JIRSS (2022)
Vol. 21, No. 02, pp 21-50
DOI:10.22034/jirss.2022.707643

Inference on Generalized Inverse Lindley Distribution under Progressive Hybrid Censoring Scheme

Suparna Basu [1], **Sanjay Kumar Singh** [2], **Umesh Singh** [2]

[1] Department of Statistics, M. M. V., Banaras Hindu University, Varanasi, UP, India.
[2] Department of Statistics, I. Sc., Banaras Hindu University, Varanasi, UP, India.

Received: 24/09/2021, Accepted: 15/09/2023, Published online: 14/01/2024

Abstract This article delineates the implementation of the product of spacings under Progressive Hybrid Type-I censoring with binomial removals for the Generalized Inverse Lindley distribution. Both point and interval estimates of the parameters have been obtained under classical as well as Bayesian paradigms using the product of spacings. The proposed estimators can be used in lieu of maximum likelihood as well as usual Bayes estimator based on likelihood function which is corroborated by a comparative simulation study. The Bayesian estimation is performed under the assumption of squared error loss function. The implicit integrals involved in the process are evaluated using Metropolis-Hastings algorithm within Gibbs sampler. We have also derived the expected total time to test statistic for the specified censoring scheme. The applicability of the proposed methodology is demonstrated by analyzing a real data set of active repair times for an airborne communication transceiver.

Keywords: Maximum Product of Spacing, Progressive Hybrid Type-I Censoring with Binomial Removals, Expected Total Time to Test, Gibbs Sampler Algorithm.

MSC: 62Nxx, 62Fxx.

1 Introduction

In life testing experiments, observing failure times of all the items put to test is unworthy in terms of cost and resource utilization. Accounting to these constraints or sometimes as per need of an experimental set-up, it is terminated prior to observing the exact

Corresponding Author: Suparna Basu (sbasu@bhu.ac.in)
Sanjay Kumar Singh (singhsk64@gmail.com)
Umesh Singh (usingh_52@yahoo.co.in)

lifetime of all the units put to test, commonly referred to as *censoring*. The observations recorded from such an experimental set-up consist of failure data related to two types of units; i.e, units that fail before the termination of experiment and those units which are removed from the experiment even though they are surviving at the time of their removal. Broadly, censoring schemes are categorized as Type-I and Type-II censoring. In the case of Type-I censoring scheme, the maximum time of the experiment is prefixed (say T) which delivers a random number of observed failures (say d) whereas in Type-II censoring scheme, we observe a fixed number of failures (say $k < n$) leading to a random termination time. Here, T needs careful inquest since the parameters are unknown and one might end up with no failure if the chosen T is too small with respect to the mean life of the event of interest. These traditional schemes, even though, are easy to implement, possess certain demerits like too few observations in the case of Type-I scheme or exceptionally long waiting times to record the prefixed number of failures for Type-II scheme. Thus, to ward off these flaws, a mixture of the two i.e. hybrid censoring scheme was proposed by Epstein (1954; 1960), which adds flexibility to the aforementioned schemes.

Further advancement and flexibility to a hybrid scheme may be brought about by subsequently removing some surviving units in the intermediate stages. The items so removed may be argued to be deliberately introduced in the testing process for reasons like reducing the effective load on the experiment, using the removed units for some other similar experiments, etc., or may occur randomly throughout the experiment accredited to situations like loss to follow up, withdrawal, etc., as usually encountered in any clinical trial.

A progressive censoring scheme extends greater flexibility at each stage of failure with the introduction of removals immediately followed by an observed failure, thereafter by removing all the surviving units at the terminal point like a usual right censoring scheme. This scheme was first discussed by Herd (1956), defining it as a "*multi-censored*" sample. Later Chen and Su (2004) detailed the importance of these schemes in life-testing experiments. The units to be removed at any particular stage may be assumed to be fixed which is feasible in the case of a controlled experiment, or random, as encountered in clinical studies (for further details see Wu et al. (2007); Mousa and Jaheen (2002). The book by Balakrishnan and Cramer (2014) provides an extensive description of all the cognate progressive censoring schemes and related inferences for some important statistical distributions.

Childs et al. (2008) introduced two generalized progressive hybrid censoring schemes and discussed exact conditional inferential procedures for the exponential mean and Kundu and Joarder (2006) performed an asymptotic analysis of the same, thereby drawing a comparison to the existing exact procedures. One of the proposed generalized progressive hybrid scheme is the Progressive Hybrid Type-I censoring (PHT-I), designed to terminate at $\tau_0 = min(X_{k:n}, \tau)$ with R_i units being removed immediately following a failure x_i; $i = 1, 2, \cdots, k$; provided $R_k = n - k - \sum_{j=1}^{k-1} R_j$. Note that, based on a sample of size n, the total number of failures k, the maximum allowable time τ and the number of removals $\boldsymbol{R} = (R_1 = r_1, R_2 = r_2, ..., R_k = r_k)$ are all fixed prior to experiment initiation.

Nevertheless, fixing the removals seems unreasonable in real situations and we argue it to be random. The assumption of randomness in the number of removals in any particular stage may be represented using a binomial distribution. Inferences on different distributions for progressive Type-II censored samples with binomial removals have been earlier explored by Hak-Keung and Siu (1996); Siu and Hak-Keung (1998); Siu et al. (2000) and Wu et al. (2007) reviewed the application of uniform distribution to the removals.

Thus, if n units are put to test and R_1 surviving units are removed immediately followed by the first failure, denoted as $X_{1:k:n}$, then, it may be reasonably claimed that $R_1 \sim bin(\overline{n-k}, p)$, where p denotes the probability of removal of an item at each stage. Similarly, at the second failure $X_{2:k:n}$, units are randomly removed from the test with a probability of removal p, thereby allowing us to claim $R_2 \sim bin(n-k-R_1, p)$. This process is continued till either the time or failure constraints are achieved. Hence, if $X_{k:k:n} < \tau$, the experiment is aborted at the k^{th} failure along with removal of all the surviving units $R_k = n - k - \sum_{i=1}^{k-1} R_i$. Otherwise, if $X_{k:k:n} > \tau$, it is terminated at τ, yielding $d < k$ observed failures $X_{d:k:n}$, with R_d $[\, 0 \le R_d \le n - k - \sum_{i=1}^{d-1} R_i]$ units randomly removed at the d^{th} failure and $R_d^* = n - k - \sum_{i=1}^{d} R_i$ removals at τ.

Here, R_k is assumed to be less than $(n - k - \sum_{j=1}^{k-1} R_j)$ in order to ensure k failures at $X_{k:k:n}$ without hindering the nature of the experiment and provide a statistically feasible life-test for further inferential analysis for a PHT-I-CBR scheme. Undoubtedly, the simple Progressive Type-I and Type-II scheme which was studied by Chen and Su (2004); Cohen (1976); Cohen and Norgaard (1977); Childs et al. (2008) may be obtained as special cases of a PHT-I scheme.

This article proposes the use of the product of spacings for the estimation of the parameters of Generalized Inverse Lindley distribution $GILD(\alpha, \theta)$ based on the data obtained through Progressive hybrid Type-I censoring with binomial removals (PHT-I-CBR). A $GILD$ variate X may be derived from an ILD variate Y using power transformation, i.e. $X = Y^{\frac{1}{\alpha}}$ or from a $Lindley(\theta)$ variate Z, i.e. $X = Z^{-\frac{1}{\alpha}}$. Also, it can be obtained as a result of a convex combination of two distributions, namely the inverse Weibull (IW) distribution with shape parameter α and scale parameter θ and a special case of the generalized inverse gamma distribution, say $f_2(x; \eta, k, \lambda, \gamma, \alpha)$ with $\eta = 2$, $k = 0$, $\lambda = 0$, $\gamma = \theta^{\frac{1}{\alpha}}$ in the proportion $\varphi = \theta/(1+\theta)$ in favor of IW distribution (see Sharma et al. (2016); Barco et al. (2016); Ghitany et al. (2008)). This generalization may be referred to as an improvement over inverse Lindley distribution owing to the flexibility in shape integrated by it.

The present study is dedicated to the development of an alternative technique to the likelihood function (LF) approach which retains the graceful properties of LF and discards the demerits of it for heavy-tailed distributions. This technique is implemented in the classical as well as Bayesian paradigm and is popularly known as the product of spacings (PS) technique. Also, several authors like Anatolye and Kosenok (2005); Cheng and Traylor (1995), among others, have elicited its efficiency and eminence, especially for those cases where MLE fails to provide consistent estimates. Furthermore, Cheng and Amin (1995); Ranneby (1984) validated possession of equivalent statistical properties of PS (referred to by them as the maximum product of spacings (MPS)) estimators and ML estimators and thus proposed the method of estimation based on PS as a suitable alternative to MLE especially when the MLEs fails to exists.

The use of the PS under Bayesian paradigm was first attempted by Coolen and Newby (1990) where they derived an approximate posterior density of observed spacings which is analogous to the usual posterior distribution by virtue of its asymptotic equivalence to the likelihood function. Later on Singh et al. (2016) applied the proposition of Coolen and Newby (1990) to PHT-II censored data from a generalized inverted exponential distribution. Basu et al. (2017) scrutinized the behavior of PS estimator using the partitions induced in the support of the random variable for Type-I censored data in the classical interface and compared it to ML method. Further, Basu et al. (2018) elaborated the PS estimator in classical and Bayesian paradigms for PHT-I censored data for $ILD(\theta)$. We extend the same principle for PHT-I-CBR and formulate the PS estimator in the classical and Bayesian paradigms and assess the computational intricacies in developing the MPS estimator for distributions with more than one parameter.

This article is organized into eight sections where Sec.2 is devoted to the explanation of the chosen censoring scheme and its corresponding likelihood function for $GILD(\alpha, \theta)$. Sec.3 discusses the classical inference based on PHT-I-CBR through maximum likelihood and PS function. Bayes estimates and their corresponding credible and HPD intervals are derived in Sec.4. We have obtained the expected total time to test statistic for the concerned censoring scheme in Sec.5. A simulation study is reported in Sec.6 which elucidates the performance of the proposed estimator. Further, in Sec.7, the applicability of the proposed methodologies is illustrated on active repair times data for an airborne communication transceiver. Sec.8 furnishes a conclusion about the proposed work.

2 Model and Censoring Scheme

The cumulative distribution function (CDF) and probability density function (PDF) of $GILD(\alpha, \theta)$, with α as the shape parameter and θ as the scale parameter may be expressed as:

$$HF(x;\alpha,\theta)=\begin{cases}\left[1+\frac{\theta}{(1+\theta)}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}} \quad ; & x,\theta,\alpha>0,\\ 0\ ; \quad \text{otherwise.}\end{cases} \tag{1}$$

$$f(x;\alpha,\theta)=\begin{cases}\dfrac{\alpha\theta^2}{1+\theta}\left(\dfrac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\theta}{x^{\alpha}}} \quad ; & x,\theta,\alpha>0,\\ 0\ ; \quad \text{otherwise.}\end{cases} \tag{2}$$

Suppose that n items whose lifetimes follow the density function given in Eq.((2)) are put to test. Furthermore, the experiment is terminated at the earliest of the pre-specified time (say, τ) or the pre-determined number of failures (say k) is observed, i.e. the termination is specified as $\tau_0 = min\,(X_{k:k:n}, \tau)$. Now, at each failure $X_{i:k:n}$, a random number of surviving units R_i is randomly removed from the experiment using the binomial law explained earlier. The probability information of each of these removals is $P(X > x_{i:k:n})=\overline{F}(x_{i:k:n};\alpha,\theta))$, owing to the identically independent nature of X_i's.

