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Statistical Inference on the CR Extropy and DCR Extropy of Equilibrium distribution of order r

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Abstract. This study develops estimators for Cumulative Residual Extropy (CR Extropy) and Dynamic Cumulative Residual Extropy (DCR Extropy) of the equilibrium distribution of order r of Lomax distribution under progressively Type-II censored data. Maximum likelihood and Bayesian estimators are obtained. The Bayesian estimators are evaluated based on LINEX loss functions. Non-informative and informative priors are considered for unknown parameters. By using Lindley's approximation and the Importance Sampling Methods, it is possible to approximate the closed-form expressions of the Bayesian estimators. The confidence intervals for the estimators are calculated using the normal approximation method and bootstrap algorithms. We have conducted simulation studies to evaluate how well the proposed estimators perform. Additionally, we have tested the practical utility of these estimators using a real data.

Keywords. Bayesian Estimation, CR Extropy, DCR Extropy, Lomax Distribution, MLE, Progressively Type-II Censoring.

MSC: 62F15, 62F25.

1 Introduction

Shannon (1948) established the concept of entropy as a way to quantify the uncertainty related to a probability distribution. For non-negative absolutely continuous random variable X having probability density function (pdf) $f(x)$, the Shannon entropy is de-

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fined as

$$H(X) = -\int_0^\infty f(x)\log f(x)dx. \tag{1.1}$$

A new addition to the area of application of entropy measures is reliability analysis. Considering X as the lifetime of a device, the random variable $X_t = (X - t|X > t)$ is the lifetime remaining to the device after attaining age t. developed the idea of entropy of residual life and is defined as

$$H(X,t) = -\int_t^\infty \frac{f(x)}{\bar{F}(x)}\log\frac{f(x)}{\bar{F}(x)}dx. \tag{1.2}$$

Rao et al. (2004) found an alternative measure of uncertainty which is more general than Shannon entropy is called cumulative residual entropy(CRE). The density function is calculated as the derivative of the distribution function. So, the cumulative distribution exhibits a higher degree of regularity than probability density function. The cumulative residual entropy is defined as

$$C(X) = -\int_0^\infty \overline{F}(x)\log\overline{F}(x)dx, \tag{1.3}$$

where $\overline{F}(x)$ is the survival function of X. Asadi and Zohrevand (2007) developed dynamic version of CRE (DCRE) defined as

$$C(X;t) = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}\log\frac{\overline{F}(x)}{\overline{F}(t)}dx. \tag{1.4}$$

Lad et al. (2015) developed the concept of extropy as a complementary dual for Shannon's entropy function and is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{1.5}$$

Also, Qiu and Jia (2018) proposed residual extropy to measure residual uncertainty of a non-negative random variables as

$$J(X,t) = -\frac{1}{2}\int_t^\infty \frac{f^2(x)}{\bar{F}^2(t)}dx. \tag{1.6}$$

Analogous to Cumulative Residual Entropy, Jahanshahi et al. (2020) proposed Cumulative Residual Extropy (CR Extropy) and is defined as

$$\xi J(X) = -\frac{1}{2}\int_0^\infty \bar{F}^2(x)dx. \tag{1.7}$$

developed dynamic version of CR Extropy (DCR Extropy), named it as Dynamic Survival Extropy and is defined as

$$\xi J(X,t) = -\frac{1}{2}\int_t^\infty \frac{\bar{F}^2(x)}{\bar{F}^2(t)}dx. \tag{1.8}$$

the concept of equilibrium distribution, first introduced in renewal theory, has been crucial in establishing numerous theoretical conclusions as well as defining aging concepts, stochastic orderings tests, replacement policies, etc. in reliability theory. Several studies have been done to extend equilibrium distributions to higher orders in order to find generalised ageing classes, stochastic orders, characterizations, and other findings helpful in reliability modelling and analysis. More details can be found in Harkness and Shantaram (1969), Gupta (1979), Pakes (1996), Nanda et al. (1996), Pakes and Navarro (2007), Nair and Preeth (2008, 2009), and Nair and Sankaran (2010).

Suppose a sequence $\{X_r\}, r = 0, 1, 2, ...$ of non-negative absolutely continuous random variables with survival function of X_r as:

$$S_r(x) = \begin{cases} \frac{1}{\mu_{r-1}} \int_x^{\infty} S_{r-1}(t)dt & x > 0, r = 1, 2, ..., \\ 0 & x \leq 0, \end{cases} \tag{1.9}$$

where $\mu_r = \int_0^{\infty} S_r(x)dx < \infty$, is the mean of the random variable X_r associated with $S_r(x)$. Here $X_0 = X, S_0(x) = \overline{F}(x)$ and $\mu_0 = E(X)$. Then the distribution specified by $S_r(x)$ of X_r as the equilibrium distribution of order r of X. Also, the survival function of $S_{r-1}(x)$, $r = 0, 1, 2, ...$ is $S_r(x)$.

Lomax (1954) developed the Lomax distribution as a model for business failure data. When the experimenter assumes that the population is heavy-tailed, it provides an alternative model to common lifetime distributions such as exponential, Weibull, or gamma distributions as discussed by Bryson (1974). It finds applications in several fields, including business, economics, actuarial science, queuing theory, and internet traffic modelling.

The probability density function of a random variable X having the Lomax distribution is given by:

$$f(x) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}; \; x > 0, \; \alpha, \lambda > 0, \tag{1.10}$$

where α is the shape parameter and λ is the scale parameter.

In life testing and reliability experiments, the experimenter may not always obtain complete information about the failure times of all experimental units. Data obtained from such experiments are called censored data. The two most common censoring schemes are termed as Type-I and Type-II censoring schemes. They can be described as follows: consider a reliability experiment in which n identical units are placed on a life test. In the conventional Type-I censoring scheme, the experimenter decides to conduct the life test for a prefixed time T. On the other hand, the conventional Type-II censoring scheme, the experimenter decides to carry out the life test until the prefixed number of units fails $m \leq n$. The mixture of Type-I and Type-II censoring schemes is known as a hybrid censoring scheme.

The fact that conventional Type-I, Type-II, or hybrid censoring schemes do not permit the removal of units at points other than the experiment's endpoint is one of their limitations. the progressively Type-II censoring scheme offers this benefit. It can be described as follows: consider n units in a study and suppose that the integer $m < n$ and $(R_1, R_2, ..., R_m)$ satisfying $R_1 + R_2 + ... + m = n$ are fixed before the experiment. At the time of the first failure, say $X_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, say $X_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the mth failure, say $X_{m:m:n}$, the rest of the R_m units are removed. The detailed explanation of the progressively Type-II censoring scheme is given by Balakrishnan et al. (2000).

this paper aims to define and study the estimators of CR Extropy and DCR Extropy of equilibrium distribution of order r of Lomax distribution under progressively Type-II censored data using different techniques.

The rest of the paper is organized as follows: We define higher order cumulative residual extropies in section 2. In section 3, we present the parameter estimation of equilibrium distribution of order r of Lomax distribution under progressively Type-II censored data. Specifically, maximum likelihood estimation and Bayesian estimation methods are carried out. We construct asymptotic confidence intervals using normal approximation (NA) of the MLE and boot-p methods in section 4. In sections 5 and 6, the simulation study and the real data analysis are carried out to illustrate theoretical results respectively.

2 Higher Order Cumulative Residual Extropies

Analogous to the definition of CRE of order r developed by Unnikrishnan et al. (2021), we propose the cumulative residual extropy of order r ($CRExtropy_r$) as

$$\xi J_r(X) = -\frac{1}{2}\int_0^\infty (S_r(x))^2 dx, \quad r = 0, 1, 2, ..., \tag{2.1}$$

for a non-negative absolutely continuous random variable with $E(X^r) < \infty$. By virtue of (2.1) the usual CR Extropy of X is ξJ_0. The density function of X_r being f_r, we also have the extropy of order r as

$$\begin{aligned} J_r(X) &= -\frac{1}{2}\int_0^\infty (S_r(x))^2 dx \\ &= -\frac{1}{2}\int_0^\infty \Big(\frac{S_{r-1}(x)}{\mu_{r-1}}\Big)^2 dx \\ &= \frac{\xi J_{r-1}(X)}{\mu_{r-1}^2}, \end{aligned} \tag{2.2}$$

developing the relationship between CR Extropy of order r and extropy of order r. In particular, $r = 0$ gives the relationship between extropy and cumulative residual extropy.

Example: The survival function of a random variable having the Lomax distribution is given by

$$\overline{F}(x) \ = \ \left(1+\frac{x}{\lambda}\right)^{-\alpha}; \ \ x>0, \ \alpha, \lambda>0,$$

where α is the shape parameter and λ is the scale parameter. Some direct calculations give

$$S_r(x) = \left(1+\frac{x}{\lambda}\right)^{-(\alpha-r)}, \ \ x>0, \ \alpha>r, \ \lambda>0, \tag{2.3}$$

and

$$\mu_r(x) = \frac{\lambda}{(\alpha-r-1)}.$$

Thus

$$\xi J_r(X) = -\frac{1}{2}\frac{\lambda}{(2(\alpha-r)-1)}. \tag{2.4}$$

And, by using (2.2)

$$J_r(X) = -\frac{1}{2}\frac{\alpha-r}{\lambda(2(\alpha-r)+1)}.$$

Similarly, the cumulative extropy of the residual life of X_r, $X_{r,t} = (X_r - t|X>t)$ is called Dynamic Cumulative Residual Extropy of order r ($DCRExtropy_r$) and is written as:

$$\xi J_r(X,t) = -\frac{1}{2}\int_t^{\infty}\left(\frac{S_r(x)}{S_r(t)}\right)^2 dx. \tag{2.5}$$

For Lomax distribution, dynamic CR Extropy of order r is

$$\xi J_r(X,t) = -\frac{1}{2}\frac{\lambda(1+(t/\lambda))}{(2(\alpha-r)-1)}. \tag{2.6}$$

3 Parametric Estimation

It is clear that for practical purposes, we need to develop some inference techniques about the measures defined in (2.4) and (2.6). Maximum likelihood method and Bayesian method are using to study the estimators of CR Extropy and DCR Extropy of equilibrium distribution of order r of Lomax distribution under progressively Type-II censored data.

