the traffic intensity parameter for the M/M/1/K queueing model is:*

$$\hat{\rho}_{EB}(c_1, c_2) = \frac{c_2 + 2T_2}{2c_1}\ln\left(\frac{c_1 + T_1}{T_1}\right)$$

Proof. By setting $a_1 = a_2 = 1$ in Theorem 3.1, the Bayesian estimator for the parameter ρ is obtained as $\hat{\rho}_B(c_1, c_2) = \frac{b_2+T_2}{b_1+T_1}$. On the other hand, according to Definition 2.1, the E-Bayesian estimator for ρ is defined as follows:

$$\begin{aligned}\hat{\rho}_{EB}(c_1, c_2) &= \int_0^{c_2}\int_0^{c_1} \hat{\rho}_B(c_1, c_2)\pi(b_1, b_2)db_1 db_2 \\ &= \frac{1}{c_1 c_2}\int_0^{c_2}\int_0^{c_1} \frac{b_2 + T_2}{b_1 + T_1}db_1 db_2 = \frac{c_2 + 2T_2}{2c_1}\ln\left(\frac{c_1 + T_1}{T_1}\right)\end{aligned}$$

□

4 Hierarchical Bayesian estimation

In this section, the hierarchical Bayesian estimator for ρ is derived using Lindley (1980) approximation. In general, the ratio of integrals of the form

$$E(u(\Lambda)|\mathbf{X}) = \frac{\int u(\Lambda)e^{Q(\Lambda)}d\Lambda}{\int e^{Q(\Lambda)}d\Lambda} \tag{4.1}$$

where $Q(\Lambda) = \ell(\Lambda) + \rho(\Lambda)$, and $\ell(\Lambda)$ represents the logarithm of the likelihood function of the observations and $\rho(\Lambda)$ represents the logarithm of the prior distribution of Λ, is approximated using Lindley (1980) approximation method as follows:

$$E(u(\Lambda)|\mathbf{X}) = \left\{ u + \frac{1}{2}\sum_i\sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right\}_{\hat{\Lambda}} \tag{4.2}$$

which is presented, where $\Lambda = (\lambda_1, \ldots, \lambda_m)$ and $i, j, k, l = 1, 2, \ldots, m$, and $\hat{\Lambda}$ is the maximum likelihood estimate of Λ. Also,

$$u = u(\Lambda), u_i = \frac{\partial u}{\partial \lambda_i}, u_{ij} = \frac{\partial^2 u}{\partial \lambda_i \partial \lambda_j}, L_{ijk} = \frac{\partial^3 \ell(\Lambda)}{\partial \lambda_i \partial \lambda_j \partial \lambda_k}, \rho_j = \frac{\partial \rho(\Lambda)}{\partial \lambda_j} \tag{4.3}$$

where σ_{ij} is the (i, j)-th entry of the inverse matrix of $\{-L_{ij}\}$. Assuming $\hat{\Lambda} = (\lambda, \mu)$, relation (4.2) becomes relation

$$E(u(\Lambda)|\mathbf{X}) = (u + Au_1 + Bu_2 + C)_{\hat{\Lambda}}, \quad \hat{\Lambda} = (\hat{\lambda}, \hat{\mu}) \tag{4.4}$$

which is transformed, where $\hat{\lambda} = \text{MLE}(\lambda)$, $\hat{\mu} = \text{MLE}(\mu)$, and

$$A = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} L_{ij}\sigma_{ij}, \quad B = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} L_{ij}\sigma_{ij}$$

$$C = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} u_{ij}\sigma_{ij}. \tag{4.5}$$

Theorem 4.1. *The Hierarchical Bayesian estimator of the traffic intensity parameter for the M/M/1/K queueing model is*

$$\hat{\rho}_{HB} = \frac{\hat{\mu}^2[(n + 1 + \hat{\lambda}\hat{\rho}_1 - \hat{\mu}\hat{\rho}_2) + \hat{\lambda} - \hat{\mu}]}{\hat{\lambda}^2[(n + 1 - \hat{\lambda}\hat{\rho}_1 + \hat{\mu}\hat{\rho}_2) - \hat{\lambda} + \hat{\mu}]}$$

Proof. Using Definition 2.2, the hierarchical Bayesian prior distributions for the parameters λ and μ are obtained as follows, respectively.