Hence, the observed ordered failures ($x_{1:k:n} < x_{2:k:n} < \cdots < x_{k:k:n}$) and subsequent removals may be obtained as:

Case-I: $(x_{1:k:n}, R_1), (x_{2:k:n}, R_2) ..., (x_{k:k:n}, R_k)$; if $x_{k:k:n} < \tau;\ 0 < k \le n;\ \sum\limits_{i=1}^{k} R_i + k = n.$

Case-II: $(x_{1:k:n}, R_1), (x_{2:k:n}, R_2) ..., (x_{d:k:n}, R_d)$; if $x_{d:k:n} < \tau;\ 0 < d < k;\ \sum\limits_{i=1}^{d} R_i + R^*_d + d = n.$

For notational simplicity, let us denote $X_{i:k:n}, F(x;\alpha,\theta), \overline{F}(x;\alpha,\theta)$ and $f(x;\alpha,\theta)$ as $X_i, F(x), \overline{F}(x)$ and $f(x)$ respectively. Following Chen and Su (2004); Balakrishnan and Cramer (2014), the conditional likelihood function of PHT-I-CBR for a fixed set of removal $\boldsymbol{R} = (R_1 = r_1, R_2 = r_2, \cdots, R_\omega = r_\omega, R^*_\omega = r^*_\omega)$ may be written as

$$L\Big(\alpha,\theta,\underset{\sim}{x}|R=r\Big) = C^* \prod_{i=1}^{\omega}\{f(x_i)[1-F(x_i)]^{r_i}\}\,[1-F(\tau_0)]^{r^*_\omega}\ ; \qquad \omega \ge 1, \tag{3}$$

where

$$\omega=\begin{cases}k\ ; & x_k<\tau,\\ d\ ; & x_d<\tau<x_{d+1};\ d<k,\end{cases}$$

$$r^*_\omega=\begin{cases}0\ ; & \omega=k,\\ n-d-\sum\limits_{j=1}^{d} r_j\ ; & \omega=d,\end{cases}$$

$$C^*=\begin{cases}n(n-r_1-1)(n-\sum\limits_{j=1}^{2} r_j-2)\cdots(n-\sum\limits_{j=1}^{k-1}[r_j+1])\ ; & \tau_0=x_k,\\ n(n-r_1-1)(n-\sum\limits_{j=1}^{2} r_j-2)\cdots(n-\sum\limits_{j=1}^{d-1}[r_j+1])(n-\sum\limits_{j=1}^{d}[r_j+1])\ ; & \tau_0=\tau,\end{cases}$$

$\forall\ r_i$, such that $0 \le r_i \le (n-k-\sum\limits_{j=1}^{i-1} r_j), \forall\ i = 1, 2, \cdots, \omega.$

Thus, the termination of the experiment at X_k results in k failures with R_i random removals, where $R_i \sim bin(n-k-\sum_{j=1}^{i-1} r_j, p)$, $i = 1, 2, \cdots, \overline{k-1}$ and likewise, when the experiment is aborted at τ, d failures are observed with R_i random removals, where $R_i \sim bin(n-k-\sum_{j=1}^{i-1} r_j, p)$, $i = 1, 2, \cdots, d$. Here,

$$P(R_1 = r_1) = \binom{n-k}{r_1} p^{r_1} (1-p)^{n-k-r_1} \; ; \qquad r_1 = 0, 1, 2, \cdots, \overline{n-k}, \tag{4}$$

and for $i = 2, 3, \cdots, \omega - 1$,

$$\begin{aligned} P(R_i = r_i | R_{i-1} = r_{i-1}, \cdots, R_1 = r_1) &= \binom{n-k-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-k-\sum_{j=1}^{i} r_j} \; ; \\ & \qquad \forall\, r_i = 0, 1, \cdots, n-k-\sum_{j=1}^{i-1} r_j, \\ P(R_d = r_d | R_{d-1} = r_{d-1}, \cdots, R_1 = r_1) &= \binom{n-k-\sum_{j=1}^{d-1} r_j}{r_d} p^{r_d} (1-p)^{n-k-\sum_{i=1}^{d} r_j} \; ; \\ & \qquad 0 \le r_d \le n-k-\sum_{j=1}^{d-1} r_j. \end{aligned} \tag{5}$$

Evidently, the assumption of independence of $X_i s$ and $R_i' s$ does not infringe statistical analysis and thus, we can rewrite the joint likelihood as

$$L(\underset{\sim}{x}; \alpha, \theta, \boldsymbol{R}, p) = L[\underset{\sim}{x}; \alpha, \theta | \boldsymbol{R}, p] \times P[\boldsymbol{R} = r; p], \tag{6}$$

where, $P[\boldsymbol{R} = r; p]$ is the joint probability of the removals. Thus,

$$P[\boldsymbol{R} = r; p] = \begin{cases} \dfrac{(n-k)!\, p^{\sum_{j=1}^{k-1} r_j} (1-p)^{(k-1)(n-k) - \sum_{j=1}^{k-1} (k-j) r_j}}{(n-k-\sum_{j=1}^{k-1} r_j)! \prod_{j=0}^{k-1} r_j!} \; ; & \tau_0 = x_k, \\[2ex] \dfrac{(n-k)!\, p^{\sum_{j=1}^{d} r_j} (1-p)^{d(n-k) - \sum_{j=1}^{d} (d-j+1) r_j}}{(n-k-\sum_{j=1}^{d} r_j)! \prod_{j=0}^{d} r_j!} \; ; & \tau_0 = \tau. \end{cases} \tag{7}$$

Also, in accordance to the censoring scheme, $P[R_k = r_k | R_{k-1}, R_{k-2}, \cdots, R_1] = 1 = P[R_d^* = r_d^* | R_d, R_{d-1}, \cdots, R_1]$. Therefore, using Eq.(3), Eq.(6) and Eq.(7), the joint likelihood function may be expressed as

$$L(\alpha, \theta, p, \underset{\sim}{x}, r) = A \times L_1(\alpha, \theta | \underset{\sim}{x}, r) \times L_2(p), \tag{8}$$

where A is a constant devoid of θ, α or p and

$$L_1(\alpha, \theta | \underset{\sim}{x}, r) = \prod_{i=1}^{\omega} \left\{ \left[\frac{\alpha\theta^2}{1+\theta} \left(\frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}} \right) e^{-\frac{\theta}{x_i^{\alpha}}} \right] \times \left[\overline{F}(x_i) \right]^{r_i} \right\} \left\{ \overline{F}(\tau_0) \right\}^{r^*_{\omega}}. \quad (9)$$

$$L_2(p) = \begin{cases} p^{\sum_{j=1}^{k-1} r_j} (1-p)^{(k-1)(n-k) - \sum_{j=1}^{k-1} (k-j) r_j}; & \tau_0 = x_k, \\ p^{\sum_{j=1}^{d} r_j} (1-p)^{d(n-k) - \sum_{j=1}^{d} (d-j+1) r_j}; & \tau_0 = \tau, \end{cases} \quad (10)$$

3 Classical Inference

3.1 Maximum Likelihood Estimation

In this section, we derive the maximum likelihood estimators (MLE) of α, θ and p for the considered censoring scheme. Evidently, $L_1(\alpha, \theta | \underset{\sim}{x}, r)$ is devoid of p as demonstrated in Eq.(9) and likewise $L_2(p)$ is independent of α and θ (see Eq.(10)). Owing to this independence, the respective MLE's may be evaluated by individually maximizing the specific likelihood equations. We calculate the partial derivatives of the logarithm of the likelihood function in Eq.(9) and equate it to 0 to obtain the estimates.

$$\frac{\partial}{\partial\theta} LogL_1 = 0 = \frac{\partial}{\partial\alpha} LogL_1,$$

$$\Rightarrow \frac{2\omega}{\theta} - \frac{\omega}{1+\theta} - \sum_{i=1}^{\omega} \frac{1}{x_i^{\alpha}} + \frac{r^*_{\omega} \left[\left(2\tau_0^{\alpha} + 1 \right) + \theta \left(\tau_0^{\alpha} + 1 \right) \right] \theta e^{-\frac{\theta}{\tau_0^{\alpha}}}}{\tau_0^{2\alpha} (1+\theta)^2 \left[1 - e^{-\frac{\theta}{\tau_0^{\alpha}}} \left(1 + \frac{\theta}{\tau_0^{\alpha}(1+\theta)} \right) \right]} + \sum_{i=1}^{\omega} \frac{r_i \left[\left(2x_i^{\alpha} + 1 \right) + \theta \left(x_i^{\alpha} + 1 \right) \right] \theta e^{-\frac{\theta}{x_i^{\alpha}}}}{x_i^{2\alpha} (1+\theta)^2 \left[1 - e^{-\frac{\theta}{x_i^{\alpha}}} \left(1 + \frac{\theta}{x_i^{\alpha}(1+\theta)} \right) \right]} = 0, \quad (11)$$

$$\Rightarrow \frac{\omega}{\alpha} - \frac{r^*_{\omega} \theta^2 \left(\tau_0^{\alpha} + 1 \right) \mathrm{e}^{-\frac{\theta}{\tau_0^{\alpha}}} log\tau_0}{(\theta+1) \tau_0^{2\alpha} \left[1 - e^{-\frac{\theta}{\tau_0^{\alpha}}} \left(1 + \frac{\theta}{\tau_0^{\alpha}(1+\theta)} \right) \right]} - \sum_{i=1}^{\omega} \frac{r_i \theta^2 \left(x_i^{\alpha} + 1 \right) \mathrm{e}^{-\frac{\theta}{x_i^{\alpha}}} log x_i}{(\theta+1) x_i^{2\alpha} \left[1 - e^{-\frac{\theta}{x_i^{\alpha}}} \left(1 + \frac{\theta}{x_i^{\alpha}(1+\theta)} \right) \right]} + \sum_{i=1}^{\omega} log x_i \left(\frac{x_i{}^{\alpha}}{1+x_i^{\alpha}} - \frac{\theta}{x_i^{\alpha}} - 2 \right) = 0. \quad (12)$$

Here, the MLE of α and θ is obtained by numerical optimization on account of its implicit nature. The initial value for the algorithm was chosen by graphical inspection of the contour plot of negative log-likelihood ($-LogL_1$) sketched with respect to α, θ.

All the numerical computations in this article have been performed in **R**.

The MLE of p is explicitly evaluated in Eq.(13) by maximizing the logarithm of Eq.(10).

$$\hat{p} = \begin{cases} \dfrac{\sum_{j=1}^{k-1} r_j}{(k-1)(n-k) - \sum_{j=1}^{k-1} (k-j-1) r_j}; & \tau_0 = x_k, \\ \dfrac{\sum_{j=1}^{d} r_j}{d(n-k) - \sum_{j=1}^{d} (d-j) r_j}; & \tau_0 = \tau. \end{cases} \tag{13}$$

3.2 Maximum Product of Spacings

The maximum product of spacings estimation procedure was proposed by Cheng and Amin (1995) as an alternative to ML estimation, for distributions with unknown scale and location. The proposition by Cheng and Amin (1995) lacked mathematical credibility which was re-established through an independent study by Ranneby (1984), where the MPS technique was developed as an approximation to the Kullback-Leibler (KL) information.

GILD is a mixture of IW and generalized inverse Gamma distribution with heavy tails which is inapt to be estimated by ML technique (see Anatolye and Kosenok (2005)). MPS estimators exhibit similar asymptotic properties to ML estimators under more liberal conditions (see Cheng and Iles (1987); Cheng and Traylor (1995)). Another favorable property of MLE is the invariance principle which is also possessed by MPS estimators (see Coolen and Newby (1990)). MPS estimators exhibit efficient small sample behavior in contrary to MLE as discussed by Anatolye and Kosenok (2005) which confers it suitable in reliability studies yielding small samples.

The spacings estimator is derived with the assumption that the density function $f(x)$ is strictly positive in any interval $(a, b) \subset \mathbb{R}$ and 0 elsewhere. In the present study, X is defined on $(0, \infty)$ with $a = 0$ and $b \equiv \infty$, $F(x) = 0 = f(x); \forall\, x < a$ and $F(x) = 1; f(x) = 0;$ $\forall\, x > b$. Based on an ordered sample of n units, $0 < x_1 < x_2 < \cdots < x_n < \infty$, the associated partitions and the spacings are defined as $(0, x_1]$, $(x_1, x_2], \cdots$, (x_n, ∞) and $D_i = F(x_i) - F(x_{i-1})$, $\forall\, i = 1, 2, \cdots, n+1$ respectively, with $F(x_0) = 0$, $F(x_{n+1}) = 1$ such that $\sum D_i = 1$. Thus, for a completely observed experiment, the product of spacings function is the geometric mean of the spacings defined above.

