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Ordering Results on Comparisons of Lifetimes of Parallel Systems through the Majorization of Unit Gamma Gompertz-F Parameters

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Abstract. This study compares two parallel systems whose components are taken from the unit gamma Gompertz-F family. The comparisons are accomplished according to various stochastic orders, including the usual stochastic order, the reversed hazard rate order, and the likelihood ratio order by way of the majorization of shape parameters. For additional research, we investigate the lifetime of system components to be dependent or independent under conditions that the baseline distribution function of components is identical or non-identical. A numerical example based on real-life data is presented as an illustration.

Keywords. Unit Distributions, Stochastic Order, Majorization, Parallel Systems.

MSC: 62N05, 62F10.

1 Introduction

Unit distributions have been discussed by many authors in recent years. These models generally are employed to describe various events such as proportions, percentages, and probabilities which take values in $(0, 1)$. One motivation for extending the application of unit distributions is the growth of combined data in diverse areas of study such as medicine, biology, meteorology, hydrology, financial modeling, and other fields. For instance, several outcomes in survival theory, such as the survival times of units and

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system lifetimes, are often larger than zero but do not overtake sufficiently large values. Accordingly, they are placed within a specific range, and the boundary models are helpful for describing such datasets. Moreover, many random variables and stochastic processes are practically bounded from above or below. In all of these circumstances, random variables can be transformed into the $(0, 1)$ interval by utilizing normalization or other transformation approaches. One advantage of unit distributions is that they could be computed using $Y = e^{-X}$ or $Y = \frac{X}{1+X}$ transformations on any statistical distribution which are defined on the $\mathbb{R}$ or $\mathbb{R}^+$ sets. Concerning this matter, several unit distributions, such as the unit Weibull by Mazucheli et al. (2020, 2018), unit Burr XII by Korkmaz and Chesneau (2021), unit gamma Gompertz by Bantan et al. (2021), unit inverse Gaussian by Ghitany et al.(2019), and unit generalized half-normal by Korkmaz (2020), have been proposed and investigated.

The Gompertz model is a statistical distribution that has a monotone hazard rate function. However, in many phenomena, the hazard rate function is not monotone in practice and pursues a bathtub curve. For this cause, various generalizations have been implemented on the Gompertz distribution. Particularly, the gamma Gompertz distribution is a generalization of the Gompertz model, which was presented by Bemmaor and Glady (2012) and discussed in more detail by Shama et al.(2022). The unit gamma Gompertz (*UGG*) distribution was proposed by Bantan et al. (2021) and was computed by applying the transformation $Y = e^{-X}$ on the Gompertz random variable. This model is absolutely flexible with increasing, decreasing, increasing-decreasing, or decreasing-increasing density functions. Furthermore, the reversed hazard rate function of *UGG* distribution is monotonically increasing or decreasing and sometimes appears as a constant function. With respect to the generalization of statistical distributions, several authors paid attention to the combined models by incorporating a distribution function of a random variable into another distribution family. In other words, by substituting the random variable X with the baseline cumulative distribution function (CDF) $F(x)$ from another family, a new distribution is constructed, which is known as the combined distribution. In these models, $F(x)$ is called the baseline CDF, and the distribution of the random variable X is known as the generator distribution.

In this regard, several distributions, such as the exponentiated half logistic by Cordeiro et al. (2014), beta by Eugene et al. (2002), and gamma by Zografos and Balakrishnan (2009), are applied as the generators in order to build up the new combined models. For any CDF $F(x)$, the unit gamma Gompertz-F, denoted in this study as $UGG-F(\alpha, \beta, b, F)$, has, respectively, the CDF and the probability density function (PDF) as below:

$$G(x) = \left(\frac{b}{b-1+F^{-\alpha}(x)}\right)^{\beta}, \ x > 0, \tag{1.1}$$

$$g(x) = \frac{\alpha\beta F^{-\alpha}(x)}{b-1+F^{-\alpha}(x)} \frac{f(x)}{F(x)} G(x), \ x > 0, \tag{1.2}$$

where, $F(x)$ is the baseline CDF, $\alpha > 0$ and $\beta > 0$ are shape parameters, and $b > 0$. As mentioned before, this study intends to compare two parallel systems with heterogeneous components of the $UGG - F$ family. Therefore, we first describe the parallel system below. It should be noted that the results of this article are obtained under condition $b > 1$.

The order statistics of a random sample $X_1, \ldots, X_n$, denoted by $X_{1:n}, \ldots, X_{n:n}$, are the sample values that are placed in ascending order. They are commonly used in various fields involving reliability engineering, survival analysis, actuarial sciences and risk management, quality control, and nonparametric statistics. As a practical example in reliability theory, the order statistics $X_{n-k+1:n}$ for $k = 1, \ldots, n$ denotes the lifetime of k out of n:G system in which the system activation depends on the working at least k components of the system. Furthermore, the 1 out of n:G system demonstrates a parallel system that is extensively used in various devices, such as the computer hard disk, brake systems, and support cables on bridges. Many applications and features of k out of n:G systems were discussed in Barlow and Proschan (1996). More details on the wide application of order statistics can be found in David and Nagaraja (2003).

Comparison of the maximum and minimum statistics of two heterogeneous random sets wherein each follows a specific distribution has been extensively discussed in recent years. In this regard, several stochastic orders have been studied by authors based on the specific distribution. In some works, the independence assumption of random variables was replaced by dependency presumption, and the copula functions could be used to demonstrate the dependency structure between the variables. Several authors have studied stochastic comparisons of systems with dependent or independent components. One may refer to Zhao and Balakrishnan (2012), Balakrishnan et al. (2020 and 2021), Barmalzan et al. (2019, 2020, 2021a, 2021b, 2021c, 2022a, 2022b, 2022c, 2023), Barmalzan and Dehsukhteh (2021), Ghanbari et al. (2020 and 2021), Haidari et al. (2022), Sattari et al. (2021), Shekari et al.(2023), and the references therein. This study stochastically compares two parallel systems with identical (nonidentical), independent (dependent), and heterogeneous components that come from a $UGG - F(\alpha, \beta, b, F)$.

This paper aims to further study how the heterogeneity of shape parameters in an independent or dependent $UGG - F$ model stochastically affects the system lifetimes. By using the majorization-type heterogeneity measuring, we show that in a parallel system with independent components, the higher heterogeneity of the shape parameters results in the larger lifetime of the parallel system in reversed hazard rate order. Similar results in likelihood ratio order are proved for two parallel systems with independent and heterogeneous components that consist of several outliers. Also, the comparing lifetimes of two parallel systems with dependent and heterogeneous $UGG - F$ components, that follow the Archimedean copula, are carried out under the same and different baseline distribution functions.

