JIRSS (2026)
Vol. 25, No. 1, pp 17-36
DOI: 10.22034/jirss.2026.2082667.1175

Testing Of Symmetry Based On Cumulative Past And Residual Extropy Of Record Values

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Received: 2025-12-31, Accepted: 2026-06-16, Published online: 2026-06-26

Abstract. This paper proposes new nonparametric tests for symmetry based on cumulative past extropy and cumulative residual extropy of record values, motivated by a recent characterization of symmetric distributions by Gupta and Chaudhary (2024). The proposed estimators are inspired by the methodology introduced by Vasicek (1976). The proposed tests do not require estimation of the centre of symmetry, making them robust and easy to implement. Their asymptotic properties and consistency are established, and critical values are obtained via Monte Carlo simulations. Power is evaluated under various asymmetric alternatives. Results show that the proposed tests perform competitively and often outperform existing symmetry tests while maintaining the nominal significance level. Application of the test to six real-world datasets confirms its effectiveness in detecting symmetric and asymmetric behavior through significant p-values.

Keywords. Cumulative past extropy; Cumulative residual extropy; Nonparametric test; Record values; Symmetry testing.

MSC: 62G30, 62E10, 62G10, 62B10.

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1 Introduction

Entropy, introduced by Shannon (1948), measures the average level of uncertainty associated with the outcomes of a random experiment. The Shannon entropy of a continuous random variable X is defined as

$$H(X) = -\int_{-\infty}^{\infty} f(x) \ln\left(f(x)\right) dx = E\left[-\ln f(X)\right]. \tag{1.1}$$

Vasicek (1976) introduced an estimator of entropy per observation and established a goodness-of-fit test for normality using sample entropy. His nonparametric entropy estimator, based on order statistics and spacing methods, provides a robust framework that avoids strong distributional assumptions. The proposed estimators in this study are inspired by this pioneering work. In a similar spirit, our methodology utilizes the structure of record values to construct estimators associated with cumulative past extropy and cumulative residual extropy. By adapting spacing-based ideas to record-based measures of extropy, we extend Vasicek's approach to a new context, thereby developing estimators suitable for symmetry testing without requiring estimation of the centre of symmetry.

The concept of k-records was introduced by Dziubdziela and Kopocinski (1976); see also Ahsanullah (1995) and Arnold et al. (1998). The probability density functions of the nth upper k-record value $U_{n,k}$ and the nth lower k-record value $L_{n,k}$ are given, respectively, by (Arnold et al., 2008; Ahsanullah, 2004)

$$f_{U_{n,k}}(x) = \frac{k^n}{(n-1)!}\left[-\log \bar{F}(x)\right]^{n-1} \bar{F}(x)^{k-1} f(x), \quad x \in \mathbb{R},$$

$$\text{and} \quad f_{L_{n,k}}(x) = \frac{k^n}{(n-1)!}\left[-\log F(x)\right]^{n-1} F(x)^{k-1} f(x), \quad x \in \mathbb{R}.$$

The corresponding cdf of $U_{n,k}$ and $L_{n,k}$ are

$$F_{U_{n,k}}(u) = 1 - \bar{F}^k(u) \sum_{i=0}^{n-1} \frac{\left[-k \log \bar{F}(u)\right]^i}{i!},$$

$$\text{and} \quad F_{L_{n,k}}(u) = F^k(u) \sum_{i=0}^{n-1} \frac{\left[-k \log F(u)\right]^i}{i!}.$$

McWilliams (1990) proposed a test for symmetry based on run statistics and studied its performance under various alternatives. Gibbons and Chakraborti (1992) provided comprehensive discussions on nonparametric methods including symmetry testing. Tajuddin (1994) developed procedures for testing symmetry in continuous distributions. Modarres and Gastwirth (1996) proposed a test for symmetry when the central value is known. Park (1999) derived the sample entropy of order statistics and proposed a goodness-of-fit test for normality based on this measure. Baklizi (2003, 2007, 2008) contributed several tests for symmetry under different settings. Cheng and Balakrishnan (2004) developed a symmetry test based on the joint information contained in the absolute ranks and signs of sample observations. Corzo and Babativa (2013) presented a modified version of the test proposed by Modarres and Gastwirth (1996) for testing symmetry when the central value is known.

Lad et al. (2015) introduced the complement dual of Shannon entropy, known as extropy. The extropy of a continuous random variable X is defined as

$$J(X) = -\frac{1}{2}\int_{-\infty}^{\infty} f^2(x)\,dx = -\frac{1}{2}E\left[f(X)\right]. \tag{1.2}$$

Jose and Sathar (2022) introduced the cumulative residual extropy of a continuous random variable X, defined as

$$\xi J(X) = -\frac{1}{2}\int_{S_X} \bar{F}^2(x)\,dx, \tag{1.3}$$

while the cumulative past extropy of X is defined by

$$\bar{\xi} J(X) = -\frac{1}{2}\int_{S_X} F^2(x)\,dx. \tag{1.4}$$

Xiong et al. (2021) explored the use of fractional Deng extropy in classification tasks. Xiong et al. (2021) established a characterization of symmetric distributions using the extropy of the kth upper and lower record values and proposed a corresponding test of symmetry. Jose and Sathar (2022) derived a characterization based on the extropy of the nth upper k-record value and the nth lower k-record value and developed a test for symmetry using this result. Tahmasebi et al. (2022) employed extropy-based measures in compressive sensing applications. Toomaj et al. (2023) showed that extropy admits a closed-form expression for finite mixture distributions, whereas such expressions are often unavailable for entropy and variance. Tahmasebi and Toomaj (2020) analyzed stock market data from OECD countries using a generalization known as negative cumulative extropy.

Motivated by the works of Xiong et al. (2021) and Jose and Sathar (2022), and by the characterization developed by Gupta and Chaudhary (2024), the present study proposes a test of symmetry based on cumulative past and cumulative residual extropy of k-record values.