3.1 Maximum Likelihood Estimation

Denote $x_i = X_{i:m:n}$, $i = 1,2,...,m$. Let $X = (x_1, x_2, ..., x_m)$ be a progressive Type-II censored sample with associated scheme $(R_1, R_2, ..., R_m)$. The likelihood function of progressively Type-II censored data is defined as

$$L(\alpha,\lambda|X) = K\prod_{i=1}^{m} f(x_i;\alpha,\lambda) \ \ [1-F(x_i;\alpha,\lambda)]^{R_i}, \tag{3.1}$$

where $\alpha, \lambda > 0$ and $K = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - \sum_{j=1}^{m-1}(R_j + 1))$. By using (2.3) and its derivative, we have the likelihood function of α^*(for simplicity we take $\alpha - r = \alpha^*$) and λ based on progressively Type-II censored sample X as follows:

$$L(\alpha^*, \lambda|X) = K\Big(\frac{\alpha^*}{\lambda}\Big)^m \exp\{-(\alpha^* + 1)Z_\lambda(X) - \alpha^* W_\lambda(X)\}. \tag{3.2}$$

where $Z_\lambda(X) = \sum_{i=1}^{m} \log(1 + \frac{x_i}{\lambda})$ and $W_\lambda(X) = \sum_{i=1}^{m} R_i \log(1 + \frac{x_i}{\lambda})$. The log-likelihood function of (3.2) as follows

$$l(\alpha^*, \lambda) = \log L(\alpha^*, \lambda) = \log K + m \log \alpha^* - m \log \lambda - (\alpha^* + 1)Z_\lambda(X) - \alpha^* W_\lambda(X). \tag{3.3}$$

Taking derivatives with respect to α^* in (3.3), and equated it to zero, we obtain the MLE of α^* as follows

$$\widehat{\alpha^*_\lambda} = \frac{m}{Z_\lambda(X) + W_\lambda(X)}. \tag{3.4}$$

Similarly, taking derivative with respect to λ in (3.3), and equated it to zero, we obtain the MLE of λ as follows

$$\widehat{\lambda} = \frac{m}{(\widehat{\alpha^*_\lambda} + 1)\sum_{i=1}^{m} \frac{x_i}{\lambda(\lambda + x_i)} + \widehat{\alpha^*_\lambda} \sum_{i=1}^{m} \frac{R_i x_i}{\lambda(\lambda + x_i)}}. \tag{3.5}$$

Note that (3.5) can be written in the form

$$\widehat{\lambda} = h(\lambda) \tag{3.6}$$

where $h(\lambda) = \frac{m}{(\widehat{\alpha^*_\lambda}+1)\sum_{i=1}^{m} \frac{x_i}{\lambda(\lambda+x_i)} + \widehat{\alpha^*_\lambda}\sum_{i=1}^{m} \frac{R_i x_i}{\lambda(\lambda+x_i)}}$. From (3.6), we use a simple iterative scheme proposed by Kundu (2007) to solve for λ. Start with an initial guess of λ, say $\lambda^{(0)}$, then obtain $\lambda^{(1)} = h(\lambda^{(0)})$ and proceed in this way iteratively to obtain $\lambda^{(n+1)} = h(\lambda^{(n)})$. Stop the iterative procedure, when $|\lambda^{(n+1)} - \lambda^{(n)}| < \epsilon$, some preassigned tolerance limit.

Thus, from (3.4) and (3.5), the maximum likelihood estimation of the CR Extropy and DCR Extropy of equilibrium distribution of order r using progressively Type-II censored data can be easily established as follows:

$$\widehat{\xi J_r}(X) = -\frac{1}{2}\frac{\widehat{\lambda}}{(2\widehat{\alpha^*_\lambda} - 1)}$$

$$\widehat{\xi J_r}(X, t) = -\frac{1}{2}\frac{\widehat{\lambda}(1 + (\frac{t}{\lambda}))}{(2\widehat{\alpha^*_\lambda} - 1)}.$$

3.2 Bayesian Estimation

Prior distributions and loss functions are significant while studying estimation problems from a Bayesian approach. In particular, loss functions are used to describe Bayesian statistical problems because they specify what constitutes a 'good' estimator or 'excellent' prediction. There are lots of symmetric and asymmetric loss functions that exist. Here we consider LINEX (linear-exponential) loss function. Let δ be an approximate estimator to an unknown parameter θ. Then, LINEX loss function is given by

$$L_l(\theta,\delta) = \exp\{p(\delta-\theta)\} - p(\delta-\theta) - 1,\ p \neq 0.$$

From the posterior distributions, the Baye estimates of the unknown parameter θ under the loss functions $L_l(\theta,\delta)$ can be obtained as

$$\widehat{\theta_l} = -p^{-1}\log\left[E_\theta(\exp\{p\theta\}|x)\right],\ p \neq 0. \tag{3.7}$$

It can be challenging to choose a prior for the unknown parameters. In fact, there is no established procedure for selecting a suitable prior for Bayesian estimation (Arnold et al. (1983)). Prior distributions can be divided into two, Informative and Non-informative. Presumably, each person has their own subjective prior and must contend with all of its features. However, it is preferable to employ informative priors, which are undoubtedly preferred over all other options, if we have sufficient knowledge of the parameters. Otherwise, using ambiguous or non-informative priors may be appropriate. First we consider non-informative priors for both α^* and λ as

$$g_1(\alpha^*) = \frac{1}{\alpha^*}, \tag{3.8}$$
$$h_1(\lambda) = \frac{1}{\lambda}. \tag{3.9}$$

The joint posterior distribution of α^* and λ can be obtained as

$$\Pi_1(\alpha^*,\lambda|X) = \frac{1}{K_1}(\alpha^*)^{m-1}\lambda^{-(m+1)}\exp\{-\alpha^*(Z_\lambda(X)+W_\lambda(X)) - Z_\lambda(X)\}, \tag{3.10}$$

where

$$K_1 = \int_0^\infty\int_0^\infty(\alpha^*)^{m-1}\lambda^{-(m+1)}\exp\{-\alpha^*(Z_\lambda(X)+W_\lambda(X))\}\ \exp\{-Z_\lambda(X)\}\,d\alpha^* d\lambda.$$

Making use of (3.7) and (3.10) , the Bayesian estimate of the $\xi J_r(X)$ with respect to the LINEX loss functions is obtained as

$$\xi J_r^l(X) = -\frac{1}{p}\Big[\int_0^\infty\int_0^\infty \exp\{-pC_r\}(\alpha^*)^{m-1}\lambda^{-(m+1)}\exp\{-\alpha^*(Z_\lambda(X)+W_\lambda(X)) - Z_\lambda(X)\}\,d\alpha^* d\lambda\Big]. \tag{3.11}$$

Secondly, due to flexibility, here we assume gamma prior for α^* and inverse gamma prior for λ as

$$g_2(\alpha^*,a,b) = \frac{b^a}{\Gamma a}(\alpha^*)^{a-1}e^{-b\alpha^*}, \alpha^* > 0,\ a,b > 0, \tag{3.12}$$
$$h_2(\lambda,c,d) = \frac{d^c}{\Gamma c}\lambda^{-(c+1)}e^{\frac{-d}{\lambda}},\ \lambda > 0,\ c,d > 0. \tag{3.13}$$

After some calculations, the joint posterior distribution of α^* and λ can be obtained as

$$\Pi_2(\alpha^*, \lambda|X) = \frac{1}{K}(\alpha^*)^{m+a-1}\lambda^{-(m+c+1)} \exp\{-\alpha^*(Z_\lambda(X) + W_\lambda(X) + b) - \frac{d}{\lambda} - Z_\lambda(X)\}, \quad (3.14)$$

where

$$K_2 = \int_0^\infty \int_0^\infty (\alpha^*)^{m-1}\lambda^{-(m+1)} \exp\{-\alpha^*(Z_\lambda(X) + W_\lambda(X))\} \exp\{-Z_\lambda(X)\} \, d\alpha^* d\lambda.$$

Making use of (3.7) and (3.14) , the Bayesian estimate of the $\xi J_r(X)$ with respect to the LINEX loss function is obtained as

$$\xi J_r^l(X) = -\frac{1}{p}\Big[\int_0^\infty \int_0^\infty \exp\{-pC_r\}(\alpha^*)^{D+a-1}\lambda^{-(D+c+1)} \times \exp\{-\alpha^*(Z_\lambda(X) + W_\lambda(X) + b) - \frac{d}{\lambda} - Z_\lambda(X)\} \, d\alpha^* d\lambda\Big]. \quad (3.15)$$

Also, we may obtain the Bayes estimates of the $\xi J_r(X, t)$ with respect to the above mentioned loss function and prior distributions by using a similar technique. It should be noted that the closed form solutions of the ratio of the integrals (3.11) and (3.15) are difficult to calculate. Thus, we use Lindley's approximation technique.

Lindley's Approximation Method: Lindley (1980) developed approximate method for evaluating the posterior expectaion of $U(\theta)$ as

$$E(U(\theta)|x) = \int U(\theta)e^{l(\theta)+\rho(\theta)}d\theta \Big/ \int e^{l(\theta)+\rho(\theta)}d\theta,$$

is the Bayes estimate of $U(\theta)$ under any of the loss function(eg: Squared error loss function, LINEX loss function, etc), where $\rho(\theta) = \log(p(\theta))$, $p(\theta)$ is arbitrary function of θ, and $l(\theta)$ is the logarithm of the likelihood function. In the two parameter case, when $\theta = (\theta_1, \theta_2)$, Lindley's approximately form reduces to the following form:

$$E(U(\theta)|x) = U(\theta) + A/2 + \rho_1 A_{12} + \rho_2 A_{21} + (1/2)\left[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}\right], \quad (3.16)$$

where $A = \sum_{i=1}^2 \sum_{i=1}^2 U_{ij}\sigma_{ij}$, $l_{\eta\epsilon} = (\partial^{\eta+\epsilon} l/\partial\theta_1^\eta \partial\theta_2^\epsilon)$, $\eta,\ \epsilon = 0, 1, 2, 3,\ \eta + \epsilon = 3$.
For $i, j = 1, 2$, $\rho_i = (\partial\rho/\partial\theta_i)$, $U_i = (\partial U/\partial\theta_i)$, $U_{ij} = (\partial^2 U/\partial\theta_i\partial\theta_j)$, $L_{ij} = (\partial^2 l/\partial\theta_i\partial\theta_j)$. $N = L_{11}L_{22} - L_{12}L_{21}$, $\sigma_{11} = -L_{22}/N$, $\sigma_{22} = -L_{11}/N$, $\sigma_{12} = \sigma_{21} = L_{12}/N$. And for $i \neq j$, $A_{ij} = U_i\sigma_{ii} + U_j\sigma_{ji}$, $B_{ij} = (U_i\sigma_{ii} + U_j\sigma_{ji})\sigma_{ii}$, $C_{ij} = 3U_i\sigma_{ii}\sigma_{ij} + U_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$.