The remainder of this paper is organized in the following manner. In Section 2, several definitions and preliminary concepts, besides several useful lemmas, are provided, which will be used throughout this paper. The parallel systems with heterogeneous and independent components from the $UGG - F$ family are compared in Section 3 through the usual stochastic order, reverse hazard rate order, and likelihood ratio order with several outliers under the identical and nonidentical baseline CDFs. In Section 4, the stochastic comparison of two parallel systems is performed under the identical and nonidentical baseline CDFs in the case that the dependent and heterogeneous components are taken from the $UGG - F$ distribution. A numerical example based on real data is taken to demonstrate the performance of the obtained results in Sections 5 and 6. Finally, the results and discussion of this work are provided in Section 7.

Throughout the paper, the term increasing (decreasing) is utilized for monotone non-decreasing (non- increasing). Besides, $\mathbb{R}$ and I^n denote, respectively, the sets of real numbers $\{x : -\infty < x < \infty\}$ and n-dimensional Euclidean space, where $I \subseteq \mathbb{R}$. Additionally, $x \overset{sign}{=} y$ is formulated to connote that x and y have an identical sign.

2 Definitions and Preliminary Concepts

In order to compare two CDFs or PDFs, several statistical indices, such as mean, median, skewness, and kurtosis, have been used previously in research works. However, comparisons based on single values did not contain sufficient information. To overcome this shortcoming, several authors, such as Mann and Whitney (1947), applied stochastic orders, which provide further information regarding the distribution structures. More detailed information on stochastic orders can be found in Shaked and Shanthikumar (2007) and Belzunce et al. (2015). In this section, we provide several stochastic order definitions consisting of the usual stochastic order, hazard rate order, reverse hazard rate order, likelihood ratio order, and the concepts of majorization theory, which will be used throughout this study. Suppose that random variables X and Y have the PDFs f and g, CDFs F and G, survival functions $\bar{F}$ and $\bar{G}$, hazard rate functions $h_f(= f/\bar{F})$ and $h_g(= g/\bar{G})$, and the reverse hazard rate functions $\tilde{r}_f(= f/F)$ and $\tilde{r}_g(= g/G)$, respectively.

Definition 2.1. (Shaked and Shanthikumar (2007)) The random variable X is said to be smaller than the random variable Y,

(i) in the usual stochastic order, if for each x, $F(x) \geq G(x)$, denoted by $X \leq_{st} Y$.

(ii) in the likelihood ratio order, if $\dfrac{g(x)}{f(x)}$ is increasing in x on the union of the support of random variables X and Y, which is denoted by $X \leq_{lr} Y$.

(iii) in the hazard rate order, if $\bar{G}(t)/\bar{F}(t)$ increases in t, which is denoted by $X \leq_{hr} Y$. If X and Y are absolutely continuous, then $X \leq_{hr} Y$ is equivalent to $h_f(t) \geq h_g(t)$ for all t.

(iv) in the reversed hazard rate order, if $G(t)/F(t)$ increases in t, denoted by $X \leq_{rhr} Y$. If X and Y are absolutely continuous, then $X \leq_{rhr} Y$ is equivalent to $\tilde{r}_f(t) \leq \tilde{r}_g(t)$ for all t.

The following definitions can be found in Marshall et al. (1979).

Definition 2.2. Suppose that $\{x_{(1)}, \ldots, x_{(n)}\}$ and $\{y_{(1)}, \ldots, y_{(n)}\}$ represent the order statistics corresponding to $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ vectors, respectively. The vector $\boldsymbol{x}$ is supposed to be greater than the vector $\boldsymbol{y}$ in

(i) majorization, denoted by $\boldsymbol{x} \overset{m}{\geq} \boldsymbol{y}$, if for any $j = 1, 2, \ldots, n-1$, it holds that $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$, and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

(ii) weakly supermajorization, denoted by $\boldsymbol{x} \overset{w}{\geq} \boldsymbol{y}$, if for any $j = 1, 2, \ldots, n$, we have $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$.

(iii) weakly submajorization, denoted by $\boldsymbol{x} \underset{w}{\geq} \boldsymbol{y}$, if for any $j = 1, 2, \ldots, n$, they satisfy $\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)}$.

Definition 2.3. For a continuous and decreasing function $\psi : (0, \infty) \to (0, 1)$ with $\psi(0) = 1$ and $\psi(+\infty) = 0$, let $\phi = \psi^{-1}$ be a pseudo-inverse. The function

$$C_{\psi}(u_1, \ldots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)), \qquad \text{for all } u_i \in (0, 1),\ i = 1, 2, \ldots, n,$$

is said to be an Archimedean copula with generator ψ, if for $k = 1, 2, \ldots, n-2$, $(-1)^k \psi^{(k)}(x) \geq 0$ and $(-1)^{n-2} \psi^{(n-2)}(x)$ are decreasing and convex.

For further details on copula theory and applications, one can refer to Nelsen (2006).

Definition 2.4. The function f is called a super-additive function if for any x and y in the support of f, $f(x + y) \geq f(x) + f(y)$.

Definition 2.5. Suppose that φ is a real valued function that is defined on $A \in \mathbb{R}$. Then, it is said to be Schur-convex (Schur-concave) on A if $\boldsymbol{x} \overset{m}{\geq} \boldsymbol{y}$ yields $\varphi(\boldsymbol{x}) \geq (\leq) \varphi(\boldsymbol{y})$.

Lemma 2.1. *(Marshal et al., 1979, p. 57) Suppose that $I \subseteq \mathbb{R}$ is an open interval. The continuous and differentiable function $h : I^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) if and only if h is a symmetric on I^n and, for all $i \neq j$ and $\boldsymbol{x} \in I^n$,*

$$(x_i - x_j)\left(\frac{\partial h(\boldsymbol{x})}{\partial x_i} - \frac{\partial h(\boldsymbol{x})}{\partial x_j}\right) \geq (\leq) 0.$$

Lemma 2.2. *(Lemma A.1 of Li and Fang (2015)) For n-dimensional Archimedean copulas C_{ψ_1} and C_{ψ_2}, if $\phi_2 \circ \psi_1$ is super-additive, then for all values of $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0, 1]^n$, $C_{\psi_1}(\boldsymbol{u}) \leq C_{\psi_2}(\boldsymbol{u})$.*

Lemma 2.3. *The function $h(x) = \frac{xp^{-x}}{b-1+p^{-x}}$ is increasing and convex in x, for all $x > 0$, $b > 1$, and $0 < p < 1$.*

Proof. Let $y = \frac{xp^{-x}}{b-1+p^{-x}}$. By simplifying, it follows that $y = \frac{x}{(b-1)p^x+1}$. The derivative of y with respect to x is denoted by y' and is obtained as follows:

$$y' = \frac{1}{(b-1)p^x + 1} + \frac{x(b-1)p^x \ln(p^{-1})}{((b-1)p^x + 1)^2}.$$