Let X be a random variable with pdf f and cumulative distribution function (cdf) F. A probability distribution is said to be symmetric about a point k if and only if there exists a finite real number k such that $f(k+x) = f(k-x)$ for all $x \in \mathbb{R}$, or equivalently, $F(k-x) + F(k+x) = 1$ for all $x \in \mathbb{R}$. The point k is referred to as the centre of symmetry and coincides with the mean of the distribution, provided it exists. In this paper, we consider the hypotheses

$$\begin{aligned} & H_0 : f(k+x) = f(k-x), \quad \text{for all } x \in \mathbb{R}, \\ \text{against} \quad & H_1 : f(k+x) \neq f(k-x), \quad \text{for some } x \in \mathbb{R}. \end{aligned}$$

The remainder of the paper is organized as follows. Section 2 derives the proposed test statistic using the characterization result and presents an estimator of the test statistic. Section 3 discusses the critical values, while Section 4 examines the size and power of the test. Section 5 provides a power comparison with existing tests. Applications to six real-life data sets are presented in Section 6, and concluding remarks along with directions for future research are given in Section 7.

2 Test of symmetry

Let $X_1, X_2, \ldots, X_N$ be a random sample of continuous random variables from a population X with cdf F and pdf f. Let $\mathbb{C}$ denote the class of all continuous pdfs f, with cdf F, such that $f\left(F^{-1}(1-u)\right) \geq (\leq) f\left(F^{-1}(u)\right)$ for all $u \in \left(0, \frac{1}{2}\right)$. A random variable X is symmetric if $f(F^{-1}(u)) = f(F^{-1}(1-u))$ for almost all $u \in \left(0, \frac{1}{2}\right)$ (see Fashandi and Ahmadi (2012)). The class $\mathbb{C}$ is non-empty and includes, but is not limited to, the power, Pareto, exponential, uniform, and standard normal distributions (Ahmadi, 2021; Gupta and Chaudhary, 2024)).

Gupta and Chaudhary (2024) showed that for $F \in \mathbb{C}$, a random variable X has a symmetric distribution if and only if, for a fixed $k \geq 1$,

$$\bar{\xi}J(L_{n,k}) = \xi J(U_{n,k}), \quad \text{for all } n \geq 1.$$

This characterization motivates the construction of a test statistic for symmetry. Accordingly, we define

$$\Delta_{n,k} = \xi J(U_{n,k}) - \bar{\xi}J(L_{n,k}).$$

Small or large values of $\Delta_{n,k}$ indicate departure from symmetry; hence, we propose a test for symmetry based on a sample estimator of $\Delta_{n,k}$.

Clearly, $\Delta_{n,k} = 0$ if and only if X follows a symmetric distribution. Therefore, for an independent and identically distributed (i.i.d.) sample of size N, the empirical counterpart $\widehat{\Delta}_{n,k}$ can be used to assess whether the underlying distribution is symmetric. To derive $\Delta_{n,k}$, we first obtain expressions for the cumulative past extropy of the nth lower k-record value and the cumulative residual extropy of the nth upper k-record value.

The cumulative past extropy of the nth lower k-record value is given by

$$\begin{aligned}
\bar{\xi}J(L_{n,k}) &= -\frac{1}{2}\int_{-\infty}^{\infty} F^2_{L_{n,k}}(x)\,dx \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} \left(F^k(x)\sum_{j=0}^{n-1}\frac{(-k\log F(x))^j}{j!}\right)^2 dx \\
&= -\frac{1}{2}\int_{0}^{1} u^{2k}\left(\sum_{j=0}^{n-1}\frac{(-k\log u)^j}{j!}\right)^2 \frac{du}{f(F^{-1}(u))} \\
&= -\frac{1}{2}\int_{0}^{1} u^{2k}\left(\sum_{j=0}^{n-1}\frac{(-k\log u)^j}{j!}\right)^2 \left(\frac{d}{du}F^{-1}(u)\right) du.
\end{aligned}$$

An estimator of $\bar{\xi}J(L_{n,k})$ is

$$\widehat{\bar{\xi}J(L_{n,k})} = -\frac{1}{2N}\sum_{i=1}^{N}\left(\frac{i}{N+1}\right)^{2k}\left(\sum_{j=0}^{n-1}\frac{\left[-k\log\left(\frac{i}{N+1}\right)\right]^j}{j!}\right)^2 \times \left(\frac{X_{i+m:N}-X_{i-m:N}}{2m/N}\right).$$

Similarly, the cumulative residual extropy of the nth upper k-record value is

$$\begin{aligned}\xi J(U_{n,k}) &= -\frac{1}{2}\int_{-\infty}^{\infty}\bar{F}^2_{U_{n,k}}(x)\,dx \\ &= -\frac{1}{2}\int_0^1 (1-u)^{2k}\left(\sum_{j=0}^{n-1}\frac{(-k\log(1-u))^j}{j!}\right)^2 \frac{d}{du}F^{-1}(u)\,du.\end{aligned}$$

An estimator of $\xi J(U_{n,k})$ is

$$\widehat{\xi J(U_{n,k})} = -\frac{1}{2N}\sum_{i=1}^{N}\left(1-\frac{i}{N+1}\right)^{2k}\left(\sum_{j=0}^{n-1}\frac{\left[-k\log\left(1-\frac{i}{N+1}\right)\right]^j}{j!}\right)^2 \times \left(\frac{X_{i+m:N}-X_{i-m:N}}{2m/N}\right),$$

where N is sample size of data and m is window size (see Vasicek (1976), Ahmadi (2021), Xiong et al. (2021), Jose and Sathar (2022)). The window size m in the entropy estimator of Vasicek (1976) is a smoothing parameter based on spacings of order statistics. For a sample of size n, the estimator uses differences of the form $X_{(i+m)} - X_{(i-m)}$, where $X_{(i)}$ denotes the i-th order statistic. The parameter m controls the bias–variance trade-off: small values of m lead to less smoothing and higher variance, whereas larger values increase smoothing but introduce bias. Thus, the proposed test statistic is

$$\Delta_{n,k} = \xi J(U_{n,k}) - \bar{\xi}J(L_{n,k}),$$

with estimator

$$\widehat{\Delta}_{n,k} = \widehat{\xi J(U_{n,k})} - \widehat{\bar{\xi}J(L_{n,k})}.$$