In our case, $\theta = (\alpha^*, \lambda)$, $p_1(\theta) = p_1(\alpha^*, \lambda) = g_1(\alpha^*; a, b)h_1(\lambda; c, d)$, where $g_1(\alpha^*; a, b)$ and $h_1(\lambda; , c, d)$ are given in (3.8) and (3.9). Thus we observe that, $\rho_1(\alpha^*, \lambda) = \log p_1(\alpha^*, \lambda) = -\log\alpha^* - \log\lambda$. And also, $p_2(\theta) = p_2(\alpha^*, \lambda) = g_2(\alpha^*; a, b)h_2(\lambda; c, d)$, where $g_2(\alpha^*; a, b)$ and $h_2(\lambda; , c, d)$ are given in (3.12) and (3.13). Similarly, $\rho_2(\alpha^*, \lambda) = \log p_2(\alpha^*, \lambda) =$

$constant + (a-1)\log\alpha^* - b\alpha^* - (c+1)\log\lambda - d/\lambda$. For the case of the LINEX loss function when estimating $\xi J_r(X)$, we have , $U(\theta) = U(\alpha^*, \lambda) = \exp\{-p\ \xi J_r(X)\}$. Further,

$$\begin{aligned}
U_1 &= \exp\Big(\frac{p\lambda}{2\alpha^*-1}\Big)\Big[\frac{-p\lambda}{(2\alpha^*-1)^2}\Big],\\
U_2 &= \exp\Big(\frac{p\lambda}{2\alpha^*-1}\Big)\Big[\frac{p}{2(2\alpha^*-1)}\Big],\\
U_{11} &= \exp\Big(\frac{p\lambda}{2\alpha^*-1}\Big)\Big[\frac{(p\lambda)^2}{(2\alpha^*-1)^4}+\frac{4p\lambda}{(2\alpha^*-1)^3}\Big],\\
U_{22} &= \exp\Big(\frac{p\lambda}{2\alpha^*-1}\Big)\Big[\frac{p^2}{4(2\alpha^*-1)}\Big],\\
U_{12} = U_{21} &= \exp\Big(\frac{p\lambda}{2\alpha^*-1}\Big)\Big[\frac{-p}{(2\alpha^*-1)^2}+\frac{-p^2\lambda}{2(2\alpha^*-1)^3}\Big].
\end{aligned}$$

Thus, the Bayes estimate of $\xi J_r(X)$ with respect to the LINEX loss function is obtained as

$$\widehat{\xi J}_r^{\,l}(X) = -\frac{1}{p}\log\Big[\exp\{-p\ \xi J_r(X)\} + A/2 + \rho_1 A_{12} + \rho_2 A_{21} + (1/2)\ [l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}]\Big]. \quad (3.17)$$

The Bayes estimates of DCR Extropy $\widehat{\xi J}_r^{\,l}(X,t)$ of Equilibrium distribution of order r with respect to the LINEX loss function can be obtained similarly.

Importance Sampling Method: For the purpose of computing Bayes estimates of the unknown parametric functions provided by (2.4) and (2.6), in this subsection, we examine the importance sampling approach. In this method, we need to rewrite the joint posterior distribution of α^* and λ in (3.14). It is given by

$$\pi(\alpha^*, \lambda|x) \propto IG_\lambda(m+c, d)\ \ G_{\alpha^*|\lambda}(m+a, Z_\lambda(X) + W_\lambda(X) + b)\ \ K(\lambda), \quad (3.18)$$

where $K(\lambda) = e^{-Z_\lambda(X)}\Big(Z_\lambda(X) + W_\lambda(X) + b\Big)^{-(a+m)}$. To obtain Bayes estimates of a function, say $g(\alpha^*, \lambda)$, with regard to the LINEX loss function, follow the procedures below.

Step 1: Generate λ from an inverse gamma distribution with shape parameter $(m+c)$ and scale parameter d^{-1} denoted by $IG_\lambda(m+c, d)$.

Step 2: For a given λ obtained in Step 1, generate α^* from a gamma distribution with shape parameter $(m+a)$ and scale parameter $(Z_\lambda(X) + W_\lambda(X) + b)^{-1}$ denoted by $G_{\alpha^*|\lambda}(m+a, Z_\lambda(X) + W_\lambda(X) + b)$.

Step 3: Repeat Step 1 and Step 2 for N times to obtain $(\alpha_1^*, \lambda_1), (\alpha_2^*, \lambda_2), ..., (\alpha_N^*, \lambda_N)$.

Step 4: The Bayes estimate of a parametric function $g(\alpha^*, \lambda)$ under LINEX loss function is given by

$$\widehat{g}^l_{be} = -\frac{1}{p} \ln\Big[\frac{\sum_{i=1}^N \exp\{-pg(\alpha_i^*, \lambda_i)\}K(\lambda_i)}{\sum_{i=1}^N K(\lambda_i)}\Big]. \tag{3.19}$$

Further, under LINEX loss function, the Bayes estimates of CR Extropy and DCR Extropy can be computed from (3.19) when $g(\alpha^*, \lambda) = \xi J(X)$ and $g(\alpha^*, \lambda) = \xi J(X, t)$, respectively.

4 Interval estimation

In the following section, we derive the asymptotic confidence intervals for the $\xi J_r(X)$ and $\xi J_r(X, t)$ using the normal approximation of the MLE (NA) method.

4.1 NA Method

For obtaining asymptotic confidence intervals, the NA approach is a helpful tool. We need the inverse of the observed Fisher information matrix of α^* and λ to get the $100(1-\delta)\%$ confidence intervals for $\xi J_r(X)$ and $\xi J_r(X, t)$. This is given by

$$\widehat{I}^{-1}(\widehat{\alpha^*}, \widehat{\lambda}) = \begin{pmatrix} -l_{20} & -l_{11} \\ -l_{11} & -l_{02} \end{pmatrix}^{-1} = \begin{pmatrix} Var(\widehat{\alpha^*}) & Cov(\widehat{\alpha*}, \widehat{\lambda}) \\ Cov(\widehat{\alpha^*}, \widehat{\lambda}) & Var(\widehat{\lambda}) \end{pmatrix} = \begin{pmatrix} \eta_{20} & \eta_{11} \\ \eta_{11} & \eta_{02} \end{pmatrix},$$

where

$$\begin{aligned} l_{20} &= \frac{m}{(\alpha^*)^2}, \\ l_{11} &= \sum_{i=1}^m (1+R_i)\frac{x_i}{\lambda(\lambda+x_i)}, \\ l_{02} &= \frac{m}{\lambda^2} - (\alpha^*+1)\sum_{i=1}^m \frac{(2\lambda+x_i)x_i}{(\lambda(\lambda+x_i))^2} - \alpha^* \sum_{i=1}^m \frac{(2\lambda+x_i)x_iR_i}{(\lambda(\lambda+x_i))^2}. \end{aligned}$$

Moreover, the approximate estimated variances of $\xi J_r(X)$ and $\xi J_r(X, t)$ are given by

$$\widehat{Var}(\widehat{\xi J}_r(X)) = \Big[Q'_{\xi J_r(X)}\widehat{I}^{-1}(\widehat{\alpha^*}, \widehat{\lambda})Q_{\xi J_r}\Big] \text{ and } \widehat{Var}(\widehat{\xi J}_r(X,t)) = \Big[Q'_{\xi J_r(X,t)}\widehat{I}^{-1}(\widehat{\alpha^*}, \widehat{\lambda})Q_{\xi J_r(X,t)}\Big].$$

Therefore, $\frac{\widehat{\xi J}_r(X) - \xi J_r(X)}{\sqrt{\widehat{Var}(\widehat{\xi J}_r(X))}}$ and $\frac{\widehat{\xi J}_r(X,t) - \xi J_r(X,t)}{\sqrt{\widehat{Var}(\widehat{\xi J}_r(X,t))}}$ asymptotically follow standard normal distribution. Thus, the $100(1-\delta)\%$ asymptotic confidence intervals for the $\xi J_r(X)$ and $\xi J_r(X, t)$ are respectively given by

$$\widehat{\xi J}_r(X) \pm Z_{\delta/2}\sqrt{\widehat{Var}(\widehat{\xi J}_r(X))} \text{ and } \widehat{\xi J}_r(X,t) \pm Z_{\delta/2}\sqrt{\widehat{Var}(\widehat{\xi J}_r(X,t))},$$

where $Z_{\delta/2}$ denotes the upper $(\delta/2)$th percentile of the standard normal distribution.

In the following subsection, we construct the percentile bootstarp (Boot-p) (see Efron and Tibshirani (1986)) confidence intervals for the CR Extropy and DCR Extropy given in (2.4) and (2.6). Although it can be used to calibrate hypothesis tests or estimate the bias and variance of an estimator, the bootstrap-based methods are frequently employed to estimate confidence intervals. We outline the procedures needed to estimate the confidence intervals using Boot-p method in the subsection below.

4.2 Boot-p Method

Step 1: From Equations (3.4) and (3.5) under the original datasets $x_i, i = 1, 2, ..., m$, we obtain $\widehat{\alpha}^*$ and $\widehat{\lambda}$, and then compute the estimates of the CR Extropy and DCR Extropy measures. The estimates for the estimands given by (2.4) and (2.6) are denoted by $\widehat{\xi J_r}(X)$ and $\widehat{\xi J_r}(X,t)$, respectively. The algorithm described in Balakrishnan and Sandhu (1995) is used for the generation purpose.

Step 2: Generate a bootstrap sample $x^* = (x_1^*, x_2^*, ..., x_m^*)$ from Step 1 based on the prespecified censoring scheme and compute the bootstrap estimates $\widehat{\xi J}_r^*(X)$ and $\widehat{\xi J}_r^*(X,t)$.

Step 3: Repeat Step 2 for N = 1000 times. The we obtain $\widehat{\xi J}_r^{*1}(X), \widehat{\xi J}_r^{*2}(X), ..., \widehat{\xi J}_r^{*1000}(X)$ and $\widehat{\xi J}_r^{*1}(X,t), \widehat{\xi J}_r^{*2}(X,t), ..., \widehat{\xi J}_r^{*1000}(X,t)$.

Step 4: Arrange $\widehat{\xi J}_r^{*i}(X)$ and $\widehat{\xi J}_r^{*i}(X,t)$, i=1, 2, ..., 1000 in ascending order and denote $\widehat{\xi J}_r^{*(1)}(X), \widehat{\xi J}_r^{*(2)}(X), ..., \widehat{\xi J}_r^{*(1000)}(X)$ and $\widehat{\xi J}_r^{*(1)}(X,t), \widehat{\xi J}_r^{*(2)}(X,t), ..., \widehat{\xi J}_r^{*(1000)}(X,t)$, respectively.

Then, the $100(1-\gamma)\%$ boot-p confidence intervals for CR Extropy and DCR Extropy entropies are respectively given by

$$\left(\widehat{\xi J}_r^{*(i\frac{\gamma}{2})}(X),\ \widehat{\xi J}_r^{*(i(1-\frac{\gamma}{2}))}(X)\right) \text{ and } \left(\widehat{\xi J}_r^{*(i\frac{\gamma}{2})}(X,t),\ \widehat{\xi J}_r^{*(i(1-\frac{\gamma}{2}))}(X,t)\right).$$

Note that when N = 1000, the bootstrap percentile confidence intervals of $\xi J_r(X)$ and $\xi J_r(X,t)$ at 95% level of confidence are $\left(\widehat{\xi J}_r^{*(25)}(X),\ \widehat{\xi J}_r^{*(975))}(X)\right)$ and $\left(\widehat{\xi J}_r^{*(25)}(X,t),\ \widehat{\xi J}_r^{*(975))}(X,t)\right)$, respectively.