It is easy to see that for $b > 1$, we have $y' \geq 0$. So y is an increasing function of x. Now we can write y' as $y' = A + xA'$, where $A = \frac{1}{(b-1)p^x+1}$ and A' is the derivative of A with respect to x. It is easy to see that $A' \geq 0$ for $b > 1$. The second derivative of y with respect to x is denoted by y'' and is obtained by $y'' = 2A' + A''x$, where A'' is the second derivative of A with respect to x. To show that y is convex, it is sufficient to show that $A'' \geq 0$ for $b > 1$. Since $A' = (b-1)p^x \ln(p^{-1})A^2$, we have $A'' = (b-1)\ln(p^{-1}) + Ap^x a_x$, where $a_x = A\ln(p) + 2A'$. Now it is enough to show that $a_x \geq 0$ for $b > 1$. We have

$$\begin{aligned} a_x &= A\ln(p) + 2A' \\ &= A\ln(p) + 2(b-1)p^x \ln(p^{-1})A^2 \\ &= A\ln(p^{-1})C_x, \end{aligned}$$

where $C_x = 2(1 - \frac{1}{(b-1)p^x+1})$ is positive for $b > 1$. □

The following notations will be used in the upcoming theorems.

(i) $D_+ = \{(x_1, x_2, \dots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$.

(ii) $\varepsilon_+ = \{(x_1, x_2, \dots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}$.

(iii) $A_n = \left\{(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} : (x_i - x_j)(y_i - y_j) \geq 0,\ i, j = 1, 2, \dots, n\right\}$.

(iv) $B_n = \left\{(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} : (x_i - x_j)(y_i - y_j) \leq 0,\ i, j = 1, 2, \dots, n\right\}$.

3 Two Parallel Systems with Independent and Heterogeneous Components

In this section, the comparing lifetimes of two parallel systems with independent and heterogeneous $UGG-F$ components are carried out under the identical and nonidentical baseline CDFs.

Assumption 3.1. Suppose that the random variables X_1 and X_2 have the CDFs $F_1(\cdot)$ and $F_2(\cdot)$, PDFs $f_1(\cdot)$ and $f_2(\cdot)$, and survival function $\bar{F}_1(\cdot)$ and $\bar{F}_2(\cdot)$, respectively. Assume that $U_i \sim UGG - F_1(\alpha_i, \beta_i, b, F_1)$ and $V_i \sim UGG - F_2(\gamma_i, \delta_i, b, F_2)$ for $i = 1, 2, \dots, n$ are two random samples of size n with the baseline CDFs F_1 and F_2, respectively. Consider functions $G_{n:n}(\cdot)$ and $H_{n:n}(\cdot)$ to be, respectively, the CDFs of the random variables $U_{n:n} = \max(U_1, \dots, U_n)$ and $V_{n:n} = \max(V_1, \dots, V_n)$. In this situation, for $x > 0$, we have

$$G_{n:n}(x) = \prod_{i=1}^{n} \left(\frac{b}{b - 1 + {F_1}^{-\alpha_i}(x)} \right)^{\beta_i}, \tag{3.1}$$

and

$$H_{n:n}(x) = \prod_{i=1}^{n} \left(\frac{b}{b - 1 + {F_2}^{-\gamma_i}(x)} \right)^{\delta_i}. \tag{3.2}$$

If $\tilde{r}_{n:n}(x)$ and $\tilde{s}_{n:n}(x)$ are the reversed hazard rate functions of $U_{n:n}$ and $V_{n:n}$, respectively, then we have

$$\tilde{r}_{n:n}(x) = \sum_{i=1}^{n} \alpha_i \beta_i \frac{f_1(x)}{F_1(x)} \frac{F_1{}^{-\alpha_i}(x)}{b - 1 + F_1{}^{-\alpha_i}(x)}, \tag{3.3}$$

and

$$\tilde{s}_{n:n}(x) = \sum_{i=1}^{n} \gamma_i \delta_i \frac{f_2(x)}{F_2(x)} \frac{F_2{}^{-\gamma_i}(x)}{b - 1 + F_2{}^{-\gamma_i}(x)}. \tag{3.4}$$

In the following results, we consider $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ that belong to I^n.

Result 1. *Under the setup of Assumption 3.1, $\beta_i = \delta_i$ for $i = 1, \ldots, n$, and $F_1 = F_2 = F$, if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in A_n$ and $(\boldsymbol{\gamma}, \boldsymbol{\beta}) \in A_n$, then $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$ implies $U_{n:n} \geq_{rhr} V_{n:n}$.*

Proof. According to Theorem A.3 in Marshall et al. (1979), it is sufficient to indicate that the reverse hazard rate function of random variable $U_{n:n}$, that is,

$$\tilde{r}_{n:n}(x) = \sum_{i=1}^{n} \alpha_i \beta_i \frac{f(x)}{F(x)} \frac{F^{-\alpha_i}(x)}{b - 1 + F^{-\alpha_i}(x)} = \frac{f(x)}{F(x)} \sum_{i=1}^{n} \beta_i \psi(\alpha_i),$$

is a Schur-convex function in $\boldsymbol{\alpha}$, where $\psi(\alpha_i) = \frac{\alpha_i F^{-\alpha_i}(x)}{b-1+F^{-\alpha_i}(x)}$. Based on Lemma 2.3, it can be concluded that $\psi(\alpha_i)$ is increasing and convex with respect to α_i for $i = 1, \ldots, n$. Therefore, according to the Theorem 3.1 (a)(i) (3.2 (b)(i)) of Kundu et al. (2016), $\tilde{r}_{n:n}(x)$ is a Schur-convex function in $\boldsymbol{\alpha}$ on $D_+(\varepsilon_+)$. □

Remark 1. Result 1 states that in a parallel system with independent components, the higher heterogeneity of the shape parameter (in majorization order) results in the higher lifetime of the parallel system in reversed hazard rate order.

Below, we examine Result 1 via a numerical example.

Examples 3.1. For $n = 3$, consider the random variables $U_i \sim UGG - F(\alpha_i, \beta_i, b, F)$ and $V_i \sim UGG - F(\gamma_i, \beta_i, b, F)$, $i = 1, 2, 3$. For $F(x) = 1 - e^{-x}$, $x > 0$, $\boldsymbol{\alpha} = (2, 2, 5) \overset{m}{\succeq} (2, 3, 4) = \boldsymbol{\gamma}$ and $\boldsymbol{\beta} = (1, 2, 3)$, the CDFs ratio plot of the random variables $U_{3:3}$ and $V_{3:3}$ for $x = -\ln y$ is demonstrated in Figure 1, where $G_{3:3}(x)$ and $H_{3:3}(x)$ are the CDF of random variables $U_{3:3}$ and $V_{3:3}$, respectively. As the figure shows, this proportion is increasing in x, which yields $U_{3:3} \geq_{rhr} V_{3:3}$. Note that to draw the CDFs of random variables $U_{3:3}$ and $V_{3:3}$ on the interval $[0, \infty)$, the transformation $x = -\ln y : [0, \infty) \to [0, 1]$ is considered.