A continuous variate might result in tied observation due to round-off errors and under such circumstances, using the geometric mean of the spacings becomes irrelevant. Shao and Hahn (1999) and Cheng and Stephens (1989) suggested a modification in the spacing function to incorporate the tied information without altering the total information content of a sample. Analogically, the spacing function can be modified to accommodate the changes brought about by the considered censoring scheme. The partitions of the support of the random variable due to PHT-I censoring scheme is either $(0, x_1]$,

$(x_1, x_2], \cdots, (x_k, \infty)$ or $(0, x_1], (x_1, x_2], \cdots, (x_d, \tau], (\tau, \infty)$. Furthermore, information of subsequent removals may be introduced in terms of the survival function $(\overline{F}(x_i))$, censored at x_i, thereby assigning equal probabilities to each unit in $R_i = r_i$, $\forall\ i = 1, 2, \cdots \omega$ (see Cheng and Traylor (1995); Basu et al. (2018)). Therefore, the revised spacings coupled with information on censored units are $D_i = \{F(x_i) - F(x_{i-1})\} \left\{\overline{F}(x_i))/r_i\right\}^{r_i} \left\{\overline{F}(\tau_0))/r^*_\omega\right\}^{r^*_\omega}$, $\forall\ i = 1, 2, \cdots \omega$.

However, the PS function requires further modification when the experiment is terminated at $\tau_0 = \tau$ with r^*_d removals. In this case, the terminal partition $(x_d, \tau]$ leads to spacing $D_\xi = \{F(\tau) - F(x_d)\}$. Now, as $\tau \longrightarrow x_d$; for a given $\epsilon > 0$, if $|\tau - x_d| < \epsilon$, D_ξ may be approximated by $f(\tau)$.Thus, the conditional spacings function for PHT-I-CBR against the given removals $\boldsymbol{R}$ is

$$S\left(\alpha, \theta, \underset{\sim}{x}|\boldsymbol{R} = r\right) \propto \begin{cases} \prod\limits_{i=1}^{\omega} D_i\ ; & \tau_0 = x_k, \\ D_\xi \cdot \prod\limits_{i=1}^{\omega} D_i\ ; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ f(\tau) \cdot \prod\limits_{i=1}^{\omega} D_i\ ; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \tag{14}$$

Our considered scheme demonstrates removal patterns governed by binomial law and since X_i and R_i are independently distributed, then, likewise, the joint spacings function may be evaluated as $S(\alpha, \theta; \boldsymbol{R}; \underset{\sim}{x}) \propto S(\alpha, \theta; \underset{\sim}{x}|\boldsymbol{R}) \times L_2(p)$. The estimate of α and θ may be computed by maximizing the logarithm of $S(\alpha, \theta; \underset{\sim}{x}|\boldsymbol{R} = r)$ since Eq.(14) is devoid of p. Hence, the normal equations are:

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta} \log S\left(\alpha, \theta|\underset{\sim}{x}\right) \\ &= \begin{cases} \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\theta(x_i) - F'_\theta(x_{i-1})}{F(x_i) - F(x_{i-1})} - \dfrac{r_i F'_\theta(x_i)}{1 - F(x_i)} \right\}; & \tau_0 = x_k, \\ \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\theta(x_i) - F'_\theta(x_{i-1})}{F(x_i) - F(x_{i-1})} - \dfrac{r_i F'_\theta(x_i)}{1 - F(x_i)} \right\} - \dfrac{r^*_\omega F'_\theta(\tau)}{1 - F(\tau)} + \dfrac{F'_\theta(\tau) - F'_\theta(x_d)}{F(\tau) - F(x_d)}; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\theta(x_i) - F'_\theta(x_{i-1})}{F(x_i) - F(x_{i-1})} - \dfrac{r_i F'_\theta(x_i)}{1 - F(x_i)} \right\} - \dfrac{r^*_\omega F'_\theta(\tau)}{1 - F(\tau)} + \dfrac{f'_\theta(\tau)}{f(\tau)}; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \end{aligned} \tag{15}$$

and

$$\begin{aligned} 0 &= \frac{\partial}{\partial\alpha} \log S\left(\alpha, \theta|\underset{\sim}{x}\right) \\ &= \begin{cases} \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\alpha(x_i) - F'_\alpha(x_{i-1})}{F(x_i) - F(x_{i-1})} + \dfrac{r_i F'_\alpha(x_i)}{1 - F(x_i)} \right\}; & \tau_0 = x_k, \\ \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\alpha(x_i) - F'_\alpha(x_{i-1})}{F(x_i) - F(x_{i-1})} + \dfrac{r_i F'_\alpha(x_i)}{1 - F(x_i)} \right\} - \dfrac{r^*_\omega F'_\alpha(\tau)}{1 - F(\tau)} + \dfrac{F'_\alpha(\tau) - F'_\alpha(x_d)}{F(\tau) - F(x_d)}; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ \sum\limits_{i=1}^{\omega} \left\{ \dfrac{F'_\alpha(x_i) - F'_\alpha(x_{i-1})}{F(x_i) - F(x_{i-1})} + \dfrac{r_i F'_\alpha(x_i)}{1 - F(x_i)} \right\} - \dfrac{r^*_\omega F'_\alpha(\tau)}{1 - F(\tau)} + \dfrac{f'_\alpha(\tau)}{f(\tau)}; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \end{aligned} \tag{16}$$

The above equations have been solved numerically to obtain estimates of θ and α with $\hat{\theta}_{ML}, \hat{\alpha}_{ML}$ as the initial guess values. The partial derivatives involved in Eq.(15) and (16) are elaborated in the Appendix.

3.3 Asymptotic Confidence Interval

The estimators proposed above, owing to their implicit form, impedes the derivation of their exact sampling distributions and thus, we resort to large sample theory to construct interval estimates of the parameters. The likelihood function is continuous over the support of X and substantiates the existence of regularity conditions for consistency and asymptotic normality of the ML estimators (see Anatolye and Kosenok (2005); Ghosh and Jammalamadaka (2001)). Under such conditions, the two estimators are asymptotically equivalent, i.e. $\hat{\theta}_{PS} = \hat{\theta}_{ML} + 0_p(n^{-\frac{1}{2}})$ with variance evaluated from the observed Fisher's information matrix. The observed Fisher information matrix can be obtained by using Eq.(9) as:

$$I(\tilde{\alpha}, \tilde{\theta}) = -\begin{bmatrix} \frac{\partial^2 logL}{\partial\alpha^2} & \frac{\partial^2 logL}{\partial\alpha\partial\theta} \\ \frac{\partial^2 logL}{\partial\theta\partial\alpha} & \frac{\partial^2 logL}{\partial\theta^2} \end{bmatrix}_{\theta=\tilde{\theta};\alpha=\tilde{\alpha}} ; \quad \tilde{\theta} = (\hat{\theta}_{ML}, \hat{\theta}_{PS}); \; \tilde{\alpha} = (\hat{\alpha}_{ML}, \hat{\alpha}_{PS}). \tag{17}$$

The second order partial derivatives with respect to the parameters α and θ are given below:

$$\frac{\partial^2 logL}{\partial\alpha^2} = \sum_{i=1}^{\omega} (logx_i)^2 \left(\frac{1}{x_i^\alpha} + \frac{x_i^\alpha}{(x_i^\alpha+1)^2} \right) - \frac{\omega}{\alpha^2} + \left\{ r_\omega^* \times \frac{\partial^2}{\partial\alpha^2} log\left[1-F(\tau_0)\right] \right\} + + \sum_{i=1}^{\omega} r_i \times \frac{\partial^2}{\partial\alpha^2} log\left[1-F(x_i)\right].$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\theta} = \frac{\partial^2 logL}{\partial\theta\partial\alpha}$$

$$= \sum_{i=1}^{\omega} \frac{logx_i}{x_i^\alpha} \left\{ \frac{r_i \times \left(x_i^\alpha+1\right)\theta \left(\left(\theta^2 + \left(1-x_i^\alpha\right)\theta - 2x_i^\alpha\right) e^{-\frac{\theta}{x_i^\alpha}} + \left(x_i^\alpha+1\right)\theta + 2x_i^\alpha \right)}{\left(\left(x_i^\alpha\theta + x_i^\alpha\right) e^{-\frac{\theta}{x_i^\alpha}} - \left(x_i^\alpha+1\right)\theta - x_i^\alpha \right)^2} - 1 \right\}$$

$$+ r_\omega^* \frac{\left(\tau_0^\alpha+1\right)\theta \left(\left(\theta^2 + \left(1-\tau_0^\alpha\right)\theta - 2\tau_0^\alpha\right) e^{-\frac{\theta}{\tau_0^\alpha}} + \left(\tau_0^\alpha+1\right)\theta + 2\tau_0^\alpha \right) log\tau_0}{\tau_0^\alpha \left(\left(\tau_0^\alpha\theta + \tau_0^\alpha\right) e^{-\frac{\theta}{\tau_0^\alpha}} - \left(\tau_0^\alpha+1\right)\theta - \tau_0^\alpha \right)^2}$$

$$\frac{\partial^2 logL}{\partial\theta^2} = \frac{\omega}{(1+\theta)^2} - \frac{2\omega}{\theta^2} + \sum_{i=1}^{\omega} r_i \times \frac{\partial^2}{\partial\theta^2} log\left[1-F(x_i)\right] + r_\omega^* \times \frac{\partial^2}{\partial\theta^2} log\left[1-F(\tau_0)\right].$$

Using the observed Fisher information matrix, a two-sided $100(1-\beta)\%$ asymptotic confidence interval for α and θ using both ML and PS may be constructed as $\tilde{\alpha} \mp Z_{1-\beta/2}\sqrt{var(\tilde{\alpha})}$ and $\tilde{\theta} \mp Z_{1-\beta/2}\sqrt{var(\tilde{\theta})}$ respectively; where $Z_{\beta/2}$ denotes the upper $\beta/2$ percentile of the standard normal distribution and $var(\tilde{\alpha}), var(\tilde{\theta})$ may be obtained from the diagonal elements of $I^{-1}(\tilde{\alpha}, \tilde{\theta})$.

4 Bayesian Inference

In this section, we propose a Bayes estimator for α and θ under the considered censoring scheme. Here, we consider independent gamma prior for α and θ owing to its flexibility in reflecting prior beliefs. Now, α and θ may be assumed to be independently distributed and thus, the joint prior density may be written as $\pi(\alpha, \theta) = \pi(\theta)\pi(\alpha)$ where;

$$\pi(\theta) \propto \mathrm{e}^{-a\theta}\theta^{b-1}; \qquad a > 0, b > 0, \theta > 0. \tag{18}$$

$$\pi(\alpha) \propto \mathrm{e}^{-c\alpha}\alpha^{s-1}; \qquad c > 0, s > 0, \alpha > 0. \tag{19}$$

Using the prior densities, the joint posterior density is obtained as:

$$\pi(\alpha, \theta|\underset{\sim}{x}, r) = \frac{\pi(\theta)\pi(\alpha) \cdot L(\alpha, \theta|\underset{\sim}{x}, r)}{\int\limits_{\theta}\int\limits_{\alpha} \pi(\theta)\pi(\alpha) \cdot L(\alpha, \theta|\underset{\sim}{x}, r) d\alpha d\theta}. \tag{20}$$

We also propose and discuss an alternative posterior density obtained as a result of replacing the likelihood function (LF) in the Bayes theorem with the PS function due to their asymptotic equivalence (refer Coolen and Newby (1990, 1994)). This proposed technique does not hinder the estimation even in the presence of censored cases, even though the posterior which is obtained by this method is quite different from any usual posterior density.