5 Simulation Study

In this section we carried out a numerical study of the measures defined in (2.5) and (2.6). The MLEs and Bayes estimates are compared based on their mean squared errors and average values. In this purpose 1000 progressively Type-II censored samples are used in each simulation. All the computations are performed using Monte Carlo simulations via the statistical software *RStudio 2022.07.2*. Further, for the generation of the progressively Type-II censored samples, we use the algorithm provided by Balakrishnan and Sandhu (1995). Here, we consider two different sample sizes ($n = 30$ and 100), and the samples are 10% and 40% censored. For the simulation purpose, the parameter values are taken to be $\alpha = 7$ and $\lambda = 2.75$. Further, the notation $(0^3, 2)$ used in tables denotes the censoring scheme $(0, 0, 0, 2)$. The Bayes estimates with respect to the LINEX loss function are computed for the values of $p = 20, 25$. The values of the hyper-parameters are taken as $a = b = 1.25$, $c = d = 1.75$ for convenience and $t = 1.8$. The simulation results are presented in Tables 1-10. Before discussing the findings based on the numerical data in the tables, each table (Tables 1-4) contains 11 columns.The first column has the value of r. The second column has the average values and MSEs (in parenthesis) of the MLEs, while the third through sixth columns provide the Bayes estimates of CR Extropy. Similarly, the table 4-8 gives the MLEs and Bayes estimates of DCR Extropy. The values of $\xi J_r(X)$ and $\xi J_r(X, t)$ can be calculated for the values from $r = 0$ to $r = \alpha - 2$, but a small portion of the outputs are tabulated in Tables 1-8 for convenience. The actual values of $\xi J_r(X)$ of orders 0,1,2, and 3 are as follows $\xi J_0(X) = -0.1096$, $\xi J_1(X) = -0.1308$, $\xi J_2(X) = -0.1603$, $\xi J_3(X) = -0.1945$. Similarly, the actual value of $\xi J_r(X, t)$ of order 0,1,2,3 respectively as follows $\xi J_0(X, t) = -0.1783$, $\xi J_1(X, t) = -0.2120$, $\xi J_2(X, t) = -0.2601$, $\xi J_3(X, t) = -0.3203$. Now, we provide findings made from the presented tables.

1. From the sensitivity analysis, we observe that MLE consistently underestimates the true values, while Lindley approximation and Important Sampling Methods tend to provide estimates closer to the true values. The choice of prior (informative or non-informative) and the parameter p also influence the accuracy of the estimates.

2. the average values and MSEs of the maximum likelihood and Bayesian estimates of the $\xi J_r(X)$ and $\xi J_r(X, t)$ for various choices of (n, m) and censoring techniques are shown in Tables 1 to 8. In terms of the MSEs, we observe that the Bayes estimates by importance sampling method perform better than the MLEs in general. In particular, Bayes estimates by importance sampling method shows better result when $p = 20$.

3. Increasing the censoring percentage does not result in significantly higher differences in the estimates.

4. In general, we observe that when the value of r increases the value of both $\xi J_r(X)$ and $\xi J_r(X, t)$ also increases.

5. We present the average length of 95% confidence intervals of the $\xi J_r(X)$ and $\xi J_r(X, t)$ in Table 9 and 10 for different choices of (n, m) and censoring schemes. The average interval lengths were determined using two different techniques. The boot-p technique performs the best among them. Also, confidence interval length increases when r value increases.

Table 1: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X)$ when $(n, m) = (30, 27)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
			Scheme= $(3, 0^{26})$				
0	-0.1782	-0.3712	-0.3265	-0.3741	-0.3296	-0.1091	-0.1504
	(0.0053)	(0.0866)	(0.0593)	(0.0875)	(0.0598)	(0.0006)	(0.0213)
1	-0.1890	-0.3834	-0.3410	-0.3882	-0.3452	-0.1324	-0.1807
	(0.0061)	(0.0813)	(0.0557)	(0.0834)	(0.0571)	(0.0009)	(0.0243)
2	-0.2359	-0.4079	-0.3667	-0.4104	-0.3690	-0.1675	-0.2509
	(0.0070)	(0.0784)	(0.0537)	(0.0776)	(0.0533)	(0.0019)	(0.0295)
3	-0.3596	-0.4224	-0.3854	-0.4260	-0.3883	-0.2131	-0.2701
	(0.0302)	(0.0649)	(0.0448)	(0.0683)	(0.0470)	(0.0066)	(0.0306)
			Scheme= $(0^{26}, 3)$				
0	-0.1297	-0.3963	-0.3469	-0.3931	-0.3448	-0.1058	-0.1592
	(0.0007)	(0.1012)	(0.0689)	(0.0981)	(0.0669)	(0.0005)	(0.0030)
1	-0.1462	-0.4028	-0.3565	-0.4046	-0.3583	-0.1255	-0.2489
	(0.0009)	(0.0931)	(0.0635)	(0.0921)	(0.0629)	(0.0008)	(0.0153)
2	-0.2285	-0.4309	-0.3853	-0.4375	-0.3907	-0.1584	-0.2726
	(0.0059)	(0.0898)	(0.0614)	(0.0925)	(0.0632)	(0.0016)	(0.0164)
3	-0.2460	-0.4490	-0.4067	-0.4560	-0.4124	-0.1948	-0.3655
	(0.0060)	(0.0793)	(0.0545)	(0.0827)	(0.0567)	(0.0031)	(0.0325)
			Scheme= $(1^3, 0^{24})$				
0	-0.1835	-0.3661	-0.3223	-0.3719	-0.3278	-0.1098	-0.1656
	(0.0061)	(0.0840)	(0.0575)	(0.0860)	(0.0588)	(0.0006)	(0.0036)
1	-0.2339	-0.3880	-0.3448	-0.3891	-0.3459	-0.1327	-0.1962
	(0.0118)	(0.0816)	(0.0560)	(0.0825)	(0.0565)	(0.0010)	(0.0051)
2	-0.2525	-0.4024	-0.3624	-0.4013	-0.3616	-0.1663	-0.2077
	(0.0199)	(0.0731)	(0.0503)	(0.0724)	(0.0498)	(0.0023)	(0.0132)
3	-0.2528	-0.4294	-0.3910	-0.4249	-0.3874	-0.2052	-0.2442
	(0.0248)	(0.0691)	(0.0476)	(0.0673)	(0.0464)	(0.0038)	(0.0138)
			Scheme= $(0^{24}, 1^3)$				
0	-0.1289	-0.3876	-0.3400	-0.3940	-0.3455	-0.1065	-0.1845
	(0.0007)	(0.0959)	(0.0654)	(0.0988)	(0.0673)	(0.0005)	(0.0064)
1	-0.1520	-0.4078	-0.3607	-0.4119	-0.3642	-0.1266	-0.1971
	(0.0009)	(0.0943)	(0.0644)	(0.0964)	(0.0658)	(0.0008)	(0.0093)
2	-0.2124	-0.4208	-0.3772	-0.4355	-0.3891	-0.1559	-0.2114
	(0.0037)	(0.0822)	(0.0564)	(0.0922)	(0.0630)	(0.0018)	(0.0105)
3	-0.3248	-0.4497	-0.4073	-0.4579	-0.4139	-0.1948	-0.2347
	(0.0191)	(0.0798)	(0.0547)	(0.0854)	(0.0584)	(0.0031)	(0.0127)

Table 2: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X)$ when $(n,m) = (30,18)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
Scheme=$(12, 0^{17})$							
0	-0.1152	-0.3859	-0.3424	-0.3604	-0.3194	-0.1123	-0.1365
	(0.0125)	(0.0946)	(0.0651)	(0.0874)	(0.0605)	(0.0012)	(0.0118)
1	-0.1731	-0.3906	-0.3518	-0.3818	-0.3393	-0.1390	-0.1502
	(0.0126)	(0.0872)	(0.0605)	(0.0833)	(0.0570)	(0.0021)	(0.0124)
2	-0.2326	-0.3931	-0.3580	-0.3974	-0.3582	-0.1734	-0.2410
	(0.0169)	(0.0725)	(0.0507)	(0.0741)	(0.0508)	(0.0084)	(0.0188)
3	-0.2921	-0.4270	-0.3886	-0.4276	-0.3895	-0.2098	-0.2749
	(0.0224)	(0.0699)	(0.0481)	(0.0680)	(0.0469)	(0.0098)	(0.0193)
Scheme=$(0^{17}, 12)$							
0	-0.1355	-0.4204	-0.3731	-0.4014	-0.3504	-0.1103	-0.1398
	(0.0011)	(0.1222)	(0.0832)	(0.1126)	(0.0764)	(0.0010)	(0.0032)
1	-0.1919	-0.4214	-0.3829	-0.4243	-0.3736	-0.1322	-0.1439
	(0.0049)	(0.1139)	(0.0793)	(0.1124)	(0.0763)	(0.0015)	(0.0108)
2	-0.2072	-0.4511	-0.4008	-0.4606	-0.4090	-0.1682	-0.1856
	(0.0136)	(0.1079)	(0.0733)	(0.1122)	(0.0762)	(0.0026)	(0.0187)
3	-0.3294	-0.4845	-0.4350	-0.4847	-0.4353	-0.2072	-0.2111
	(0.0229)	(0.1054)	(0.0717)	(0.1034)	(0.0704)	(0.0048)	(0.0195)
Scheme= $(4^3, 0^{15})$							
0	-0.1781	-0.3830	-0.3404	-0.3677	-0.3243	-0.1129	-0.2457
	(0.0055)	(0.0942)	(0.0650)	(0.0906)	(0.0623)	(0.0010)	(0.0207)
1	-0.1946	-0.3903	-0.3526	-0.3889	-0.3453	-0.1421	-0.2717
	(0.0109)	(0.0866)	(0.0603)	(0.0852)	(0.0583)	(0.0031)	(0.0229)
2	-0.2015	-0.3998	-0.3678	-0.4071	-0.3660	-0.1830	-0.2846
	(0.0130)	(0.0799)	(0.0563)	(0.0772)	(0.0530)	(0.0045)	(0.0239)
3	-0.3398	-0.4272	-0.3887	-0.4276	-0.3895	-0.2056	-0.3229
	(0.0266)	(0.0704)	(0.0484)	(0.0681)	(0.0469)	(0.0132)	(0.0240)
Scheme=$(0^{15}, 4^3)$							
0	-0.1449	-0.4164	-0.3683	-0.4151	-0.3618	-0.1111	-0.1692
	(0.0118)	(0.1184)	(0.0813)	(0.1205)	(0.0816)	(0.0009)	(0.0143)
1	-0.1522	-0.4239	-0.3763	-0.4238	-0.3734	-0.1338	-0.1869
	(0.0120)	(0.1074)	(0.0750)	(0.1080)	(0.0794)	(0.0015)	(0.0144)
2	-0.1954	-0.4508	-0.4003	-0.4647	-0.4122	-0.1662	-0.1999
	(0.0124)	(0.1104)	(0.0749)	(0.1069)	(0.0792)	(0.0025)	(0.0229)
3	-0.2664	-0.4859	-0.4360	-0.4824	-0.4335	-0.2043	-0.2554
	(0.0216)	(0.1052)	(0.0716)	(0.1026)	(0.0698)	(0.0050)	(0.0261)