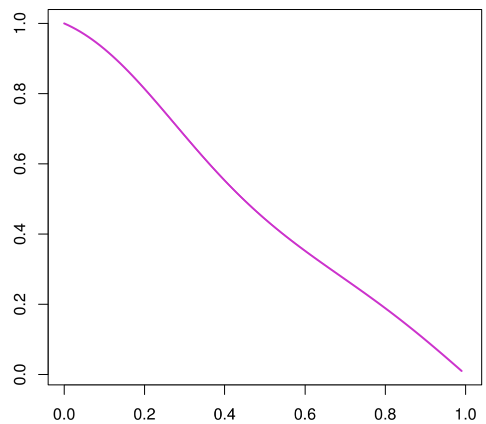

Figure 1: Plot of $\frac{G_{3:3}(x)}{H_{3:3}(x)}$ for $x = -\ln y$.

Result 2. *Under the setup of Assumption 3.1, $\gamma_i = \alpha_i$ for $i = 1, \ldots, n$, and $F_1 = F_2 = F$, we have if $(\boldsymbol{\alpha}, \boldsymbol{\delta}) \in A_n$, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in A_n$, and $\boldsymbol{\beta} \overset{m}{\geq} \boldsymbol{\delta}$, then $U_{n:n} \geq_{rhr} V_{n:n}$.*

Proof. The proof is similar to the proof of Result 1. □

Combining Results 1 and 2, we obtain more general results as follows.

Result 3. *Under the setup of Assumption 3.1 and $F_1 = F_2$, if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in A_n$, $(\boldsymbol{\alpha}, \boldsymbol{\delta}) \in A_n$, and $(\boldsymbol{\gamma}, \boldsymbol{\beta}) \in A_n$, then $\boldsymbol{\beta} \overset{m}{\geq} \boldsymbol{\delta}$ and $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\gamma}$ imply $U_{n:n} \geq_{rhr} V_{n:n}$.*

In the following theorem, we compare two parallel systems with independent and heterogeneous components that consist of several outliers.

Theorem 3.1. *Under the setup of Assumption 3.1, $\gamma_i = \alpha_i$ for $i = 1, 2$, and $F_1 = F_2 = F$, if*

$$(\underbrace{\beta_1, \beta_1, \ldots, \beta_1}_{n_1}, \underbrace{\beta_2, \beta_2, \ldots, \beta_2}_{n_2}) \overset{m}{\geq} (\underbrace{\delta_1, \delta_1, \ldots, \delta_1}_{n_1}, \underbrace{\delta_2, \delta_2, \ldots, \delta_2}_{n_2}),$$

where $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \delta_1 \leq \delta_2$, then $U_{n:n} \geq_{lr} V_{n:n}$.

Proof. Based on the equation $\frac{g_{n:n}(x)}{h_{n:n}(x)} = \frac{G_{n:n}(x)}{H_{n:n}(x)} \cdot \frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)}$ and Result 1, it is sufficient to indicate that the term $\frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)}$ is increasing in x. To prove this point, we have

$$\frac{\tilde{r}_{n:n}(x)}{\tilde{s}_{n:n}(x)} = \frac{\sum_{i=1}^{2} \frac{\alpha_i \beta_i F^{-\alpha_i - 1}(x)}{(b-1)+F^{-\alpha_i}(x)}}{\sum_{i=1}^{2} \frac{\alpha_i \delta_i F^{-\alpha_i - 1}(x)}{(b-1)+F^{-\alpha_i}(x)}} = \frac{n_1\beta_1 u_1 + n_2\beta_2 u_2}{n_1\delta_1 u_1 + n_2\delta_2 u_2} \overset{def}{=} \eta(x),$$

where $u_i = \frac{\alpha_i F^{-\alpha_i - 1}(x)}{b-1+F^{-\alpha_i}(x)}$, $i = 1, 2$. Differentiating the term $\eta(x)$ with respect to x yields

$$\eta'(x) \overset{sign}{=} (n_1\beta_1 u_1' + n_2\beta_2 u_2')(n_1\delta_1 u_1 + n_2\delta_2 u_2) - (n_1\beta_1 u_1 + n_2\beta_2 u_2)(n_1\delta_1 u_1' + n_2\delta_2 u_2').$$

To show $\eta'(x) > 0$, it should be indicated that the symmetric function $\psi_1(\beta_1, \beta_2) = \frac{n_1\beta_1 u_1' + n_2\beta_2 u_2'}{n_1\beta_1 u_1 + n_2\beta_2 u_2}$ is Schur-convex in (β_1, β_2), where $u_i' = \frac{1}{F(x)} u_i v_i$. For this aim, we have

$$\frac{\partial\psi_1}{\partial\beta_1} - \frac{\partial\psi_1}{\partial\beta_2} = n_1 n_2 u_1 u_2 (\beta_1 + \beta_2)(v_1 - v_2),$$

where $v_i = \frac{(b-1)(-\alpha-1) - F^{-\alpha_i}(x)}{b-1+F^{-\alpha_i}(x)}$ for $i = 1, 2$. Now, since $\alpha \in \varepsilon_+$ and v_i is increasing in α_i, it can be concluded that $v_1 - v_2 \leq 0$ for all $\alpha_1 < \alpha_2$. Therefore, according to Lemma 2.1, ψ_1 is a Schur-convex function in $\boldsymbol{\beta}$, and $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$ implies that $\eta'(x) \geq 0$, which completes the proof. □

Remark 2. It is worth noting that, under the same baseline distribution function and for the weak supermajorization order, the results related to the usual stochastic order can be found in Theorem 3.2 and Corollary 3.5 of Nadeb et al. (2021).

4 Two Parallel Systems with Dependent and Heterogeneous Components

In this section, the comparing lifetimes of two parallel systems with dependent and heterogeneous $UGG{-}F$ components are carried out under the identical and nonidentical baseline CDFs.