Let $\underset{\sim}{x} = (x_1, x_2, \cdots, x_n)$ be a random sample from Eq.(2). The joint posterior density of (α, θ) with the usual LF, expressed up to proportionality, is obtained as:

$$\pi_1(\alpha, \theta|\underset{\sim}{x}, r) \propto \frac{\alpha^{\omega+s-1} \cdot \theta^{2\omega+b-1}}{(1+\theta)^{\omega}} e^{-(\sum_{i=1}^{\omega} \frac{\theta}{x_i^{\alpha}} + \theta a + c\alpha)} \left[\overline{F}(\tau_0)\right]^{r_{\omega}^{*}} \times \prod_{i=1}^{\omega} \left\{ \left(\frac{1 + x_i^{\alpha}}{x_i^{2\alpha+1}} \right) \left[\overline{F}(x_i)\right]^{r_i} \right\}; \tag{21}$$

The posterior density using PS function, expressed up to proportionality is:

$$\pi_2(\alpha, \theta|\underset{\sim}{x}, r) \propto \begin{cases} \pi(\theta)\pi(\alpha) \cdot \prod\limits_{i=1}^{\omega} D_i\,; & \tau_0 = x_k, \\ \pi(\theta)\pi(\alpha) \cdot D_{\xi} \cdot \prod\limits_{i=1}^{\omega} D_i; & \tau_0 = \tau;\ |\tau - x_d| > \epsilon, \\ \pi(\theta)\pi(\alpha) \cdot f(\tau) \cdot \prod\limits_{i=1}^{\omega} D_i\,; & \tau_0 = \tau;\ |\tau - x_d| < \epsilon. \end{cases} \tag{22}$$

We conduct this study under the assumption of a squared error loss function for both the parameters. Consequently, the Bayes estimator of α and θ for the considered loss are their respective posterior means. Thus, the usual Bayes estimators are:

$$\hat{\theta}_{LF} = K_1^{-1} \int_{\theta} \frac{\alpha^{\omega+s-1}\theta^{2\omega+b}}{(1+\theta)^{\omega}} e^{-(\sum_{i=1}^{\omega} \frac{\theta}{x_i^{\alpha}} + \theta a + c\alpha)} \left[\overline{F}(\tau_0)\right]^{r_{\omega}^*} \prod_{i=1}^{\omega} \left\{ \left(\frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}} \right) \left[\overline{F}(x_i)\right]^{r_i} \right\} d\theta. \tag{23}$$

$$\hat{\alpha}_{LF} = K_2^{-1} \int_{\alpha} \frac{\alpha^{\omega+s}\theta^{2\omega+b-1}}{(1+\theta)^{\omega}} e^{-(\sum_{i=1}^{\omega} \frac{\theta}{x_i^{\alpha}} + \theta a + c\alpha)} \left[\overline{F}(\tau_0)\right]^{r_{\omega}^*} \prod_{i=1}^{\omega} \left\{ \left(\frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}} \right) \left[\overline{F}(x_i)\right]^{r_i} \right\} d\alpha. \tag{24}$$

Also, the Bayes estimators using PS function are:

$$\hat{\theta}_{BPS} = \begin{cases} K_3^{-1} \int_{\theta} \theta^{b} \alpha^{s-1} e^{-a\theta - c\alpha} \cdot \prod_{i=1}^{\omega} D_i \, d\theta; & \tau_0 = x_k, \\ K_4^{-1} \int_{\theta} \theta^{b} \alpha^{s-1} e^{-a\theta - c\alpha} \cdot D_{\xi} \cdot \prod_{i=1}^{\omega} D_i \, d\theta; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ K_5^{-1} \int_{\theta} \theta^{b} \alpha^{s-1} e^{-a\theta - c\alpha} \cdot f(\tau) \cdot \prod_{i=1}^{\omega} D_i \, d\theta; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \tag{25}$$

$$\hat{\alpha}_{BPS} = \begin{cases} K_6^{-1} \int_{\alpha} \theta^{b-1} \alpha^{s} e^{-a\theta - c\alpha} \cdot \prod_{i=1}^{\omega} D_i \, d\alpha; & \tau_0 = x_k, \\ K_7^{-1} \int_{\alpha} \theta^{b-1} \alpha^{s} e^{-a\theta - c\alpha} \cdot D_{\xi} \cdot \prod_{i=1}^{\omega} D_i \, d\alpha; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ K_8^{-1} \int_{\alpha} \theta^{b-1} \alpha^{s} e^{-a\theta - c\alpha} \cdot f(\tau) \cdot \prod_{i=1}^{\omega} D_i \, d\alpha; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \tag{26}$$

Apparently, Eqs.(23),(24),(25) and (26) are not mathematically tractable and to solve these implicit integrals, we use Markov chain Monte-Carlo method with Gibbs sampler technique via Metropolis-Hatings (M-H) algorithm to generate samples from the desired posterior densities. The marginalizing constants in the Eqs.(23),(24),(25) and (26) have been expressed in Appendix.

Once a sample is generated from the posteriors obtained in Eq.(21) and (22), the sample means will provide us with an estimate of the concerned parameters for the considered loss function. However, to proceed with Gibbs sampler via M-H algorithm, we re-write the joint posterior in terms of full conditionals and then use an arbitrary proposal density to generate samples from these full conditionals. A detailed layout of this technique may be found in Roberts and Smith (1994); Chib and Greenberg (1995); Gelfand and Smith (1990). The required full conditional posteriors based on the LF are:

$$\pi_1^{\alpha}(\alpha|\theta, \underset{\sim}{x}, r) \propto \alpha^{\omega+s-1} \cdot \mathrm{e}^{-(c\alpha + \sum_{i=1}^{\omega} \frac{\theta}{x_i^{\alpha}})} \left\{\overline{F}(\tau_0)\right\}^{r_{\omega}^*} \prod_{i=1}^{\omega} \left[\left\{ \frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}} \right\} \left\{\overline{F}(x_i)\right\}^{r_i} \right]. \tag{27}$$

$$\pi_1^{\theta}(\theta|\alpha, \underset{\sim}{x}, r) \propto \frac{\theta^{2\omega+b-1}}{(1+\theta)^{\omega}} \cdot \mathrm{e}^{-(a\theta + \sum_{i=1}^{\omega} \frac{\theta}{x_i^{\alpha}})} \left\{\overline{F}(\tau_0)\right\}^{r_{\omega}^*} \prod_{i=1}^{\omega} \left\{\overline{F}(x_i)\right\}^{r_i}. \tag{28}$$

Subsequently, the full conditional posteriors based on PS are:

$$\pi_2^{\alpha}(\alpha|\theta, \underset{\sim}{x}, r) \propto \begin{cases} \pi(\alpha) \cdot \prod_{i=1}^{\omega} D_i \, ; & \tau_0 = x_k, \\ \pi(\alpha) \cdot D_{\xi} \cdot \prod_{i=1}^{\omega} D_i \, ; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ \pi(\alpha) \cdot f(\tau) \cdot \prod_{i=1}^{\omega} D_i \, ; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \tag{29}$$

$$\pi_2^{\theta}(\theta|\alpha, \underset{\sim}{x}, r) \propto \begin{cases} \pi(\theta) \cdot \prod_{i=1}^{\omega} D_i; & \tau_0 = x_k, \\ \pi(\theta) \cdot D_{\xi} \cdot \prod_{i=1}^{\omega} D_i \, ; & \tau_0 = \tau; |\tau - x_d| > \epsilon, \\ \pi(\theta) \cdot f(\tau) \cdot \prod_{i=1}^{\omega} D_i \, ; & \tau_0 = \tau; |\tau - x_d| < \epsilon. \end{cases} \tag{30}$$

$10,000$ samples from each of the aforementioned full conditionals were generated by the M-H algorithm, by considering the asymptotic normal distributions of their classical counterparts as their proposal densities and thence, Bayes point estimates along with their 95% highest posterior density (HPD) intervals of α and θ were evaluated respectively.

Note that, $N_0 = 1500$ generated units from every simulation were discarded to ensure that each chain attained its corresponding stationary distribution. The HPD intervals for the underlying parameters are obtained through the technique suggested by Chen and Shao (1999) with $\left(\alpha^{(j)}, \theta^{(j)}\right); \forall\, j = 1, 2, ..., N^*$ ordered MCMC samples from the desired posterior densities. For each simulation, $100(1-\beta)\%$ credible interval of (α, θ) are computed as $\left(\alpha^{(j)}, \alpha^{(j+[(1-\beta)N^*])}\right)$ and $\left(\theta^{(j)}, \theta^{(j+[(1-\beta)N^*])}\right)$. Consequently, the HPD interval for θ is $\left(\theta^{(j^*)}, \theta^{(j^*+[(1-\beta)N^*])}\right)$ where j^* is chosen so that it yields the interval of minimum length amongst all the credible intervals. Likewise, the HPD interval for α may be obtained.

5 Expected Total Time to Test

In practical situations, cognizance of the duration of a life test is quite desirable, since the cost of an experiment is directly proportional to the total time to test (TTT). Under progressive Type-II censoring with a fixed number of removals, this may be derived according to the suggestion by Kamps and Cramer (2001). In the case of PHT-I-CBR, the expected TTT consists of both τ and k with d $(< k)$ failures in the former case and k failures in the latter case. Without loss of generality, the conditional expectation of total time for a given R is:

$$E(\tau^*|R) = \tau P\left(X_k > \tau\right) + X_k P\left(X_k < \tau\right). \tag{31}$$

Thus, to obtain the expected TTT, the unconditional expectation of Eq.(31) may be obtained as:

$$\begin{aligned} E(\tau^*) &= E_R\left[E(\tau^*|R)\right] \\ &= \tau \times E_R\left\{P\left(X_k > \tau\right)|R\right\} + E_R\left\{X_k \cdot P\left(X_k < \tau\right)|R\right\}. \end{aligned} \tag{32}$$

A detailed note of the expected TTT may be found in Basu et al. (2018) and owing to its implicitness, we use a simulation technique to obtain an estimate of the expected total time to test; i.e. $\widehat{E(\tau^*)} = \left[N_1 \cdot \tau + (N - N_1) \cdot x_k\right]/N$; where, N denotes the total number of simulations, N_1 is the number of times the experiment terminates at τ out of N simulations.

6 Simulation Study

In this section, we investigate the performance of the proposed estimators through a simulation study. A detailed simulation study is carried out for $p = 0.50$ and combinations of k and τ for a sample of size $30, 40$ and 50, reported in Tabs.3-5. Evidently, this specific choice of k ensures 60% and 40% censoring in the absence of time constraints and similarly, the chosen τ's furnishes around 40% and 20% censored data in the absence of failure constraints respectively, in connection to the true distribution.

The first two moment equations (i.e. $\theta = b/a$ and $v_1 = b/a^2$; $\alpha = s/c$ and $v_2 = s/c^2$) were used simultaneously to determine the hyperparameters by considering the prior mean as the true value of α and θ respectively. The hyperparameters are chosen to reflect our belief on the true mean with variance ranging from small to large, yielding to an informative prior for small variance and a non-informative prior corresponding to a large variance. We have documented average point estimates (denoted as $\widehat{\Theta}$) of all the methods with their respective simulated risks (denoted as $R(\widehat{\Theta})$) and bias (denoted as $b(\widehat{\Theta})$) along-with their average lengths of 95% confidence and HPD intervals, in addition to the coverage probabilities based on $M = 1000$ simulations in Tabs.3- 8 for varying p and n, for a hypothetical choice of the parameters.

This extensive simulation study indicates consistency of the proposed estimators (see Tabs.3-5), in addition to an insight into the performance of the Bayes procedure for both LF and PS approach transcending their classical counterparts in terms of simulated risks and length of HPD intervals.

Undoubtedly, as the prior variance of both parameters increases (denoted by σ^2), the Bayes estimators and classical estimators behave alike which is justified, since a flat prior emphasizes strongly on the observed data just like the likelihood function. Thus, an informative prior generates HPD intervals shorter than the asymptotic intervals and a non-informative prior provides intervals of more or less equal width to the classical intervals, which is quite apparent in Tabs.6-8. Evidently, the performances of the Bayes PS estimator are the best followed by Bayes LF estimator, MPS estimator and ML estimator.

The expected total time to test shows an increasing trend with increasing sample size and increasing p respectively for a given censoring scheme (see Tab.1). Heuristically, it may be reasoned through the interpretation of p which determines the number of removals at a particular stage. Thus, a small p (say 0.2) designs the experiment to remove less number of units at the initial stages and a large volume of units is retained till the final termination point with k failures, if in fact, realized X_k is much less than τ for the considered distribution and censoring scheme. However, with an increase in p (say $p = 0.8$), the bulk of experimental units are removed at the initial stages leading to the termination at τ (*iff* $\tau < X_k$).

We have also assessed the behavior of the proposed estimators for varying probabilities of removal. The simulated risks of all the estimators increase as p increases for a particular choice of censoring scheme (see Tab.2). Also, a similar behavior is depicted by the interval estimates presented in Tabs.6 - 8, wherein the lengths of confidence and HPD intervals increase with an increase in p for a given (τ, k). It may be noted that, expectedly, the coverage probabilities also increase for such cases. Apparently, PS estimators in both the classical and Bayesian approaches are consistent.

It must be noted that the average point estimates, simulated risks, bias and confidence intervals of all the estimators in the classical paradigm are independent of the role of prior variance and yet they are reported in Tabs.3-8 only for comparative purposes (the different values against varying σ^2 can be accredited to sampling fluctuations).