Table 3: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X)$ when $(n, m) = (100, 90)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
				Scheme= $(10, 0^{89})$			
0	-0.1633	-0.3843	-0.3382	-0.3747	-0.3205	-0.1103	-0.1953
	(0.0031)	(0.0890)	(0.0609)	(0.0806)	(0.0554)	(0.0002)	(0.0076)
1	-0.2198	-0.3845	-0.3401	-0.3814	-0.3400	-0.1311	-0.1982
	(0.0083)	(0.0753)	(0.0518)	(0.0759)	(0.0522)	(0.0003)	(0.0088)
2	-0.2254	-0.3851	-0.3406	-0.3975	-0.3444	-0.1624	-0.2167
	(0.0107)	(0.0576)	(0.0399)	(0.0609)	(0.0421)	(0.0006)	(0.0090)
3	-0.3021	-0.3915	-0.3478	-0.3988	-0.3486	-0.1974	-0.245
	(0.0124)	(0.0426)	(0.0298)	(0.0455)	(0.0319)	(0.0009)	(0.0131)
				Scheme= $(0^{89}, 10)$			
0	-0.1577	-0.4122	-0.3605	-0.3914	-0.3418	-0.1042	-0.1630
	(0.0025)	(0.1024)	(0.0699)	(0.0935)	(0.0626)	(0.0002)	(0.0031)
1	-0.1675	-0.4249	-0.3749	-0.4025	-0.3550	-0.1245	-0.1711
	(0.0036)	(0.1000)	(0.0682)	(0.0915)	(0.0619)	(0.0003)	(0.0049)
2	-0.2109	-0.4385	-0.3917	-0.4342	-0.3882	-0.1528	-0.1956
	(0.0011)	(0.0897)	(0.0614)	(0.0868)	(0.0595)	(0.0004)	(0.0115)
3	-0.2162	-0.4396	-0.3976	-0.4435	-0.4024	-0.1848	-0.2425
	(0.0109)	(0.0716)	(0.0493)	(0.0772)	(0.0530)	(0.0007)	(0.0128)
				Scheme= $(5^2, 0^{88})$			
0	-0.1731	-0.3867	-0.3401	-0.3717	-0.3113	-0.1095	-0.1626
	(0.0043)	(0.0935)	(0.0638)	(0.0823)	(0.0565)	(0.0002)	(0.0030)
1	-0.2413	-0.3815	-0.3408	-0.3752	-0.3350	-0.1321	-0.1740
	(0.0127)	(0.0759)	(0.0522)	(0.0710)	(0.0490)	(0.0003)	(0.0021)
2	-0.2498	-0.3766	-0.3419	-0.3790	-0.3441	-0.1615	-0.2222
	(0.0128)	(0.0588)	(0.0407)	(0.0604)	(0.0418)	(0.0005)	(0.0042)
3	-0.3159	-0.3675	-0.3420	-0.3876	-0.3446	-0.1992	-0.2352
	(0.0156)	(0.0457)	(0.0321)	(0.0449)	(0.0316)	(0.0010)	(0.0121)
				Scheme= $(0^{88}, 5^2)$			
0	-0.1567	-0.4261	-0.3717	-0.3915	-0.3424	-0.1043	-0.1510
	(0.0024)	(0.1150)	(0.0782)	(0.0955)	(0.0689)	(0.0002)	(0.0019)
1	-0.1585	-0.4283	-0.3736	-0.4104	-0.3639	-0.1246	-0.1912
	(0.0030)	(0.0968)	(0.0662)	(0.0919)	(0.0646)	(0.0002)	(0.0039)
2	-0.1971	-0.4407	-0.3934	-0.4340	-0.3813	-0.1520	-0.2000
	(0.0036)	(0.0908)	(0.0622)	(0.0910)	(0.0622)	(0.0004)	(0.0106)
3	-0.2382	-0.4419	-0.4011	-0.4478	-0.4059	-0.1852	-0.2311
	(0.0118)	(0.0736)	(0.0506)	(0.0774)	(0.0532)	(0.0123)	(0.0007)

Table 4: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X)$ when $(n, m) = (100, 60)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
			Scheme=$(0^{59}, 40)$				
0	-0.1449	-0.3859	-0.3394	-0.3787	-0.3137	-0.1100	-0.1644
	(0.0116)	(0.0905)	(0.0619)	(0.0850)	(0.0583)	(0.0003)	(0.0033)
1	-0.1680	-0.3932	-0.3494	-0.3923	-0.3387	-0.1313	-0.2246
	(0.0117)	(0.0834)	(0.0571)	(0.0817)	(0.0561)	(0.0004)	(0.0090)
2	-0.2406	-0.4037	-0.3637	-0.3961	-0.3576	-0.1627	-0.2520
	(0.0171)	(0.0717)	(0.0494)	(0.0674)	(0.0465)	(0.0008)	(0.0091)
3	-0.2576	-0.4039	-0.3695	-0.3986	-0.3662	-0.2021	-0.2807
	(0.0174)	(0.0518)	(0.0360)	(0.0538)	(0.0374)	(0.0017)	(0.0098)
			Scheme=$(40, 0^{59})$				
0	-0.1519	-0.4554	-0.3951	-0.4006	-0.3413	-0.1075	-0.1301
	(0.0020)	(0.1362)	(0.0921)	(0.1081)	(0.0791)	(0.0002)	(0.0006)
1	-0.1595	-0.4754	-0.4153	-0.4765	-0.3662	-0.1298	-0.1698
	(0.0026)	(0.1365)	(0.0914)	(0.1075)	(0.0780)	(0.0004)	(0.0018)
2	-0.1844	-0.5034	-0.4436	-0.4794	-0.4004	-0.1625	-0.2023
	(0.0029)	(0.1344)	(0.0910)	(0.1046)	(0.0748)	(0.0006)	(0.0022)
3	-0.2060	-0.5218	-0.4652	-0.4802	-0.4339	-0.1980	-0.2100
	(0.0036)	(0.1221)	(0.0829)	(0.1012)	(0.0623)	(0.0011)	(0.0037)
			Scheme= $(0^{58}, 20^2)$				
0	-0.1739	-0.3821	-0.3363	-0.3655	-0.3191	-0.1103	-0.1618
	(0.0045)	(0.0887)	(0.0607)	(0.0895)	(0.0613)	(0.0003)	(0.0030)
1	-0.1923	-0.3889	-0.3460	-0.3873	-0.3448	-0.1311	-0.1833
	(0.0056)	(0.0792)	(0.0544)	(0.0777)	(0.0535)	(0.0004)	(0.0031)
2	-0.1996	-0.3968	-0.3582	-0.3969	-0.3583	-0.1633	-0.2177
	(0.0097)	(0.0682)	(0.0471)	(0.0692)	(0.0477)	(0.0008)	(0.0038)
3	-0.2003	-0.3954	-0.3637	-0.3957	-0.3640	-0.2028	-0.2442
	(0.0109)	(0.0531)	(0.0368)	(0.0520)	(0.0362)	(0.0017)	(0.0039)
			Scheme= $(20^2, 0^{58})$				
0	-0.1436	-0.4549	-0.3947	-0.4068	-0.3562	-0.1084	-0.1269
	(0.0014)	(0.1356)	(0.0938)	(0.1183)	(0.0735)	(0.0002)	(0.0005)
1	-0.1705	-0.4762	-0.4160	-0.4169	-0.36416	-0.1309	-0.1617
	(0.0019)	(0.1343)	(0.0929)	(0.1165)	(0.0724)	(0.0004)	(0.0012)
2	-0.1871	-0.506	-0.4457	-0.4598	-0.4108	-0.1627	-0.1641
	(0.0023)	(0.1339)	(0.0920)	(0.1109)	(0.0717)	(0.0006)	(0.0013)
3	-0.2027	-0.5164	-0.4609	-0.4804	-0.4272	-0.1989	-0.2285
	(0.0035)	(0.1176)	(0.0799)	(0.1009)	(0.0667)	(0.0010)	(0.0018)

Table 5: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X,t)$ when $(n,m) = (30,27)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
			Scheme=$(3,0^{26})$				
0	-0.2388	-0.5018	-0.4459	-0.5029	-0.4468	-0.1929	-0.2716
	(0.0341)	(0.1210)	(0.0821)	(0.1223)	(0.0830)	(0.0009)	(0.0104)
1	-0.3229	-0.5264	-0.4723	-0.5284	-0.4740	-0.2047	-0.3047
	(0.0791)	(0.1151)	(0.0782)	(0.1171)	(0.0795)	(0.0009)	(0.0107)
2	-0.4858	-0.5689	-0.5160	-0.5698	-0.5168	-0.2557	-0.2720
	(0.2445)	(0.1117)	(0.0760)	(0.1108)	(0.0754)	(0.0015)	(0.0116)
3	-0.7373	-0.6119	-0.5625	-0.6139	-0.5642	-0.3901	-0.2831
	(0.9722)	(0.0976)	(0.0667)	(0.1012)	(0.0691)	(0.0082)	(0.0129)
			Scheme=$(0^{26},3)$				
0	-0.2508	-0.5229	-0.4628	-0.5177	-0.4586	-0.1402	-0.1723
	(0.0134)	(0.1364)	(0.0923)	(0.1326)	(0.0898)	(0.0018)	(0.0019)
1	-0.3130	-0.5418	-0.4846	-0.5420	-0.4848	-0.1580	-0.2701
	(0.0272)	(0.1267)	(0.0859)	(0.1258)	(0.0853)	(0.0033)	(0.0034)
2	-0.4235	-0.5882	-0.5315	-0.5924	-0.5348	-0.2477	-0.2959
	(0.0849)	(0.1235)	(0.0838)	(0.1263)	(0.0851)	(0.0036)	(0.0038)
3	-0.5579	-0.6329	-0.5794	-0.6387	-0.5840	-0.2666	-0.3970
	(0.1830)	(0.1120)	(0.0763)	(0.1159)	(0.0788)	(0.0044)	(0.0097)
			Scheme=$(1^3,0^{24})$				
0	-0.2557	-0.4964	-0.4416	-0.5000	-0.4444	-0.1987	-0.1794
	(0.0423)	(0.1176)	(0.0799)	(0.1201)	(0.0815)	(0.0011)	(0.0016)
1	-0.3327	-0.5303	-0.4754	-0.5296	-0.4749	-0.2535	-0.2127
	(0.0987)	(0.1159)	(0.0788)	(0.1163)	(0.0791)	(0.0030)	(0.0040)
2	-0.4908	-0.5629	-0.5113	-0.5605	-0.5093	-0.2739	-0.225
	(0.2994)	(0.1056)	(0.0720)	(0.1046)	(0.0713)	(0.0034)	(0.0042)
3	-0.6380	-0.6180	-0.5674	-0.6131	-0.5635	-0.2740	-0.2649
	(0.5206)	(0.1023)	(0.0698)	(0.1002)	(0.0684)	(0.0037)	(0.0046)
			Scheme=$(0^{24},1^3)$				
0	-0.2497	-0.5154	-0.4568	-0.5188	-0.4595	-0.1392	-0.1999
	(0.0143)	(0.1305)	(0.0884)	(0.1336)	(0.0904)	(0.0019)	(0.0023)
1	-0.3165	-0.5474	-0.4891	-0.5494	-0.4907	-0.1643	-0.1373
	(0.0298)	(0.1290)	(0.0874)	(0.1310)	(0.0887)	(0.0027)	(0.0059)
2	-0.4163	-0.5781	-0.5234	-0.5914	-0.5341	-0.2301	-0.2052
	(0.1044)	(0.1150)	(0.0782)	(0.1263)	(0.0856)	(0.0028)	(0.0042)
3	-0.5659	-0.6340	-0.5802	-0.6415	-0.5862	-0.3523	-0.2546
	(0.1895)	(0.1129)	(0.0768)	(0.1192)	(0.0810)	(0.0035)	(0.0056)