$$\pi(\lambda) = \int_0^{c_1} \pi(\lambda|b_1)\pi(b_1)db_1 = \frac{1 - (1 + c_1\lambda)e^{-c_1\lambda}}{c_1\lambda^2}, \tag{4.6}$$

$$\pi(\mu) = \int_0^{c_2} \pi(\mu|b_2)\pi(b_2)db_2 = \frac{1 - (1 + c_2\mu)e^{-c_2\mu}}{c_2\mu^2} \tag{4.7}$$

Therefore, based on relations (3.3), (4.6), and (4.7), the hierarchical posterior distribution of λ and μ conditional on $\mathbf{X}$ is as follows, where $S(\lambda, \mu) = \left(1 - (1 + c_1\lambda)e^{-c_1\lambda}\right)(1 - (1 + c_2\mu)e^{-c_2\mu})$.

$$\pi^{**}(\lambda, \mu|\mathbf{X}) = \frac{\lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda, \mu)}{\int_0^\infty\int_0^\infty \lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda, \mu)d\lambda d\mu} \tag{4.8}$$

Using relations (2.8) and (4.8), the hierarchical Bayesian estimator for ρ is obtained as follows:

$$\hat{\rho}_{HB} = \frac{\frac{\int_0^\infty\int_0^\infty \left(\frac{\lambda}{\mu}\right)\lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda,\mu)d\lambda d\mu}{\int_0^\infty\int_0^\infty \lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda,\mu)d\lambda d\mu}}{\frac{\int_0^\infty\int_0^\infty \left(\frac{\mu}{\lambda}\right)\lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda,\mu)d\lambda d\mu}{\int_0^\infty\int_0^\infty \lambda^{a_1+n-3}\mu^{a_2+n-3}e^{-\lambda(b_1+T_1)-\mu(b_2+T_2)}S(\lambda,\mu)d\lambda d\mu}} = \frac{I_1}{I_2} \tag{4.9}$$

Now, Lindley (1980) approximation method is used to obtain the numerator and denominator of $\hat{\rho}_{HB}$. Assuming $\Lambda = (\lambda, \mu)$, $u(\lambda, \mu) = \frac{\lambda}{\mu}$, $\rho(\Lambda) \propto \ln S(\lambda, \mu) - 2\ln\lambda - 2\ln\mu$ and $\ell(\Lambda) \propto (n-1)\ln\lambda - b_1T_1 + (n-1)\ln\mu - b_2T_2$ the following relations are obtained.

$$\rho_1 = \frac{c_1\lambda^2 + 2c_1\lambda - 2)e^{-c_1\lambda} - 2}{\lambda[1-(1+c_1\lambda)e^{-c_1\lambda}]}, \quad \rho_2 = \frac{c_2\mu^2 + 2c_2\mu - 2)e^{-c_2\mu} - 2}{\mu[1-(1+c_2\mu)e^{-c_2\mu}]}$$

$$u_{11} = 0, u_{22} = \frac{2\lambda}{\mu^3}, u_{12} = u_{21} = -\frac{1}{\mu^2}, u_1 = \frac{1}{\mu}, u_2 = -\frac{\lambda}{\mu^2}$$

$$L_{11} = -\frac{n}{\lambda^2}, L_{22} = -\frac{n}{\mu^2}, L_{12} = L_{21} = 0, \sigma_{12} = \sigma_{21} = 0$$

$$\sigma_{11} = \frac{\lambda^2}{n}, \sigma_{22} = \frac{\mu^2}{n}, A = \frac{\lambda^2\rho_1 - 1}{n}, B = \frac{\mu^2\rho_2 - 1}{n}, C = \frac{\lambda}{n\mu}$$

Therefore, with the help of relation (4.4), we have

$$I_1 = \frac{\hat{\lambda}\hat{\mu}(n + 1 + \hat{\lambda}\hat{\mu}_1 - \hat{\mu}\hat{\rho}_2) + \hat{\lambda} - \hat{\mu}}{n\hat{\mu}^2}$$

