JIRSS (2023)
Vol. 22, No. 02, pp 1-33
DOI: 10.22034/jirss.2023.1999147.10120

Developing Flexible Classes of Distributions to Account for both Skewness and Bimodality

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Received: 28/03/2023, Accepted: 15/10/2023, Published online: 30/10/2024

Abstract. We develop two novel approaches for constructing flexible skewed and bimodal distributions that can effectively generalize classical symmetric distributions. We illustrate the application of introduced techniques by extending normal, student-t, and Laplace distributions. We also study the properties of the newly constructed distributions. The method of maximum likelihood is proposed for estimating the model parameters. Furthermore, the application of new distributions is represented using real-life data.

Keywords. Bimodal Distributions, ML Estimation, Skewed Distributions, Symmetric Density Functions.

MSC: 62J07, 62F15, 62F03, 62H05.

1 Introduction

In many sciences, the observed data of interesting phenomena often exhibit asymmetry, skewness, and bimodality. For example, in astronomy, duration times of gamma-ray bursts are bimodal (Sharifipanah et al., 2020). In sport, the lean body mass of athletes is skewed Sastry and Bhati (2016). In medicine, plasma lipid levels of heart attacks are skewed (Riediger et al., 2011), and gene expression patterns in breast cancer are bimodal (Ertel, 2010). Many other examples can be addressed in engineering, meteorology, environmental studies, and economics. Compared to well-known families of

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parametric distributions, modeling such data needs more flexible distributions with additional parameters to control the mentioned features.

In recent decades, there is growing interest in developing distributions that can account for skewness and/or bimodality. Some of the proposed distributions bellow:

a) Azzalini (1985) introd uced a family of densities as $2g(x)F(\lambda x)$ for skewness modeling, where $\lambda \in \mathbb{R}$ controls the skewness, g is a probability density function (pdf) symmetric about 0, and F denotes the cumulative distribution function (cdf) of a symmetric distribution on $\mathbb{R}$. When g is the density of F, the distributions are usually denoted as skew-F; skew-normal (SN; Azzalini (1985)) and skew-student-t (SSt; Arellano-Valle and Azzalini (2013)) distributions are two well-known examples.

b) A family of skewed distributions based on order statistics was introduced by Jones (2004). This broad family of distributions is generated from the beta distribution. Let G is a continuous cdf with density g. Then, the univariate family of distributions generated by G and the positive real parameters a and b, has the following pdf:

$$\frac{1}{B(a,b)}g(x)G(x)^{a-1}(1-G(x))^{b-1},$$

where a and b are two shape parameters governing skewness and tail weight, and $B(a,b)$ is the complete beta function. Beta-normal (BN; Eugene et al. (2002)) distribution is an example of beta-generated distributions.

c) Arnold and Lin (2004) proposed the skew-symmetric family of distributions whose pdf is given by $2\pi(x)g(x)$ where $\pi(x)$ is a Lebesgue measurable function satisfying $0 \leq \pi(x) \leq 1$ and $\pi(x) + \pi(-x) = 1$, $\forall x \in \mathbb{R}$, and g is a symmetric pdf around 0. A similar class of distributions was proposed by Genton and Loperfido (2005) Genton and Loperfido (2005).

d) The odd log-logistic (OLL) family of distributions was worked out by Gleaton and Lynch (2006), and its pdf is given by

$$\frac{\alpha G(x)^{\alpha-1}\left(1-G(x)\right)^{\alpha-1}}{\left[G(x)^{\alpha}+(1-G(x))^{\alpha}\right]^{2}}g(x),$$

where $\alpha > 0$ is a shape parameter, and G and g denote, respectively, the cdf and pdf of a symmetric distribution. The OLL normal (OLLN) distribution of Duarte et al. (2018) is an example of this family.

e) Bolfarine et al. (2018) constructed the power log-Dagum (PLD) distribution worked out by .

Recently, the modeling of data bimodality has received significant attention, and the use of finite mixture distributions is the predominant approach. However, this approach may suffer significant from computational burdens and identifiability problems in estimation, such as label switching, in both classical and Bayesian frameworks (McLachlan and Peel (2000); Lin et al. (2007)). Furthermore, compared to finite mixtures of distributions, constructing new flexible parametric families that could adapt as nearly as possible to real data, particularly in the case of bimodal data modeling, provides more parsimonious models. Therefore, statistical inferences are implemented more straightforward. To this end, several efforts have been made so far that are not possible to mention all. We can refer to Elal-Olivero et al. (2009), Hassan and El-Bassiouni (2016), Vila et al. (2020), and Gómez-Déniz et al. (2021), to name a few. mm

As an innovative new contribution in this way, our aim is threefold. First, we present two general constructive approaches for developing flexible univariate skewed and bimodal distributions. We then use these approaches to construct novel classes extending normal, student-t, and Laplace distributions. Finally, we use these classes to analyze real-life data and briefly compare them with previously proposed bimodal and skewed distributions.

This paper is organized as follows. Section 2 introduces two proposed constructive representations of skewed/bimodal distributions and several theoretical properties based on these representations. Section 3 presents new classes of skewed/bimodal distributions extending well-known symmetric families, including normal, Student-t, and Laplace distributions. Some of their characteristics and maximum likelihood (ML) estimation of the parameters are also given. Section 4 uses the various families for the analysis of three real datasets. Section 5 provides two extensions to create 1) positive support distributions and 2) a linear regression framework where the error term follows the proposed families. The final section addresses some concluding remarks. All technical details and proofs are relegated to the Appendix.

2 Constructing Bimodal and Skewed Families

2.1 The Weighted Approach

One way to generalize distributions to flexible ones is to multiply two pdfs or their functionals. The following theorem illustrates how this idea can be applied to construct symmetric bimodal distributions.

Theorem 2.1. *Let $g(\cdot)$ be a symmetric pdf about c with support on $\mathbb{R}$, $w(\cdot)$ be a strictly positive convex symmetric function about b, and $\mathrm{E}_g[w(X)] < \infty$. Hence, for $b = c$, the function*

$$f(x) = \frac{w(x)}{\mathrm{E}_g[w(X)]} g(x), \tag{2.1}$$

i) *is symmetric about c.*

ii) *displays a bimodal pdf.*

iii) *the sum of two modes is equal to* $2c$*.*

Theorem 2.1 implies that if $w(\cdot)$ is convex, then we could have skew bimodal distributions when $b \neq c$. We can conclude that when $w(\cdot)$ is concave, (2.1) will be unimodal. It is also symmetric for $b = c$ and skewed for $b \neq c$.

Examples 2.1. Let X have the standard normal distribution with pdf $\phi(x)$, and $w(x) = \exp(k|x|)$ that is convex for $k > 0$, then

$$f(x) = \frac{\exp(k|x|)}{\mathrm{E}_\phi[\exp(k|X|)]}\phi(x),$$

is symmetric around zero, and its modes are at $\mp k$.

To use the given representation in Theorem 2.1 for constructing flexible distributions, we must obtain the normalizing constant $\mathrm{E}_g[w(X)]$, which could be troublesome. The following theorem provides the conditions under which there is no difficulty in calculating the normalizing constant.

Theorem 2.2. *Let* X *be a symmetric random variable around zero, with the cdf* $G(x)$ *and pdf* $g(x)$*. Then*

$$f(x) = \frac{k}{2(e^k - e^{\frac{k}{2}})} g(x) e^{kG(|x|)},$$

and

$$f(x) = \frac{k+1}{2(1 - \frac{1}{2^{k+1}})} g(x) G(|x|)^k,$$

provide two symmetric pdfs. They are also bimodal for $k > 0$*.*

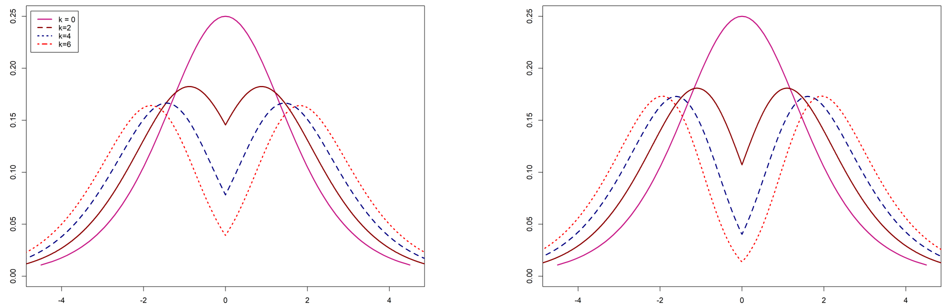
Examples 2.2. Let X follow the standard logistic distribution. By applying Theorem 2.2, we can construct two following symmetric pdfs named Bimodal Logistic type I and type II densities, respectively:

$$f(x) = \frac{k}{2\left(e^k - e^{\frac{k}{2}}\right)} \left\{\frac{e^{-x}}{(1+e^{-x})^2}\right\} e^{\left\{\frac{k}{1+e^{-|x|}}\right\}},$$

and

$$f(x) = \frac{k+1}{2\left(1 - \frac{1}{2^{k+1}}\right)} \left\{\frac{e^{-x}}{(1+e^{-x})^2}\right\} \left\{\frac{1}{1+e^{-|x|}}\right\}^k.$$

Figure 1 depicts the densities of both types of Bimodal Logistic distributions for some values of k.



(a) (b)

Figure 1: Bimodal Logistic densities for different values of k, type I (a) and type II (b).

2.2 The Composition Approach

Suppose that $g(x)$ is a pdf, and $h(\cdot)$ is any positive measurable function for which $\int_{-\infty}^{\infty} g\,(h(w))\,dw < \infty$. Our second representation for constructing new flexible distributions is given by

$$f(x) = \frac{g\,(h(x))}{\int_{-\infty}^{\infty} g\,(h(w))\,dw}. \tag{2.2}$$

The following theorem shows how to apply this approach to build symmetric bimodal distributions from any unimodal distribution with support $\mathbb{R}$.

Theorem 2.3. *Given the representation in* (2.2), *let $g(x)$ has a mode at k and $h(\cdot)$ is a strictly convex symmetric function around* 0 *for which $h(0) = 0$, then*

i) *$f(x)$ is unimodal with mode at* 0, *when $k = 0$.*

ii) *$f(x)$ is bimodal for $k \neq 0$.*

iii) *the mode(s) is(are) obtained by solving the equation $h(x) = k$.*

Examples 2.3. Let X follow a Cauchy distribution with the location parameter k and the scale parameter 1, $X \sim \mathrm{C}(k, 1)$, and $h(x) = |x|$, then

$$f(x) = \frac{1}{2G(k)} \frac{1}{\pi} \frac{1}{1 + (|x| - k)^2}, \tag{2.3}$$

is symmetric around zero, and its modes are at $\mp k$, where $G(\cdot)$ is the cdf of $\mathrm{C}(0, 1)$. The shapes of f for particular choices of k are shown in Figure 2 (panel a).