Table 6: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X,t)$ when $(n,m) = (30,18)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
Scheme=$(12,0^{17})$							
0	-0.3059	-0.5019	-0.4458	-0.4935	-0.4498	-0.1454	-0.1549
	(0.1100)	(0.1231)	(0.0835)	(0.1205)	(0.0816)	(0.0010)	(0.0012)
1	-0.4602	-0.5253	-0.4714	-0.5239	-0.4784	-0.1969	-0.1876
	(0.2978)	(0.1162)	(0.0789)	(0.1161)	(0.0793)	(0.0012)	(0.0046)
2	-0.5933	-0.5565	-0.5062	-0.5572	-0.5087	-0.2656	-0.2750
	(0.3160)	(0.1031)	(0.0703)	(0.1057)	(0.0690)	(0.0022)	(0.0032)
3	-0.6240	-0.6164	-0.5663	-0.6150	-0.5517	-0.3335	-0.2992
	(0.3760)	(0.1025)	(0.0699)	(0.1006)	(0.0602)	(0.0039)	(0.0158)
Scheme= $(0^{17},12)$							
0	-0.2565	-0.5323	-0.4701	-0.5257	-0.5008	-0.1536	-0.1819
	(0.0161)	(0.1494)	(0.1007)	(0.1461)	(0.1149)	(0.0012)	(0.0019)
1	-0.3161	-0.5608	-0.4998	-0.5592	-0.5373	-0.2188	-0.1936
	(0.0250)	(0.1458)	(0.0983)	(0.1460)	(0.1176)	(0.0016)	(0.0031)
2	-0.4155	-0.6057	-0.5455	-0.6117	-0.5799	-0.2364	-0.3263
	(0.0499)	(0.1407)	(0.0950)	(0.1456)	(0.1118)	(0.0024)	(0.0083)
3	-0.5315	-0.6634	-0.6038	-0.6607	-0.6296	-0.3768	-0.2294
	(0.0973)	(0.1382)	(0.0935)	(0.1355)	(0.1057)	(0.0093)	(0.0101)
Scheme=$(4^3,0^{15})$							
0	-0.2970	-0.4965	-0.4415	-0.5002	-0.455	-0.2028	-0.2805
	(0.0389)	(0.1213)	(0.0822)	(0.1245)	(0.0850)	(0.0017)	(0.0132)
1	-0.3226	-0.5219	-0.4687	-0.5309	-0.4747	-0.2791	-0.3105
	(0.0597)	(0.1145)	(0.0778)	(0.1185)	(0.0764)	(0.0071)	(0.0136)
2	-0.4382	-0.5643	-0.5124	-0.5659	-0.5098	-0.2998	-0.3474
	(0.0746)	(0.1110)	(0.0755)	(0.1093)	(0.0704)	(0.0081)	(0.0064)
3	-0.5276	-0.6161	-0.5660	-0.6131	-0.5495	-0.3880	-0.3981
	(0.0988)	(0.1026)	(0.0700)	(0.0997)	(0.0588)	(0.0116)	(0.0167)
Scheme=$(0^{15},4^3)$							
0	-0.2573	-0.5282	-0.4669	-0.5392	-0.5053	-0.1646	-0.1928
	(0.0140)	(0.1468)	(0.0990)	(0.1554)	(0.1185)	(0.0009)	(0.0012)
1	-0.3220	-0.5492	-0.4905	-0.5587	-0.5373	-0.1848	-0.2131
	(0.0266)	(0.1369)	(0.0985)	(0.1413)	(0.1169)	(0.0096)	(0.0098)
2	-0.4125	-0.6062	-0.5459	-0.6158	-0.5800	-0.2230	-0.2282
	(0.0491)	(0.1336)	(0.0969)	(0.1406)	(0.1126)	(0.0129)	(0.0130)
3	-0.5234	-0.6653	-0.6053	-0.6591	-0.6381	-0.2893	-0.2918
	(0.0997)	(0.1315)	(0.0936)	(0.1349)	(0.1110)	(0.0182)	(0.0198)

Table 7: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X,t)$ when $(n,m) = (100,90)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
			Scheme=$(10, 0^{89})$				
0	-0.1857	-0.5146	-0.4562	-0.5011	-0.4456	-0.1665	-0.1992
	(0.0029)	(0.1260)	(0.0855)	(0.1164)	(0.0792)	(0.0003)	(0.0007)
1	-0.2346	-0.5269	-0.4728	-0.5269	-0.4728	-0.2241	-0.1939
	(0.0103)	(0.1108)	(0.0755)	(0.1114)	(0.0758)	(0.0005)	(0.0006)
2	-0.3059	-0.5458	-0.4976	-0.5502	-0.5011	-0.2298	-0.1984
	(0.0324)	(0.0914)	(0.0626)	(0.0951)	(0.0651)	(0.0014)	(0.0051)
3	-0.4142	-0.5725	-0.5310	-0.5792	-0.5364	-0.3081	-0.2498
	(0.0831)	(0.0732)	(0.0505)	(0.0773)	(0.0532)	(0.0020)	(0.0055)
			Scheme=$(0^{89}, 10)$				
0	-0.2283	-0.5362	-0.4735	-0.5083	-0.4552	-0.1608	-0.1663
	(0.0038)	(0.1390)	(0.0941)	(0.1206)	(0.0852)	(0.0005)	(0.0008)
1	-0.2854	-0.5625	-0.5013	-0.5326	-0.4814	-0.1708	-0.1744
	(0.0085)	(0.1361)	(0.0922)	(0.1157)	(0.0819)	(0.0016)	(0.0017)
2	-0.3662	-0.5969	-0.5385	-0.5915	-0.5320	-0.2252	-0.1994
	(0.0180)	(0.1251)	(0.0850)	(0.1147)	(0.0827)	(0.0016)	(0.0040)
3	-0.4615	-0.6297	-0.5767	-0.6354	-0.5813	-0.2405	-0.2473
	(0.0325)	(0.1063)	(0.0726)	(0.1128)	(0.0768)	(0.0104)	(0.0158)
			Scheme=$(5^2, 0^{88})$				
0	-0.1861	-0.5166	-0.4579	-0.4961	-0.4434	-0.1765	-0.1658
	(0.0037)	(0.1305)	(0.0884)	(0.1184)	(0.0805)	(0.0003)	(0.0004)
1	-0.2386	-0.5265	-0.4725	-0.5201	-0.4673	-0.2461	-0.1774
	(0.0134)	(0.1112)	(0.0757)	(0.1054)	(0.0719)	(0.0017)	(0.0024)
2	-0.3125	-0.5470	-0.4985	-0.5505	-0.5014	-0.2540	-0.2266
	(0.0372)	(0.0926)	(0.0634)	(0.0949)	(0.0650)	(0.0025)	(0.0026)
3	-0.4094	-0.5803	-0.5372	-0.5801	-0.5371	-0.3221	-0.2399
	(0.0896)	(0.0784)	(0.0540)	(0.0772)	(0.0532)	(0.0029)	(0.0069)
			Scheme=$(0^{88}, 5^2)$				
0	-0.2288	-0.5503	-0.4848	-0.5087	-0.4755	-0.1598	-0.1539
	(0.0040)	(0.1532)	(0.1034)	(0.1424)	(0.0964)	(0.0006)	(0.0008)
1	-0.2824	-0.5610	-0.5001	-0.5464	-0.4890	-0.1616	-0.1950
	(0.0075)	(0.1329)	(0.0901)	(0.1303)	(0.0836)	(0.0027)	(0.0036)
2	-0.3644	-0.5991	-0.5402	-0.5910	-0.5335	-0.2010	-0.2035
	(0.0168)	(0.1266)	(0.0860)	(0.1258)	(0.0820)	(0.0038)	(0.0061)
3	-0.4566	-0.6343	-0.5805	-0.6385	-0.5838	-0.2429	-0.2357
	(0.0304)	(0.1092)	(0.0745)	(0.1130)	(0.0769)	(0.0064)	(0.0076)

Table 8: Average values and MSEs of the MLEs & Bayes estimates by Lindley's approximation method and Importance Sampling Method (ISM) of $\xi J_r(X, t)$ when $(n, m) = (100, 60)$.

	MLE	Bayesian Estimation					
		Lindley's approximation				ISM	
		Non- informative prior		Informative prior			
r		p=20	p=25	p=20	p=25	p=20	p=25
				Scheme=$(40, 0^{59})$			
0	-0.2023	-0.5137	-0.4555	-0.5066	-0.4392	-0.1617	-0.1695
	(0.0101)	(0.1263)	(0.0857)	(0.1201)	(0.0817)	(0.0024)	0.0004
1	-0.2487	-0.5348	-0.4791	-0.5139	-0.4703	-0.1733	-0.2317
	(0.0219)	(0.1183)	(0.0804)	(0.1067)	(0.0788)	(0.0028)	0.0039
2	-0.3466	-0.5672	-0.5147	-0.5558	-0.5067	-0.2482	-0.2600
	(0.0699)	(0.1062)	(0.0724)	(0.1010)	(0.0720)	(0.0029)	0.0048
3	-0.4981	-0.5934	-0.5477	-0.5984	-0.5652	-0.2658	-0.2896
	(0.2249)	(0.0850)	(0.0584)	(0.0878)	(0.0687)	(0.0037)	0.0079
				Scheme=$(0^{59}, 40)$			
0	-0.2403	-0.5750	-0.5046	-0.5103	-0.4649	-0.1567	-0.1341
	(0.0056)	(0.1741)	(0.1172)	(0.1407)	(0.0985)	(0.0007)	(0.0021)
1	-0.2978	-0.6069	-0.5368	-0.5407	-0.4985	-0.1652	-0.1751
	(0.0104)	(0.1740)	(0.1170)	(0.1348)	(0.0988)	(0.0035)	(0.0037)
2	-0.3861	-0.6527	-0.5831	-0.6087	-0.5503	-0.1902	-0.2087
	(0.0210)	(0.1711)	(0.1152)	(0.1366)	(0.0983)	(0.0053)	(0.0081)
3	-0.4809	-0.6980	-0.6313	-0.6558	-0.6016	-0.2125	-0.2166
	(0.0356)	(0.1579)	(0.1065)	(0.1267)	(0.0917)	(0.0121)	(0.0123)
				Scheme=$(20^2, 0^{58})$			
0	-0.2000	-0.5096	-0.4522	-0.4931	-0.4446	-0.1794	-0.1669
	(0.0101)	(0.1240)	(0.0841)	(0.1202)	(0.0843)	(0.0003)	(0.0004)
1	-0.2570	-0.5308	-0.4759	-0.5292	-0.4739	-0.2087	-0.1890
	(0.0272)	(0.1139)	(0.0775)	(0.1122)	(0.0705)	(0.0005)	(0.0009)
2	-0.3548	-0.5610	-0.5097	-0.5611	-0.5037	-0.2152	-0.2246
	(0.0745)	(0.1022)	(0.0698)	(0.1032)	(0.0700)	(0.0060)	(0.0068)
3	-0.5022	-0.5950	-0.5490	-0.5957	-0.5436	-0.2757	-0.2519
	(0.2161)	(0.0866)	(0.0594)	(0.0856)	(0.0581)	(0.0082)	(0.0104)
				Scheme=$(0^{58}, 20^2)$			
0	-0.2421	-0.5743	-0.5040	-0.5259	-0.4758	-0.1481	-0.1309
	(0.0058)	(0.1734)	(0.1167)	(0.1462)	(0.1047)	(0.0011)	(0.0024)
1	-0.3009	-0.6080	-0.5376	-0.5475	-0.4982	-0.1758	-0.1668
	(0.0108)	(0.1751)	(0.1178)	(0.1337)	(0.1034)	(0.0017)	(0.0028)
2	-0.3864	-0.6558	-0.5855	-0.6089	-0.5536	-0.1821	-0.1692
	(0.0213)	(0.1732)	(0.1166)	(0.1271)	(0.1016)	(0.0090)	(0.0096)
3	-0.4832	-0.6926	-0.6270	-0.6464	-0.6003	-0.2691	-0.2358
	(0.0355)	(0.1530)	(0.1033)	(0.1247)	(0.0912)	(0.0108)	(0.0128)