Also, assuming $u(\lambda, \mu) = \frac{\mu}{\lambda}$ and with the help of Lindley (1980) approximation and then with the help of relation (4.4), we have

$$I_2 = \frac{\hat{\lambda}\hat{\mu}(n + 1 - \hat{\lambda}\hat{\mu}_1 + \hat{\mu}\hat{\rho}_2) - \hat{\lambda} + \hat{\mu}}{n\hat{\lambda}^2}$$

Therefore, according to relation (4.9), the Hierarchical Bayesian estimator for ρ is equal to

$$\hat{\rho}_{HB} = \frac{\hat{\lambda}^2[(n + 1 + \hat{\lambda}\hat{\mu}_1 - \hat{\mu}\hat{\rho}_2) + \hat{\lambda} - \hat{\mu}]}{\hat{\mu}^2[n + 1 - \hat{\lambda}\hat{\mu}_1 + \hat{\mu}\hat{\rho}_2) - \hat{\lambda} + \hat{\mu}]}$$

□

5 *EE*-Bayesian estimation

Generally, the Bayesian estimate of a parameter, say θ, depends on a hyperparameter, say b, in the prior distribution of θ. To make the Bayesian estimate of θ independent of the value of b, an estimate known as the E-Bayesian estimate is derived. To compute the E-Bayesian estimate, b is considered a uniform random variable with an upper bound, such as c, and then the E-Bayesian estimate is obtained, whose value depends on c. As is clear in all articles related to E-Bayesian estimation, a specific value for c is considered to calculate its numerical value, which results in different values for the E-Bayesian estimator for different values of c. Therefore, to make the value of the E-Bayesian estimator independent of a specific value of c and to obtain a general estimator, c is considered a continuous random variable, and a new estimator, called the EE-Bayesian estimator, is derived for θ.

In Section 3, the hyperparameters b_1 and b_2 were considered as $0 < b_1 < c_1$ and $0 < b_2 < c_2$, where c_1 and c_2 were taken as constants. However, in this section, it is assumed that c_1 and c_2 are continuous random variables, and a new EE-Bayesian estimate for the parameter ρ is obtained.

Theorem 5.1. *If c_1 and c_2 are continuous, independent random variables with the Gamma distribution having the probability density function*

$$f_{c_i}(t) = \frac{\beta^i}{\Gamma(\alpha_i)} c_i^{\alpha_i - 1} e^{-\beta_i c_i}, c_i > 0, \alpha_i > 0, \beta_i > 0, \quad i = 1, 2,$$

then under the condition $c_1 < T_1$, the EE-Bayesian estimation for the traffic intensity parameter of the $M/M/1/K$ queuing model is.

$$\hat{\rho}_{EEB} = \frac{\alpha_2 + 2\beta_2 T_2}{2\beta_2 \Gamma(\alpha_1)} \sum_{j=1}^{\infty} \left(-\frac{1}{\beta_1 T_1} \right)^j \frac{\Gamma(j + \alpha_1 + 1)\Gamma(j + \alpha_1 + 1, \beta_1 T_1)}{j}$$

where $\Gamma(a, t) = \int_0^t u^{a-1} e^{-u} du$.

Proof. According to Definition 2.1 and Theorem 4.1, the EE-Bayesian estimate of ρ is defined and calculated as follows.

$$\begin{aligned}
\hat{\rho}_{EEB} &= \int_0^{\infty} \int_0^{\infty} \hat{\rho}_{EB}(c_1, c_2) f(c_1, c_2) dc_1 dc_2 \\
&= \frac{\beta_1^{\alpha_1}}{2\Gamma(\alpha_1)} \int_0^{\infty} (c_2 + 2T_2) \left(\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \int_0^{T_1} \log\left(1 + \frac{c_1}{T_1}\right) c_1^{\alpha_1 - 2} e^{-\beta_1 c_1} dc_1 \right) c_2^{\alpha_2 - 1} e^{-\beta_2 c_2} dc_2
\end{aligned}$$

Under the condition $c_1 < T_1$ and with the help of Maclaurin expansion of $\log\left(1 + \frac{c_1}{T_1}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} c_1^j}{j T_1^j}$, we have:

$$\hat{\rho}_{EEB} = \frac{\alpha_2 + 2\beta_2 T_2}{2\beta_2 \Gamma(\alpha_1)} \sum_{j=1}^{\infty} \left(-\frac{1}{\beta_1 T_1} \right)^j \frac{\Gamma(j + \alpha_1 + 1)\Gamma(j + \alpha_1 + 1, \beta_1 T_1)}{j}$$

□

6 Simulation and data analysis

In this section, using Monte Carlo simulation and a real data set, estimators of the traffic intensity parameter are compared to determine the most appropriate estimator. The appropriate estimator is the one that minimizes the average waiting time in the customers' queue as well as the system cost associated with that waiting time.