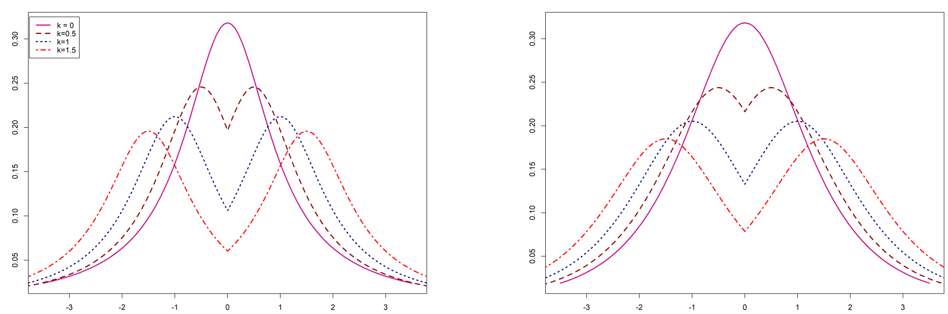
Examples 2.4. In the previous example, let X follow a hyperbolic secant distribution (Fisher, 1921) with the pdf

$$g(x) = \frac{1}{2}\mathrm{sech}(\frac{\pi}{2}(x - k)),$$

in which k is the location parameter and the scale parameter is equal to one, HS$(k, 1)$, then

$$f(x) = \frac{1}{2G(k)} \frac{2}{\pi} \frac{1}{e^{(|x|-k)} + e^{-(|x|-k)}}, \tag{2.4}$$

is symmetric about zero and its modes are at $\mp k$, where $G(\cdot)$ is the cdf of HS$(0, 1)$. Panel (b) in Figure 2 shows the shapes of f for different values of k.



(a) (b)

Figure 2: Examples of the pdf given in (2.3) (a) and (2.4) (b) for different choices of k.

The following theorem explains how to construct symmetric bimodal distributions while avoiding the complexities associated with calculating the normalizing constant.

Theorem 2.4. *Suppose that $g(x)$ is a symmetric pdf about $k \neq 0$, and $G(x)$ is the corresponding cdf, then $f(x) = \frac{1}{2}\frac{g(|x|)}{G_{X-k}(k)}$ is a symmetric bimodal pdf.*

We can apply the skewing mechanism of Azzalini to extend the results of Theorem 2.4 for introducing skewness and bimodality in families of symmetric distributions. The result stated next shows how we can use this idea.

Corollary 2.1. *With the same notation as in Theorem 2.4, $f(x) = \frac{g(|x|)}{G_{X-k}(k)}F(\lambda x)$ is a pdf where $F(\cdot)$ is a cdf having a symmetric pdf about zero, and $\lambda \in \mathbb{R}$ controls the skewness.*

Examples 2.5. Let $X \sim$ N$(k, 1)$, then

$$f(x) = \phi(|x| - k)\frac{\Phi(\lambda x)}{\Phi(k)}, \tag{2.5}$$

is a flexible pdf, where ϕ and Φ denote, respectively, the pdf and the cdf of the standard normal distribution. Densities for particular choices of parameters for (2.5) are shown in Figure 3.

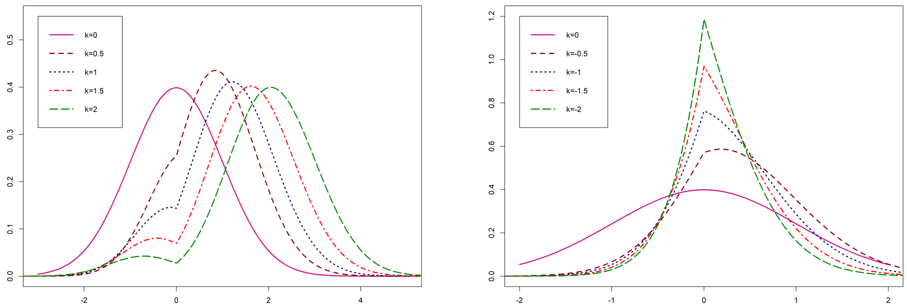


Figure 3: Examples of the pdf gievn in (2.5) for different choices of k and $\lambda = 1$.

3 Particular Cases of Flexible Bimodal Distributions

In this section, we use the theoretical results presented in Section 2 to generate classes of distributions based on well-known symmetric distributions, useful for practical modeling of skewed and bimodal data. Specifically, we focus on three distributions with different tail behavior: normal, Student-t, and Laplace distributions. The new families of distributions are flexible enough to support both unimodal and bimodal densities as well as symmetric and asymmetric ones.

3.1 A generalized Normal Distribution

Applying the representation provided in Theorem 2.1, we can introduce a new generalization for the normal distribution, named unimodal-bimodal normal (UBN), whose pdf is given by

$$f(x) \quad = \quad c_{\sigma,k,a}^{-1} \exp\left(k\left|\frac{x-\mu}{\sigma}\right|\right)\frac{1}{\sigma}\phi\left(\frac{x-\mu-a}{\sigma}\right), \qquad x \in \mathbb{R}, \tag{3.1}$$

where $c_{\sigma,k,a}^{-1} = \exp\left(\frac{ka}{\sigma} + \frac{k^2}{2}\right)\Phi\left(k + \frac{a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma} + \frac{k^2}{2}\right)\Phi\left(k - \frac{a}{\sigma}\right)$, $\mu \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameters, respectively and $a \in \mathbb{R}$ and $k \in \mathbb{R}$ are the shape parameters that control skewness and bimodality. The UBN distribution with pdf (3.1) is denoted by UBN(μ, σ, k, a). This distribution is symmetric when a is zero, denoted by UBN(μ, σ, k). Indeed, this family has the normal distribution as a particular member whenever both k and a are zero. Furthermore, the bimodal area is specified by $\sigma k > |a|$. The shapes of the UBN pdf for particular choices of parameter values are shown in Figure 4.

Proposition 3.1. *Modes of the UBN pdf are at*

$$\begin{cases} \mu + a \pm k\sigma & k\sigma > |a| \\ \mu + a + \operatorname{sign}(a)k\sigma & -|a| < k\sigma < |a| \\ \mu & k\sigma \le -|a|. \end{cases}$$

For simplicity, in the following, we will use the standardized case of the UBN distribution where $Z = \frac{X-\mu}{\sigma} \sim \mathrm{UBN}(0, 1, k, \frac{a}{\sigma})$. The cdf of Z is given by

$$F(z) = \begin{cases} \dfrac{\mathrm{e}^{-\frac{ka}{\sigma}}\ \Phi\big(z+k-\frac{a}{\sigma}\big)}{\mathrm{e}^{\frac{ka}{\sigma}}\Phi\big(k+\frac{a}{\sigma}\big)+\mathrm{e}^{-\frac{ka}{\sigma}}\Phi\big(k-\frac{a}{\sigma}\big)}, & z \le 0, \\ \dfrac{\mathrm{e}^{-\frac{ka}{\sigma}}\ \Phi\big(k-\frac{a}{\sigma}\big)+\mathrm{e}^{\frac{ka}{\sigma}}\big[\Phi\big(z-k-\frac{a}{\sigma}\big)-\Phi\big(-k-\frac{a}{\sigma}\big)\big]}{\mathrm{e}^{\frac{ka}{\sigma}}\Phi\big(k+\frac{a}{\sigma}\big)+\mathrm{e}^{-\frac{ka}{\sigma}}\Phi\big(k-\frac{a}{\sigma}\big)}, & z > 0. \end{cases}$$

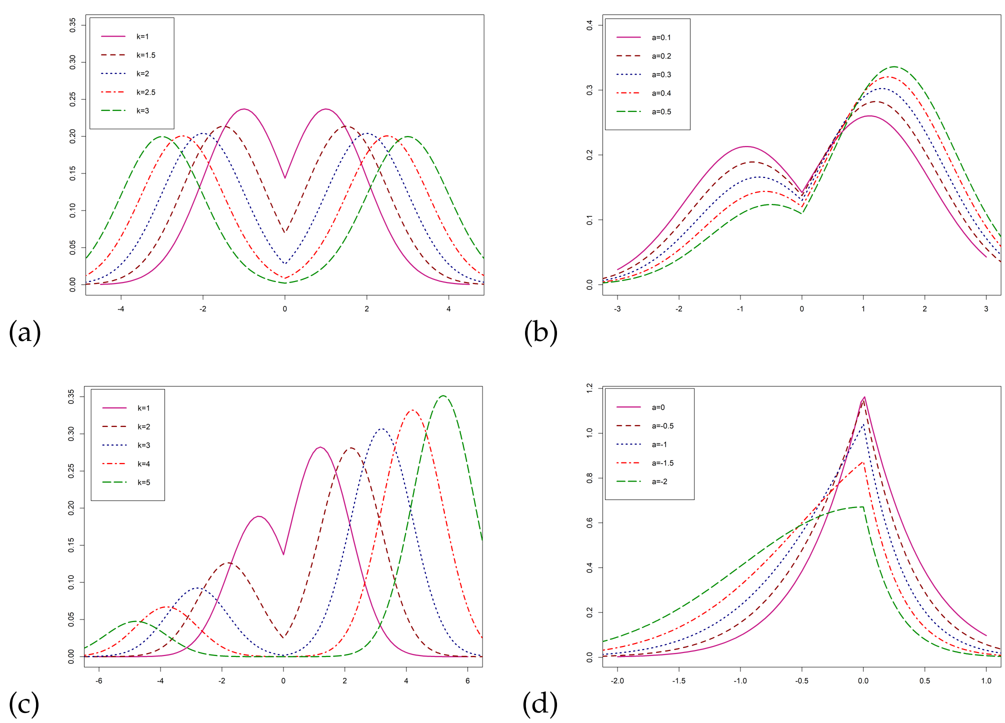


Figure 4: Examples of the UBN pdf for different choices of parameters: (a) $\mu = 0, \sigma = 1$ and $a = 0$, (b) $\mu = 0, \sigma = 1$ and $k = 1$, (c) $\mu = 0, \sigma = 1$ and $a = 0.2$ and (d) $\mu = 0, \sigma = 1$ and $k = -2$

Moreover, the moment generating function of Z is

$$M(t) = \frac{\mathrm{e}^{(k+t)\frac{a}{\sigma}+\frac{(k+t)^2}{2}}\Phi\left(k+\frac{a}{\sigma}+t\right)+\mathrm{e}^{-(k-t)\frac{a}{\sigma}+\frac{(k-t)^2}{2}}\Phi\left(k-\frac{a}{\sigma}-t\right)}{\mathrm{e}^{\frac{ka}{\sigma}+\frac{k^2}{2}}\Phi\left(k+\frac{a}{\sigma}\right)+\mathrm{e}^{-\frac{ka}{\sigma}+\frac{k^2}{2}}\Phi\left(k-\frac{a}{\sigma}\right)}.$$

It follows that the first four moments of the standard UBN distribution are obtained by

$$\begin{aligned}
\mathrm{E(Z)} &= \frac{p_k p_t - n_k n_t}{\delta},\\
\mathrm{E(Z^2)} &= \frac{(1+p_k^2)p_t + (1+n_k^2)n_t + 2kn_d}{\delta},\\
\mathrm{E(Z^3)} &= \frac{(3p_k+p_k^3)p_t - (3n_k+n_k^3)n_t + 4\frac{ka}{\sigma}n_d}{\delta},\\
\mathrm{E(Z^4)} &= \frac{\left(3+6p_k^2+p_k^4\right)p_t + \left(3+6n_k^2+n_k^4\right)n_t + \left[2k^3+10k+6(\frac{a}{\sigma})^2k\right]n_d}{\delta},
\end{aligned}$$

where $p_t = \mathrm{e}^{\frac{ka}{\sigma}}\Phi\left(k+\frac{a}{\sigma}\right)$, $n_t = \mathrm{e}^{\frac{-ka}{\sigma}}\Phi\left(k-\frac{a}{\sigma}\right)$, $\delta = p_t + n_t$, $p_k = k+\frac{a}{\sigma}$, $n_k = k-\frac{a}{\sigma}$, and $n_d = \mathrm{e}^{\frac{-ka}{\sigma}}\phi\left(k-\frac{a}{\sigma}\right)$. We can use these moments, for instance, to compute the skewness and excess of kurtosis coefficients when the UBN pdf has one mode.