Table 9: Average length of the interval estimates of $\xi J_r(X)$ and $\xi J_r(X,t)$ when $n = 30$.

r	(n, m)=(30, 27)				(n, m)=(30, 18)			
	$\xi J_r(X)$		$\xi J_r(X,t)$		$\xi J_r(X)$		$\xi J_r(X,t)$	
	NA	Boot p	NA	Boot p	NA	Boot p	NA	Boot p
	Scheme=$(3, 0^{26})$				Scheme = $(12, 0^{17})$			
0	0.0896	0.0808	0.0952	0.0875	0.3567	0.1227	0.2133	0.1388
1	0.1109	0.1083	0.1187	0.1143	0.4111	0.2683	0.2877	0.1439
2	0.1367	0.1306	0.1484	0.1388	0.5027	0.1711	0.3233	0.1495
3	0.1907	0.1856	0.1866	0.1783	0.5901	0.1870	0.4618	0.2797
	Scheme=$(0^{26}, 3)$				Scheme=$(0^{17}, 12)$			
0	0.0874	0.0778	0.1079	0.0607	0.1316	0.0792	0.1703	0.0615
1	0.1097	0.1084	0.1303	0.0682	0.1589	0.1097	0.2051	0.0684
2	0.1319	0.1237	0.1628	0.1163	0.1987	0.1450	0.2564	0.1170
3	0.1642	0.1620	0.2027	0.0939	0.2488	0.1630	0.3231	0.1952
	Scheme=$(1^3, 0^{24})$				Scheme= $(4^3, 0^{15})$			
0	0.0761	0.0374	0.0988	0.0927	0.3090	0.1383	0.1306	0.1253
1	0.0987	0.0833	0.1201	0.1104	0.4337	0.1237	0.1617	0.1430
2	0.1242	0.1163	0.1494	0.1304	0.4952	0.1460	0.2038	0.1727
3	0.1738	0.1573	0.1853	0.1544	0.6781	0.2169	0.2610	0.2150
	Scheme=$(0^{24}, 1^3)$				Scheme= $(0^{15}, 4^3)$			
0	0.0873	0.0866	0.1078	0.0802	0.1306	0.0905	0.1705	0.0808
1	0.1053	0.1042	0.1373	0.1351	0.1578	0.1136	0.2052	0.1351
2	0.1317	0.1233	0.1628	0.1549	0.1978	0.1546	0.2560	0.1555
3	0.1637	0.1273	0.2087	0.2066	0.2473	0.2267	0.3236	0.2063

Table 10: Average length of the interval estimates of $\xi J_r(X)$ and $\xi J_r(X,t)$ when $n = 100$.

r	(n, m)=(100, 90)				(n, m)=(100, 60)			
	$\xi J_r(X)$		$\xi J_r(X,t)$		$\xi J_r(X)$		$\xi J_r(X,t)$	
	NA	Boot p	NA	Boot p	NA	Boot p	NA	Boot p
	Scheme=$(10, 0^{89})$				Scheme = $(40, 0^{59})$			
0	0.0586	0.0421	0.2958	0.0424	0.0680	0.0658	0.1231	0.0652
1	0.0721	0.0533	0.3227	0.0593	0.0731	0.0718	0.2389	0.0834
2	0.0996	0.0646	0.3588	0.0647	0.0928	0.0910	0.3103	0.1248
3	0.1143	0.1111	0.3753	0.1505	0.1196	0.1135	0.4558	0.2132
	Scheme =$(0^{89}, 10)$				Scheme = $(0^{59}, 40)$			
0	0.0485	0.0355	0.0727	0.0364	0.0595	0.0228	0.0938	0.0232
1	0.0586	0.0371	0.0877	0.0377	0.0719	0.0290	0.1128	0.0319
2	0.0731	0.0504	0.1096	0.0502	0.0899	0.0351	0.1410	0.07255
3	0.0909	0.0712	0.1379	0.0716	0.1118	0.0489	0.1779	0.1488
	Scheme=$(5^2, 0^{88})$				Scheme = $(20^2, 0^{58})$			
0	0.0595	0.0484	0.3025	0.0491	0.0544	0.0472	0.0939	0.0850
1	0.0720	0.0635	0.3247	0.0635	0.0672	0.0595	0.1270	0.0895
2	0.0895	0.0838	0.3587	0.0839	0.0845	0.0775	0.2002	0.1175
3	0.1164	0.1066	0.4063	0.1454	0.1109	0.1104	0.2191	0.1306
	Scheme= $(0^{88}, 5^2)$				Scheme= $(0^{58}, 20^2)$			
0	0.0484	0.0361	0.0729	0.0369	0.0595	0.0247	0.0937	0.0251
1	0.0586	0.0470	0.0877	0.0472	0.0719	0.0286	0.1127	0.0290
2	0.0732	0.0418	0.1094	0.0491	0.0899	0.0369	0.1411	0.0327
3	0.0910	0.0736	0.1378	0.0742	0.1119	0.0395	0.1774	0.0697

6 Real Data Analysis

In this section, we consider a real data set from Lawless (1982) to illustrate the proposed estimators. In this data set, 36 appliances were put through an automatic life test and the results include failure or censoring times for each unit. The dataset is given below.

Dataset: 11, 35, 49, 170, 329, 381, 708, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451, 2471, 2551, 2565, 2568, 2694, 2702, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329, 6367, 6976, 7846, 13403.

In order to determine which distribution fits best for the data, we take into consideration of various distributions, including Lomax, Gamma, Lindley, and Logistic distribution. The histogram of the real dataset related to the failure time of appliances and the density plots of four lifetime distributions are presented in Figure 1.

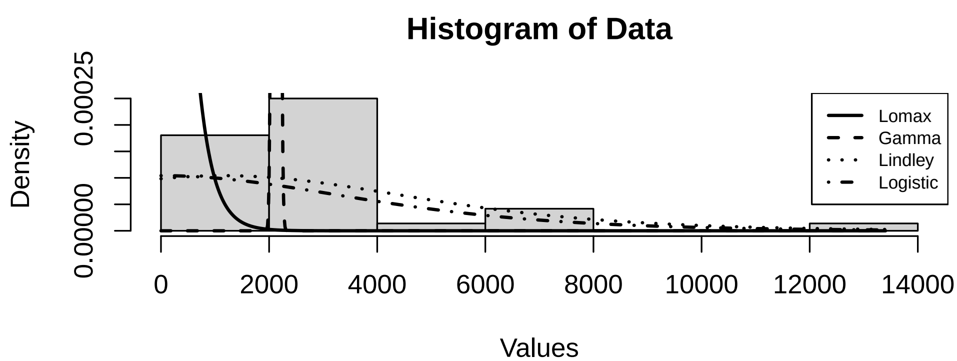

Figure 1: The histogram of the real data considered in Section 6 and the plots of the probability density functions of the fitted Lomax, gamma, Lindley and logistic models

Also, we consider various methods to apply goodness of fit test statistic and information criteria. These are Kolmogorov-Smirnov(KS) statistic, negative log-likelihood criterion, Akaikes-Information Criterion ($AIC = 2k - 2\log L$), the associated second-order information criterion ($AICc = AIC + \frac{2k(k+1)}{n-k-1}$) and Bayesian Information Criterion ($BIC = k\log n - 2\log L$), where k is the number of parameters of the model, n is the number of observations in the dataset, L is the maximum value of the likelihood function for the model. The values of the MLEs and five goodness of fit test statistics are presented in Table 11. The best distribution corresponds to the lowest values for BIC, AICc, AIC, $-\log L$, and KS statistic. In contrast to other distributions, the numerical values in Table 11 indicate that the Lomax distribution fits the data well.

The data set has been analysed by Kundu and Joarder (2006) and they created artificial progressively Type-II censored data from this data set. The artificially created

Type-II progressively Type-II censored dataset is given below:

Type-II progressively censored dataset: {11, 35, 49, 170, 329, 958, 1925, 2223, 2400, 2568} , where $(n, m) = (36, 10)$, $T = 2600$, $D = m = 10$ and Scheme=(2, 2, 2, 2, 2, 2, 2, 2, 2, 8).

Table 11: The MLEs, BIC, AICc, AIC, $-\log L$ and KS statistic for the real data set.

Model	$\widehat{\alpha}$	$\widehat{\lambda}$	BIC	AICc	AIC	$-\log L$	KS
Lomax	977.6461	2692.858×10^3	649.5444	646.7410	646.3774	321.1887	0.10585
Gamma	1.1381	2422.322	650.1319	647.3285	646.9649	321.4824	0.17441
Lindley	0.00073		660.8157	659.3499	659.2322	328.6161	0.47170
Logistic	2399.374	1153.39	663.3787	660.5753	660.2117	328.1059	0.45466

We deduced from the simulation analysis that Bayes estimates by importance sampling method are superior to MLEs. As a result, we solely present Bayes estimates by importance sampling method here and we took $t = 50$. In Table 12, we presented Bayes estimates by importance sampling method of $\xi J_r(X)$ and $\xi J_r(X, t)$ for the Type-II progressively censored dataset. In Table 13, we present the interval estimate of $\xi J_r(X)$ and $\xi J_r(X, t)$ of Type-II progressively censored dataset by using the NA and Boot-p method.

Table 12: Bayes estimates by importance sampling method of $\xi J_r(X)$ and $\xi J_r(X, t)$.

r	$\xi J_r(X)$	$\xi J_r(X, t)$
0	-1185.87	-1185.90
1	-1190.06	-1190.08
2	-1194.27	-1194.29
3	-1198.50	-1198.52

Table 13: Interval estimate of $\xi J_r(X)$ and $\xi J_r(X, t)$ by using NA and Boot p method.