6.1 Monte Carlo simulation

In this section, the estimator for the traffic intensity parameter, i.e., ρ, is estimated using Monte Carlo simulation. The simulation steps are as follows:

1. Two random numbers are randomly generated from the distribution $f(x) = e^{-(x-1)}, x > 1$, and are denoted by c_1 and c_2, respectively.
2. Two numbers are randomly generated from the distributions $U(0, c_1)$ and $U(0, c_2)$, and denoted by b_1 and b_2, respectively.
3. Two numbers are randomly generated from the distribution $f(x) = e^{-x}, x > 0$ and are denoted by a_1 and a_2, respectively.
4. Two random samples of size 30 are randomly generated from the distributions $Exp(\lambda = 2.5)$ and $Exp(\mu = 3.5)$, and the sums of the observations in each sample, i.e., T_1 and T_2, are calculated.
5. Four random numbers are randomly generated from the distribution $f(x) = e^{-x}, x > 0$ and are denoted by α_1, α_2, β_1, and β_2, respectively.
6. Using Theorems 3.1, 3.2, 4.1, and 5.1, the Bayesian estimator, E-Bayesian estimator, Hierarchical Bayesian estimator, and EE-Bayesian estimator for the traffic intensity parameter of the $M/M/1/K$ queueing model are derived.

Steps 1-6 are repeated $10,000$ times, and the average of the Bayesian, E-Bayesian, Hierarchical Bayesian, and EE-Bayesian estimators for ρ were computed for different values of K and d; the results are reported in Tables 1 and 2. In these tables, the estimates of ρ under Bayesian, E-Bayesian, hierarchical Bayesian, and EE-Bayesian estimation methods are indicated by the symbols $\hat{\rho}_B$, $\hat{\rho}_{EB}$, $\hat{\rho}_{HB}$, and $\hat{\rho}_{EEB}$, respectively. Using these estimated values of ρ, the average waiting time in the queue for each customer was calculated by Equation (2.5) and recorded in Tables 3 and 4. In these tables, the average waiting time in the queue under Bayesian, E-Bayesian, hierarchical Bayesian, and EE-Bayesian estimation methods are indicated by the symbols $\hat{W}_{q.B}$, $\hat{W}_{q.EB}$, $\hat{W}_{q.HB}$, and $\hat{W}_{q.EEB}$, respectively. Assuming $C_1 = 250$, $C_2 = 300$, and $C_3 = 400$, the system cost was obtained from Equation (2.7) and is shown in Tables 5 and 6. Figure 1 plots the average customer waiting time in the queue versus ρ for different estimators; it shows that the average waiting time corresponding to the EE-Bayesian estimator is the smallest among the estimators. Similarly, Figure 2 plots the system cost versus ρ and indicates that the EE-Bayesian estimator yields the lowest system cost. Both figures confirm the simulation results.

Table 1: Values of different estimates of ρ for the simulated data

		Estimation			
K	λ	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$	$\hat{\rho}_{EEB}$
	0.5	0.2513	0.2067	0.5437	0.2323
	1.0	0.2494	0.2107	0.5483	0.2306
	1.5	0.2318	0.2172	0.5945	0.2064
	2.0	0.2340	0.2218	0.6308	0.2107
2	2.5	0.2387	0.2276	0.6518	0.2124
	3.0	0.2388	0.2295	0.6590	0.2120
	3.5	0.2371	0.2271	0.6561	0.2104
	4.0	0.2339	0.2246	0.6465	0.2068
	0.5	0.2692	0.2296	0.6294	0.2491
	1.0	0.2488	0.2227	0.6281	0.2243
	1.5	0.2465	0.2303	0.6467	0.2202
	2.0	0.2435	0.2343	0.6968	0.2264
3	2.5	0.2356	0.2244	0.6245	0.2083
	3.0	0.2490	76	0.5282	0.2198
	3.5	0.2436	0.2356	0.6750	0.2160
	4.0	0.2442	0.2373	0.6807	0.2165
	0.5	0.2371	0.1976	0.5282	0.2198
	1.0	0.2279	0.2022	0.5644	0.2053
	1.5	0.2282	0.2134	0.5895	0.2038
	2.0	0.2289	0.2143	0.6118	0.2041
4	2.5	0.2283	0.2189	0.6146	0.2029
	3.0	0.2344	0.2245	0.6528	0.2089
	3.5	0.2295	0.2235	0.6312	0.2034
	4.0	0.2279	0.2211	0.6277	0.2017