3.1.1 Stochastic Representation

The UBN distribution can be stochastically represented as a mixture of two truncated normal random variables without overlapping. Then, we can easily generate data from the UBN distribution.

Proposition 3.2. *The UBN pdf is a mixture of two truncated normal pdfs as*

$$pX_1 + (1-p)X_2,$$

where $p = \frac{e^{-\frac{ka}{\sigma}}\Phi\left(k-\frac{a}{\sigma}\right)}{e^{\frac{ka}{\sigma}}\Phi\left(k+\frac{a}{\sigma}\right)+e^{-\frac{ka}{\sigma}}\Phi\left(k-\frac{a}{\sigma}\right)}$. *Further,* X_1 *and* X_2 *are the pdfs of the truncated normal distributions with scale parameters* 1 *and location parameters* $\frac{a}{\sigma}-k$ *and* $\frac{a}{\sigma}+k$ *on intervals* $(-\infty, 0)$ *and* $(0,\infty)$*, respectively.*

An immediate consequence of Proposition 3.2 is that the distribution has a closed-form density and also avoids the label switching problem.

3.1.2 ML Estimation

Here we obtain the ML estimators of the parameters of the UBN family. Let $x_1, \ldots, x_n$ be the observed values of a random sample of size n from $X \sim \mathrm{UBN}(\mu, \sigma, k, a)$. The log-likelihood function for $\boldsymbol{\theta}^{\mathrm{UBN}} = (\mu, \sigma, k, a)$ can be expressed as

$$\ell = \ell(\boldsymbol{\theta}^{\mathrm{UBN}}) = -\frac{nk^2}{2} - n\log(\sigma) - n\log(\delta) + \frac{k}{\sigma}\sum_{i=1}^{n}\left|x_i-\mu\right| - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu-a}{\sigma}\right)^2.$$

Maximization of the log-likelihood function must be performed numerically. To improve the efficiency, we can provide the vector of partial derivatives of the log-likelihood with respect to parameters, i.e. the score vector $\boldsymbol{U}_n^{\text{UBN}} = \left(\frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial k}, \frac{\partial \ell}{\partial a}\right)$, to an optimization algorithm. The components of $\boldsymbol{U}_n^{\text{UBN}}$ are given in the appendix. The ML estimates can be obtained by equating a system of equations $\frac{\partial \ell}{\partial \mu}$, $\frac{\partial \ell}{\partial \sigma}$, $\frac{\partial \ell}{\partial k}$ and $\frac{\partial \ell}{\partial a}$ to zero and solve them simultaneously. Moreover, to implement statistical inferences, we need to approximate the standard errors of ML estimates of parameters, $\hat{\boldsymbol{\theta}}^{\text{UBN}}$. We can obtain them by taking the square roots of the diagonal elements of the inverse of the observed information matrix evaluated at $\hat{\boldsymbol{\theta}}^{\text{UBN}}$. The observed information matrix is computed by evaluating the negative of the Hessian matrix. Indeed, the observed information matrix can also be obtained numerically using software packages such as the `optim` function in the `R` environment (R-Team, 2021) (R-Team, 2021). Both the Hessian and information matrices of UBN distribution are provided in the appendix.

3.2 A generalized Student-t Distribution

The Student-t distribution could be the most popular model for the analysis of heavy-tail data. A few extensions of this distribution have been introduced in the literature. For example, see Jones and Faddy (2003) Jones and Faddy (2003), Aas and Haff (2006) Aas and Haff (2006), and Huang et al. (2019) Huang et al. (2019). Here, by applying Theorem 2.3, we propose a new flexible distribution, named unimodal-bimodal Student-t (UBSt), whose pdf is given by

$$f(x) = \frac{d_\nu\left(\left|\frac{x-\mu}{\sigma}\right| - k\right)}{2\sigma D_\nu(k)} = \frac{\left\{d_\nu\left(\frac{x-\mu}{\sigma} - k\right)\right\}^{I(x\geq\mu)} \left\{d_\nu\left(\frac{x+\mu}{\sigma} + k\right)\right\}^{I(x<\mu)}}{2\sigma D_\nu(k)}, \tag{3.2}$$

where $I(\cdot)$ is the indicator function, $d_\nu(\cdot)$ and $D_\nu(\cdot)$ denote, respectively, the pdf and the cdf of the Student-t distribution with ν degrees of freedom, $\mu \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameters, respectively, and $k \in \mathbb{R}$ is the shape parameter that controls bimodality. Indeed, the UBSt pdf is bimodal for $k > 0$, unimodal for $k < 0$ and leads to classical Student-t distribution when $k = 0$.

The proposed UBSt distribution with pdf (3.2), denoted by UBSt(μ, σ, k, ν), is symmetric. Therefore, we extend it to a more flexible distribution with the following pdf to consider the skewness as well:

$$f(x) = \frac{\left\{s_-^{-\frac{\nu+1}{2}} d_\nu\left(\frac{x-\mu-(a+k)}{\sqrt{s_-}}\right)\right\}^{I(x\geq\mu)} \left\{s_+^{-\frac{\nu+1}{2}} d_\nu\left(\frac{x-\mu-(a-k)}{\sqrt{s_+}}\right)\right\}^{I(x<\mu)}}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}, \tag{3.3}$$

where $a \in \mathbb{R}$ is a further shape parameter to control the skewness, $s_- = \frac{\nu\sigma^2 - 2ak}{\nu}$ and $s_+ = \frac{\nu\sigma^2 + 2ak}{\nu}$. When $a = 0$, the pdf (3.3) can be simplified to (3.2). The UBSt distribution with pdf (3.3) is also denoted by UBSt(μ, σ, k, a, ν). The shapes of the UBSt pdf for some selected values of parameters are shown in Figure 5 that illustrate its ability to accommodate different shapes in terms of skewness, kurtosis, and bimodality.

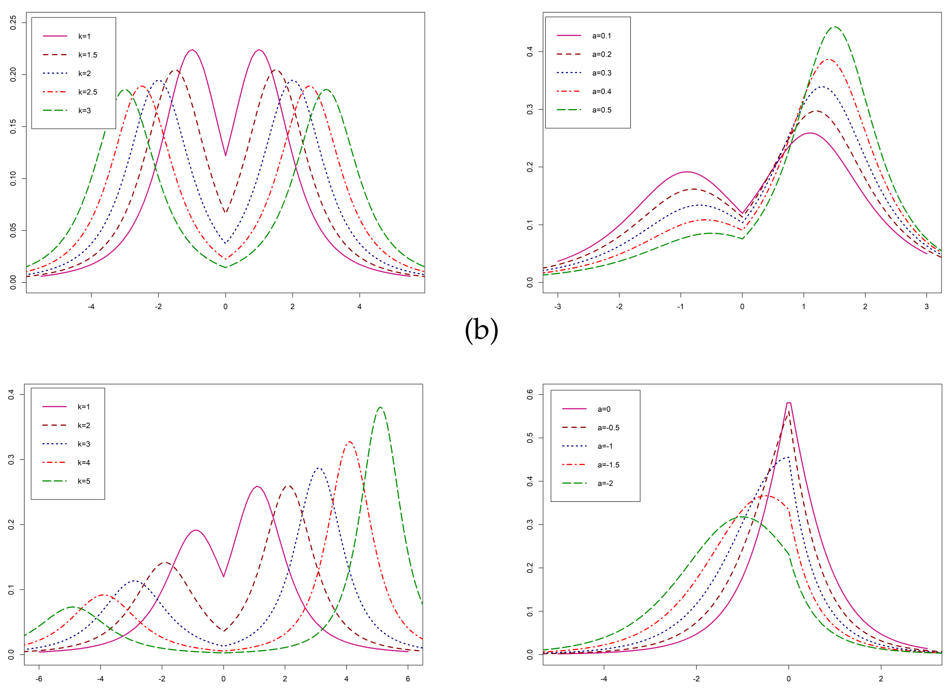
Proposition 3.3. *Modes of the UBSt distribution are at*

$$\begin{cases} \mu + a \pm k & k > |a| \\ \mu + a + \mathrm{sign}(a)k & -|a| < k < |a| \\ \mu & k \leq -|a|. \end{cases}$$

The following result shows that some of the properties of classical symmetric distribution will be retained by the newly proposed family.

Proposition 3.4. *Let* $X \sim \mathrm{UBSt}(\mu, \sigma, k, a, \nu)$*. If* $\nu \to \infty$*, then* $X \xrightarrow{dist.} \mathrm{UBN}(\mu, \sigma, k, a)$*.*

The UBSt distribution could also be represented as a mixture of two truncated Student-t random variables without overlapping.



(a) (b) (c) (d)

Figure 5: Examples of the UBSt pdf for some particular values of parameters: (a) $\mu = 0$, $\sigma = 1$, $a = 0$ and $\nu = 2$, (b) $\mu = 0$, $\sigma = 1$, $k = 1$ and $\nu = 2$, (c) $\mu = 0$, $\sigma = 1$, $a = 0.1$ and $\nu = 2$ and (d) $\mu = 0$, $\sigma = 1$, $k = -1$ and $\nu = 5$

Proposition 3.5. *The UBSt distribution could be written as a mixture of two truncated Student-t distributions as follows:*

$$R_1X_1 + R_2X_2,$$

where $X_1 \sim \mathrm{Tt}_{(\mu,\infty)}(\mu + (a+k), \sqrt{s_-}; \nu)$, $X_2 \sim \mathrm{Tt}_{(-\infty,\mu)}(\mu + (a-k), \sqrt{s_+}; \nu)$ *and*

$$\begin{aligned} R_1 &= \frac{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right)}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}, \\ R_2 &= \frac{s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}. \end{aligned}$$

From Kim (2008)Kim (2008), the pdfs of X_1 *and* X_2 *are, respectively, given by*

$$f_{X_1}(x_1) = \frac{s_-^{-\frac{1}{2}} d_\nu\left(\frac{x_1-\mu-(a+k)}{\sqrt{s_-}}\right)}{D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right)},$$

and

$$f_{X_2}(x_2) = \frac{s_+^{-\frac{1}{2}} d_\nu\left(\frac{x_2-\mu-(a-k)}{\sqrt{s_+}}\right)}{D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}.$$

We can use Proposition 3.5 to generate random samples and to calculate moments of the UBSt distribution. The rth moment of UBSt(μ, σ, k, a, ν), for $r = 1, 2, 3, 4$, is given by

$$\mathrm{E}(X^r) = \sum_{i=0}^{r} \binom{r}{i} R_1^i R_2^{r-i} \mathrm{E}(X_1^i)\mathrm{E}(X_2^{r-i}).$$