	$\xi J_r(X)$		$\xi J_r(X, t)$	
r	NA	Boot p	NA	Boot p
0	(-1613.11, -758.63)	(-1668.12, -680.06)	(-1613.14, -758.65)	(-1668.15, -680.07)
1	(-1617.74, -762.38)	(-1689.16, -688.20)	(-1617.77, -762.40)	(-1689.19, -688.21)
2	(-1622.39, -766.15)	(-1664.20, -666.45)	(-1622.42, -766.17)	(-1664.23, -666.46)
3	(-1631.75, -773.76)	(-1661.17, -680.38)	(-1631.78, -773.77)	(-1661.20, -680.39)

7 Conclusion

This study uses the progressively Type-II censored sample to examine various approaches for estimating the CR Extropy and DCR Extropy measures of the equilibrium distribution of order r of Lomax distribution. Many experimenters believe that using this kind of censored sample efficiently reduces the time and expense involved in a study. We utilized maximum likelihood and Bayesian estimators for estimation. LINEX loss function was taken into account when establishing Bayesian estimators. We have observed that the proposed Bayesian estimators can not be obtained in explicit forms. Because of this, Lindley's approximation and Importance Sampling Methods are used. Additionally, the confidence intervals are constructed for the estimators. The NA and boot-p methods are used to obtain the confidence intervals. We use the Monte Carlo Simulation approach to conduct a thorough simulation study to compare the performance of the given estimators. It has been found that Bayes estimate by importance sampling method performs better than MLEs. A real dataset is considered to demonstrate an application of the suggested theoretical results for illustrative purposes.

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References

Abd-Elfattah, A.M., Alaboud, F.M. and Alharby, A.H. (2007). On sample size estimation for Lomax distribution, *Australian Journal of Basic and Applied Sciences*, **1**(4), 373–378.

Asgharzadeh, A. and Valiollahi, R. (2010). Estimation of the scale parameter of the Lomax distribution under progressive censoring, *International Journal of Statistics Economics*, **6**(S11), 37–48.

Cramer, E. and Schmiedt, A.B. (2011). Progressively Type-II censored competing risks data from Lomax distributions, *Computational Statistics Data Analysis*, **55**(3), 1285–1303.

Helu, A. and Samawi, H. (2021). Statistical analysis based on adaptive progressive hybrid censored data from Lomax distribution, *Statistics, Optimization Information Computing*, **9**(4), 789.

Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data, *Journal of the American Statistical Association*, **49**(268), 847–852.

Bryson, M.C. (1974). Heavy-tailed distributions: properties and tests, *Technometrics*, **16**(1), 61–68.

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate, *Computational Statistics Data Analysis*, **53**(12), 4433–4440.

Johnson, N.L., Kotz, S. and Balakrishan, N. (1994). Continuous Univariate Distributions, Volume 2. Wiley.

Rajab, M., Aleem, M., Nawaz, T. and Daniyal, M. (2013). On five parameter beta Lomax distribution, *Journal of Statistics*, **20**(1).

El-Bassiouny, A.H., Abdo, N.F. and Shahen, H.S. (2015). Exponential Lomax distribution, *International Journal of Computer Applications*, **121**(13).

Cordeiro, G.M., Ortega, E.M.M. and Popović, B.V. (2015). The gamma-Lomax distribution, *Journal of Statistical Computation and Simulation*, **85**(2), 305–319.

Al-Zahrani, B. and Sagor, H. (2014). The Poisson-Lomax distribution, *Revista Colombiana de Estadística*, **37**(1), 225–245.

Tahir, M.H., Cordeiro, G.M., Mansoor, M. and ZUBAİR, M. (2015). The Weibull-Lomax distribution: properties and applications, *Hacettepe Journal of Mathematics and Statistics*, **44**(2), 455–474.

Kilany, N.M. (2016). Weighted Lomax distribution, *SpringerPlus*, **5**(1), 1–18.

Balakrishnan, N., Balakrishnan, N. and Aggarwala, R. (2000). Progressive censoring: theory, methods, and applications. Springer Science Business Media.

Kundu, D. and Joarder, A. (2006). Analysis of Type-II progressively hybrid censored data, *Computational Statistics Data Analysis*, **50**(10), 2509–2528.

Helu, A., Samawi, H. and Raqab, M.Z. (2015). Estimation on Lomax progressive censoring using the EM algorithm, *Journal of Statistical Computation and Simulation*, **85**(5), 1035–1052.

Harris, C.M. (1968). The Pareto distribution as a queue service discipline, *Operations Research*, **16**(2), 307–313.

Ebrahimi, N. (1996). How to measure uncertainty in the residual life time distribution, *Sankhyā: The Indian Journal of Statistics, Series A*, pages 48–56.

Rao, M., Chen, Y. and Vemuri, B.C. (2004). Cumulative residual entropy: a new measure of information, *IEEE Transactions on Information Theory*, **50**(6), 1220–1228.

Rao, M. (2005). More on a new concept of entropy and information, *Journal of Theoretical Probability*, **18**(4), 967–981.

Asadi, M. and Zohrevand, Y. (2007). On the dynamic cumulative residual entropy, *Journal of Statistical Planning and Inference*, **137**(6), 1931–1941.

Navarro, J., del Aguila, Y. and Asadi, M. (2010). Some new results on the cumulative residual entropy, *Journal of Statistical Planning and Inference*, **140**(1), 310–322.

Nair, N.U. and Preeth, M. (2009). On some properties of equilibrium distributions of order n, *Statistical Methods and Applications*, **18**(4), 453–464.

Nair, N.U. and Preeth, M. (2008). Multivariate equilibrium distributions of order n, *Statistics Probability Letters*, **78**(18), 3312–3320.

Unnikrishnan Nair, N., Sunoj, S.M. and Rajesh, G. (2021). Cumulative residual entropy of equilibrium distribution of order n, *Communications in Statistics-Theory and Methods*, pages 1–13.

Shannon, C.E. (1948). A mathematical theory of communication, *The Bell System Technical Journal*, **27**(3), 379–423.

Ebrahimi, N., Kirmani S.N.U.A. and Soofi E.S. (2007). Multivariate dynamic information, *Journal of Multivariate Analysis*, **98**(2), 328–349.

Arnold, B.C. (2014). Pareto distribution, *Wiley StatsRef: Statistics Reference Online*, pages 1–10.

Rytgaard, M. (1990). Estimation in the Pareto distribution, *ASTIN Bulletin: The Journal of the IAA*, **20**(2), 201–216.

Kundu D. (2007). On hybrid censored Weibull distribution ,*Journal of Statistical Planning and Inference*, **137**(7),2127-2142.

Lindley,D.V.(1980) Approximate bayesian methods ,*Trabajos de estadística y de investigación operativa*,**31**(1),223-245.

Arnold,B.C., Press,S.J.and others(1983) Bayesian inference for Pareto populations ,*Journal of Econometrics*,**21**(3),287-306.

Kundu,D.and Joarder,A.(2006) Analysis of Type-II progressively hybrid censored data ,*Computational Statistics Data Analysis*, **50**(10),2509-2528.

Zaman,R.and Nasiri,P.(2019) Estimation of the Parameters of the Lomax Distribution using the EM Algorithm and Lindley Approximation ,*Mathematical Researches*, **4**(2),201-210.

Lawless,J.F.(1982) Statistical models and methods for lifetime data Wiley ,New York.

Kundu,D.and Joarder,A.(2006) Analysis of Type-II progressively hybrid censored data ,*Computational Statistics Data Analysis*, **50**(10),2509-2528.

Greene, W.H. (2003). Econometric analysis. Pearson Education India.

Ma, Y. and Shi, Y. (2013). Inference for Lomax distribution based on type-II progressively hybrid censored data. Vidyasagar University, Midnapore, West-Bengal, India.

Renjini, K.R., Abdul Sathar, E.I. and Rajesh, G. (2018). Bayesian estimation of dynamic cumulative residual entropy for classical Pareto distribution, *American Journal of Mathematical and Management Sciences*, **37**(1), 1–13.

Renjini, K.R., Sathar, E.I.A. and Rajesh, G. (2016). Bayes estimation of dynamic cumulative residual entropy for Pareto distribution under type-II right censored data, *Applied Mathematical Modelling*, **40**(19-20), 8424–8434.

Yu, J., Gui, W. and Shan, Y. (2019). Statistical inference on the Shannon entropy of inverse Weibull distribution under the progressive first-failure censoring, *Entropy*, **21**(12), 1209.

Maiti, K., Kayal, S. and Kundu, D. (2022). Statistical Inference on the Shannon and Rényi Entropy Measures of Generalized Exponential Distribution Under the Progressive Censoring, *SN Computer Science*, **3**(4), 1–21.

Ahmadini, A.A.H., Hassan, A.S., Zaky, A.N. and Alshqaq, S.S. (2020). Bayesian inference of dynamic cumulative residual entropy from Pareto II distribution with application to COVID-19, *AIMS Math*, **6**(3), 2196–2216.

Shannon, C.E. (1948). A mathematical theory of communication, *The Bell System Technical Journal*, **27**(3), 379–423.

Harkness, W. and Shantaram, R. (1969). Convergence of a sequence of transformations of distribution functions, *Pacific Journal of Mathematics*, **31**(2), 403–415.

Gupta, R.C. (1979). On the characterization of survival distributions in reliability by properties of their renewal densities, *Communications in Statistics-Theory and Methods*, **8**(7), 685–697.

Pakes, A.G. (1996). Length biasing and laws equivalent to the log-normal, *Journal of Mathematical Analysis and Applications*, **197**(3), 825–854.

Nanda, A.K., Jain, K. and Singh, H. (1996). Properties of moments for s-order equilibrium distributions, *Journal of Applied Probability*, **33**(4), 1108–1111.

Pakes, A.G. and Navarro, J. (2007). Distributional characterizations through scaling relations, *Australian New Zealand Journal of Statistics*, **49**(2), 115–135.

Nair N.U. and Sankaran P.G. (2010). Properties of a mean residual life function arising from renewal theory ,*Naval Research Logistics (NRL)*,**57**(4),373-379.

Lad F., Sanfilippo G., Agro G. (2015) Extropy: Complementary dual of entropy ,*Statistical Science*,**30**(1),40-58.

Rao M., Chen Y., Vemuri B.C., Wang F.(2004) Cumulative residual entropy: a new measure of information ,*IEEE Transactions on Information Theory*, **50**(6),1220-1228.

Jahanshahi S.M.A., Zarei H., Khammar A.H.(2020) On cumulative residual extropy ,*Probability in the Engineering and Informational Sciences*, **34**(4),605-625.

Asadi M.and Zohrevand Y.(2007) On the dynamic cumulative residual entropy ,*Journal of Statistical Planning and Inference*, **137**(6),1931-1941.

Muliere P., Parmigiani G.and Polson N.G.(1993) A note on the residual entropy function ,*Probability in the Engineering and Informational Sciences*, **7**(3),413-420.

Qiu G.and Jia K.(2018) The residual extropy of order statistics ,*Statistics Probability Letters*, **133**,15-22.

Abdul Sathar E.I.and Nair R.Dhanya(2021) On dynamic survival extropy ,*Communications in Statistics-Theory and Methods*, **50**(6),1295-1313.

Arnold B.C., Press S.J.and others(1983) Bayesian inference for Pareto populations ,*Journal of Econometrics*, **21**(3),287-306.

Efron B.and Tibshirani R.(1986) Bootstrap Methods for Standard Errors ,Confidence Intervals,and Other Measures of ,*Statistical Science*, **1**(1),54-75.

Balakrishnan N.and Sandhu R.A.(1995) A simple simulational algorithm for generating progressive Type-II censored samples ,*The American Statistician*, **49**(2),229-230.