Table 2: Values of different estimates of ρ for the simulated data

		Estimation			
K	λ	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$	$\hat{\rho}_{EEB}$
	0.5	0.2461	0.2051	0.5417	0.2381
	1.0	0.2362	0.2130	0.5901	0.2132
	1.5	0.2352	0.2172	0.6141	0.2105
	2.0	0.2332	0.2181	0.6217	0.2079
5	2.5	0.2388	0.2271	0.6587	0.2128
	3.0	0.2439	0.2360	0.6877	0.2175
	3.5	0.2267	0.2214	0.6094	0.2001
	4.0	0.2242	0.2192	0.6069	0.1978
	0.5	0.3075	0.2504	0.6972	0.2839
	1.0	0.2746	0.2420	0.7195	0.2472
	1.5	0.2797	0.2608	0.7675	0.2498
	2.0	0.2686	0.2508	0.7329	0.2391
6	2.5	0.2718	0.2620	0.7576	0.2417
	3.0	0.2686	0.2557	0.7494	0.2384
	3.5	0.2605	0.2549	0.7202	0.2308
	4.0	0.2659	0.2568	0.7469	0.2358
	0.5	0.3269	0.2725	0.7225	0.3031
	1.0	0.2883	0.2558	0.7376	0.2598
	1.5	0.2792	0.2614	0.7548	0.2497
	2.0	0.2786	0.2582	0.7562	0.2458
7	2.5	0.2722	0.2610	0.7484	0.2419
	3.0	0.2776	0.2659	0.7786	0.2474
	3.5	0.2704	0.2632	0.7508	0.2399
	4.0	0.2676	0.2595	0.7416	0.2367

Table 3: The Estimated system cost for different estimates ρ and based Table 1

		Estimation			
K	λ	$\hat{W}_{q.B}$	$\hat{W}_{q.EB}$	$\hat{W}_{q.HB}$	$\hat{W}_{q.EEB}$
	0.5	0.09609	0.08394	0.32144	0.06839
	1.0	0.04804	0.04132	0.1626	0.03537
	1.5	0.02786	0.02487	0.12096	0.02274
	2.0	0.02124	0.01935	0.09807	0.01769
2	2.5	0.01759	0.01619	0.08183	0.01435
	3.0	0.01467	0.01369	0.06915	0.01192
	3.5	0.01242	0.01152	0.05444	0.01008
	4.0	0.01061	0.00989	0.05061	0.00856
	0.5	0.16381	0.11885	0.78669	0.14015
	1.0	0.06991	0.05668	0.39204	0.05586
	1.5	0.04574	0.03986	0.41070	0.03639
	2.0	0.03347	0.03096	0.23053	0.02888
3	2.5	0.02505	0.02269	0.15537	0.01950
	3.0	0.02334	0.02082	0.15034	0.01823
	3.5	0.01914	0.01789	0.12547	0.01499
	4.0	0.01683	0.01588	0.11122	0.01319
	0.5	0.14102	0.11563	0.79437	0.09479
	1.0	0.0645	0.06246	0.45399	0.04983
	1.5	0.04324	0.03735	0.32586	0.03379
	2.0	0.03664	0.02827	0.26592	0.02542
4	2.5	0.02597	0.02369	0.21462	0.02008
	3.0	0.02292	0.02086	0.20053	0.02004
	3.5	0.01876	0.01771	0.16132	0.01442
	4.0	0.01617	0.01513	0.13867	0.01239