According to Kim (2008)Kim (2008), we have

$$\mathrm{E}(X_1^m) = \sum_{j=0}^{m} \binom{m}{j} (\mu + a + k)^{m-j} (s_-)^{\frac{j}{2}} \eta_m,$$

and

$$\mathrm{E}(X_2^l) = \sum_{j=0}^{l} \binom{l}{j} (\mu + a - k)^{l-j} (s_+)^{\frac{j}{2}} \lambda_l,$$

where, for $m, l = 1,2,3,4$,

$$
\begin{aligned}
\eta_1 &= G_\nu(1)\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 1, \\
\eta_2 &= \frac{\nu}{\nu-2} - \frac{a+k}{\sqrt{s_-}} G_\nu(1)\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 2, \\
\eta_3 &= G_\nu(3)\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-3}{2}} + \frac{(a+k)^2}{s_-} G_\nu(1)\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 3, \\
\eta_4 &= 3\left\{\frac{\nu^2}{(\nu-2)(\nu-4)} - \frac{G_\nu(3)}{2}\frac{a+k}{\sqrt{s_-}}\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-3}{2}}\right\}, \\
&\quad -\frac{(a+k)^3}{(s_-)^{\frac{3}{2}}} G_\nu(1)\left(\nu + \frac{(a+k)^2}{s_-}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 4,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_1 &= -G'_\nu(1)\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 1, \\
\lambda_2 &= \frac{\nu}{\nu-2} + \frac{a-k}{\sqrt{s_+}} G'_\nu(1)\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 2, \\
\lambda_3 &= -G'_\nu(3)\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-3}{2}} - \frac{(a-k)^2}{s_+} G'_\nu(1)\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 3, \\
\lambda_4 &= 3\left\{\frac{\nu^2}{(\nu-2)(\nu-4)} + \frac{G'_\nu(3)}{2}\frac{a-k}{\sqrt{s_+}}\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-3}{2}}\right\}, \\
&\quad +\frac{(a-k)^3}{(s_+)^{\frac{3}{2}}} G'_\nu(1)\left(\nu + \frac{(a-k)^2}{s_+}\right)^{-\frac{\nu-1}{2}}, \quad \nu > 4,
\end{aligned}
$$

and for $s = 1,2$

$$
G_\nu(s) = \frac{\Gamma\left(\frac{\nu-s}{2}\right)\nu^{\frac{\nu}{2}}}{2D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)},
$$

and

$$
G'_\nu(s) = \frac{\Gamma\left(\frac{\nu-s}{2}\right)\nu^{\frac{\nu}{2}}}{2D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)}.
$$

The cdf of (3.3) is also given by

$$F(x) = \begin{cases} \dfrac{s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{x-\mu-(a-k)}{\sqrt{s_+}}\right)}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}, & x \leq \mu \\ \dfrac{s_-^{-\frac{\nu}{2}}\left(D_\nu\left(\frac{x-\mu-(a+k)}{\sqrt{s_-}}\right) - D_\nu\left(\frac{-k-a}{\sqrt{s_-}}\right)\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}, & x > \mu. \end{cases}$$

The log-likelihood of the UBSt distribution, corresponding to a random sample of size n, is given by

$$\begin{aligned} \ell = \ell(\boldsymbol{\theta}^{\mathrm{UBSt}}) \;=\;& -n\log(\delta) - \frac{\nu+1}{2}\log(s_-)\sum_{i=1}^{n} I(x_i \geq \mu) - \frac{\nu+1}{2}\log(s_+)\sum_{i=1}^{n} I(x_i < \mu) \\ & + n\log\left(\Gamma\left(\frac{\nu+1}{2}\right)\right) - n\log\left(\Gamma\left(\frac{\nu}{2}\right)\right) - n\log\left(\sqrt{\nu\pi}\right) \\ & - \frac{\nu+1}{2}\sum_{i=1}^{n}\left\{\log\left(1 + \frac{1}{\nu}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2\right) I(x_i \geq \mu)\right\} \\ & - \frac{\nu+1}{2}\sum_{i=1}^{n}\left\{\log\left(1 + \frac{1}{\nu}\left(\frac{u_i^+}{\sqrt{s_+}}\right)^2\right) I(x_i < \mu)\right\}, \end{aligned}$$

where $\boldsymbol{\theta}^{\mathrm{UBSt}} = (\mu, \sigma, k, a, \nu)$ denotes the vector of parameters. Similar to the UBN model, maximization of the log-likelihood function must be accomplished numerically. The elements of the score vector $\boldsymbol{U}_n^{\mathrm{UBSt}} = (\frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial k}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial \nu})$ are given in the appendix. Obtaining the Hessian and information matrices of the UBSt distribution is impossible due to complicated derivatives and computing the corresponding expectations. Therefore, they are calculated numerically.

3.3 A Generalized Laplace Distribution

We, here, apply our second proposed representation to the Laplace distribution to construct another flexible skewed/bimodal distribution, denoted as the unimodal-bimodal Laplace (UBL) distribution. The classical Laplace distribution has many applications in different scientific disciplines, and several generalized forms of the distribution have been introduced in the literature. See, for example, Nekoukhou and Alamatsaz (2012) Nekoukhou and Alamatsaz (2012), Shams Harandi and Alamatsaz (2013)Shams Harandi and Alamatsaz (2013), and Shah et al. (2019)Shah et al. (2019). The pdf of UBL distribution is given by

$$f(x) = \frac{k}{\sigma c}\left(1 + \left(z - \frac{a}{\sigma}\right)^2\right) \mathrm{e}^{-k|z|}, \tag{3.4}$$

where $z = \frac{x-\mu}{\sigma}$, $c = 2(1 + \frac{a^2}{\sigma^2} + \frac{2}{k^2})$, $\mu \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameters, respectively, $a \in \mathbb{R}$ and $k \in \mathbb{R}$ are also two shape parameters that control skewness and bimodality. The UBL distribution with pdf (3.4) is denoted by UBL(μ, σ, k, a). It is symmetric for $a = 0$ and will be denoted by UBL(μ, σ, k). The cdf of UBL distribution can be obtained as

$$F(x) = \begin{cases} \frac{k}{c}\left(\frac{z^2}{k} - \frac{2}{k}\left(\frac{a}{\sigma}+\frac{1}{k}\right)\left(z-\frac{1}{k}\right)+\left(\frac{a^2}{\sigma^2 k}+\frac{1}{k}\right)\right)\mathrm{e}^{kz} & x \geq \mu \\ \frac{k}{c}\left(\frac{2}{k^2}\left(\frac{a}{\sigma}+\frac{1}{k}\right)+\left(\frac{a^2}{\sigma^2 k}+\frac{1}{k}\right)\right) & \\ +\frac{k}{c}\left(\left(\frac{a^2}{\sigma^2 k}+\frac{1}{k}\right)\left(1-\mathrm{e}^{-kz}\right)-\frac{z^2}{k}\mathrm{e}^{-kz}\right) & x < \mu. \\ -\frac{2k}{c}\left(\frac{a}{\sigma}-\frac{1}{k}\right)\left(\frac{1-\mathrm{e}^{-kz}-kz\mathrm{e}^{-kz}}{k^2}\right), & \end{cases}$$

Figure 6 shows various plots of the UBL pdf.

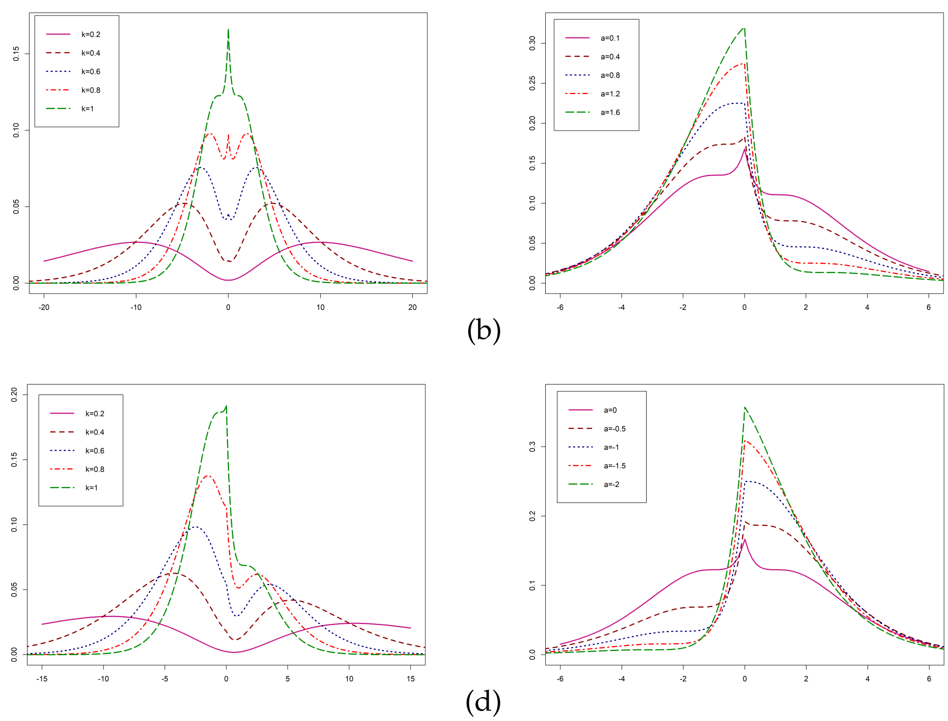


Figure 6: Various shapes of the pdfs of UBL distribution: (a) $\mu = 0, \sigma = 1$ and $a = 0$, (b) $\mu = 0, \sigma = 1$ and $k = 1$, (c) $\mu = 0, \sigma = 1$ and $a = 0.5$ and (d) $\mu = 0, \sigma = 1$ and $k = 1$

Proposition 3.6. *When $k \leq 1$, the modes of the UBL distribution are given by μ as well as $\mu + a \pm \sigma\left(\frac{1}{k} + \sqrt{\frac{1}{k^2} - 1}\right)$. Otherwise, it has one mode located at μ.*

Let $Z = \dfrac{X-\mu}{\sigma} \sim$ UBL$(0, 1, k, \frac{a}{\sigma})$, then the rth moment of Z is given by

$$\mathrm{E}(Z^r) = \frac{2}{c}\left\{(1+\frac{a^2}{\sigma^2})\mathrm{E}(W_L^r) - 2\frac{a}{\sigma}\mathrm{E}(W_L^{r+1}) + \mathrm{E}(W_L^{r+2})\right\},$$

where W_L follows a Laplace distribution with the location zero and the scale $\frac{1}{k}$. It is clear that the sth moment of W_L is $\dfrac{s!}{2k^s}(1+(-1)^s)$.

The log-likelihood of the UBL distribution, corresponding to a random sample of size n, is written as

$$\begin{aligned}\ell = \ell(\boldsymbol{\theta}^{\mathrm{UBL}}) &= n\log\left(\frac{k}{2\sigma}\right) - n\log\left(1+\frac{a^2}{\sigma^2}+\frac{2}{k^2}\right) - k\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\sigma}\right| \\ &\quad + \sum_{i=1}^{n}\log\left(1+\left(\frac{x_i-\mu-a}{\sigma}\right)^2\right),\end{aligned}$$

where $\boldsymbol{\theta}^{\mathrm{UBL}} = (\mu, \sigma, k, a)$. Similarly, we must maximize the log-likelihood function using numerical optimization techniques. We supply the elements of the score vector $\boldsymbol{U}_n^{\mathrm{UBL}} = (\frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial k}, \frac{\partial \ell}{\partial a})$ in the appendix. Similarly, obtaining the Hessian and information matrices of the UBL distribution is impossible, and hence, they are numerically computed.

4 Empirical Applications

We used three data sets to compare the fits of the proposed UB distributions with those mentioned in the introduction, i.e., SN, SSt, BN, OLLN, and PLD. In each case, the estimation of the parameters was carried out by maximizing the corresponding likelihood function, employing the BFGS quasi-Newton method (proposed by Broyden (1970) Broyden (1970), Fletcher (1970)Fletcher (1970), Goldfarb (1970) Goldfarb (1970), and Shanno (1970)Shanno (1970) independently) in the `R` environment. This optimization method can be accessed through the `optim()` function. Then, the ML estimates (and the corresponding standard errors in parentheses) of the parameters were reported. Moreover, the Kolmogorov-Smirnov (K-S) test was used to assess the goodness of fit, while the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) were used for model comparisons. The lower the value of AIC and BIC, the better the fit. Finally, we prepared the histograms of the data sets to compare the fitted densities visually.