The theorem is derived using the nth upper and lower k-record values and holds for all $n \geq 1$ and $k \geq 1$. Hence, any specific values of n and k may be chosen for implementation. For computational simplicity, we select $n = 2$ and $k = 2$, following Jose and Sathar (2022) and Xiong et al. (2021). Larger values could also be used, particularly in simulations; however, $n = 2$ and $k = 2$ provide a convenient and commonly adopted choice. We may consider any particular value of n and k for computation. Consider the statistic $\widehat{\Delta}_{2,2}$, given by

$$\begin{aligned}\widehat{\Delta}_{2,2} = -\frac{1}{2N}\sum_{i=1}^{N}\Bigg[&\left(1-\frac{i}{N+1}\right)^4\left(1-2\log\left(1-\frac{i}{N+1}\right)\right)^2 \\ &-\left(\frac{i}{N+1}\right)^4\left(1-2\log\left(\frac{i}{N+1}\right)\right)^2\Bigg]\frac{X_{i+m:N}-X_{i-m:N}}{2m/N}.\end{aligned}$$

The same procedure applies for other choices of n and k. The following theorem establishes the consistency of $\widehat{\Delta}_{2,2}$.

Theorem 2.1. *Let $X_1, X_2, \ldots, X_N$ be a random sample from a population with pdf f and cdf F, and assume that the variance of X is finite. Then*

$$\widehat{\Delta}_{2,2} \xrightarrow{P} \Delta_{2,2} \quad \text{as } N \to \infty,\ m \to \infty,\ \text{and } \frac{m}{N} \to 0.$$

Proof. Following the arguments of Theorem 1 in Vasicek (1976), we have $\widehat{\xi J(U_{2,2})} \xrightarrow{P} \xi J(U_{2,2})$ and $\widehat{\bar{\xi} J(L_{2,2})} \xrightarrow{P} \bar{\xi} J(L_{2,2})$. Hence, $\widehat{\Delta}_{2,2} \xrightarrow{P} \Delta_{2,2}$. □

Note that the test statistics proposed by Jose and Sathar (2022), Park (1999), and Xiong et al. (2021) are also consistent under the framework of Vasicek (1976).

Theorem 2.2. *Let $X_1, X_2, \ldots, X_N$ be i.i.d. random variables and define $Y_i = aX_i + b$, where $a > 0$ and $b \in \mathbb{R}$. Let $\widehat{\Delta}^X_{2,2}$ and $\widehat{\Delta}^Y_{2,2}$ denote the estimators based on $\{X_i\}$ and $\{Y_i\}$, respectively. Then*

(i) $E(\widehat{\Delta}^Y_{2,2}) = a\, E(\widehat{\Delta}^X_{2,2})$,

(ii) $\mathrm{Var}(\widehat{\Delta}^Y_{2,2}) = a^2\, \mathrm{Var}(\widehat{\Delta}^X_{2,2})$,

(iii) $\mathrm{MSE}(\widehat{\Delta}^Y_{2,2}) = a^2\, \mathrm{MSE}(\widehat{\Delta}^X_{2,2})$.

Proof. Since $Y_{i+m:N} - Y_{i-m:N} = a(X_{i+m:N} - X_{i-m:N})$, it follows directly that $\widehat{\Delta}^Y_{2,2} = a\, \widehat{\Delta}^X_{2,2}$. The stated results then follow from standard properties of expectation, variance, and mean squared error. □

Table 1: Critical values of $|\widehat{\Delta}_{2,2}|$ at significance level $\alpha = 0.10$.

$m\backslash N$	5	10	20	30	40	50	100
2	0.3151	0.5250	0.5743	0.5777	0.5738	0.5590	0.5197
3	–	0.4141	0.5179	0.5092	0.5176	0.5071	0.4824
4	–	0.3167	0.4581	0.4783	0.4818	0.4762	0.4572
5	–	–	0.4183	0.4501	0.4535	0.4588	0.4401
6	–	–	0.3778	0.4256	0.4436	0.4360	0.4187
7	–	–	0.3330	0.3907	0.4131	0.4148	0.4149
8	–	–	0.2948	0.3652	0.3901	0.4076	0.3955
9	–	–	0.2584	0.3464	0.3747	0.3929	0.3874
10	–	–	–	0.3190	0.3621	0.3735	0.3862
11	–	–	–	0.2989	0.3409	0.3681	0.3676
12	–	–	–	0.2808	0.3291	0.3483	0.3682
13	–	–	–	0.2542	0.3099	0.3428	0.3672
14	–	–	–	0.2324	0.2991	0.3311	0.3631
15	–	–	–	–	0.2814	0.3150	0.3528
16	–	–	–	–	0.2667	0.3075	0.3442
17	–	–	–	–	0.2497	0.2906	0.3373
18	–	–	–	–	0.2379	0.2868	0.3358
19	–	–	–	–	0.2251	0.2738	0.3375
20	–	–	–	–	–	0.2589	0.3329
21	–	–	–	–	–	0.2500	0.3205
22	–	–	–	–	–	0.2395	0.3163
23	–	–	–	–	–	0.2305	0.3126
24	–	–	–	–	–	0.2212	0.3097
25	–	–	–	–	–	–	0.3111
26	–	–	–	–	–	–	0.2977
27	–	–	–	–	–	–	0.2957
28	–	–	–	–	–	–	0.2972
29	–	–	–	–	–	–	0.2896
30	–	–	–	–	–	–	0.2827
40	–	–	–	–	–	–	0.2409