6.2 Numerical Example

Consider an insurance company with a single employee dedicated to processing car-accident claim payments in a designated room. The room has a limited capacity, accommodating up to 10 individuals at any time. The time intervals (in minutes) between successive customer arrivals and the service times are recorded in Table 7. The Kolmogorov-Smirnov test statistic with the corresponding p-value for the first data in Table 7 is 0.0876 and 0.8765, respectively, which indicates that the data for time intervals between inputs follows the Exp(λ = 0.2788) distribution. Also, the Kolmogorov-Smirnov test statistic with the corresponding p-value for the second data in Table 7 is 0.1125 and 0.7896, respectively, which indicates that the data for time intervals between services follows the Exp(μ = 0.4494) distribution. Based on the data in Table 7 and the assumed parameter values $a_1 = 2$, $b_1 = 2.5$, $a_2 = 3$, $b_2 = 4$, $c_1 = 3$, $c_2 = 4.5$, $\alpha_1 = \alpha_2 = 1.5$, $\beta_1 = \beta_2 = 2.5$, $C_1 = 250$, $C_2 = 300$, and $C_3 = 350$, the different estimates of ρ were obtained. Using these estimates and applying the criteria described in Table 8, we computed the system cost (C) and the customer waiting time (W_q) for each estimator. The *EE*-Bayesian estimate of ρ produced the lowest values of both the system cost and average waiting time, indicating its superiority over the other

Table 4: The Estimated system cost for different estimates ρ and based Table 2

		Estimation			
K	λ	$\hat{W}_{q.B}$	$\hat{W}_{q.EB}$	$\hat{W}_{q.HB}$	$\hat{W}_{q.EEB}$
	0.5	0.15834	0.10506	0.99325	0.04691
	1.0	0.07213	0.05728	0.6031	0.05716
	1.5	0.04763	0.03981	0.43939	0.03711
	2.0	0.03504	0.03014	0.33861	0.02707
5	2.5	0.02958	0.02640	0.30718	0.02282
	3.0	0.02586	0.02399	0.28052	0.01997
	3.5	0.01879	0.01781	0.18513	0.01421
	4.0	0.01603	0.01524	0.16052	0.00981
	0.5	0.26981	0.16652	2.0447	0.22324
	1.0	0.10321	0.07696	1.1004	0.08082
	1.5	0.07185	0.06101	0.84960	0.05519
	2.0	0.04401	0.04178	0.057414	0.03742
6	2.5	0.04030	0.03699	0.49519	0.03069
	3.0	0.03267	0.02913	0.40267	0.02478
	3.5	0.02607	0.02479	0.31512	0.01972
	4.0	0.02393	0.02207	0.25972	0.01813
	0.5	0.31562	0.20369	2.5234	0.2626
	1.0	0.11644	0.08779	1.3304	0.09104
	1.5	0.07192	0.06157	0.94018	0.05532
	2.0	0.05367	0.04486	0.70843	0.04005
7	2.5	0.04063	0.03681	0.52214	0.03084
	3.0	0.03574	0.03204	0.50784	0.02707
	3.5	0.02857	0.02682	0.39758	0.02161
	4.0	0.02439	0.02269	0.33719	0.01833

Table 5: The Estimated system cost for different estimates ρ and based Table 1

		Estimation			
K	λ	$C(\hat{\rho}_B)$	$C(\hat{\rho}_{EB})$	$C(\hat{\rho}_{HB})$	$C(\hat{\rho}_{EEB})$
	0.5	151.99	140.44	319.28	124.33
	1.0	159.12	146.43	350.09	133.01
	1.5	154.55	143.92	408.26	136.08
	2.0	163.59	154.04	465.22	145.39
2	2.5	174.88	165.66	515.30	152.99
	3.0	182.77	174.44	556.96	158.91
	3.5	188.35	179.36	550.71	163.68
	4.0	193.24	183.92	617.64	166.32
	0.5	156.49	163.00	366.19	143.43
	1.0	157.22	143.93	383.78	130.09
	1.5	146.45	135.99	435.88	128.36
	2.0	149.76	144.44	489.69	140.79
3	2.5	152.71	155.08	491.92	133.20
	3.0	165.08	144.23	516.79	133.46
	3.5	162.69	156.22	588.73	140.68
	4.0	165.12	157.59	460.04	142.47
	0.5	148,14	135.93	335.38	120.61
	1.0	141.98	126.12	425.77	123.99
	1.5	142.56	132.04	458.64	125.33
	2.0	144.09	132.97	491.07	125.78
4	2.5	143.37	137.20	504.53	125.18
	3.0	148.24	140.95	527.86	129.68
	3.5	144.99	140.55	545.43	126.01
	4.0	144.17	139.12	552.45	125.03