Assumption 4.1. Suppose that $U_i \sim UGG{-}F_1(\alpha_i, \beta_i, b, F_1)$ and $V_i \sim UGG{-}F_2(\gamma_i, \delta_i, b, F_2)$, for $i = 1, 2, \ldots, n$, are two sets of dependent random variables from the UGG distribution family that follow the Archimedean copula with generators ψ_1 and ψ_2, respectively. The CDFs of random variables $U_{n:n}$ and $V_{n:n}$ are respectively,

$$\begin{aligned} G_{n:n}(x) &= \psi_1\left(\textstyle\sum_{k=1}^n \phi_1\left\{\left(\frac{b}{b-1+F_1{}^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right), \ x > 0, \\ H_{n:n}(x) &= \psi_2\left(\textstyle\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F_2{}^{-\gamma_k}(x)}\right)^{\delta_k}\right\}\right), \ x > 0. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Under the setup of Assumption 4.1, $\beta_i = \delta_i$ for $i = 1, \ldots, n$, and $F_1 = F_2 = F$, if $\phi_2 \circ \psi_1$ is super-additive, ψ_1 or ψ_2 is log-convex, $(\boldsymbol{\gamma}, \boldsymbol{\beta}) \in A_n$, and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in A_n$, then $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$ implies $U_{n:n} \geq_{st} V_{n:n}$.*

Proof. By the super-additive property of function $\phi_2 \circ \psi_1$, to achieve the necessary result, it is enough to indicate that

$$\psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right) \leq \psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\gamma_k}(x)}\right)^{\beta_k}\right\}\right). \tag{4.2}$$

Suppose $J(\boldsymbol{\alpha}) = \psi_2\left(\sum_{k=1}^{n}\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right)$. We indicate that the function $J(\boldsymbol{\alpha})$ is a Schur-concave function in $\boldsymbol{\alpha}$. The partial derivative with respect to α_i is

$$\frac{\partial J(\boldsymbol{\alpha})}{\partial \alpha_i} = \psi_2'\left(\sum_{k=1}^{n}\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right)\left(\ln F(x)^{\beta_i}\right)\left(\frac{F^{-\alpha_i}(x)}{b-1+F^{-\alpha_i}(x)}\right) \times\frac{\psi_2\left(\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right)^{\beta_i}\right\}\right)}{\psi_2'\left(\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right)^{\beta_i}\right\}\right)}.$$

According to the assumptions of the theorem, for $i \leq j$, $\alpha_i \geq \alpha_j,\ \beta_i \geq \beta_j$ (or $\alpha_i \leq \alpha_j,\ \beta_i \leq \beta_j$), since the function $\frac{F^{-\alpha_i}(x)}{b-1+F^{-\alpha_i}(x)}$ is increasing in α_i by Lemma 2.3, the functions $\phi_2(x)$ and $\psi_2(x)$ are decreasing in x, and $\psi_2(x)$ is log-convex ($\frac{\psi_2(x)}{\psi_2'(x)}$ is decreasing in x), therefore, we have

$$\left(\alpha_i - \alpha_j\right)\left(\frac{\partial J(\boldsymbol{\alpha})}{\partial \alpha_i} - \frac{\partial J(\boldsymbol{\alpha})}{\partial \alpha_j}\right) \leq 0.$$

Therefore, by Lemma 2.1, $J(\boldsymbol{\alpha})$ is Schur-concave in $\boldsymbol{\alpha}$, and if $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\gamma}$, then $J(\boldsymbol{\alpha}) \leq J(\boldsymbol{\gamma})$. Therefore, $G_{n:n}(x) \leq H_{n:n}(x)$, and the proof is completed. □

Remark 3. Comparing Result 1 and Theorem 4.1, it can be observed that for independent cases, when $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\gamma}$, the stochastic ordering exists between $U_{n:n}$ and $V_{n:n}$ under the less restrictive condition than reversed hazard rate ordering, which is expected.

In the following example, the correctness of Theorem 4.1 has been investigated.

Examples 4.1. Let $n = 2$, let the CDF be $F(x) = 1 - e^{-x},\ x > 0$, and let generator functions be $\psi_1(x) = e^{\left(\frac{1-e^x}{0.2}\right)}$ and $\psi_2(x) = (4x+1)^{-0.25}$. Furthermore, assume $\boldsymbol{\alpha} = (2,5) \overset{m}{\geq} (3,4) = \boldsymbol{\gamma}$. Simply, it can be denoted that $\psi_2(x)$ is a log-convex function. It can also be seen that

$$\frac{d^2}{dx^2}[\phi_2 \circ \psi_1(x)] = e^x\left(\exp\left(\frac{1-e^x}{0.2}\right)\right)^{-4}(0.2 + 4e^x)(0.2)^{-2} \geq 0.$$

Therefore, $\phi_2 \circ \psi_1(x)$ is convex in x, which implies that $\phi_2 \circ \psi_1(x)$ is a super-additive function. The curves of CDFs of random variables $U_{2:2}$ and $V_{2:2}$ on the interval $[0,\infty)$ are depicted in Figure 2 based on the transformation $(1+x)^{-1} : [0,\infty) \to [0,1]$. Thus, $(1+U_{2:2})^{-1} \leq_{st} (1+V_{2:2})^{-1}$, which is equivalent to $U_{2:2} \geq_{st} V_{2:2}$. Figure 2 shows $G_{(1+U_{2:2})^{-1}}(x) \geq H_{(1+V_{2:2})^{-1}}(x)$, and this implies that $U_{2:2} \geq_{st} V_{2:2}$

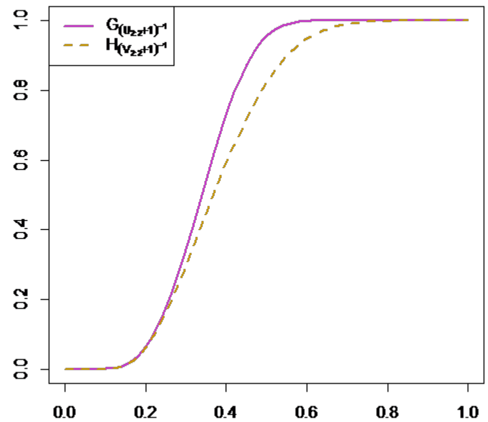

Figure 2: plots of CDFs of $(1+U_{2:2})^{-1}$ and $(1+V_{2:2})^{-1}$.

Theorem 4.2. *Under the setup of Assumption 4.1, $\alpha_i = \gamma_i$ for $i = 1, \ldots, n$, and $F_1 = F_2 = F$, if $\phi_2 \circ \psi_1$ is a super-additive function, ψ_1 or ψ_2 is log-convex, $(\boldsymbol{\alpha}, \boldsymbol{\delta}) \in A_n$, and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in A_n$, then $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$ implies $U_{n:n} \geq_{st} V_{n:n}$.*

Proof.