4.1 Devices Lifetime Data

This data set was reported and studied by Aarset (1987)Aarset (1987). The data represent the times of failure for 50 devices put on life test at time 0. These data were recently analyzed by Bakouch et al. (2019) using the PLD distribution that exhibits a skew bimodal density. The histogram of the data appears in Figure 7 (top-left), suggesting the presence of bimodality in the data and then revealing that the use of bimodal distributions can be a reasonable choice to analyze them.

The ML estimates, AIC, BIC, and the K-S test statistic as well as its corresponding p-value (in the parentheses) for each distribution, are given in Table 1. It is noteworthy that we take $a = 0$ for UB models due to the symmetric appearance of the empirical distribution of the data. According to the K-S tests, all the models passed the goodness of fit test. Both information criteria point out that the UBN distribution is the best model. However, they imply that the OLLN and UBSt models are also comparable. Indeed, the estimate of $\nu = 44.829$ in the UBSt model confirms the acceptable convergence to the UBN model, according to Proposition 3.4. Moreover, the estimation of k in all three UB models confirms the bimodality of data that is in agreement with the empirical distribution of the data.

A visual comparison of the histogram of the data with the fitted densities could also be helpful. The fitted pdf curves indicate a satisfactory fit to the data provided by the UBN, OLLN, and UBSt models. The difference between UBN and OLLN models has also vanished almost perfectly.

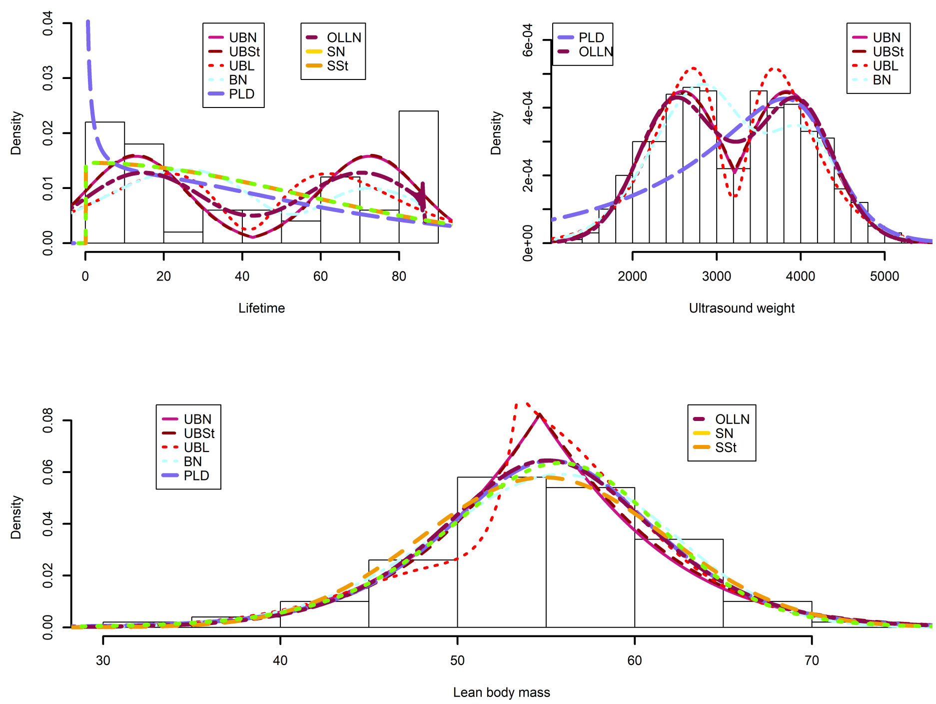
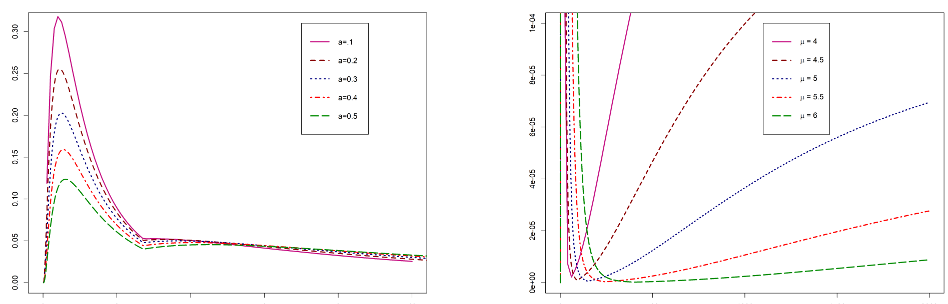


Figure 7: Histogram of the data and the fitted competing densities for lifetime data (top-left), ultrasound weight data (top-right), and lean body mass data (bottom).

4.2 Ultrasound Weight Data

In this example, the data include 500 observations from the ultrasound weight (fetal weight in grams). These data are freely available at http://www.mat.uda.cl and studied previously by Bolfarine et al. (2018)Bolfarine et al. (2018). The histogram of the data, displayed in Figure 8 (top-right), shows that a symmetric bimodal distribution could fit the data well.

The ML inferences are reported in Table 1, which shows that all the models, excluding the PLD model, exhibit the goodness of fit, based on the K-S test. The lowest values of the information criteria are also obtained for this data set when fitting the UBN distribution. The result of the UBSt model is very close to the best model, which is expected due to the large estimate of its degrees of freedom. For these data, the estimation of k in all UB models points out the bimodality in the data as well. Figure 8 shows the histogram of the data and the fitted densities, which further indicates the agreement between the observed data and fitted UBN and UBSt distributions.



(a) (b)

Figure 8: Shapes of the log-UBN pdf for $\mu = 1$, $\sigma = 1$ and $k = 1$ (a) and hazard function for $\sigma = 1$, $k = 4$ and $a = 0.1$ (b).

4.3 Lean Body Mass Data

The third data set includes the lean body mass of 202 athletes that have been collected at the Australian Institute of Sport. The data were analyzed by Cook and Weisberg (2009) Cook and Weisberg (2009) and Sastry and Bhati (2016)Sastry and Bhati (2016). Figure 8 (bottom) shows the histogram of the data that suggests a skewed unimodal distribution for analyzing them. Comparing UB models with skewed distributions, mainly the SN and SSt models, might be interesting here. The results of the K-S tests prove that all models fit the data well. Further, both information criteria disclose the UBL distribution as the best model. Such a result is also verified by the fitted densities shown in Figure 8 (bottom). Indeed, the estimate of $k = 1.224$, in the UBSt model, confirms that the distribution should be unimodal.

5 Extensions and Further Applications

5.1 Multivariate Distributions

The extension to the multivariate case is relatively straightforward. As an example, we demonstrate how to construct a multivariate version of the UBN distribution.

Let q independent random variables Z_i follow an identical UBN$(0,1,k,a)$ distribution. Therefore, the random vector $\boldsymbol{Z} = \left(Z_1,\ldots,Z_q\right)$ has the probability density function as $f(\boldsymbol{z}) = \Pi_{i=1}^{q} f(z_i)$. In a matrix notation, it could be shown that

$$f(\boldsymbol{z}) = c_{k,a}^{q}\,(2\pi)^{-\frac{q}{2}}\,e^{k\mathbf{J}_q'|\boldsymbol{z}|}e^{-\frac{1}{2}\left(\boldsymbol{z}-a\mathbf{J}_q\right)'\left(\boldsymbol{z}-a\mathbf{J}_q\right)} \tag{5.1}$$

where k and a are the shape parameters, $\mathbf{J}_q$ is a q dimensional vector of ones, and $|\cdot|$ represents the elementwise absolute value of a vector. Now, let $\boldsymbol{X} = \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{Z} + \boldsymbol{\mu}$, where $\boldsymbol{\mu} \in \mathbb{R}^q$ is the location vector, $\boldsymbol{\Sigma}$ is a $q \times q$ positive definite scale matrix, and $\boldsymbol{\Sigma}^{\frac{1}{2}}$ means the square root of matrix $\boldsymbol{\Sigma}$. Therefore, $\boldsymbol{X}$ follows a multivariate UBN distribution and has the following probability density

$$f(\boldsymbol{x}) = \frac{c_{k,a}^{q}e^{k\mathbf{J}_q'\left|\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{x}-\boldsymbol{\mu})\right|}}{(2\pi)^{\frac{q}{2}}\,(\det\boldsymbol{\Sigma})^{\frac{1}{2}}}e^{-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu}-a\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{J}_q\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}-a\boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{J}_q\right)}. \tag{5.2}$$

The multivariate normal distribution is a special case of the multivariate UBN family when $k = a = 0$.

5.2 Linear Regression with UB-distributed Errors

In many applications, the error term in a linear regression model is skewed and/or bimodal. Our proposed UB distributions could be useful in such situations. To explain the idea, suppose that the observed responses y_i, $i = 1,\ldots,n$, are generated from

$$y_i = \boldsymbol{x}_i^t\boldsymbol{\mu} + \sigma\epsilon_i,$$

where $\boldsymbol{x}_i$ is a p-dimensional vector of explanatory variables, $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_p)^t$ is a vector of regression coefficients, and σ is a scale parameter. We consider the cases where ϵ_i is iid with a distribution from one of the UB classes in Section 3. We can develop both the mean and modal regression here.

5.3 Log-UB Distributions

Let $X = \log(T)$ follow a distribution from one of the UB families, e.g., the UBN family, then T is said to follow the log-UB distribution. Log-UB distributions could adequately be used for fatigue lifetimes processes, cure rate proportional hazard models, and survival regression models, to name a few. These extensions can inherit flexibility and suitability of original distributions as well. As an example, we briefly explain constructing a log-UBN distribution in the following.

Let X follow a UBN distribution. Then, $T = \exp(X)$ will follow a log-UBN distribution, denoted by log-BUN(μ, σ, k, a), with the pdf

$$f(t) \quad = \quad c_{\sigma,k,a} \exp\left(k \left|\frac{\log(t) - \mu}{\sigma}\right|\right) \frac{1}{t\sigma} \phi\left(\frac{\log(t) - \mu - a}{\sigma}\right), \qquad t > 0,$$

where $c^{-1}_{\sigma,k,a} = \exp\left(\frac{ka}{\sigma} + \frac{k^2}{2}\right)\Phi\left(k + \frac{a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma} + \frac{k^2}{2}\right)\Phi\left(k - \frac{a}{\sigma}\right)$. It is easy to show that

$$\mathrm{E}(T^r) = M(r) = \exp\left(-\frac{r\mu}{\sigma}\right) M_Z\left(\frac{r}{\sigma}\right),$$

where $M_Z(\cdot)$ is the moment generating function of a standard UBN random variable. Also, $\mathrm{P}(T \leq t) = F_X(\log(t))$ where $F_X(\cdot)$ is the cdf of the UBN distribution. Figure 8 displays various plots of the log-UBN pdf (left) as well as hazard function (right) for some given parameter values, offering its potential flexibility in modeling time-to-event data.

6 Conclusion

In this article, we introduced two new constructive representations to develop flexible distributions that could account for both skewness and bimodality. The representations proposed here provide a powerful framework for developing new classes of skewed and/or bimodal distributions, as we illustrated in Section 3. We used the maximum likelihood technique to estimate the parameters of the models. We also empirically proved the usefulness of the proposed models through three real data sets. We believe that our proposals could be practically relevant in tackling real problems. R code is available from the first author on request.