Table 1: Effect of removal probability on $\widehat{E(\tau)}$ for $\alpha = 2; \theta = 3$

n	τ	k	$p = 0.2$	$p = 0.5$	$p = 0.8$
30	2.5	12	2.375	2.493	2.495
		18	2.499	2.500	2.500
	4	12	3.034	3.790	3.858
		18	3.850	3.947	3.959
40	2.5	16	2.478	2.498	2.500
		24	2.500	2.500	2.500
	4	16	3.537	3.924	3.934
		24	3.980	3.983	3.988
50	2.5	20	2.497	2.500	2.500
		30	2.500	2.500	2.500
	4	20	3.809	3.968	3.968
		30	3.995	3.995	3.995

Table 2: Effect of removal probability on risks of estimators for $n = 30$, $\alpha = 2$, $\theta = 3$ & $v_1 = v_2 = 0.5$

p	τ	k	θ				α			
			ML	PS	BLF	BPS	ML	PS	BLF	BPS
0.2	2.5	12	0.6004	0.3796	0.1331	0.1298	0.3173	0.2214	0.1181	0.1044
		18	0.5382	0.3426	0.1366	0.1260	0.2307	0.1757	0.1118	0.1015
	4	12	0.4938	0.3250	0.1313	0.1236	0.3079	0.2426	0.1224	0.1130
		18	0.5641	0.3440	0.1452	0.1265	0.2185	0.1676	0.1089	0.0985
0.8	2.5	12	0.6494	0.3876	0.1418	0.1275	0.4082	0.2999	0.1358	0.1158
		18	0.5911	0.3654	0.1480	0.1348	0.2651	0.2014	0.1254	0.1137
	4	12	0.5272	0.3600	0.1400	0.1307	0.3199	0.2447	0.1391	0.1191
		18	0.5974	0.3849	0.1442	0.1340	0.2410	0.1895	0.1252	0.1129

Table 3: Average point estimates, simulated risks and bias for $p = 0.5$ & $\alpha = 2; \theta = 3$, with highly informative prior ($\sigma^2 = 1$)

n	τ	k		θ				α			
				$\widehat{\theta}_{ML}$	$\widehat{\theta}_{PS}$	$\widehat{\theta}_{LF}$	$\widehat{\theta}_{BPS}$	$\widehat{\alpha}_{ML}$	$\widehat{\alpha}_{PS}$	$\widehat{\alpha}_{LF}$	$\widehat{\alpha}_{BPS}$
30	2.5	12	$\widehat{\Theta}$	3.2551	2.9632	3.0913	2.8880	2.2593	2.0454	2.1640	1.9893
			$R(\widehat{\Theta})$	0.6782	0.4307	0.2470	0.2459	0.4044	0.2915	0.2282	0.1858
			$b(\widehat{\Theta})$	0.2551	-0.0368	0.0913	-0.1120	0.2593	0.0454	0.1640	-0.0107
		18	$\widehat{\Theta}$	3.2406	2.9632	3.1094	2.9042	2.1536	1.9766	2.1076	1.9510
			$R(\widehat{\Theta})$	0.5711	0.3493	0.2284	0.2002	0.2505	0.1900	0.1854	0.1572
			$b(\widehat{\Theta})$	0.2406	-0.0368	0.1094	-0.0958	0.1536	-0.0234	0.1076	-0.0490
	4	12	$\widehat{\Theta}$	3.1881	2.9029	3.0685	2.8537	2.2197	2.0273	2.1482	1.9757
			$R(\widehat{\Theta})$	0.5009	0.3365	0.2162	0.2062	0.3577	0.3132	0.2257	0.2049
			$b(\widehat{\Theta})$	0.1881	-0.0971	0.0685	-0.1463	0.2197	0.0273	0.1482	-0.0243
		18	$\widehat{\Theta}$	3.1649	2.8998	3.0589	2.8544	2.1221	1.9449	2.0901	1.9270
			$R(\widehat{\Theta})$	0.4875	0.3312	0.2215	0.2132	0.1929	0.1589	0.1504	0.1361
			$b(\widehat{\Theta})$	0.1649	-0.1002	0.0589	-0.1456	0.1221	-0.0551	0.0901	-0.0730
40	2.5	16	$\widehat{\Theta}$	3.1464	2.9238	3.0759	2.8937	2.1490	1.9747	2.1074	1.9517
			$R(\widehat{\Theta})$	0.3705	0.2578	0.2288	0.1912	0.2595	0.1964	0.1947	0.1628
			$b(\widehat{\Theta})$	0.1464	-0.0762	0.0759	-0.1063	0.1490	-0.0253	0.1074	-0.0483
		24	$\widehat{\Theta}$	3.0766	2.8692	3.0241	2.8469	2.1069	1.9629	2.0870	1.9517
			$R(\widehat{\Theta})$	0.3082	0.2404	0.2036	0.1907	0.1668	0.1361	0.1446	0.1245
			$b(\widehat{\Theta})$	0.0766	-0.1308	0.0241	-0.1531	0.1069	-0.0371	0.0870	-0.0483
	4	16	$\widehat{\Theta}$	3.1121	2.8897	3.0479	2.8643	2.1030	1.9332	2.0748	1.9157
			$R(\widehat{\Theta})$	0.3347	0.2405	0.1951	0.1816	0.2096	0.1869	0.1692	0.1608
			$b(\widehat{\Theta})$	0.1121	-0.1103	0.0479	-0.1357	0.1030	-0.0668	0.0748	-0.0843
		24	$\widehat{\Theta}$	3.1426	2.9267	3.0834	2.9001	2.1270	1.9773	2.1110	1.9679
			$R(\widehat{\Theta})$	0.3298	0.2378	0.2070	0.1815	0.1492	0.1174	0.1318	0.1082
			$b(\widehat{\Theta})$	0.1426	-0.0733	0.0834	-0.0999	0.1270	-0.0227	0.1110	-0.0321
50	2.5	20	$\widehat{\Theta}$	3.1584	2.9626	3.0980	2.9334	2.1262	1.9772	2.1008	1.9621
			$R(\widehat{\Theta})$	0.3443	0.2382	0.1941	0.1727	0.2017	0.1607	0.1689	0.1423
			$b(\widehat{\Theta})$	0.1584	-0.0374	0.0980	-0.0666	0.1262	-0.0228	0.1008	-0.0379
		30	$\widehat{\Theta}$	3.1119	2.9269	3.0742	2.9088	2.0706	1.9469	2.0596	1.9406
			$R(\widehat{\Theta})$	0.2630	0.1976	0.1894	0.1632	0.1124	0.0984	0.1029	0.0935
			$b(\widehat{\Theta})$	0.1119	-0.0731	0.0742	-0.0912	-0.9294	-1.0531	-0.9404	-1.0594
	4	20	$\widehat{\Theta}$	3.1052	2.9153	3.0615	2.8966	2.1185	1.9638	2.1014	1.9540
			$R(\widehat{\Theta})$	0.2750	0.2048	0.1867	0.1654	0.1501	0.1219	0.1315	0.1118
			$b(\widehat{\Theta})$	0.1052	-0.0847	0.0615	-0.1034	0.1185	-0.0362	0.1014	-0.0460
		30	$\widehat{\Theta}$	3.0808	2.8999	3.0447	2.8830	2.0886	1.9586	2.0802	1.9537
			$R(\widehat{\Theta})$	0.2808	0.2199	0.2070	0.1803	0.1130	0.0933	0.1052	0.0893
			$b(\widehat{\Theta})$	0.0808	-0.1001	0.0447	-0.1170	0.0886	-0.0414	0.0802	-0.0463

Table 4: Average point estimates, simulated risks and bias for $p = 0.5$ & $\alpha = 2; \theta = 3$, with highly informative prior ($\sigma^2 = 5$)

n	τ	k		θ				α			
				$\widehat{\theta}_{ML}$	$\widehat{\theta}_{PS}$	$\widehat{\theta}_{LF}$	$\widehat{\theta}_{BPS}$	$\widehat{\alpha}_{ML}$	$\widehat{\alpha}_{PS}$	$\widehat{\alpha}_{LF}$	$\widehat{\alpha}_{BPS}$
30	2.5	12	$\widehat{\Theta}$	3.1459	2.8797	3.0969	2.8526	2.2462	2.0337	2.2036	2.0039
			$R(\widehat{\Theta})$	0.6717	0.3968	0.5241	0.3451	0.3998	0.2733	0.3475	0.2493
			$b(\widehat{\Theta})$	0.1459	-0.1203	0.0969	-0.1474	0.2462	0.0337	0.2036	0.0039
		18	$\widehat{\Theta}$	3.1834	2.9157	3.1447	2.8924	2.1434	1.9683	2.1236	1.9526
			$R(\widehat{\Theta})$	0.5622	0.3672	0.4604	0.3283	0.2290	0.1765	0.2113	0.1690
			$b(\widehat{\Theta})$	0.1834	-0.0843	0.1447	-0.1076	0.1434	-0.0317	0.1236	-0.0474
	4	12	$\widehat{\Theta}$	3.1785	2.8928	3.1371	2.8666	2.2129	2.0212	2.1845	1.9974
			$R(\widehat{\Theta})$	0.5938	0.3971	0.4798	0.3470	0.3512	0.3056	0.3089	0.2736
			$b(\widehat{\Theta})$	0.1785	-0.1072	0.1371	-0.1334	0.2129	0.0212	0.1845	-0.0026
		18	$\widehat{\Theta}$	3.1472	2.8829	3.1167	2.8637	2.1425	1.9642	2.1282	1.9530
			$R(\widehat{\Theta})$	0.4277	0.2956	0.3743	0.2770	0.2026	0.1622	0.1906	0.1569
			$b(\widehat{\Theta})$	0.1472	-0.1171	0.1167	-0.1363	0.1425	-0.0358	0.1282	-0.0470
40	2.5	16	$\widehat{\Theta}$	3.1834	2.9534	3.0994	2.9161	2.1500	1.9764	2.1088	1.9532
			$R(\widehat{\Theta})$	0.4059	0.2684	0.2266	0.1919	0.2454	0.1875	0.1868	0.1569
			$b(\widehat{\Theta})$	0.1834	-0.0466	0.0994	-0.0839	0.1500	-0.0236	0.1088	-0.0468
		24	$\widehat{\Theta}$	3.1347	2.9208	3.1154	2.9081	2.1245	1.9795	2.1137	1.9707
			$R(\widehat{\Theta})$	0.3192	0.2309	0.2911	0.2205	0.1576	0.1247	0.1509	0.1300
			$b(\widehat{\Theta})$	0.1347	-0.0792	0.1154	-0.0919	0.1245	-0.0205	0.1137	-0.0293
	4	16	$\widehat{\Theta}$	3.1551	2.9269	3.1297	2.9110	2.1710	1.9972	2.1561	1.9849
			$R(\widehat{\Theta})$	0.3347	0.2405	0.4043	0.2939	0.2334	0.1884	0.2191	0.1802
			$b(\widehat{\Theta})$	0.1551	-0.0731	0.1297	-0.0890	0.1710	-0.0028	0.1561	-0.0151
		24	$\widehat{\Theta}$	3.1914	2.9681	3.1695	2.9539	2.1249	1.9752	2.1167	1.9685
			$R(\widehat{\Theta})$	0.4019	0.2689	0.3580	0.2511	0.1533	0.1240	0.1473	0.1210
			$b(\widehat{\Theta})$	0.1914	-0.0319	0.1695	-0.0461	0.1249	-0.0248	0.1167	-0.0315
50	2.5	20	$\widehat{\Theta}$	3.1055	2.9159	3.0902	2.9056	2.1355	1.9836	2.1237	1.9738
			$R(\widehat{\Theta})$	0.2824	0.2072	0.2590	0.1976	0.2003	0.1559	0.1909	0.1516
			$b(\widehat{\Theta})$	0.1055	-0.0841	0.0902	-0.0944	0.1355	-0.0164	0.1237	-0.0262
		30	$\widehat{\Theta}$	3.1203	2.9346	3.1068	2.9252	2.1104	1.9839	2.1034	1.9781
			$R(\widehat{\Theta})$	0.2553	0.1914	0.2395	0.1846	0.1475	0.1208	0.1431	0.1189
			$b(\widehat{\Theta})$	0.1203	-0.0654	0.1068	-0.0748	0.1104	-0.0161	0.1034	-0.0219
	4	20	$\widehat{\Theta}$	3.0773	2.8908	3.0632	2.8812	2.1213	1.9666	2.1125	1.9593
			$R(\widehat{\Theta})$	0.2469	0.1933	0.2326	0.1881	0.1746	0.1421	0.1681	0.1388
			$b(\widehat{\Theta})$	0.0773	-0.1092	0.0632	-0.1188	0.1213	-0.0334	0.1125	-0.0407
		30	$\widehat{\Theta}$	3.1124	2.9311	3.1006	2.9225	2.0778	1.9492	2.0729	1.9453
			$R(\widehat{\Theta})$	0.2335	0.1774	0.2211	0.1723	0.1076	0.0910	0.1053	0.0902
			$b(\widehat{\Theta})$	0.1124	-0.0689	0.1006	-0.0775	0.0778	-0.0508	0.0729	-0.0547