3 Critical values

In order to provide a more comprehensive comparison, we also considered several additional asymmetric alternatives, including distributions from the family of generalized lambda distributions. For each test, the empirical power was computed using 10,000 Monte Carlo replications. The critical values were obtained under the null distribution of symmetry. For the proposed test, the rejection rule is given by

$$\left|\widehat{\Delta}_{2,2}\right| > c_\alpha,$$

where c_α is the empirical critical value corresponding to the significance level α. The critical values for $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$ were obtained as the 0.90, 0.95, and 0.99 quantiles, respectively, of the simulated values of $\left|\widehat{\Delta}_{2,2}\right|$ under the null distribution.

For the computation of critical values, it is essential to examine the distribution of the newly proposed test statistic. However, deriving the asymptotic distribution of $\widehat{\Delta}_{2,2}$

Table 2: Critical values of $|\widehat{\Delta}_{2,2}|$ at significance level $\alpha = 0.05$.

$m\backslash N$	5	10	20	30	40	50	100
2	0.3637	0.6093	0.6673	0.6703	0.6658	0.6474	0.5969
3	–	0.4787	0.5833	0.5936	0.6011	0.5857	0.5611
4	–	0.3641	0.5333	0.5539	0.5553	0.5387	0.5284
5	–	–	0.4776	0.5216	0.5287	0.5305	0.5054
6	–	–	0.4362	0.4872	0.5074	0.4979	0.4794
7	–	–	0.3848	0.4536	0.4785	0.4718	0.4750
8	–	–	0.3460	0.4207	0.4543	0.4647	0.4573
9	–	–	0.2951	0.4044	0.4271	0.4491	0.4488
10	–	–	–	0.3642	0.4188	0.4235	0.4405
11	–	–	–	0.3440	0.3953	0.4259	0.4242
12	–	–	–	0.3254	0.3775	0.4053	0.4249
13	–	–	–	0.2948	0.3561	0.3947	0.4251
14	–	–	–	0.2751	0.3515	0.3821	0.4203
15	–	–	–	–	0.3239	0.3626	0.4157
16	–	–	–	–	0.3057	0.3521	0.3967
17	–	–	–	–	0.2868	0.3433	0.3901
18	–	–	–	–	0.2769	0.3356	0.3820
19	–	–	–	–	0.2583	0.3182	0.3883
20	–	–	–	–	–	0.3011	0.3779
21	–	–	–	–	–	0.2909	0.3646
22	–	–	–	–	–	0.2777	0.3705
23	–	–	–	–	–	0.2697	0.3607
24	–	–	–	–	–	0.2533	0.3575
25	–	–	–	–	–	–	0.3592
26	–	–	–	–	–	–	0.3446
27	–	–	–	–	–	–	0.3358
28	–	–	–	–	–	–	0.3427
29	–	–	–	–	–	–	0.3355
30	–	–	–	–	–	–	0.3258
40	–	–	–	–	–	–	0.2797

as $N \to \infty$ is analytically challenging, since the window size m depends on the sample size N.

To obtain the empirical critical values of $\widehat{\Delta}_{2,2}$, 10,000 random samples of size N were generated from the standard normal distribution. For each sample, the corresponding value of $\left|\widehat{\Delta}_{2,2}\right|$ was computed. From these 10,000 values, the $(1-\alpha)$th empirical quantile was taken as the critical value of the test at significance level α.

The empirical critical values of $\widehat{\Delta}_{2,2}$, based on 10,000 samples of different sizes generated from the standard normal distribution, are reported in Tables 1, 2, and 3 for significance levels $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$, respectively. The results are presented for sample sizes $N = 5, 10, 20, 30, 40, 50$, and 100, with window sizes m ranging from 2 to 40. The next section presents a simulation study to evaluate the power performance of the proposed test statistic.

Table 3: Critical values of $|\widehat{\Delta}_{2,2}|$ at significance level $\alpha = 0.01$.

$m\backslash N$	5	10	20	30	40	50	100
2	0.4569	0.7690	0.8663	0.8474	0.8501	0.8292	0.7548
3		0.6042	0.7349	0.7590	0.7570	0.7447	0.6997
4		0.4739	0.6810	0.7185	0.6946	0.6843	0.6553
5			0.6158	0.6570	0.6695	0.6629	0.6488
6			0.5636	0.6427	0.6256	0.6502	0.6257
7			0.4903	0.5819	0.6082	0.5989	0.6017
8			0.4428	0.5273	0.5784	0.6020	0.5835
9			0.3757	0.5063	0.5343	0.5828	0.5844
10				0.4594	0.5488	0.5394	0.5560
11				0.4536	0.4971	0.5462	0.5438
12				0.4146	0.4845	0.5288	0.5504
13				0.3862	0.4462	0.5049	0.5480
14				0.3467	0.4364	0.4878	0.5433
15					0.4213	0.4523	0.5259
16					0.4091	0.4747	0.5016
17					0.3727	0.4471	0.5187
18					0.3572	0.4239	0.4955
19					0.3285	0.4134	0.4912
20						0.3869	0.4794
21						0.3777	0.4761
22						0.3587	0.4931
23						0.3505	0.4606
24						0.3241	0.4648
25							0.4524
26							0.4469
27							0.4388
28							0.4405
29							0.4270
30							0.4086
40							0.3650

4 Power and size

Deriving the exact distribution of the statistic $\widehat{\Delta}_{2,2}$ is analytically intractable, as it depends on the window size m, which itself varies with the sample size N. Therefore, critical values of the test statistic $|\widehat{\Delta}_{2,2}|$ are obtained using Monte Carlo simulation with 10,000 replications. Tables 1, 2, and 3 report the critical values for various sample sizes at significance levels $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$, respectively. A similar simulation-based approach has been adopted in Xiong et al. (2021).