Acknowledgement

The authors wish to thank an Associate Editor and two anonymous reviewers for careful reading and constructive suggestions which led to this improved version of the article. The Authors declare that there is no conflict of interest.

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All Technical Details and Proofs

Proof of Theorem 2.1

(i) It follows immediately from symmetric properties of $g(\cdot)$ and $w(\cdot)$.
(ii) Since $f(x)$ is a pdf with support on $\mathbb{R}$, then it has at least one mode. Let m_1 is one of the modes. Without loss of generality, let $m_1 > c$. Hence, we can write

$$w'(m_1)g(m_1) + w(m_1)g'(m_1) = -w'(2c - m_1)g(2c - m_1) - w(2c - m_1)g'(2c - m_1) = 0.$$

Therefore, the second mode is $m_2 = 2c - m_1$.
(iii) From (ii) we have $m_1 + m_2 = m_1 + 2c - m_1 = 2c$.

Proof of Theorem 2.2

We must establish that both presented $f(x)$ in the theorem are pdfs. Then their bimodal properties are concluded from Theorem 2.1 and the convexity of $e^{kG(|x|)}$ and $G(|x|)^k$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{kG(|x|)} g(x)dx &= \int_{-\infty}^{0} e^{kG(-x)} g(x)dx + \int_{0}^{\infty} e^{kG(x)} g(x)dx \\
&= \int_{-\infty}^{0} e^{k(1-G(x))} g(x)dx + \int_{0}^{\infty} e^{kG(x)} g(x)dx \\
&= \int_{0}^{\frac{1}{2}} e^{k(1-u)} du + \int_{\frac{1}{2}}^{1} e^{ku} du = \frac{2\left(e^{k} - e^{\frac{k}{2}}\right)}{k},
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} G(|x|)^k g(x)dx &= \int_{-\infty}^{0} G(-x)^k g(x)dx + \int_{0}^{\infty} G(x)^k g(x)dx \\
&= \int_{-\infty}^{0} (1 - G(x))^k g(x)dx + \int_{0}^{\infty} G(x)^k g(x)dx \\
&= \int_{-\infty}^{0} (1 - u)^k du + \int_{0}^{\infty} u^k du = \frac{2\left(1 - \frac{1}{2^{k+1}}\right)}{k+1}.
\end{aligned}$$

Hence, the proof is completed.

Proof of Theorem 2.3

i) Taking the derivative of $f(x) \propto g(h(x))$ and equating it to zero, we have $h'(x)g'(h(x)) = 0$. Hence, $h'(x) = 0$ or $g'(h(x)) = 0$. Since $h(x)$ is convex and symmetric around zero, the solution of the former, $h'(x) = 0$, is zero that is the mode of (2.1). For the latter, $g'(h(x)) = 0$, $g(x)$ has a mode at zero, and then $h(x) = 0$. Therefore, $x = 0$ is the only mode of (2.1).

ii) Let $k \neq 0$. Similar to i), the solution of $h'(x) = 0$ is zero. For $g'(h(x)) = 0$, we should solve the equation $h(x) = k$. It has two solutions due to the convexity of h. By using the second derivative, it is easy to show that the two solutions are local maximums (modes). Specifically, by letting $h(x) = k$, we have

$$f''(x) = h''(x)g'(k) + h'(x)g''(k).$$

Since k is the mode of $g(\cdot)$, then $g'(k) = 0$ and $g''(k) > 0$. Therefore,

$$f''(x) = h'(x)g''(k) > 0.$$

iii) It follows from ii).

Proof of Theorem 2.4

Let $h(x) = |x|$, then $f(x) = \dfrac{g(|x|)}{\int_{-\infty}^{\infty} g(|w|)dw}$. We can rewrite f as follows

$$f(x) = \begin{cases} \dfrac{g(x)}{\int_0^{\infty} g(w)\, dw} & \text{if} \quad x \geq 0; \\[2ex] \dfrac{g(-x)}{\int_{-\infty}^{0} g(-w)\, dw} & \text{if} \quad x < 0. \end{cases}$$

or equivalently as

$$f(x) = \begin{cases} \dfrac{g(x)}{1 - G_X(0)} & \text{if} \quad x \geq 0; \\[2ex] \dfrac{g(-x)}{G_{-X}(0)} & \text{if} \quad x < 0. \end{cases}$$

where G_X and G_{-X} are the cdfs of X and $-X$, respectively. Then we have

$$f(x) = \begin{cases} \dfrac{g(x)}{1 - G_X(0)} & \text{if} \quad x \geq 0; \\[2ex] \dfrac{g(-x)}{1 - G_X(0)} & \text{if} \quad x < 0. \end{cases}$$

or equivalently $f(x) = \frac{1}{2}\dfrac{g(|x|)}{G_{(X-k)}(k)}$. Hence, the proof is completed.

Proof of Proposition 3.1

We should examine the critical points of (3.1). The pdf in (3.1) is not differentiable at $x = \mu$. This point could be the mode whenever the sign of the derivative is positive for $x < \mu$ and negative for $x > \mu$. We could write

$$f'(x) \propto k\sigma \text{sign}(x-\mu) - x + \mu + a.$$

Hence, it is positive for $x < \mu$ when $-k\sigma + \mu + a > x$ that leads to $k\sigma \leq a$. By similar arguments, we can show that $f'(x)$ is negative for $x > \mu$ when $k\sigma \leq -a$. Hence, if $k\sigma \leq -|a|$, the pdf in (3.1) has a mode at $x = \mu$. Now, we explore other states, i.e. $k\sigma > -|a|$. We must check other critical points of (3.1). Then, we must solve the following equation

$$k\sigma \text{sign}(x-\mu) - x + \mu + a = 0.$$

The possible solutions are

$$\begin{cases} x = \mu + a + k\sigma, & x > \mu, \\ x = \mu + a - k\sigma, & x < \mu. \end{cases}$$

The pdf has a mode at $x = \mu + a + k\sigma$, when $x > \mu$. It is true whenever $k\sigma > -a$. Combining $k\sigma > -|a|$ with this condition, we have $-a < k\sigma < a$. Similarly, the pdf has a mode at $x = \mu + a - k\sigma$, whenever $a < k\sigma < -a$. Hence, if $k\sigma > |a|$, the pdf has two modes at $x = \mu + a \pm k\sigma$ and the proof is completed.

Proof of Proposition 3.2

We could express the UBN pdf in (3.1) as follows

$$f(x) = \begin{cases} c_{\sigma,k,a} \exp\left(-k\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma} - \frac{a}{\sigma}\right), & x < \mu, \\ c_{\sigma,k,a} \exp\left(k\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma} - \frac{a}{\sigma}\right), & x > \mu. \end{cases}$$

It is easy to show that

$$f(z) = \begin{cases} \dfrac{\exp\left(-\frac{ka}{\sigma}\right)\Phi\left(k - \frac{a}{\sigma}\right)}{\exp\left(\frac{ka}{\sigma}\right)\Phi\left(k + \frac{a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma}\right)\Phi\left(k - \frac{a}{\sigma}\right)} \dfrac{\phi\left(\frac{x-\mu}{\sigma} + k - \frac{a}{\sigma}\right)}{\sigma\Phi\left(k - \frac{a}{\sigma}\right)}, & x < \mu, \\ \dfrac{\exp\left(\frac{ka}{\sigma}\right)\Phi\left(k + \frac{a}{\sigma}\right)}{\exp\left(\frac{ka}{\sigma}\right)\Phi\left(k + \frac{a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma}\right)\Phi\left(k - \frac{a}{\sigma}\right)} \dfrac{\phi\left(\frac{x-\mu}{\sigma} - k - \frac{a}{\sigma}\right)}{\sigma\Phi\left(k + \frac{a}{\sigma}\right)}, & x > \mu, \end{cases}$$

where $z = \frac{x-\mu}{\sigma}$. Then, we have

$$f(z) = p\frac{\phi\left(\frac{x-\mu}{\sigma} + k - \frac{a}{\sigma}\right)}{\sigma\Phi\left(k - \frac{a}{\sigma}\right)} + (1-p)\frac{\phi\left(\frac{x-\mu}{\sigma} - k - \frac{a}{\sigma}\right)}{\sigma\Phi\left(k + \frac{a}{\sigma}\right)}.$$

Proof of Proposition 3.3

The proof is similar to the proof of Proposition 3.1.

Proof of Proposition 3.4

We have

$$
\begin{aligned}
\lim_{\nu\to\infty} s_- = \lim_{\nu\to\infty} s_+ &= \sigma^2,\\
\lim_{\nu\to\infty} s_-^{-\frac{\nu}{2}} &= \exp\left(\frac{ak}{\sigma^2}\right),\\
\lim_{\nu\to\infty} s_+^{-\frac{\nu}{2}} &= \exp\left(-\frac{ak}{\sigma^2}\right),\\
\lim_{\nu\to\infty} s_-^{-\frac{1}{2}} d_\nu\left(\frac{x-\mu-(a+k)}{\sqrt{s_-}}\right) &= \frac{1}{\sigma}\phi\left(\frac{x-\mu-(a+k)}{\sigma}\right),\\
\lim_{\nu\to\infty} s_+^{-\frac{1}{2}} d_\nu\left(\frac{x-\mu-(a-k)}{\sqrt{s_+}}\right) &= \frac{1}{\sigma}\phi\left(\frac{x-\mu-(a-k)}{\sigma}\right),\\
\lim_{\nu\to\infty} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) &= \Phi\left(\frac{a+k}{\sigma}\right)\\
\lim_{\nu\to\infty} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right) &= \Phi\left(\frac{k-a}{\sigma}\right).
\end{aligned}
$$

Hence,

$$
\lim_{\nu\to\infty} \frac{\left\{s_+^{-\frac{\nu+1}{2}} d_\nu\left(\frac{x-\mu-(a-k)}{\sqrt{s_+}}\right)\right\}^{I(x<\mu)} \left\{s_-^{-\frac{\nu+1}{2}} d_\nu\left(\frac{x-\mu-(a+k)}{\sqrt{s_-}}\right)\right\}^{I(x\geq\mu)}}{s_-^{-\frac{\nu}{2}} D_\nu\left(\frac{a+k}{\sqrt{s_-}}\right) + s_+^{-\frac{\nu}{2}} D_\nu\left(\frac{k-a}{\sqrt{s_+}}\right)}
$$

$$
= \frac{\left\{\exp\left(-\frac{ak}{\sigma^2}\right)\frac{1}{\sigma}\phi\left(\frac{x-\mu-(a-k)}{\sigma}\right)\right\}^{I(x<\mu)} \left\{\exp\left(\frac{ak}{\sigma^2}\right)\frac{1}{\sigma}\phi\left(\frac{x-\mu-(a+k)}{\sigma}\right)\right\}^{I(x\geq\mu)}}{\exp\left(\frac{ak}{\sigma^2}\right)\Phi\left(\frac{a+k}{\sigma}\right) + \exp\left(-\frac{ak}{\sigma^2}\right)\Phi\left(\frac{k-a}{\sigma}\right)}
$$

$$
= \begin{cases}
\dfrac{\exp\left(-\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k-a}{\sigma}\right)}{\exp\left(\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k+a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k-a}{\sigma}\right)} \dfrac{\phi\left(\frac{x-\mu-(a-k)}{\sigma}\right)}{\sigma\Phi\left(\frac{k-a}{\sigma}\right)}, & x<\mu,\\
\dfrac{\exp\left(\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k+a}{\sigma}\right)}{\exp\left(\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k+a}{\sigma}\right) + \exp\left(-\frac{ka}{\sigma^2}\right)\Phi\left(\frac{k-a}{\sigma}\right)} \dfrac{\phi\left(\frac{x-\mu-(a+k)}{\sigma}\right)}{\sigma\Phi\left(\frac{k+a}{\sigma}\right)}, & x\geq\mu.
\end{cases}
$$

According to Proposition 3.2, the last equation is the pdf of UBN$(\mu, \sigma, \frac{k}{\sigma}, a)$ distribution.