Table 5: Average point estimates, simulated risks and bias for $p = 0.5$ & $\alpha = 2; \theta = 3$, with highly informative prior ($\sigma^2 = 50$)

n	τ	k		θ				α			
				$\widehat{\theta}_{ML}$	$\widehat{\theta}_{PS}$	$\widehat{\theta}_{LF}$	$\widehat{\theta}_{BPS}$	$\widehat{\alpha}_{ML}$	$\widehat{\alpha}_{PS}$	$\widehat{\alpha}_{LF}$	$\widehat{\alpha}_{BPS}$
30	2.5	12	$\widehat{\Theta}$	3.2526	2.9650	3.2287	2.9458	2.2959	2.0796	2.2672	2.0549
			$R(\widehat{\Theta})$	0.6930	0.4200	0.6653	0.4090	0.4886	0.3399	0.4581	0.3241
			$b(\widehat{\Theta})$	0.2526	-0.0350	0.2287	-0.0542	0.2959	0.0796	0.2672	0.0549
		18	$\widehat{\Theta}$	3.2300	2.9555	3.2043	2.9362	2.2001	2.0204	2.1820	2.0050
			$R(\widehat{\Theta})$	0.6191	0.3911	0.5732	0.3743	0.2667	0.1920	0.2519	0.1860
			$b(\widehat{\Theta})$	0.2300	-0.0445	0.2043	-0.0638	0.2001	0.0204	0.1820	0.0050
	4	12	$\widehat{\Theta}$	3.1431	2.8659	3.1185	2.8477	2.1989	2.0075	2.1768	1.9882
			$R(\widehat{\Theta})$	0.4804	0.3338	0.4515	0.3259	0.3214	0.2832	0.3000	0.2705
			$b(\widehat{\Theta})$	0.1431	-0.1341	0.1185	-0.1523	0.1989	0.0075	0.1768	-0.0118
		18	$\widehat{\Theta}$	3.2353	2.9539	3.2119	2.9364	2.1494	1.9712	2.1368	1.9603
			$R(\widehat{\Theta})$	0.6081	0.3702	0.5715	0.3552	0.2330	0.1837	0.2231	0.1790
			$b(\widehat{\Theta})$	0.2353	-0.0461	0.2119	-0.0636	0.1494	-0.0288	0.1368	-0.0397
40	2.5	16	$\widehat{\Theta}$	3.1539	2.9281	3.1378	2.9154	2.1617	1.9861	2.1460	1.9728
			$R(\widehat{\Theta})$	0.4114	0.2816	0.3964	0.2762	0.2407	0.1772	0.2306	0.1737
			$b(\widehat{\Theta})$	0.1539	-0.0719	0.1378	-0.0846	0.1617	-0.0139	0.1460	-0.0272
		24	$\widehat{\Theta}$	3.1451	2.9263	3.1303	2.9148	2.1187	1.9725	2.1094	1.9646
			$R(\widehat{\Theta})$	0.3618	0.2584	0.3462	0.2515	0.1746	0.1397	0.1693	0.1378
			$b(\widehat{\Theta})$	0.1451	-0.0737	0.1303	-0.0852	0.1187	-0.0275	0.1094	-0.0354
	4	16	$\widehat{\Theta}$	3.1312	2.9065	3.1151	2.8939	2.1315	1.9549	2.1193	1.9443
			$R(\widehat{\Theta})$	0.3893	0.2752	0.3728	0.2688	0.2157	0.1792	0.2069	0.1756
			$b(\widehat{\Theta})$	0.1312	-0.0935	0.1151	-0.1061	0.1315	-0.0451	0.1193	-0.0557
		24	$\widehat{\Theta}$	3.1494	2.9310	3.1351	2.9200	2.0997	1.9505	2.0926	1.9447
			$R(\widehat{\Theta})$	0.3698	0.2583	0.3550	0.2531	0.1413	0.1169	0.1377	0.1158
			$b(\widehat{\Theta})$	0.1494	-0.0690	0.1351	-0.0800	0.0997	-0.0495	0.0926	-0.0553
50	2.5	20	$\widehat{\Theta}$	3.1104	2.9192	3.0992	2.9106	2.1161	1.9657	2.1058	1.9569
			$R(\widehat{\Theta})$	0.2716	0.1999	0.2644	0.1982	0.1796	0.1432	0.1747	0.1415
			$b(\widehat{\Theta})$	0.1104	-0.0808	0.0992	-0.0894	0.1161	-0.0343	0.1058	-0.0431
		30	$\widehat{\Theta}$	3.1065	2.9226	3.0962	2.9149	2.0924	1.9665	2.0863	1.9611
			$R(\widehat{\Theta})$	0.2511	0.1922	0.2428	0.1892	0.1199	0.1001	0.1175	0.0993
			$b(\widehat{\Theta})$	0.1065	-0.0774	0.0962	-0.0851	0.0924	-0.0335	0.0863	-0.0389
	4	20	$\widehat{\Theta}$	3.1099	2.9196	3.0977	2.9103	2.1380	1.9827	2.1300	1.9760
			$R(\widehat{\Theta})$	0.2784	0.2068	0.2685	0.2037	0.1778	0.1402	0.1726	0.1378
			$b(\widehat{\Theta})$	0.1099	-0.0804	0.0977	-0.0897	0.1380	-0.0173	0.1300	-0.0240
		30	$\widehat{\Theta}$	3.0686	2.8901	3.0595	2.8832	2.0566	1.9286	2.0523	1.9251
			$R(\widehat{\Theta})$	0.2250	0.1816	0.2188	0.1799	0.0919	0.0818	0.0905	0.0817
			$b(\widehat{\Theta})$	0.0686	-0.1099	0.0595	-0.1168	0.0566	-0.0714	0.0523	-0.0749

Table 6: Average length of Confidence Intervals (Coverage probabilities in 2^{nd} row of each (τ, k)) for $n = 40$, $\alpha = 2$; $\theta = 3$ and $\sigma^2 = 1$ against varying p

σ^2	p	(τ, k)	θ				α			
			$L_{\theta_{ML}}$	$L_{\theta_{PS}}$	$L_{\theta_{LF}}$	$L_{\theta_{BPS}}$	$L_{\alpha_{ML}}$	$L_{\alpha_{PS}}$	$L_{\alpha_{LF}}$	$L_{\alpha_{BPS}}$
1	0.2	2.5, 16	2.018	1.828	1.534	1.455	1.708	1.604	1.279	1.243
			0.932	0.908	0.922	0.884	0.950	0.940	0.920	0.888
		2.5, 24	1.973	1.796	1.497	1.428	1.485	1.404	1.079	1.068
			0.954	0.904	0.906	0.890	0.952	0.942	0.890	0.872
		4, 16	2.014	1.822	1.536	1.452	1.629	1.563	1.207	1.207
			0.950	0.912	0.920	0.894	0.958	0.934	0.912	0.882
		4, 24	1.947	1.775	1.486	1.414	1.360	1.286	0.967	0.959
			0.962	0.950	0.946	0.916	0.958	0.934	0.884	0.850
	0.5	2.5, 16	2.058	1.843	1.585	1.487	1.755	1.639	1.328	1.283
			0.956	0.900	0.930	0.894	0.956	0.936	0.916	0.892
		2.5, 24	2.026	1.829	1.546	1.455	1.484	1.403	1.082	1.071
			0.958	0.916	0.936	0.892	0.956	0.922	0.882	0.862
		4, 16	2.115	1.893	1.623	1.522	1.633	1.540	1.222	1.192
			0.948	0.910	0.934	0.902	0.950	0.924	0.900	0.862
		4, 24	2.040	1.844	1.556	1.471	1.363	1.285	0.973	0.962
			0.964	0.928	0.928	0.910	0.954	0.932	0.888	0.842
	0.8	2.5, 16	2.214	1.954	1.682	1.564	1.750	1.638	1.322	1.283
			0.952	0.890	0.926	0.878	0.958	0.930	0.910	0.890
		2.5, 24	2.071	1.856	1.582	1.486	1.476	1.395	1.077	1.062
			0.972	0.930	0.948	0.912	0.964	0.930	0.890	0.856
		4, 16	2.232	1.970	1.688	1.575	1.607	1.509	1.198	1.169
			0.956	0.910	0.936	0.908	0.960	0.938	0.924	0.880
		4, 24	2.039	1.833	1.555	1.462	1.345	1.268	0.955	0.946
			0.938	0.886	0.912	0.862	0.966	0.942	0.864	0.840

Table 7: Average length of Confidence Intervals (Coverage probabilities in 2^{nd} row of each (τ, k)) $n = 40$, $\alpha = 2$; $\theta = 3$ and $\sigma^2 = 5$ against varying p

σ^2	p	(τ, k)	θ				α			
			$L_{\theta_{ML}}$	$L_{\theta_{PS}}$	$L_{\theta_{LF}}$	$L_{\theta_{BPS}}$	$L_{\alpha_{ML}}$	$L_{\alpha_{PS}}$	$L_{\alpha_{LF}}$	$L_{\alpha_{BPS}}$
5	0.2	2.5, 16	2.048	1.852	1.706	1.593	1.705	1.605	1.350	1.317
			0.958	0.930	0.908	0.894	0.958	0.962	0.920	0.896
		2.5, 24	1.937	1.767	1.606	1.513	1.481	1.401	1.123	1.110
			0.982	0.940	0.932	0.882	0.960	0.956	0.894	0.870
		4, 16	2.032	1.836	1.695	1.583	1.614	1.550	1.257	1.261
			0.940	0.896	0.876	0.842	0.952	0.944	0.890	0.856
		4, 24	1.948	1.776	1.617	1.519	1.369	1.296	1.009	1.002
			0.948	0.936	0.916	0.898	0.950	0.924	0.846	0.840
	0.5	2.5, 16	2.153	1.922	1.815	1.673	1.759	1.645	1.410	1.361
			0.958	0.916	0.918	0.886	0.940	0.932	0.890	0.862
		2.5, 24	2.048	1.848	1.705	1.589	1.497	1.416	1.139	1.128
			0.972	0.922	0.922	0.882	0.974	0.966	0.884	0.874
		4, 16	2.146	1.918	1.811	1.669	1.636	1.534	1.283	1.251
			0.958	0.926	0.926	0.890	0.948	0.916	0.878	0.844
		4, 24	1.998	1.809	1.663	1.558	1.349	1.272	0.991	0.980
			0.964	0.932	0.918	0.898	0.942	0.906	0.860	0.830
	0.8	2.5, 16	2.261	1.992	1.913	1.738	1.766	1.653	1.419	1.376
			0.962	0.934	0.940	0.912	0.952	0.952	0.898	0.892
		2.5, 24	2.062	1.849	1.714	1.591	1.476	1.396	1.120	1.109
			0.970	0.928	0.928	0.890	0.968	0.946	0.884	0.870
		4, 16	2.232	1.969	1.880	1.713	1.608	1.511	1.258	1.229
			0.976	0.928	0.946	0.900	0.952	0.916	0.884	0.846
		4, 24	2.059	1.849	1.713	1.588	1.351	1.274	0.994	0.983
			0.954	0.920	0.902	0.870	0.942	0.922	0.846	0.846

Table 8: Average length of Confidence Intervals (Coverage probabilities in 2^{nd} row of each (τ, k)) $n = 40$, $\alpha = 2; \theta = 3$ and $\sigma^2 = 50$ against varying p