To compute the size of the test, samples of size N are generated from the standard normal distribution, and the proportion of rejections over 10,000 replications is recorded. The power of the test is defined as this empirical rejection probability under alternative distributions. Tables 4, 5, and 6 present the power of the proposed test when the alternative distribution is the chi-square distribution with one degree of freedom, $\chi^2(1)$, for $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$, respectively. Since the $\chi^2(1)$ distribution is asymmetric, the proposed test effectively detects departures from symmetry.

From Tables 4, 5, and 6, it is observed that the power increases with the sample size

Table 4: Powers of the statistic $\widehat{\Delta}_{2,2}$ against the alternative $\chi^2_{(1)}$ at significance level $\alpha = 0.10$.

$m\backslash N$	5	10	20	30	40	50	100
2	0.3168	0.6397	0.9151	0.9822	0.9966	0.9998	1.0000
3	–	0.6285	0.9186	0.9870	0.9971	0.9996	1.0000
4	–	0.6127	0.9220	0.9885	0.9976	0.9999	1.0000
5	–	–	0.9166	0.9856	0.9990	0.9999	1.0000
6	–	–	0.9171	0.9848	0.9970	0.9996	1.0000
7	–	–	0.9125	0.9851	0.9979	0.9998	1.0000
8	–	–	0.9106	0.9850	0.9984	0.9996	1.0000
9	–	–	0.8980	0.9841	0.9981	0.9996	1.0000
10	–	–	–	0.9846	0.9970	0.9998	1.0000
11	–	–	–	0.9836	0.9978	0.9993	1.0000
12	–	–	–	0.9804	0.9984	1.0000	1.0000
13	–	–	–	0.9763	0.9972	0.9995	1.0000
14	–	–	–	0.9776	0.9969	0.9996	1.0000
15	–	–	–	–	0.9963	0.9997	1.0000
16	–	–	–	–	0.9957	0.9994	1.0000
17	–	–	–	–	0.9960	0.9996	1.0000
18	–	–	–	–	0.9952	0.9991	1.0000
19	–	–	–	–	0.9948	0.9992	1.0000
20	–	–	–	–	–	0.9994	1.0000
21	–	–	–	–	–	0.9987	1.0000
22	–	–	–	–	–	0.9992	1.0000
23	–	–	–	–	–	0.9987	1.0000
24	–	–	–	–	–	0.9982	1.0000
25	–	–	–	–	–	–	1.0000
26	–	–	–	–	–	–	1.0000
27	–	–	–	–	–	–	1.0000
28	–	–	–	–	–	–	1.0000
29	–	–	–	–	–	–	1.0000
30	–	–	–	–	–	–	1.0000
40	–	–	–	–	–	–	1.0000

N. In particular, when $N = 100$, the power equals 1.000 for all considered values of m, indicating excellent performance of the test for large samples.

Table 7 reports the power of the test against several alternative distributions, namely $\chi^2(1)$, $\chi^2(2)$, $\chi^2(3)$, and the standard normal distribution $N(0,1)$. Since $N(0,1)$ is symmetric, the power under this distribution corresponds to the size of the test. Ideally, the power of $\widehat{\Delta}_{2,2}$ should be close to the nominal significance level when the underlying distribution is symmetric. Tables 7 and 8 confirm this behavior, as the empirical powers are approximately equal to 0.05 when the data follow the standard normal distribution. This demonstrates that the proposed test maintains its nominal level.

Finally, the simulation results indicate that smaller values of the window size m generally provide better performance across all sample sizes. Based on these findings, Table 9 presents recommended values of the window size m for different sample sizes.

Table 5: Powers of the statistic $\widehat{\Delta}_{2,2}$ against the alternative $\chi^2_{(1)}$ at significance level $\alpha = 0.05$.

$m \backslash N$	5	10	20	30	40	50	100
2	0.2649	0.5644	0.8759	0.9685	0.9924	0.9984	1.0000
3	–	0.5624	0.8859	0.9703	0.9922	0.9991	1.0000
4	–	0.5534	0.8769	0.9749	0.9957	0.9997	1.0000
5	–	–	0.8794	0.9694	0.9952	0.9990	1.0000
6	–	–	0.8691	0.9784	0.9941	0.9991	1.0000
7	–	–	0.8756	0.9756	0.9945	0.9995	1.0000
8	–	–	0.8536	0.9759	0.9952	0.9989	1.0000
9	–	–	0.8556	0.9680	0.9949	0.9993	1.0000
10	–	–	–	0.9679	0.9936	0.9991	1.0000
11	–	–	–	0.9708	0.9937	0.9986	1.0000
12	–	–	–	0.9609	0.9928	0.9991	1.0000
13	–	–	–	0.9589	0.9945	0.9991	1.0000
14	–	–	–	0.9586	0.9913	0.9989	1.0000
15	–	–	–	–	0.9924	0.9991	1.0000
16	–	–	–	–	0.9902	0.9988	1.0000
17	–	–	–	–	0.9902	0.9978	1.0000
18	–	–	–	–	0.9903	0.9972	1.0000
19	–	–	–	–	0.9870	0.9984	1.0000
20	–	–	–	–	–	0.9987	1.0000
21	–	–	–	–	–	0.9977	1.0000
22	–	–	–	–	–	0.9971	1.0000
23	–	–	–	–	–	0.9964	1.0000
24	–	–	–	–	–	0.9958	1.0000
25	–	–	–	–	–	–	1.0000
26	–	–	–	–	–	–	1.0000
27	–	–	–	–	–	–	1.0000
28	–	–	–	–	–	–	1.0000
29	–	–	–	–	–	–	1.0000
30	–	–	–	–	–	–	1.0000
40	–	–	–	–	–	–	1.0000

Table 6: Powers of the statistic $\widehat{\Delta}_{2,2}$ against the alternative $\chi^2_{(1)}$ at significance level $\alpha = 0.01$.