Proof of Proposition 3.5

The proof follows similarly to the proof of Proposition 3.2.

Proof of Proposition 3.6

Taking the derivative of (3.4) and equating it to zero, we have

$$\begin{cases} t^2 - 2\frac{\sigma}{k}t + \sigma^2 = 0, & x > \mu, \\ t^2 + 2\frac{\sigma}{k}t + \sigma^2 = 0, & x < \mu, \end{cases}$$

where $t = x - \mu - a$. For both cases, we have $\Delta = \dfrac{4\sigma^2}{k^2} - 4\sigma^2$. When $k \leq 1$, $\Delta \geq 0$. Therefore, the solutions are

$$\begin{cases} x = \mu + a + \sigma\left(\frac{1}{k} + \sqrt{\frac{1}{k^2} - 1}\right), & x > \mu, \\ x = \mu + a + \sigma\left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1}\right), & x > \mu, \\ x = \mu + a - \sigma\left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1}\right), & x < \mu, \\ x = \mu + a - \sigma\left(\frac{1}{k} + \sqrt{\frac{1}{k^2} - 1}\right), & x < \mu. \end{cases} \tag{.-15}$$

Using the second derivative, we could find out the local maximums (modes). The second derivative is proportional to

$$\begin{cases} 2 - 2\frac{k}{\sigma}t, & x > \mu, \\ 2 + 2\frac{k}{\sigma}t, & x < \mu. \end{cases}$$

By evaluating the solutions in (.-15), we have

$$\begin{cases} x = \mu + a + \sigma\left(\frac{1}{k} + \sqrt{\frac{1}{k^2} - 1}\right), & x > \mu, \\ x = \mu + a - \sigma\left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1}\right), & x < \mu. \end{cases}$$

Then, for $k \leq 1$, the modes are $\mu + a \pm \sigma\left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1}\right)$.

Elements of the score vector for the UBN distribution

$$\begin{aligned} \frac{\partial \ell}{\partial \mu} &= -\frac{k}{\sigma}\sum_{i=1}^{n} \operatorname{sign}(x_i - \mu) + \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i - \mu - a}{\sigma}\right), \\ \frac{\partial \ell}{\partial \sigma} &= -\frac{n}{\sigma} + n\frac{kas}{\sigma^2} - \frac{k}{\sigma^2}\sum_{i=1}^{n}\left|x_i - \mu\right| + \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i - \mu - a}{\sigma}\right)^2, \\ \frac{\partial \ell}{\partial k} &= -nk - n\frac{as}{\sigma} - n\rho + \sum_{i=1}^{n}\left|\frac{x_i - \mu}{\sigma}\right|, \\ \frac{\partial \ell}{\partial a} &= -n\frac{ks}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i - \mu - a}{\sigma}\right), \end{aligned}$$

where

$$\begin{aligned}
\delta &= e^{\frac{ka}{\sigma}}\Phi\left(k+\frac{a}{\sigma}\right)+e^{-\frac{ka}{\sigma}}\Phi\left(k-\frac{a}{\sigma}\right),\\
s &= \frac{e^{\frac{ka}{\sigma}}\Phi\left(k+\frac{a}{\sigma}\right)-e^{-\frac{ka}{\sigma}}\Phi\left(k-\frac{a}{\sigma}\right)}{\delta},\\
\rho &= \frac{2e^{\frac{ka}{\sigma}}\phi\left(k+\frac{a}{\sigma}\right)}{\delta}.
\end{aligned}$$

Components of the hessian matrix of the UBN distribution

$$\begin{aligned}
\frac{\partial^2\ell}{\partial\mu^2} &= -\frac{\mathrm{n}}{\sigma^2},\\
\frac{\partial^2\ell}{\partial\mu\partial\sigma} &= \frac{k}{\sigma^2}\sum_{i=1}^{n}\mathrm{sign}\,(x_i)-\frac{2}{\sigma^2}\sum_{i=1}^{n}x_{ij}\left(\frac{x_i-a}{\sigma}\right),\\
\frac{\partial^2\ell}{\partial\mu\partial k} &= -\frac{1}{\sigma}\sum_{i=1}^{n}\mathrm{sign}\,(x_i)\,,\\
\frac{\partial^2\ell}{\partial\mu\partial a} &= -\frac{\mathrm{n}}{\sigma^2},\\
\frac{\partial^2\ell}{\partial\sigma^2} &= \frac{n}{\sigma^2}+\frac{2k}{\sigma^3}\sum_{i=1}^{n}|x_i|-3\sum_{i=1}^{n}\frac{(x_i-a)^2}{\sigma^4}+n\frac{ka}{\sigma^4}\left(kas^2-ka-a\rho-2\sigma s\right),\\
\frac{\partial^2\ell}{\partial\sigma\partial k} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i|+\frac{na}{\sigma^2}\left(s+k\left(\frac{a}{\sigma\delta}-\frac{as^2}{\sigma}+\rho s\right)\right),\\
\frac{\partial^2\ell}{\partial\sigma\partial a} &= -\frac{2}{\sigma^3}\sum_{i=1}^{n}(x_i-a)+\frac{nk}{\sigma^2}\left(s+a\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right)\right),\\
\frac{\partial^2\ell}{\partial k^2} &= -n-\frac{na}{\sigma}\left(\frac{a}{\sigma\delta}-\frac{as^2}{\sigma}\right)+n\rho\,(k+\rho)\,,\\
\frac{\partial^2\ell}{\partial k\partial a} &= -\frac{n}{\sigma}\left(s+a\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right)\right)+n\rho\left(\frac{a}{\sigma^2}+\frac{ks}{\sigma}\right),\\
\frac{\partial^2\ell}{\partial a^2} &= -\frac{n}{\sigma^2}-\frac{nk}{\sigma}\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right).
\end{aligned}$$

Components of the information matrix of the UBN distribution

$$\begin{aligned}
\mathrm{E}\left(-\frac{\partial^2\ell}{\partial\mu^2}\right) &= \frac{\mathrm{n}}{\sigma^2},\\
\mathrm{E}\left(-\frac{\partial^2\ell}{\partial\mu\partial\sigma}\right) &= n\frac{ks}{\sigma^2},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial\mu\partial k}\right) &= n\frac{\mathrm{s}}{\sigma}, \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial\mu\partial a}\right) &= n\frac{1}{\sigma^2}, \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial\sigma^2}\right) &= n\frac{1}{\sigma^2}\left(2+k^2+k\rho\right) - n\frac{ka^2}{\sigma^4}\left(ks^2-k-\rho\right). \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial\sigma\partial k}\right) &= n\frac{1}{\sigma}\left(k+\rho\right) - n\frac{ka}{\sigma^2}\left(\frac{a}{\sigma\delta}-\frac{as^2}{\sigma}+\rho s\right), \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial\sigma\partial a}\right) &= n\frac{ks}{\sigma^2} - n\frac{ka}{\sigma^2}\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right), \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial k^2}\right) &= n + \frac{na}{\sigma}\left(\frac{a}{\sigma\delta}+\frac{as^2}{\sigma}\right) - n\rho\left(k+\rho\right), \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial k\partial a}\right) &= \frac{n}{\sigma}\left(s + a\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right)\right) - n\rho\left(\frac{a}{\sigma^2}+\frac{ks}{\sigma}\right), \\
\mathrm{E}\left(-\frac{\partial^2 \ell}{\partial a^2}\right) &= \frac{n}{\sigma^2} + \frac{nk}{\sigma}\left(\frac{k}{\sigma}+\frac{\rho}{\delta}-\frac{ks^2}{\sigma}\right).
\end{aligned}
$$