σ^2	p	(τ, k)	θ				α			
			$L_{\theta_{ML}}$	$L_{\theta_{PS}}$	$L_{\theta_{LF}}$	$L_{\theta_{BPS}}$	$L_{\alpha_{ML}}$	$L_{\alpha_{PS}}$	$L_{\alpha_{LF}}$	$L_{\alpha_{BPS}}$
50	0.2	2.5, 16	1.982	1.798	1.701	1.584	1.697	1.596	1.363	1.327
			0.974	0.934	0.928	0.900	0.952	0.952	0.898	0.880
		2.5, 24	1.978	1.802	1.678	1.576	1.484	1.404	1.137	1.124
			0.966	0.932	0.904	0.892	0.958	0.938	0.864	0.856
		4, 16	2.022	1.827	1.734	1.609	1.601	1.540	1.262	1.269
			0.958	0.918	0.904	0.880	0.946	0.936	0.874	0.846
		4, 24	1.905	1.740	1.609	1.517	1.380	1.303	1.029	1.022
			0.944	0.918	0.892	0.848	0.944	0.946	0.846	0.834
	0.5	2.5, 16	2.166	1.931	1.884	1.722	1.766	1.653	1.435	1.392
			0.956	0.940	0.928	0.920	0.950	0.940	0.894	0.876
		2.5, 24	1.994	1.802	1.698	1.580	1.482	1.401	1.137	1.126
			0.964	0.926	0.920	0.868	0.940	0.930	0.866	0.856
		4, 16	2.158	1.924	1.876	1.711	1.776	1.660	1.448	1.400
			0.946	0.924	0.900	0.896	0.940	0.934	0.900	0.874
		4, 24	1.960	1.778	1.668	1.557	1.351	1.275	0.999	0.991
			0.962	0.920	0.912	0.882	0.956	0.946	0.884	0.870
	0.8	2.5, 16	2.247	1.980	1.960	1.770	1.725	1.617	1.398	1.356
			0.950	0.924	0.928	0.888	0.930	0.918	0.874	0.856
		2.5, 24	2.095	1.878	1.794	1.658	1.486	1.406	1.142	1.131
			0.980	0.940	0.920	0.900	0.960	0.958	0.882	0.876
		4, 16	2.162	1.914	1.881	1.709	1.612	1.516	1.278	1.248
			0.942	0.892	0.912	0.860	0.932	0.910	0.850	0.828
		4, 24	2.070	1.860	1.769	1.635	1.341	1.266	0.989	0.983
			0.954	0.920	0.908	0.878	0.960	0.944	0.860	0.854

7 Real Data Analysis

This section demonstrates the applicability of the proposed methodologies to suitable data. Sharma et al. (2016) verified the applicability of $GILD(\alpha, \theta)$ for modified bathtub shaped hazard data and demonstrated it on maximum flood level (in millions of cubic feet per second) for the Susquehanna river at Harrisburg, Pennsylvania over 20 four-year periods from 1890 to 1969 and observed $GILD$ to be the model of best fit among several competing models. We illustrate the proposed methodology on the data of active repair times (in hours) for an airborne communication transceiver, which was initially reported and analyzed by Alven and William (1964) using log-normal distribution by virtue of its modified bath-tub hazard function. In our study, we observed, $GILD(\alpha, \theta)$ is also suitable to analyze this data set which is graphically quite evident from Figure 1 and also from the K-S distance $D = 0.0799$ (tabulated value at 5% level of significance is $D_{n,\beta} \approx 0.2002$).

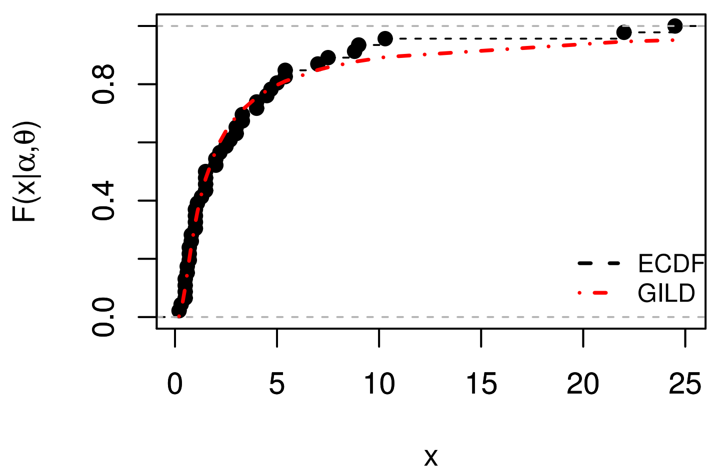

Figure 1: Fitting of $GILD(\alpha, \theta)$ on the data of active repair times (in hours) for an airborne communication transceiver

However, we do not emphasize unearthing the model of best fit, instead, we proceed to the analysis of this data through *GILD* for some hypothetical censoring schemes and obtain the parameter estimates. Here, we resort to the estimates obtained for the complete sample, i.e. $(\hat{\alpha}_{ML}, \hat{\theta}_{ML} = 0.938, 1.602)$ and $(\hat{\alpha}_{PS}, \hat{\theta}_{PS} = 0.869, 1.572)$ as an initial guess for the optimization of the likelihood and PS function for a particular censoring scheme.

The Bayesian analysis is performed with the assumption of vague prior and convergence of the chains were validated for varying initial chain values. The generated sequences of α and θ from the corresponding posterior densities are presented in Figure 2. These generated sequences reveal a slightly positively skewed well-mixed sample. The MLE, MPS and Bayes estimate using both the LF and PS functions along with their asymptotic and HPD intervals are given in Tab.9. The interval estimates for the real data analysis cognate with the simulation study and thus, we observe the shortest lengths for Bayes PS estimators followed by Bayes LF, classical PS and lastly MLE.

Table 9: Estimates and length of confidence intervals of α, θ based on real data

τ	k	θ				α				$\widehat{E(\tau)}$
		$\hat{\theta}_{ML}$	$\hat{\theta}_{PS}$	$\hat{\theta}_{LF}$	$\hat{\theta}_{BPS}$	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{PS}$	$\hat{\alpha}_{LF}$	$\hat{\alpha}_{BPS}$	
3	15	1.6026	1.5893	1.6236	1.6098	0.8133	0.7413	0.8021	0.7287	3
		1.1986	1.1421	1.0744	1.0013	0.6694	0.6225	0.5643	0.4999	
	30	1.8136	1.7626	1.8296	1.7652	0.6991	0.6535	0.6920	0.6475	3
		1.0068	0.9631	0.9000	0.8651	0.4768	0.4519	0.3562	0.3341	
10	15	1.4304	1.4504	1.4546	1.4828	0.9284	0.8341	0.9168	0.8246	10
		1.0511	1.0186	0.9246	0.8841	0.6656	0.6121	0.5328	0.5026	
	30	1.5873	1.5511	1.5797	1.5521	0.9219	0.8581	0.9163	0.8524	10
		0.8759	0.8426	0.7853	0.7448	0.4774	0.4517	0.3622	0.3372	
25	15	1.6545	1.6392	1.6628	1.6538	0.8285	0.7512	0.8209	0.7493	9
		1.1785	1.1175	1.0779	1.0560	0.5584	0.5168	0.4380	0.4036	
	30	1.5426	1.5195	1.5333	1.5137	0.9226	0.8532	0.9189	0.8495	22
		0.8474	0.8206	0.7338	0.7175	0.4818	0.4528	0.3695	0.3331	

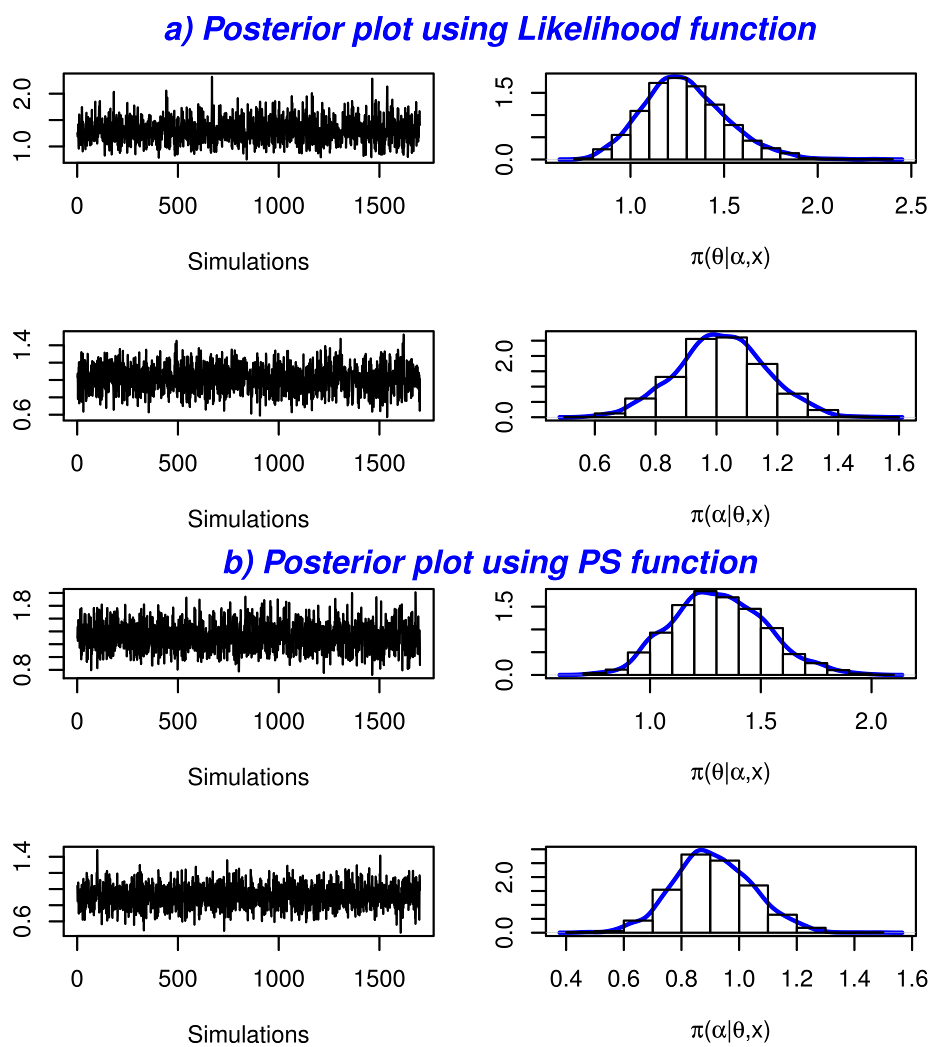

Figure 2: Real data posterior and trace plot of θ and α

8 Conclusion

In this article, we have considered the classical and Bayesian inference for generalized inverse Lindley distribution with PHT-I-CBR. The simulation study successfully delivers a palpable justification that the PS estimators in both paradigms surpass the other estimators in terms of simulated risks and length of confidence intervals. Although, the classical and Bayesian estimators under the non-informative scenario exhibit an analogous nature, yet, the Bayes PS estimators outshine their classical counterparts as well as the other estimators in the presence of suitable prior information.

References

Alven, Von and William, H. (1964), Reliability engineering, Prentice Hall.

Anatolyev, S. and Kosenok, G. (2005), An Alternative to Maximum Likelihood Based on Spacings, *Econometric Theory*, **21**(2), 472-476.

Balakrishnan, N., and Kundu, D. (2013), Hybrid censoring: Models, Inferential results and Applications, *Comput. Stat. Data Anal.*, **57**(1), 166-209.

Balakrishnan, N. and Cramer, E. (2014), *The Art of Progressive Censoring*, Birkhäuser Basel, New York.

Barco, K. V. P., Mazucheli, J., and Janeiro, V. (2016), The inverse power Lindley distribution, *Communications in Statistics - Simulation and Computation*, 1-16.

Basu, S., Singh S. K., and Singh, U. (2017), Parameter estimation of inverse Lindley distribution for Type-I censored data, *Computational Statistics*, **32**(1), 367-385.

Basu, S., Singh S. K., and Singh, U. (2018), Bayesian Inference using Product of Spacings function for Progressive Hybrid Type-I censoring scheme, *Statistics*, **52**(2), 1623-1650.

Berger, J. O. (2013), *Statistical Decision theory and Bayesian analysis*, Springer Science and Business Media.

Chen, M. H., and Shao, Q. M. (1999), Monte Carlo estimation of Bayesian Credible and HPD Intervals, *Journal of Computational and Graphical Statistics*, **8**(1), 69-92.

Chen, Y. and Su, C. (2006), Finite time ruin probability with heavy-tailed insurance and financial risks, *Statistics and Probability Letters*, **76**(16), 1812-1820.

Chen, S. M., and Bhattacharyya, G. K. (1987), Exact confidence bounds for an Exponential parameter under Hybrid Censoring, *Communications in Statistics-Theory and Methods*, **19**(8), 2429-2442.

Cheng, R. C. H. and Iles, T. C. (1987), Corrected maximum likelihood in non-regular problems, *Journal of the Royal Statistical Society. Series B (Methodological)*, 95-101.

Cheng, R. C. H., and Stephens, M. A. (1989), A Goodness-Of-Fit Test Using Moran's Statistic with Estimated Parameters, *Biometrika*, **76**(2), 385-392.