$m \backslash N$	5	10	20	30	40	50	100
2	0.1924	0.4437	0.8671	0.9162	0.9714	0.9978	1.0000
3	–	0.4483	0.7861	0.9273	0.9760	0.9981	1.0000
4	–	0.4114	0.7799	0.9215	0.9825	0.9989	1.0000
5	–	–	0.7713	0.9318	0.9789	0.9991	1.0000
6	–	–	0.7566	0.9186	0.9830	0.9991	1.0000
7	–	–	0.7611	0.9264	0.9818	0.9994	1.0000
8	–	–	0.7496	0.9398	0.9800	0.9991	1.0000
9	–	–	0.7502	0.9302	0.9802	0.9991	1.0000
10	–	–	–	0.9283	0.9728	0.9990	1.0000
11	–	–	–	0.9104	0.9813	0.9992	1.0000
12	–	–	–	0.9127	0.9744	0.9989	1.0000
13	–	–	–	0.8990	0.9795	0.9933	1.0000
14	–	–	–	0.8929	0.9738	0.9918	1.0000
15	–	–	–	–	0.9721	0.9942	1.0000
16	–	–	–	–	0.9611	0.9900	1.0000
17	–	–	–	–	0.9649	0.9904	1.0000
18	–	–	–	–	0.9565	0.9897	1.0000
19	–	–	–	–	0.9588	0.9887	1.0000
20	–	–	–	–	–	0.9979	1.0000
21	–	–	–	–	–	0.9984	1.0000
22	–	–	–	–	–	0.9972	1.0000
23	–	–	–	–	–	0.9951	1.0000
24	–	–	–	–	–	0.9974	1.0000
25	–	–	–	–	–	–	1.0000
26	–	–	–	–	–	–	1.0000
27	–	–	–	–	–	–	1.0000
28	–	–	–	–	–	–	1.0000
29	–	–	–	–	–	–	1.0000
30	–	–	–	–	–	–	1.0000
40	–	–	–	–	–	–	1.0000

Table 7: Power of $\widehat{\Delta}_{2,2}$ at significance level $\alpha = 0.05$.

m	$\chi^2_{(1)}$	$\chi^2_{(2)}$	$\chi^2_{(3)}$	$N(0,1)$
		$N = 20$		
2	0.8759	0.8861	0.8627	0.0518
3	0.8859	0.8976	0.8728	0.0481
4	0.8769	0.8956	0.8762	0.0519
5	0.8794	0.8943	0.8715	0.0478
6	0.8691	0.8941	0.8732	0.0502
7	0.8756	0.8889	0.8716	0.0489
8	0.8536	0.8814	0.8643	0.0491
9	0.8556	0.8757	0.8555	0.0487
		$N = 50$		
2	0.9984	0.9981	0.9955	0.0507
4	0.9997	0.9991	0.9969	0.0516
7	0.9995	0.9987	0.9964	0.0521
9	0.9993	0.9986	0.9960	0.0494
15	0.9991	0.9985	0.9961	0.0466
17	0.9978	0.9986	0.9955	0.0502
20	0.9987	0.9979	0.9944	0.0488
22	0.9971	0.9976	0.9946	0.0497
		$N = 100$		
2	1.0000	1.0000	1.0000	0.0496
4	1.0000	1.0000	1.0000	0.0512
5	1.0000	1.0000	1.0000	0.0494
7	1.0000	1.0000	1.0000	0.0515
10	1.0000	1.0000	1.0000	0.0502
15	1.0000	1.0000	1.0000	0.0473
20	1.0000	1.0000	1.0000	0.0524
30	1.0000	1.0000	1.0000	0.0530
40	1.0000	1.0000	1.0000	0.0520

Table 8: Size of $\widehat{\Delta}_{2,2}$ at significance level $\alpha = 0.05$

N	m	$N(0,1)$	N	m	$N(0,1)$	N	m	$N(0,1)$
20	2	0.0518	50	2	0.0507	100	2	0.0496
	3	0.0481		3	0.0467		3	0.0494
	4	0.0519		5	0.0491		5	0.0494
	5	0.0478		8	0.0514		8	0.0484
	6	0.0502		10	0.0513		10	0.0502
	7	0.0489		15	0.0466		15	0.0473
	8	0.0491		20	0.0488		20	0.0524
	9	0.0487		24	0.0503		30	0.0530

Table 9: Recommended window size m

Sample size N	Window size m
≤ 10	2
11–50	6
51–100	8
≥ 101	10

5 Power Comparison

We recall that Jose and Sathar (2022), Park (1999), and Xiong et al. (2021) also extended Vasicek (1976) idea to propose tests for symmetry. Specifically, their approaches are based on the entropy of order statistics, the extropy of the kth upper and lower record values, and the extropy of the nth upper and lower k-record values, respectively. Let T_1 and T_2 denote the test statistics proposed by Xiong et al. (2021) and Jose and Sathar (2022), respectively. The powers of T_1 and T_2 shown in Table 10 are taken from the respective papers.

Park (1999) demonstrated that his test outperformed others, while Xiong et al. (2021) compared their test with Park's. For alternative distributions such as $\chi^2_{(1)}$, $\chi^2_{(2)}$, or $N(0,1)$, the tests of Park (1999) and Xiong et al. (2021) show similar performance for moderate and large sample sizes.