Elements of the score vector for the UBSt distribution

$$
\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n} \frac{\frac{\nu+1}{\nu}\frac{u_i^-}{s_-} I\left(x_i \geq \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2} + \sum_{i=1}^{n} \frac{\frac{\nu+1}{\nu}\frac{u_i^+}{s_+} I\left(x_i < \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^+}{\sqrt{s_+}}\right)^2},
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \sigma} = {} & n\nu\sigma \frac{s_-^{-\frac{\nu+2}{2}} D_\nu^- + s_+^{-\frac{\nu+2}{2}} D_\nu^+}{\delta} \\
& - n\sigma \frac{(a+k)\, s_-^{-\frac{\nu+3}{2}} d_\nu^- + (k-a)\, s_+^{-\frac{\nu+3}{2}} d_\nu^+}{\delta} \\
& + \frac{\nu+1}{\nu} \sum_{i=1}^{n} \frac{\frac{\sigma}{s_-^2}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2 I\left(x_i \geq \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2} - (\nu+1)\,\frac{\sigma}{s_-}\sum_{i=1}^{n} I\left(x_i \geq \mu\right) \\
& + \frac{\nu+1}{\nu} \sum_{i=1}^{n} \frac{\frac{\sigma}{s_+^2}\left(\frac{u_i^+}{\sqrt{s_+}}\right)^2 I\left(x_i < \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^+}{\sqrt{s_-}}\right)^2} - (\nu+1)\,\frac{\sigma}{s_+}\sum_{i=1}^{n} I\left(x_i < \mu\right)
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial \ell}{\partial k} = & -n\frac{as_-^{-\frac{\nu+2}{2}}D_\nu^- + s_-^{-\frac{\nu+1}{2}}\left((a+k)\frac{as_-^{-1}}{\nu}+1\right)d_\nu^-}{\delta} \\
& -n\frac{as_+^{-\frac{\nu+2}{2}}D_\nu^+ + s_+^{-\frac{\nu+1}{2}}\left((k-a)\frac{as_+^{-1}}{\nu}-1\right)d_\nu^+}{\delta} \\
& -\frac{\nu+1}{\nu}\sum_{i=1}^{n}\frac{\left(\frac{u_i^-}{\sqrt{s_-}}\right)\left(\frac{-1+\frac{as_-^{-1}}{\nu}\left(u_i^-\right)}{\sqrt{s_-}}\right)I\left(x_i \geq \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2} \\
& -\frac{\nu+1}{\nu}\sum_{i=1}^{n}\frac{\left(\frac{u_i^+}{\sqrt{s_+}}\right)\left(\frac{1-\frac{as_+^{-1}}{\nu}\left(u_i^+\right)}{\sqrt{s_+}}\right)I\left(x_i < \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^+}{\sqrt{s_+}}\right)^2} \\
& +a\frac{\nu+1}{\nu s_-}\sum_{i=1}^{n}I\left(x_i \geq \mu\right) - a\frac{\nu+1}{\nu s_+}\sum_{i=1}^{n}I\left(x_i < \mu\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial a} = & -n\frac{ks_-^{-\frac{\nu+2}{2}}D_\nu^- + s_-^{-\frac{\nu+1}{2}}\left((a+k)\frac{ks_-^{-1}}{\nu}+1\right)d_\nu^-}{\delta} \\
& -n\frac{ks_+^{-\frac{\nu+2}{2}}D_\nu^+ + s_+^{-\frac{\nu+1}{2}}\left((k-a)\frac{ks_+^{-1}}{\nu}-1\right)d_\nu^+}{\delta} \\
& -\frac{\nu+1}{\nu}\sum_{i=1}^{n}\frac{\left(\frac{u_i^-}{\sqrt{s_-}}\right)\left(\frac{-1+\frac{ks_-^{-1}}{\nu}\left(u_i^-\right)}{\sqrt{s_-}}\right)I\left(x_i \geq \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^-}{\sqrt{s_-}}\right)^2} \\
& +\frac{\nu+1}{\nu}\sum_{i=1}^{n}\frac{\left(\frac{u_i^+}{\sqrt{s_+}}\right)\left(\frac{1+\frac{ks_+^{-1}}{\nu}\left(u_i^+\right)}{\sqrt{s_+}}\right)I\left(x_i < \mu\right)}{1+\frac{1}{\nu}\left(\frac{u_i^+}{\sqrt{s_+}}\right)^2} \\
& +k\frac{\nu+1}{\nu s_-}\sum_{i=1}^{n}I\left(x_i \geq \mu\right) - k\frac{\nu+1}{\nu s_+}\sum_{i=1}^{n}I\left(x_i < \mu\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \nu} = &-n\frac{\left(-\frac{\log\left(s_{-}\right)}{2}-\frac{ak}{\nu s_{-}}\right)s_{-}^{-\frac{\nu}{2}}D_{\nu}^{-}+s_{-}^{-\frac{\nu+5}{2}}\frac{(a+k)\,ak}{\nu^{2}}d_{\nu}^{-}}{\delta}\\
&-n\frac{\left(-\frac{\log\left(s_{+}\right)}{2}+\frac{ak}{\nu s_{+}}\right)s_{+}^{-\frac{\nu}{2}}D_{\nu}^{+}-s_{+}^{-\frac{\nu+5}{2}}\frac{(k-a)\,ak}{\nu^{2}}d_{\nu}^{+}}{s_{+}^{-\frac{\nu}{2}}D_{\nu}\left(\frac{a+k}{\sqrt{s_{-}}}\right)+s_{+}^{-\frac{\nu}{2}}D_{\nu}\left(\frac{k-a}{\sqrt{s_{+}}}\right)}\\
&-\frac{\log\left(s_{-}\right)}{2}\sum_{i=1}^{n}I\left(x_{i}\geq\mu\right)-\frac{\nu+1}{\nu^{2}}\frac{ak}{s_{-}}\sum_{i=1}^{n}I\left(x_{i}\geq\mu\right)\\
&-\frac{\log\left(s_{+}\right)}{2}\sum_{i=1}^{n}I\left(x_{i}<\mu\right)+\frac{\nu+1}{\nu^{2}}\frac{ak}{s_{+}}\sum_{i=1}^{n}I\left(x_{i}<\mu\right)\\
&+n\left(\psi\left(\frac{\nu+1}{2}\right)-\psi\left(\frac{\nu}{2}\right)-\frac{1}{\nu}\right)\\
&-\frac{1}{2}\sum_{i=1}^{n}\left\{\log\left(1+\frac{1}{\nu}\left(\frac{u_{i}^{-}}{\sqrt{s_{-}}}\right)^{2}\right)I\left(x_{i}\geq\mu\right)\right\}\\
&-\frac{\nu+1}{2}\sum_{i=1}^{n}\frac{-\frac{1}{\nu^{2}}\left(\frac{u_{i}^{-}}{\sqrt{s_{-}}}\right)^{2}+\frac{2ak}{\nu^{3}s_{-}}\left(\frac{u_{i}^{-}}{\sqrt{s_{-}}}\right)^{2}I\left(x_{i}\geq\mu\right)}{\left(1+\frac{1}{\nu}\left(\frac{u_{i}^{-}}{\sqrt{s_{-}}}\right)^{2}\right)}\\
&-\frac{1}{2}\sum_{i=1}^{n}\left\{\log\left(1+\frac{1}{\nu}\left(\frac{u_{i}^{-}}{\sqrt{s_{+}}}\right)^{2}\right)I\left(x_{i}<\mu\right)\right\}\\
&-\frac{\nu+1}{2}\sum_{i=1}^{n}\frac{-\frac{1}{\nu^{2}}\left(\frac{u_{i}^{-}}{\sqrt{s_{+}}}\right)^{2}-\frac{2ak}{\nu^{3}s_{-}}\left(\frac{u_{i}^{-}}{\sqrt{s_{+}}}\right)^{2}I\left(x_{i}<\mu\right)}{\left(1+\frac{1}{\nu}\left(\frac{u_{i}^{-}}{\sqrt{s_{+}}}\right)^{2}\right)},
\end{aligned}
$$

where

$$
\delta = s_{-}^{-\frac{\nu}{2}}D_{\nu}\left(\frac{a+k}{\sqrt{s_{-}}}\right)+s_{+}^{-\frac{\nu}{2}}D_{\nu}\left(\frac{k-a}{\sqrt{s_{+}}}\right),\ u_{i}^{-}=x_{i}-\mu-a-k,\ u_{i}^{+}=x_{i}-\mu-a+k,
$$

$$
D_{\nu}^{-}=D_{\nu}\left(\frac{a+k}{\sqrt{s_{-}}}\right),\ D_{\nu}^{+}=D_{\nu}\left(\frac{k-a}{\sqrt{s_{+}}}\right),\ d_{\nu}^{-}=d_{\nu}\left(\frac{a+k}{\sqrt{s_{-}}}\right)\text{ and }d_{\nu}^{+}=d_{\nu}\left(\frac{k-a}{\sqrt{s_{+}}}\right).
$$

Elements of the score vector for the UBL distribution

$$
\frac{\partial \ell}{\partial \mu} \quad = \quad \frac{k}{\sigma}\sum_{i=1}^{n}\operatorname{sign}\left(x_{i}-\mu\right)-\frac{2}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_{i}-\mu-a}{\sigma}\right)}{1+\left(\frac{x_{i}-\mu-a}{\sigma}\right)^{2}},
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{2na^2}{\sigma^3\left(1+\frac{a^2}{\sigma^2}+\frac{2}{k^2}\right)} + \frac{k}{\sigma^2}\sum_{i=1}^{n}\left|x_i-\mu\right| - \frac{2}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu-a}{\sigma}\right)^2}{1+\left(\frac{x_i-\mu-a}{\sigma}\right)^2}, \\
\frac{\partial \ell}{\partial k} &= \frac{n}{k} + \frac{4n}{k^3\left(1+\frac{a^2}{\sigma^2}+\frac{2}{k^2}\right)} - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\sigma}\right|, \\
\frac{\partial \ell}{\partial a} &= -\frac{2na}{\sigma^2\left(1+\frac{a^2}{\sigma^2}+\frac{2}{k^2}\right)} - \frac{2}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu-a}{\sigma}\right)}{1+\left(\frac{x_i-\mu-a}{\sigma}\right)^2}.
\end{aligned}
$$

Table 1: ML estimates, information criteria, and K-S test

Devices lifetimes			estimates			AIC	BIC	K-S (p-value)
UBN(μ,σ,k)	42.6800	12.7632	2.3325			467.78	473.52	0.0728 (0.59)
	(1.8066)	(1.4428)	(0.3199)					
UBSt(μ,σ,k,ν)	42.7163	12.5585	29.8799	44.8296		470.03	477.66	0.0737 (0.58)
	(1.8131)	(1.4853)	(1.9252)	(90.0821)				
UBL(μ,σ,k)	41.5776	3.5453	0.3399			485.51	491.24	0.0961 (0.40)
	(3.3617)	(2.0864)	(0.1884)					
BN(μ,σ,a,b)	52.4717	4.1059	0.0221	0.0380		481.98	489.63	0.1097 (0.30)
	(0.0028)	(0.0030)	(0.0035)	(0.0073)				
PLD(ν,ρ,ζ)	0.0246	0.1572	1710.2743			474.02	479.75	0.1510 (0.10)
	(0.0054)	(0.0592)	(6077.8982)					
OLLN(α,μ,σ)	42.6193	5.3097	0.0665			469.19	474.93	0.0903 (0.44)
	(0.0023)	(0.0022)	(0.0076)					
SN(μ,σ,λ)	0.0986	54.6253	132914			478.45	484.19	0.1376 (0.15)
	(0.0013)	(5.4771)	(11864.8050)					
SSt(μ,σ,λ,ν)	0.0989	54.5568	188650	195063		480.45	488.10	0.1376 (0.15)
	(0.0008)	(5.4599)	(11863.4442)	(11863.2845)				
Ultrasound weight			estimates			AIC	BIC	K-S (p-value)
UBN(μ,σ,k)	3213.3166	498.4040	1.2402			8087.44	8100.08	0.0209 (0.64)
	(22.2890)	(24.8634)	(0.1161)					
UBSt(μ,σ,k,ν)	3212.6529	505.2531	612.6168	417.0372		8089.62	8106.48	
	(22.6309)	(26.2388)	(34.9919)	(1130.2158)				
UBL(μ,σ,k)	3201.2857	100.5165	0.3976			8099.21	8111.85	0.0149 (0.8)
	(21.8584)	(18.5496)	(0.0678)					
BN(μ,σ,a,b)	3506.5995	208.7271	0.0773	0.1329		8117.62	8134.48	0.0273 (0.47)
	(0.0229)	(0.0229)	(0.0040)	(0.0081)				
PLD(ν,ρ,ζ)	0.0037	-0.0558	0.2001			8288.33	8300.98	0.0644 (0.02)
	(0.00005)	(0.0009)	(0.0125)					
OLLN(α,μ,σ)	3226.2404	252.1888	0.1894			8096.70	8109.35	0.0319 (0.36)
	(0.0326)	(0.0326)	(0.0071)					
Lean boddy mass			estimates			AIC	BIC	K-S (p-value)
UBN(μ,σ,k,a)	54.6299	11.7357	-1.4901	0.7996		673.29	683.71	0.0324 (0.81)
	(0.0913)	(4.9644)	(1.2654)	(2.1223)				
UBSt(μ,σ,k,a,ν)	54.6295	10.0016	-12.6934	0.8034	33.1377	675.57	688.60	0.0416 (0.71)
	(0.1059)	(4.1002)	(16.9004)	(1.7828)	(57.4507)			
UBL(μ,σ,k,a)	53.4097	3.2203	1.2239	-1.5922		666.60	677.02	0.0461 (0.65)
	(0.0558)	(0.8701)	(0.2481)	(0.5358)				
BN(μ,σ,a,b)	65.8970	10.2662	1.0495	4.5749		673.74	681.56	0.0709 (0.37)
	(25.2460)	(17.5231)	(2.5262)	(18.4501)				
PLD(ν,ρ,ζ)	0.5103	-1.8475	1.0909			702.66	710.47	0.0568 (0.52)
	(0.1775)	(0.0376)	(0.2074)					
OLLN(α,μ,σ)	55.0707	17.3130	2.7999			673.92	681.73	0.0534 (0.56)
	(0.6687)	(27.6627)	(4.6383)					
SN(μ,σ,λ)	54.7805	6.8832	0.0201			675.73	683.54	0.0580 (0.51)
	(11.4491)	(0.5207)	(2.0816)					
SSt(μ,σ,λ,ν)	59.0816	7.2329	-0.9021	9.9109		675.48	685.90	0.0626 (0.46)
	(3.1412)	(2.3248)	(0.8717)	(12.2811)				