Table 6: The Estimated system cost for different estimates ρ and based Table 2

		Estimation			
K	λ	$C(\hat{\rho}_B)$	$C(\hat{\rho}_{EB})$	$C(\hat{\rho}_{HB})$	$C(\hat{\rho}_{EEB})$
	0.5	155.17	149.34	419.39	125.96
	1.0	148.07	131.66	476.10	131.52
	1.5	147.44	134.54	509.19	129.83
	2.0	146.08	135.25	524.59	128.06
5	2.5	150.27	141.76	577.90	131.56
	3.0	154.14	148.31	638.21	134.96
	3.5	141.63	137.81	527.95	122.77
	4.0	139.90	136.31	530.64	121.22
	0.5	202.85	184.06	631.17	158.54
	1.0	176.89	156.20	668.86	152.37
	1.5	180.91	166.40	746.53	158.16
	2.0	172,38	158.94	704.03	150.29
6	2.5	174.89	167.39	760.69	152.23
	3.0	172.48	162.67	746.81	149.84
	3.5	166.34	161.10	707.82	144.33
	4.0	170.50	163.58	760.46	147.98
	0.5	219.25	199.53	705.95	175.34
	1.0	187.69	163.40	751.93	147.87
	1.5	180.67	166.88	771.91	158.10
	2.0	180.11	164.47	781.09	155.22
7	2.5	175.16	166.60	847.13	152.35
	3.0	179.37	170.34	836.72	156.42
	3.5	173.80	168.29	791.09	150.89
	4.0	171.67	165.49	779.89	148.56

Table 7: Corresponding data for the insurance company (Numerical example Section 6

	3.98	2.35	2.91	2.67	2.27	2.32	3.15	5.40
	4.71	3.46	4.84	6.52	2.12	3.08	4.34	4.11
Time intervals between inputs (First data)	4.17	2.68	3.82	3.50	3.32	3.97	4.95	5.00
	3.68	4.38	2.52	1.77	2.37	3.25		
	3.44	0.73	1.50	0.63	0.11	0.64	3.86	2.62
	0.17	1.51	2.80	1.97	5.17	0.56	1.90	2.15
Time intervals between services (Second data)	0.49	10.00	3.15	4.47	2.93	0.34	0.94	0.18
	1.86	4.25	0.10	2.94	3.15	2.19		

Table 8: The Estimated system cost for different estimates ρ and based Table 7

	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$	$\hat{\rho}_{E^2B}$
Estimates	2.2936	2.4821	2.3505	2.6572
L	8.2294	8,3264	8.2615	8.3971
L_q	7,2298	7.3266	7.2617	7.3972
$\bar{\lambda}$	2.4848	2. 2963	2.4247	2.1312
$C(\hat{\rho})$	2613.9	2653.3	2626.8	2683.1
W_q	5.5613	5.6358	5.5859	5.6902

estimators in this example.