The CDFs of random variables $U_{n:n}$ and $V_{n:n}$ are, respectively,

$$\begin{aligned} G_{n:n}(x) &= \psi_1\left(\textstyle\sum_{k=1}^n \phi_1\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right), \qquad x > 0, \\ H_{n:n}(x) &= \psi_2\left(\textstyle\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\delta_k}\right\}\right), \qquad x > 0. \end{aligned} \tag{4.3}$$

The super-additive property of $\phi_2 \circ \psi_1$ yields

$$\psi_1\left(\sum_{k=1}^n \phi_1\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right) \leq \psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right). \tag{4.4}$$

To attain the required results, it must be shown that

$$\psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right) \leq \psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\delta_k}\right\}\right).$$

Suppose $J_1(\boldsymbol{\beta}) = \psi_2\left(\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right)$. We will show that the function $J_1(\boldsymbol{\beta})$ is Schur-concave in $\boldsymbol{\beta}$. The partial derivative of $J_1(\boldsymbol{\beta})$ with respect to β_i is given by

$$\begin{aligned} \frac{\partial J_1(\beta)}{\partial \beta_i} = {} & \psi_2'\left(\textstyle\sum_{k=1}^n \phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_k}(x)}\right)^{\beta_k}\right\}\right) \log\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right) \\ & \times \frac{\psi_2\left(\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right)^{\beta_i}\right\}\right)}{\psi_2'\left(\phi_2\left\{\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right)^{\beta_i}\right\}\right)}. \end{aligned}$$

Now, if, for every $i \le j$, $\alpha_i \le \alpha_j,\ \beta_i \le \beta_j$ (or $\alpha_i \ge \alpha_j,\ \beta_i \ge \beta_j$), then

$$\log\left(\frac{b}{b-1+F^{-\alpha_i}(x)}\right) \ge (\le) \log\left(\frac{b}{b-1+F^{-\alpha_j}(x)}\right).$$

On the other hand, $\phi_2(x)$ and $\frac{\psi_2(x)}{\psi_2'(x)}$ are decreasing in x, so that

$$\left(\beta_i-\beta_j\right)\left(\frac{\partial J_1(\boldsymbol{\beta})}{\partial \beta_i}-\frac{\partial J_1(\boldsymbol{\beta})}{\partial \beta_j}\right) \le 0.$$

Therefore, by Lemma 2.1, $J_1(\boldsymbol{\beta})$ is Schur-concave in $\boldsymbol{\beta}$, and the proof is completed. □

Remark 4. Comparing Theorem 4.2 and Result 2, it can be observed that for independent cases, when $\boldsymbol{\beta} \overset{m}{\ge} \boldsymbol{\delta}$, the stochastic ordering exists between $U_{n:n}$ and $V_{n:n}$ under the less restrictive condition than reversed hazard rate ordering, which is expected.

Theorem 4.3. *Under the setup of Assumption 4.1 and $\beta_i = \delta_i$ for $i = 1,\dots,n$, suppose that $\phi_2 \circ \psi_1$ is a super-additive function, ψ_1 or ψ_2 is log-convex, $(\boldsymbol{\gamma},\boldsymbol{\beta}) \in A_n$, $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in A_n$, and $\boldsymbol{\alpha} \overset{m}{\ge} \boldsymbol{\gamma}$. If $X_1 \ge_{st} X_2$, then $U_{n:n} \ge_{st} V_{n:n}$.*

Proof. We define a new random variable, say W_i, such that for $i = 1,2,\dots,n$, $W_i \sim UGG-F_1(\gamma_i,\beta_i,b,F_1)$. According Theorem 4.1, we have $W_{n:n} \le_{st} U_{n:n}$. Moreover, by the assumption $F_1 \le F_2$ and the decreasing property of functions ϕ_2 and ψ_2, we have

$$\psi_2\left(\sum_{k=1}^{n}\phi_2\left\{\left(\frac{b}{b-1+{F_1}^{-\gamma_k}(x)}\right)^{\beta_k}\right\}\right) \le \psi_2\left(\sum_{k=1}^{n}\phi_2\left\{\left(\frac{b}{b-1+{F_2}^{-\gamma_k}(x)}\right)^{\beta_k}\right\}\right). \tag{4.5}$$

By Equation 4.5, we get $G_{n:n}(x) \le H_{n:n}(x)$. Therefore, $U_{n:n} \ge_{st} V_{n:n}$, and the proof is completed. □

In order to illustrate the correctness of Theorem 4.3, a numerical example is provided below.

Examples 4.2. Suppose that $F_1(x) = e^{-\frac{2}{x}}$, $x > 0$ and $F_2(x) = 1 - \frac{1}{1+x}$, $x > 0$ are two baseline CDFs. Furthermore, consider $\boldsymbol{\gamma} = (3,4) \overset{m}{\le} (2,5) = \boldsymbol{\alpha}$, $\boldsymbol{\beta} = (1,2)$, $\psi_1(x) = e^{\left(\frac{1-e^x}{0.8}\right)}$, and $\psi_2(x) = (2x+1)^{-0.5}$. It can be seen that the conditions of Theorem 4.3 hold. To draw the curves of CDFs of random variables $U_{2:2}$ and $V_{2:2}$ on the interval $[0,\infty)$, we apply the transformation $(1+x)^{-1} : [0,\infty) \to [0,1]$. Figure 3 represents that the CDF of $(1+U_{2:2})^{-1}$ is greater than that of $(1+V_{2:2})^{-1}$, which implies that $U_{2:2} \ge_{st} V_{2:2}$.

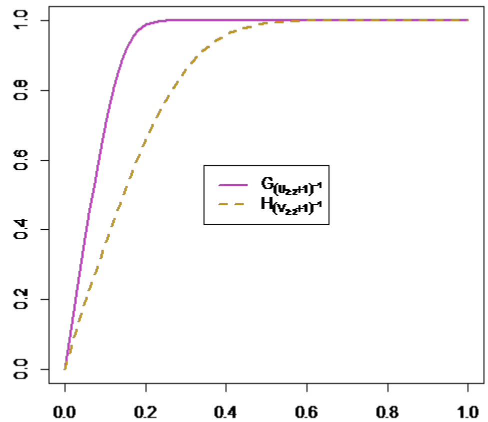

Figure 3: Plots of CDFs of $(1 + U_{2:2})^{-1}$ and $(1 + V_{2:2})^{-1}$.

Under the assumptions $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in B_n$ and $(\boldsymbol{\gamma}, \boldsymbol{\beta}) \in B_n$, the next counterexample represents that the ordering result in Theorem 4.3 may not be obtained.