Table 10 presents a comparison of the power of our proposed test with those of Jose and Sathar (2022) and Xiong et al. (2021). Since $\chi^2_{(2)}$ is asymmetric, a higher power indicates a better test. Overall, our proposed test demonstrates superior power compared to the tests of Jose and Sathar (2022) and Xiong et al. (2021), except in a few isolated cases. For the symmetric distribution $N(0,1)$, the power corresponds to the significance level (size) of the test, which remains close to α as shown in Tables 7, 8, and 10. Table 10 provide comparison of powers for $\widehat{\Delta}_{2,2}$, T_1, and T_2 for $N = 20, 50, 100$ at significance level $\alpha = 0.05$ against alternative distributions $\chi^2_{(2)}$ and $N(0,1)$.

Table 10: Comparison of powers

N	m	T_1 ($\chi^2_{(2)}$)	T_2 ($\chi^2_{(2)}$)	$\widehat{\Delta}_{2,2}$ ($\chi^2_{(2)}$)	T_1 ($N(0,1)$)	$\widehat{\Delta}_{2,2}$ ($N(0,1)$)
20	2	0.3999	0.4765	0.8861	0.0549	0.0518
	3	0.5133	0.6842	0.8976	0.0501	0.0481
	4	0.5962	0.5082	0.8956	0.0501	0.0519
	6	0.6157	0.5187	0.8641	0.0440	0.0502
	7	0.6234	0.4431	0.8889	0.0535	0.0489
	8	0.5650	0.5673	0.8814	0.0518	0.0491
50	5	0.9999	0.9813	0.9988	0.0569	0.0491
	8	0.9995	0.9874	0.9990	0.0558	0.0514
	20	0.9971	0.9770	0.9979	0.0494	0.0488
100	8	0.9989	0.9936	1.0000	0.0497	0.0484
	10	0.9992	0.9954	1.0000	0.0509	0.0502
	15	0.9987	0.9994	1.0000	0.0541	0.0473
	20	0.9978	0.9974	1.0000	0.0513	0.0524

We may, therefore, conclude that our suggested test, which is based on the cumulative past and residual extropy of the nth lower and upper k-record value, works satisfactorily in the simulation study. Our test performs better than the tests proposed by Jose and Sathar (2022) and Xiong et al. (2021) in terms of power comparison. Test proposed by Jose and Sathar (2022) and Xiong et al. (2021) performs better than many tests avilable in literature as claimed in manuscript. We, therefore, anticipate that the proposed test will be superior to the memsined competing tests in many real-world applications.

Table 11: Empirical powers when $N = 10$ and $m = 2$.

Alternative distribution	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
Chi-square(3)	0.7270	0.6735	0.5736
Exponential(1)	0.4484	0.3567	0.2218
Gamma(2,1)	0.4921	0.4106	0.2795
Gamma(3,1)	0.4954	0.4169	0.3028
Lognormal(0,1)	0.7101	0.6483	0.5406
Weibull(2)	0.0071	0.0019	0.0001
Beta(2,5)	0.0000	0.0000	0.0000
Beta(5,2)	0.0000	0.0000	0.0000
GLD(0,1,0.14,0.14)	0.0000	0.0000	0.0000
GLD(0,1,0.10,0.30)	0.0000	0.0000	0.0000
GLD(0,1,0.30,0.10)	0.0000	0.0000	0.0000
GLD(0,1,-0.10,0.20)	0.0004	0.0000	0.0000
GLD(0,1,0.20,-0.10)	0.0004	0.0000	0.0000

The proposed test shows high power against several common skewed alternatives, such as chi-square, gamma, exponential, and lognormal distributions. The empirical rejection probabilities under symmetric distributions remain small, supporting the validity of the test under symmetry. However, for some bounded or selected generalized lambda alternatives, the rejection probabilities are low when $N = 10$, indicating that the power depends on the type and strength of asymmetry as well as the sample size.

6 Real Data Application

Dataset 1: Dataset 1 from Montgomery et al. (2021) is known to follow a normal distribution (symmetric model). Gupta and Chaudhary (2024) and Jose and Sathar (2022) also used this dataset for their proposed tests of symmetry.

Dataset 1: 15.5, 23.75, 8.0, 17.0, 5.5, 19.0, 24.0, 2.5, 7.5, 11.0, 13.0, 3.75, 25.0, 9.75, 22.0, 18.0, 6.0, 12.5, 2.0, 21.5.

Since the normal distribution is symmetric, our test verifies this fact. The value of the test statistic $\widehat{\Delta}_{2,2}$ is 0.1531 with an estimated p-value of 0.2969 for window size $m = 2$ and sample size $N = 20$. Our test fails to reject the null hypothesis even at a 10% significance level, as expected.

Dataset 2: Dataset 2 from Qiu and Jia (2018) represents active repair times (in hours) for an airborne communication transceiver. Gupta and Chaudhary (2024) and Xiong et al. (2021) also used this dataset for testing symmetry.

Dataset 2: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

This dataset can be fitted by an inverse Gamma (IG) distribution as pointed out by Qiu and Jia (2018), which is asymmetric. Our test confirms this: $\widehat{\Delta}_{2,2} = 3.6678$ with p-value = 0 for $m = 20$ and $N = 45$. Thus, the null hypothesis is rejected even at 1% significance level, demonstrating the effectiveness of our test.

Dataset 3: Taken from Sathar and Jose (2020), this dataset has been modeled as a normal distribution (symmetric).

Dataset 3: 1.42, 0.84, 2.32, 1.84, 2.4, 0.9, 1.49, 0.87, 1.36, 1.25, 1.25, 1.8, 0.86, 0.04, 0.49, 2.08, 0.58, 0.22, 0.06, 1.7, 2.67, 2.39, 2.32, 2.98, 3.21, 1.99, 1.3, 1.25, 1.76, 1.67, 1.36, 1.57, 1.21, 1.24, 1.62, 0.93, 1.32, 0.86, 1.48, 0.85, 1.23, 1.23, 2.14.