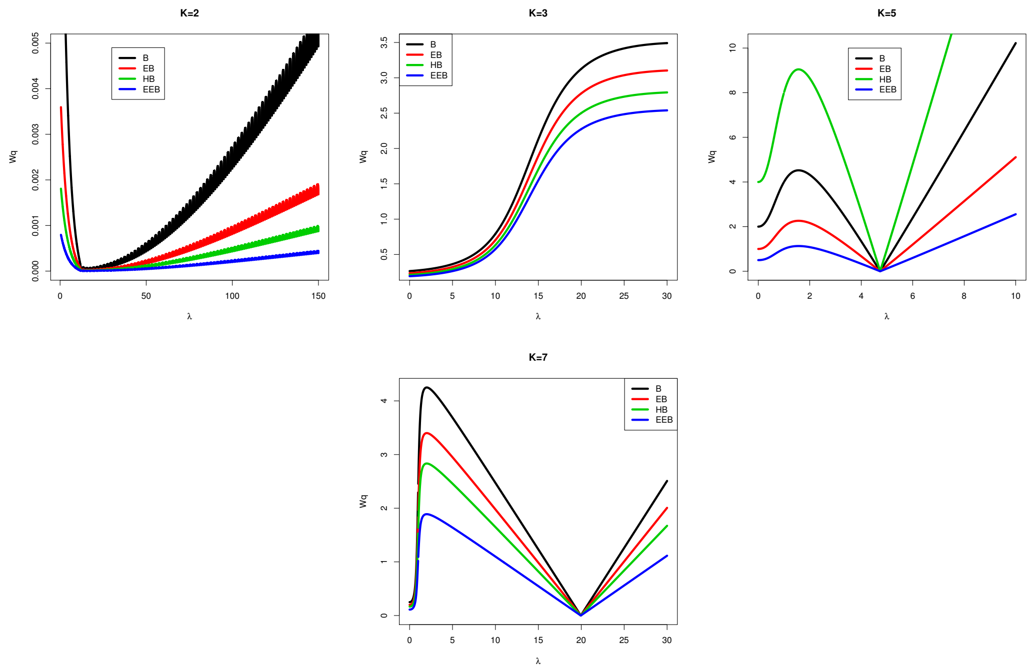

Figure 1: Chart of average waiting time in customer queue (W_q) in terms of λ for different K.

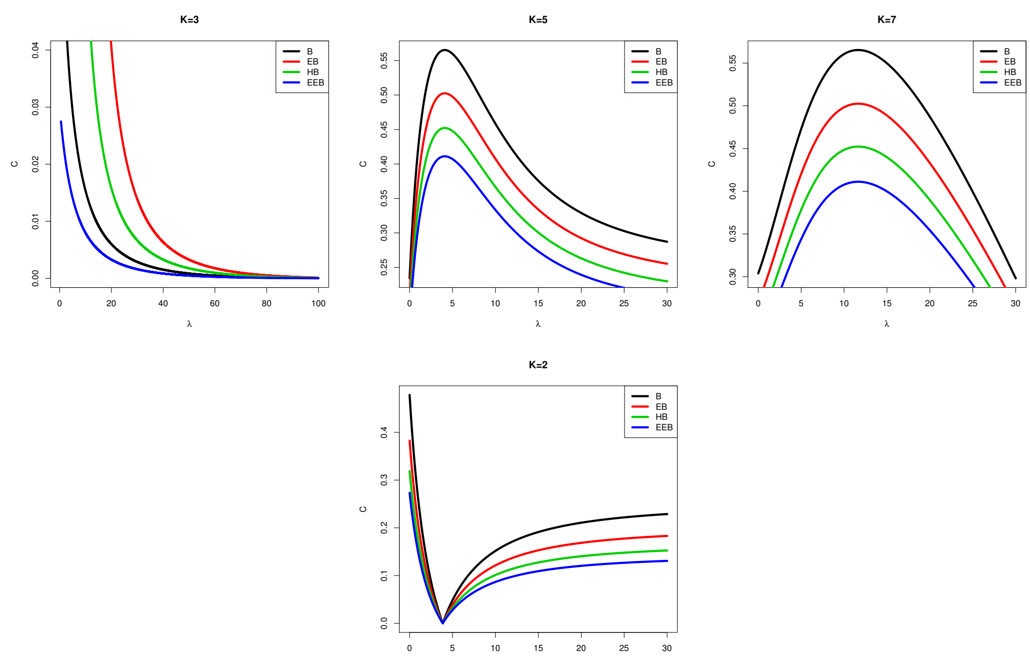

Figure 2: Chart of system cost function (C) in terms of λ for different K.

7 Conclusion

In this paper, we considered the $M/M/1/k$ queueing model, where customer interarrival times are exponentially distributed with parameter λ and service times are exponentially distributed with parameter μ. The traffic intensity parameter $\rho = \frac{\lambda}{\mu}$ plays a crucial role in evaluating queue performance. We estimated ρ using four methods Bayesian, *E*-Bayesian, hierarchical Bayesian, and the proposed *EE*-Bayesian estimator and compared their performance using Monte Carlo simulation and a numerical example. Performance metrics included average customer waiting time and system cost. The numerical and simulation results indicate that the *EE*-Bayesian estimator minimizes both the average waiting time and the system cost while maintaining a desirable level of service; therefore, it was selected as the preferred estimator in this study.

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