Counterexample 1. Let $F_1(x) = 1 - e^{-6x}, x > 0$ and let $F_2(x) = 1 - e^{-7x}, x > 0$. Clearly, $X_1 \leq_{st} X_2$. Suppose that $\boldsymbol{\gamma} = (2.3, 3, 3.7)$ and $\boldsymbol{\alpha} = (1.3, 2, 5.7)$ such that $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \varepsilon_+$, $\boldsymbol{\beta} = (3, 1, 1) \in D_+$, and $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$. Furthermore, assume that $\psi_1(x) = (2x+1)^{-\frac{1}{2}}$ and $\psi_2(x) = (5x+1)^{-\frac{1}{5}}$. Simply, it can be denoted that $\psi_1(x)$ is a log-convex function. It can also be seen that

$$\frac{d^2}{dx^2}[\phi_2 \circ \psi_1(x)] = \frac{3}{5}(2x+1)^{-\frac{5}{2}} \geq 0.$$

Therefore, $\phi_2 \circ \psi_1(x)$ is convex in x, which implies that $\phi_2 \circ \psi_1(x)$ is a super-additive function. The plot of $G_{3:3}(x) - H_{3:3}(x)$ is demonstrated in Figure 4, which is equivalent to $U_{3:3} \nleq_{st} V_{3:3}$ and $V_{3:3} \nleq_{st} U_{3:3}$.

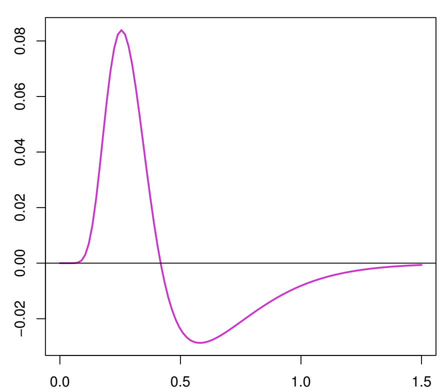

Figure 4: Plots of $G_{3:3}(x) - H_{3:3}(x)$.

The next theorem states that the ordering result holds between the largest order statistics $X_{n:n}$ and $X_{n^*:n^*}$, where $n < n^*$, according to the usual stochastic ordering. It is assumed that the samples share Archimedean copula with a common generator and that the samples are collected from multiple-outlier UGG-F models.

Theorem 4.4. *Let $U_1, \ldots, U_n$ be nonnegative dependent random variables that follow a Archimedean copula with generator function ψ such that $U_i \sim UGG - F_1(\alpha_1, \beta, b, F_1, \psi)$ for $i = 1, \ldots, n_1$ and $U_j \sim UGG - F_2(\alpha_2, \beta, b, F_2, \psi)$ for $j = n_1 + 1, \ldots, n_1 + n_2 (= n)$. Furthermore, suppose that there exist two natural numbers n_1^* and n_2^* such that $n_2 < n_2^* < n_1 < n_1^*$, $n_1 + n_2 = n$, and $n_1^* + n_2^* = n^*$. Then for $(\alpha_1, \alpha_2) \in \varepsilon_+$ and $F_1 \geq F_2$, the condition $(n_1^*, n_2^*) \overset{m}{\preceq} (n_1, n_2)$ implies that $U_{n:n} \leq_{st} U_{n^*:n^*}$.*

Proof. The CDFs of random variables $U_{n:n}$ and $U_{n^*:n^*}$ are, respectively, defined as follows:

$$
\begin{aligned}
G_{n:n}(x) &= \psi\Big(n_1\phi(A_1(\alpha_1)) + n_2\phi(A_2(\alpha_2))\Big) \\
&\overset{def}{=} \eta_1(n_1, n_2, \phi), \qquad x > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
H_{n^*:n^*}(x) &= \psi\Big(n_1^*\phi(A_1(\alpha_1)) + n_2^*\phi(A_2(\alpha_2))\Big) \\
&\overset{def}{=} \eta_2(n_1^*, n_2^*, \phi), \qquad x > 0,
\end{aligned}
$$

where $A_i(\alpha_i) = \left(\frac{b}{b-1+F_i^{-\alpha_i}(x)}\right)^{\beta}$, $i = 1, 2$. To achieve the desired result, it must be shown that $\eta_1(n_1, n_2, \phi)$ is Schur-convex in (n_1, n_2). For this purpose, the partial derivative of the function $\eta_1(n_1, n_2, \phi)$ with respect to n_1 and n_2 can be, respectively, obtained as follows:

$$\frac{\partial \eta_1(n_1, n_2, \phi)}{\partial n_1} = \phi(A_1(\alpha_1))\psi'\Big(n_1\phi(A_1(\alpha_1)) + n_2\phi(A_2(\alpha_2))\Big),$$

and

$$\frac{\partial \eta_1(n_1, n_2, \phi)}{\partial n_2} = \phi(A_2(\alpha_2))\psi'\Big(n_1\phi(A_1(\alpha_1)) + n_2\phi(A_2(\alpha_2))\Big).$$

According to the assumptions $\alpha_1 \leq \alpha_2$ and $F_1 \geq F_2$, it can be easily shown $A_1(\alpha_1) \geq A_2(\alpha_2)$. Also, from the increasing property of functions ψ and ϕ, it can be concluded that $\phi(A_1(\alpha_1)) \leq \phi(A_2(\alpha_2))$. Finally, it follows from Lemma 2.1, for $n_1 \geq n_2$, that

$$(n_1 - n_2)\Big(\frac{\partial \eta_1(n_1, n_2, \phi)}{\partial n_1} - \frac{\partial \eta_1(n_1, n_2, \phi)}{\partial n_2}\Big) \geq 0.$$

By applying the Schur-convex property of the function η_1, the proof is completed.

□

5 Application of the Real Data in the Parallel System Reliability

In this section, we attempt to apply the results of this study to a real dataset in order to analyze two parallel systems with dependent components. For this purpose, we employed the wind speed dataset that was collected by Xie et al. (2012) from two different wind farms, which are denoted as sites A and B. The basic statistics of wind speeds at the two sites are shown in Table 1.

Table 1: The basic statistics of wind speeds at the two sites.