Our test statistic is $\widehat{\Delta}_{2,2} = 0.1545$ with p-value = 0.2821 for $m = 3$ and $N = 43$, failing to reject the null hypothesis at 10% significance level, as expected for a symmetric distribution.

Dataset 4: From Thomas and Jose (2021), this dataset follows a Burr-type XII distribution (skewed).

Dataset 4: 99, 61, 86, 113, 96, 99, 83, 57, 80, 79, 75, 70, 15, 62, 87, 95, 81, 71, 44, 13, 52, 97, 146, 52, 52, 29, 108, 135, 102, 48, 66, 90, 22, 72, 176, 107, 84, 83, 37, 67, 83, 36, 49, 39, 102, 66, 154, 72, 63, 83, 77.

Here, $\widehat{\Delta}_{2,2} = 6.2144$ with p-value = 0.0 for $m = 25$ and $N = 51$, successfully rejecting the null hypothesis at 1% significance level, confirming the skewness.

Dataset 5: Transformed vinyl chloride data (uniform via probability integral transformation, see Xiong et al. (2022)).

Dataset 5: 0.0518, 0.0518, 0.1009, 0.1009, 0.1917, 0.1917, 0.1917, 0.2336, 0.2336, 0.2336, 0.2733, 0.2733, 0.3467, 0.3805, 0.3805, 0.4126, 0.4431, 0.4719, 0.4719, 0.4993, 0.6162, 0.6550, 0.6550, 0.7059, 0.7211, 0.7356, 0.7623, 0.7863, 0.8178, 0.8810, 0.9337, 0.9404, 0.9732, 0.9858.

Our test statistic is $\widehat{\Delta}_{2,2} = 0.0247$ with p-value = 0.4425 for $m = 11$ and $N = 34$, failing to reject the null hypothesis at 10%, consistent with uniformity.

Dataset 6: From Lawless (2011), representing 1000-cycle-to-failure data for electrical appliances.

Dataset 6: 0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100.

Our test statistic is $\widehat{\Delta}_{2,2} = 0.5776$ with p-value = 0.0210 for $m = 2$ and $N = 50$, successfully rejecting the null hypothesis at 5% significance level. See Table 12 for the values of the test statistic and corresponding p-values for the different datasets, based on the specific window size m and sample size N of each dataset. Table 11 shows value of test statistics and p-values for various datasets based on the proposed $\widehat{\Delta}_{2,2}$ test

At a 5% level of significance, a p-value less than 0.05 indicates that the data exhibit asymmetry, whereas a p-value greater than 0.05 suggests that the data are symmetric. As shown in Table 12, the proposed test effectively identifies whether the distribution of a random sample is symmetric or asymmetric. Specifically, the p-values for datasets 2, 4, and 6 are below 0.05, indicating asymmetry in these samples. In contrast, the

Table 12: P-values of datasets

Dataset	N	m	$\widehat{\Delta}_{2,2}$	p-**value**
Dataset 1	20	2	0.1531	0.2969
Dataset 2	45	20	3.6678	0.0000
Dataset 3	43	3	0.1545	0.2821
Dataset 4	51	25	6.2144	0.0000
Dataset 5	34	11	0.0247	0.4425
Dataset 6	50	2	0.5776	0.0210

moderate p-values for datasets 1, 3, and 5 support the acceptance of symmetry in their distributions. These results confirm that the test statistic successfully detects the symmetry or asymmetry in the underlying distributions of the datasets.

7 Conclusion and Future Work

Gupta and Chaudhary (2024) showed that the cumulative past extropy of the nth lower k-record value is equal to the cumulative residual extropy of the nth upper k-record value if and only if the underlying distribution is continuous and symmetric. Building on this result, we proposed a new test for symmetry. Critical values and the power of the test against the $\chi^2_{(1)}$ distribution were provided for different sample sizes at significance levels $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$. Additionally, the power of the test was evaluated against $\chi^2_{(1)}$, $\chi^2_{(2)}$, $\chi^2_{(3)}$, and $N(0,1)$ distributions at a 5% significance level for sample sizes $N = 20, 50$, and 100. Comparisons with two competing tests demonstrate that the proposed test generally exhibits superior power. Application of the test to six real-world datasets further confirms its effectiveness in detecting symmetric and asymmetric behavior through significant p-values.

For future work, other characteristics of symmetric distributions, as provided by Gupta and Chaudhary (2024), could be explored to develop additional symmetry tests based on extropy. Moreover, similar approaches could be employed to construct tests for exponentiality, uniformity, and normality using the cumulative past and residual extropy of record values. Such methods may yield improved goodness-of-fit tests compared to existing procedures in the literature.

Acknowledgment

The authors are thankful to the referees and editor for their valuable suggestions, which significantly improved the paper.

Disclosure statement

There are no conflicts of interest among authors about this work and its publication.

Data availability statement

The manuscript includes all of the used data sets.

Funding statement

No funding received for this research.

Ethical statement

The work has not been previously published or submitted in any other format, language, etc. The submission is entirely original.

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Appendix-1

The following steps were used to determine the critical values and compute the power of our proposed test and that of other tests for symmetry at significance level $\alpha = 0.10,\ \alpha = 0.05, \alpha = 0.01$:
(1) we defined a function to calculate the absolute value of $\widehat{\Delta}_{2,2}$.
(2) Generate a sample of size N from the null distribution and compute the test statistics for the sample data;
(3) Repeat Step 2 for 10,000 times and determine the 950th, 975th and 995th quantile respectively of the test statistics as the critical value;
(4) Generate a sample of size N from the alternative distribution and check if the absolute value of the test statistic is greater than the critical value;
(5) Repeat Step 4 for 10,000 times and the percentage of rejection is the power of the test.