Cheng, R.C.H., and Traylor, L. (1995), Non-regular Maximum Likelihood problems, *Journal of the Royal Statistical Society. Series B (Methodological)*, 3-44.

Cheng, R. C. H., and Amin, N. A. K. (1995), Estimating Parameters in Continuous Univariate distributions with a shifted origin, *Journal of the Royal Statistical Society: Series B (Methodological)*, 394-403.

Chhikara, R. S., and Folks, J. L. (1977), The Inverse Gaussian Distribution as a Lifetime Model, *Technometrics*, **19**(4), 461-468.

Chib, S., and Greenberg, E. (1995), Understanding the Metropolis-Hastings Algorithms, *The american statistician*, **49**(4), 327-335.

Childs, A., Chandrasekar, B., and Balakrishnan, N. (2008), *Statistical Models and Methods for Biomedical and Technical Systems*, Birkhäuser Boston, 319-330.

Cohen, A. C. (1963), Progressively Censored Samples in Life Testing, *Technometrics*, **5**(3), 327-339.

Cohen, A. C. (1976), Progressively Censored Sampling in the Three Parameter Log-Normal Distribution, *Technometrics*, **18**(1), 99-103.

Cohen, A. C. and Norgaard, N. J.(1977), Progressively Censored Sampling in the Three-Parameter Gamma Distribution, *Technometrics*, **19**(3), 333-340.

Coolen, F. P. A., and Newby, M. J. (1990), A note on the use of the product of spacings in Bayesian Inference, *Technische Universiteit Eindhoven*.

Coolen, F. P. A. and Newby, M. J. (1994), Bayesian estimation of location parameters in life distributions, *Reliability Engineering and System Safety*, **45**(3), 293-298.

Dumonceaux, R., and Antle, C. E. (1973), Discrimination between the Log-Normal and the Weibull Distributions, *Technometrics*, **15**(4), 923-926.

Epstein, B. (1954), Truncated Life Tests in the Exponential Case, *Ann. Math. Stat.*, **25**(3), 555-564.

Epstein, B. (1960), Estimation from Life Test Data, *Technometrics*, **2**(4), 447-454.

Gelfand, Alan E. and Smith, Adrian F. M. (1990), Sampling-based approaches to calculating marginal densities, *Journal of the American statistical association*, **85**, 398-409.

Ghitany, M. E., Atieh, B., and Nadarajah, S. (2008), Lindley distribution and its application, *Mathematics and computers in simulation*, **78**(4), 493-506.

Ghosh, K., and Jammalamadaka, S. R. (2001), A General Estimation method using Spacings, *Journal of Statistical Planning and Inference*, **93**(1), 71-82.

Hak-Keung, Y., and Siu, K. T. (1996), Parameters estimation for Weibull distributed lifetimes under progressive censoring with random removals, *Journal of Statistical Computation and Simulation*, **55**, 57-71.

Herd, G. R. (1956), Estimation of the parameters of a population from a multi-censored sample, *Retrospective Theses and Dissertations*, http://lib.dr.iastate.edu/rtd/12873.

Huzurbazar, V. S. (1947), The Likelihood equation, Consistency and the Maxima of the Likelihood function, *Annals of Eugenics*, **14**(1), 185-200

Kamps, U., and Cramer, E. (2001), On distributions Of generalized order statistics, *Statistics*, **35**(3), 269-280.

Krishna, H., and Kumar, K. (2011), Reliability estimation in Lindley distribution with progressively type {II} right censored sample, *Mathematics and Computers in Simulation*, **82**(2), 281 - 294.

Kundu, D. and Joarder, A. (2006), Analysis of Type-II Progressively hybrid censored data, *Computational Statistics and Data Analysis*, **50**(10), 2509-2528.

Lin, C. T., Chou, C. C., and Huang, Y. L. (2012), Inference for the Weibull distribution with progressive hybrid censoring, *Computational Statistics and Data Analysis*, **56**(3), 451-467.

Mousa, M. A. M. A., and Jaheen, Z. F. (2002), Statistical inference for the Burr model based on progressively censored data, *Computers and Mathematics with Applications*, **43**(10), 1441-1449.

Ng, H. K. T., Chan, P. S., and Balakrishnan, N. (2002), Estimation of parameters from progressively censored data using EM algorithm, *Computational Statistics and Data Analysis*, **39**(4), 371-386.

Ranneby, B. (1984), The Maximum Spacing Method: An Estimation Method Related to the Maximum Likelihood Method, *Scand. J. Stat.*, **11**(2), 93-112.

Roberts, G. O., and Smith, A. F. M. (1994), Simple conditions for the convergence of the Gibbs sampler and Metropolis-Hastings algorithms, *Stochastic Processes and their Applications*, **49**(2), 207-216.

Shao, Y. (2001), Consistency of the Maximum Product of Spacings Method and Estimation of a Unimodal Distribution, *Statistica Sinica*, 1125-1140.

Shao, Y., and Hahn, M. G. (1999), Maximum product of spacings method: A unified formulation with illustration of Strong Consistency, *Illinois Journal of Mathematics*, **43**(3), 489-499.

Sharma, V. K., Singh, S. K., Singh, U., and Agiwal, V. (2015), The Inverse Lindley Distribution : A Stress-Strength Reliability model with Application to Head and Neck cancer Data, *J. Ind. Prod. Eng.*, **32**(3), 162-173.

Sharma, V. K., Singh, S. K., Singh, U., and Merovci, F. (2016), The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub data, *Communications in Statistics - Theory and Methods*, **45**(19), 5709-5729.

Singh, S. K., Singh, U., and Sharma, V. K. (2013), Expected Total Test Time and Bayesian Estimation for Generalized Lindley Distribution Under Progressively Type-II Censored Sample Where Removals Follow the Beta-binomial Probability Law, *Appl. Math. Comput.*, **222**, 402-419.

Sinha, S. K. (1986), Bayesian estimation of the reliability function of the inverse gaussian distribution, *Statistics and Probability Letters*, **4**(6), 319 -323.

Singh, S. K., Singh, U., and Kumar, M. (2013), Estimation of Parameters of Generalized Inverted Exponential distribution for progressive Type-II censored sample with Binomial removals, *Journal of Probability and Statistics*, **2013**, 1-12.

Singh, R. K., Singh, S. K., and Singh, U. (2016), Maximum product spacings method for the estimation of parameters of generalized inverted exponential distribution under Progressive Type II Censoring, *Journal of Statistics and Management Systems*, **19**(2), 219-245.

Siu, K. T., and Hak-Keung, Y. (1998), Expected experiment times for the Weibull distribution under progressive censoring with random removals, *Journal of Applied Statistics*, **25**(1), 75-83.

Siu, K. T., Chunyan, Y. and Hak-Keung, Y. (2000), Statistical analysis of Weibull distributed lifetime data under Type II progressive censoring with binomial removals, *Journal of Applied Statistics*, **27**(8), 1033-1043.

Upadhyay, S. K., and Peshwani, M. (2003), Choice Between Weibull and Lognormal Models: A Simulation Based Bayesian Study, *Communications in Statistics - Theory and Methods*, **32**(2), 381-405.

Wu, S. J., Chen, Y. J., and Chang, C. T. (2007), Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, *Journal of Statistical Computation and Simulation*, **77**(1), 19-27.

Appendix

$$F'_\theta(x)=\frac{\partial}{\partial\theta}F(x)=-\frac{e^{-\frac{\theta}{x^\alpha}}}{x^{2\alpha}(1+\theta)^2}\left\{\theta^2(x^\alpha+1)+\theta(2x^\alpha+1)\right\};$$

$$f'_\theta(x)=\frac{\partial}{\partial\theta}f(x)=\frac{\alpha\theta(x^\alpha+1)\left(2x^\alpha-\theta^2-\theta[1-x^\alpha]\right)\mathrm{e}^{-\frac{\theta}{x^\alpha}}}{x^{3\alpha+1}(\theta+1)^2};$$

$$F'_\alpha(x)=\frac{\theta^2 logx(x^\alpha+1)\mathrm{e}^{-\frac{\theta}{x^\alpha}}}{(\theta+1)x^{2\alpha}};$$

$$f'_\alpha(x)=\frac{\theta^2\left\{x^{2\alpha}+x^\alpha-\alpha logx\left(x^{2\alpha}+(2-\theta)x^\alpha-\theta\right)\right\}\mathrm{e}^{-\frac{\theta}{x^\alpha}}}{(\theta+1)x^{3\alpha+1}};$$

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}\left\{log(\overline{F}(x;\alpha,\theta))\right\}=&\frac{(x^\alpha+1)\theta}{x^\alpha(\theta+1)\left((x^\alpha\theta+x^\alpha)\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(x^\alpha+1)\theta-x^\alpha\right)}\\&+\frac{(x^\alpha+1)\theta+2x^\alpha+1}{x^\alpha(\theta+1)\left((x^\alpha\theta+x^\alpha)\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(x^\alpha+1)\theta-x^\alpha\right)}\\&-\frac{\theta((x^\alpha+1)\theta+2x^\alpha+1)}{x^\alpha(\theta+1)^2\left((x^\alpha\theta+x^\alpha)\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(x^\alpha+1)\theta-x^\alpha\right)}\\&-\frac{\theta((x^\alpha+1)\theta+2x^\alpha+1)\left(\frac{(x^\alpha\theta+x^\alpha)\mathrm{e}^{-\frac{\theta}{x^\alpha}}}{x^\alpha}+x^\alpha\mathrm{e}^{-\frac{\theta}{x^\alpha}}-x^\alpha-1\right)}{x^\alpha(\theta+1)\left((x^\alpha\theta+x^\alpha)\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(x^\alpha+1)\theta-x^\alpha\right)^2};\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\left\{log(\overline{F}(x;\alpha,\theta))\right\}=&\frac{\theta^2(logx)^2}{(\theta+1)x^\alpha\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(\theta+1)x^\alpha-\theta}\left\{\frac{1}{x^\alpha}\right\}\\&+\frac{\theta^2 logx(x^\alpha+1)\left(\mathrm{e}^{-\frac{\theta}{x^\alpha}}(\theta+1)x^\alpha logx-\theta\mathrm{e}^{-\frac{\theta}{x^\alpha}}(\theta+1)logx+(\theta+1)x^\alpha logx\right)}{x^\alpha\left((\theta+1)x^\alpha\mathrm{e}^{-\frac{\theta}{x^\alpha}}-(\theta+1)x^\alpha-\theta\right)^2};\end{aligned}$$

$$K_1=\int_\theta\frac{\alpha^{\omega+s-1}\cdot\theta^{2\omega+b-1}}{(1+\theta)^\omega}e^{-(\sum_{i=1}^{\omega}\frac{\theta}{x_i^\alpha}+\theta a+c\alpha)}\left[\overline{F}(\tau_0)\right]^{r^*_\omega}\prod_{i=1}^{\omega}\left\{\left(\frac{1+x_i^\alpha}{x_i^{2\alpha+1}}\right)\left[\overline{F}(x_i)\right]^{r_i}\right\}d\theta;$$

$$K_2=\int_\alpha\frac{\alpha^{\omega+s-1}\cdot\theta^{2\omega+b-1}}{(1+\theta)^\omega}e^{-(\sum_{i=1}^{\omega}\frac{\theta}{x_i^\alpha}+\theta a+c\alpha)}\left[\overline{F}(\tau_0)\right]^{r^*_\omega}\prod_{i=1}^{\omega}\left\{\left(\frac{1+x_i^\alpha}{x_i^{2\alpha+1}}\right)\left[\overline{F}(x_i)\right]^{r_i}\right\}d\alpha;$$

$$K_3=\int_\theta\theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha}\cdot\prod_{i=1}^{\omega}D_i\,d\theta;$$

$$K_4=\int_\theta\theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha}\cdot D_\xi\cdot\prod_{i=1}^{\omega}D_i\,d\theta;$$

$$K_5=\int_\theta\theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha}\cdot f(\tau)\cdot\prod_{i=1}^{\omega}D_i\,d\theta;$$

$$K_6 = \int_{\alpha} \theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha} \cdot \prod_{i=1}^{\omega} D_i \, d\alpha;$$

$$K_7 = \int_{\alpha} \theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha} \cdot D_{\xi} \cdot \prod_{i=1}^{\omega} D_i \, d\alpha;$$

$$K_8 = \int_{\alpha} \theta^{b-1}\alpha^{s-1}e^{-a\theta-c\alpha} \cdot f(\tau) \cdot \prod_{i=1}^{\omega} D_i \, d\alpha.$$