Statistics	Site A	Site B
Mean	4.25	4.14
Standard deviation	2.33	2.35

Eryilmaz and Kan (2020) considered the Rayleigh distribution with CDF, $F_i(x) = 1 - e^{\frac{-x^2}{2\xi_i^2}}, x > 0, i = 1, 2$, as the marginal distribution of the wind speed in the two farms. the maximum likelihood estimation of the parameters at sites A and B for the observed data was $\hat{\xi}_1 = 3.36$ and $\hat{\xi}_2 = 3.42$, respectively. Herein, we contemplated the Rayleigh model to be the baseline CDF with the vectors of shape parameters as $\boldsymbol{\beta}_1 = (0.04, 2.1, 5.66)$, $\boldsymbol{\beta}_2 = (0.65, 2.5, 4.65)$, and $\boldsymbol{\beta}_3 = (2, 2.6, 3.2)$. It should be noted that the selection of these values for shape parameters is arbitrary. Xie et al. (2012) also considered the Gumbel copula

$$c(u_1, u_2, u_3) = e^{-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta} + (-\ln u_3)^{\theta}\right]^{\frac{1}{\theta}}}, \theta \geq 1,$$

for describing the dependency structure of wind speeds. Xie et al. (2012) computed the MLE of this parameter, which was computed $\hat{\theta} = 3.32$. Therefore, based on the real data, the CDF ($U_{3:3}$) for $b = 2$ and $\alpha = 3$ can be expressed as

$$\begin{aligned} G_{3:3}(x) &= \psi\left(\sum_{k=1}^{3} \phi\left(\left(\frac{2}{1 + (F_1(x))^{-3}}\right)^{\beta_k}\right)\right) \\ &= e^{-\left[(-\ln u_1)^{3.32} + (-\ln u_2)^{3.32} + (-\ln u_3)^{3.32}\right]^{\frac{1}{3.32}}}, \end{aligned}$$

where, $u_i = \left(\frac{2}{1+(F_1(x))^{-3}}\right)^{\beta_k}$ for $i = 1, 2, 3$ and $F_1(x) = 1 - e^{\frac{-x^2}{2(3.32)^2}}$, $x > 0$. The curves of this CDF for different shape parameters ($\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$) are demonstrated in Figure 5. As illustrated in this figure, it can be realized that based on the majorization order, the higher the heterogeneity between the parameters, the higher the reliability of the parallel systems.

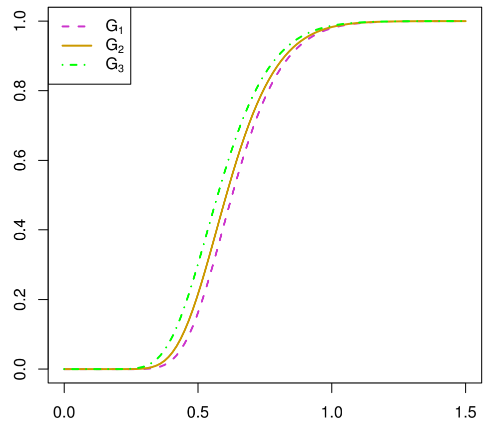

Figure 5: Plot of CDFs of $U_{3:3}$ for different shape parameters $\boldsymbol{\beta}_1 = (0.04, 2.1, 5.66)$, $\boldsymbol{\beta}_2 = (0.65, 2.5, 4.65)$, and $\boldsymbol{\beta}_3 = (2, 2.6, 3.2)$, denoted by G_1, G_2, and G_3, respectively.

6 Analysis of the Obtained Results

In order to compare the results of the experiment described in Section 6 with the obtained theoretical data, we applied the Clayton copula

$$c(u_1, u_2) = \left[\max(u_1^{\theta} + u_2^{\theta} - 1, 0)\right]^{\frac{1}{\theta}}, \theta \geq 0,$$

and Rayleigh distribution as baseline distributions. Wang and Li (2018) estimated the Claytone copula parameters $\hat{\theta} = 0.64$ and Eryilmaz and Kan (2020) obtained the MLE of Rayleigh distribution parameter $\hat{\xi}_1 = 3.36$ and $\hat{\xi}_2 = 3.42$ for wind speed data. Based on the obtained theoretical results of Theorem 6, we plot the curves of reliability functions of $U_{3:3}$ and $V_{3:3}$ under the different parameters as shown in Figure 6. As figure shows, two curves do not interrupt each other which means that the theoretical approach is feasible. Furthermore, for different shape and copula parameters, the stronger heterogeneity among shape parameters according to the majorization order leads to a larger reliability of the parallel system. Therefore, we can conclude that the influence of the results of the experiment and obtained theoretical data on the reliabilities of parallel systems is consistent with theorem 6, so we believe that the theoretical characteristics of the research can be used as the evaluation of system performance in reliability engineering.

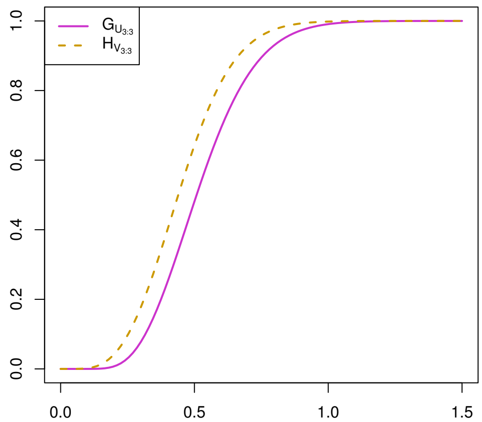

Figure 6: Plots of CDFs of $U_{3:3}$ and $V_{3:3}$ for $\boldsymbol{\beta} = (0.04, 2.1, 5.66)$ and $\boldsymbol{\delta} = (2, 2.6, 3.2)$.

It should be noted that in the first part of this section, the Gumbel copula is used with different shape parameters for the random variable $U_{3:3}$ with the Rayleigh baseline CDF. Whereas, in the second part of this section, the common Clayton copula ($\psi_1 = \psi_2 = \psi$, which yields $\phi_2 \circ \psi_1(x) = x$ is a super-additive function) is used with the generator function $\psi(x) = (0.64x + 1)^{-\frac{1}{0.64}}$. The correctness of Theorem 4.2 with these assumptions is shown in Figure 6.

7 Discussion and Conclusion

In this study, we investigated the stochastic comparison of two parallel systems with independent (dependent) and heterogeneous components from the unit gamma Gompertz distribution under the identical and nonidentical baseline CDFs of the system components. These comparisons are performed based on the usual stochastic order, the reversed hazard rate order, and likelihood ratio order concepts. We also provided a condition for holding the likelihood order in the case where the system components are independent with several outliers. Furthermore, the stochastic comparison of two parallel systems was performed based on the usual stochastic order in the case that the dependency structure between the system components follows the Archimedean copula. In the future study, for samples with dependent unit gamma Gompertz-F components, we can generalize the results of Ma (1997), which compare lifetimes of two $(n-k+1)$ -out-of-n:G systems with heterogeneous dependent and homogeneous dependent populations. We discussed how the heterogeneity of shape parameters in a dependent $UGG - F$ model that follows the Archimedean copula affects the system lifetimes. However, in practice, sometimes system components do not follow the Archimedean copula and have another dependence structure. It is interesting to investigate whether the comparing results under the Archimedean copula scenario still hold with the presence of the Farley–Gamble–Morgenstern (FGM) copula. Naturally, the answer needs to be discussed in a new study. As one important and widely used copula, the FGM copula can model relatively weak dependence. For future study, the stochastic comparison of the $UGG-F$ model could be performed in a situation where the mutual dependency between system components follows the FGM copula.

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