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Inference on Quantiles of Several Exponential Populations with a Common Location: Hypothesis Testing and Interval Estimation

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Abstract. This article deals with the problems of testing the hypothesis and interval estimation of the p-th quantile, $\xi = \mu + \eta\sigma_1$, where $\eta = -\log(1-p), (0 < p < 1)$ of the first population when samples are available from several exponential populations with a common location and possibly different scales. Several test procedures, such as tests using a generalized variable approach, tests based on parametric bootstrap method, and tests using a computational approach to test the null hypothesis against a suitable alternative, have been proposed. Besides several interval estimators for the quantile ξ, such as confidence intervals based on generalized variable approach, parametric bootstrap approach and Bayesian intervals using Markov chain Monte Carlo (MCMC) method have been suggested. The confidence intervals are compared through their coverage probabilities and average lengths, whereas the test statistics are compared in terms of powers and sizes numerically. The application of our model problem has been shown using real-life data sets, and conclusions have been made there.

Keywords. Average Length, Coverage Probability, Generalized Variable Method, Maximum Likelihood Estimator (MLE), Parametric Bootstrap Method, Computational Approach Test (CAT), Markov Chain Monte Carlo (MCMC) Method, Numerical Comparison, Power and Size of a Test.

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1 Introduction

Applications of exponential quantiles are seen in the study of reliability, life testing, survival analysis and related areas. Quantiles are also useful for comparing several groups or populations. One may refer to Epstein, B. (1962), Albers and Lohnberg (1984) and Rudolfer and Campbell (1985) for some practical applications of quantiles in various fields of study.

Suppose we have $\Pi_1, \Pi_2, \ldots, \Pi_k$ exponential populations with a common location parameter μ and possibly different scale parameters $\sigma_1, \sigma_2, \ldots, \sigma_k$ respectively. Specifically, let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ be a random sample from the i-th exponential population $\Pi_i \sim \text{Exp}(\mu, \sigma_i); i = 1, 2, \ldots, k$. The population $\text{Exp}(\mu, \sigma_i)$ has the probability density function

$$f(x|\mu, \sigma_i) = \frac{1}{\sigma_i} \exp\Big\{ -\Big(\frac{x-\mu}{\sigma_i}\Big)\Big\},\ x \geq \mu,\ \sigma_i > 0,\ i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i. \quad (1.1)$$

The parameter μ is also known as the 'minimum guarantee time' and σ_i denotes the 'residual life time' in the context of reliability and life testing experiments.

The main focus of this present study can be broadly divided into two major parts. In first part, we test the hypothesis regarding the p-th quantile of the first population, that is, $\xi = \mu + \eta\sigma_1$, where $\eta = -\log(1-p) > 0; 0 < p < 1$. Mathematically, we state the problem as, test the null hypothesis

$$H_0 : \xi = \xi_0 \text{ against } H_a : \xi \neq \xi_0, \quad (1.2)$$

where ξ_0 is a known constant for a given value of η. In the second part, the target will be to obtain several confidence intervals for the quantile $\xi = \mu + \eta\sigma_1$, utilizing some of the existing methodologies. Note that, in the first part, the other one-sided alternatives, such as $H_a^* : \xi > \xi_0$ or $H_a^{'} : \xi < \xi_0$ can also be considered and similar types of results may be derived by doing little modification in the procedures that we have adopted in this paper.

Inference on quantiles (mostly point estimation) in the case of shifted exponential distribution has been a focus of interest by several researchers due to its real-life applications and the theoretical challenges involved in it. In this connection, several authors have investigated the problem from classical and decision-theoretic points of view and derived some nice theoretical results. In the case of a single shifted exponential population, Rukhin and Strawderman (1982) and Rukhin (1986) derived certain decision-theoretic results, such as admissibility and minimaxity of the best affine equivariant estimator for the quantile.

When samples are available from two or more shifted exponential populations with a common location and different scale parameters, Sharma and Kumar (1994), and Kumar and Sharma (1996) derived certain decision-theoretic results, such as proving inadmissibility of the best affine equivariant estimators, improving upon the maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE) on estimating the quantiles of the first population. Jin and Crouse (1998) considered the same model problem, and using an identity, they compared the performances of the MLE and the UMVUE of the quantiles. For some related results and review on estimating exponential quantiles, we refer readers to Jena and Tripathy (2019), and the references cited therein. Malekzadeh and Jafari (2018) considered the problem of testing equality of quantiles for several shifted exponential populations. Malekzadeh and Kharrati-Kopaei (2020) considered the simultaneous interval estimation of differences of quantiles using progressive type-II censoring samples from shifted exponential populations. The interval estimation of quantiles for a single shifted exponential population has been considered by Balakrishnan et al. (2015), Krishnamoorthy and Xia (2018) and Malekzadeh (2022). Inference of quantiles in the case of normal distribution also has been considered by some researchers in the literature. One may refer to Nagamani and Tripathy (2020), Khatun et al. (2020) and the references cited therein for some recent updates on the inference on normal quantiles.

Under the current model set-up, Sharma and Kumar (1994) considered the point estimation of the quantile $\xi = \mu + \eta\sigma_1$ and obtained certain decision-theoretic results. To the best of our knowledge, the problems of hypothesis testing and interval estimation of the quantile $\xi = \mu + \eta\sigma_1$ have not been studied so far in the literature. In this study, our target is to cover these two aspects. Let us consider a practical situation where modeling of the problem leads to inferring the quantile of several exponential populations with a common location and different scale parameters. Suppose there are several brands of cellphones to be launched in the market and the lifetimes of each of the cellphones follow exponential distributions. It is pretty evident that due to the market competition, the minimum guarantee times of each of the brands of cellphones will remain the same. In contrast, their mean residual lifetimes will vary due to several factors such as methodologies used, the durability of the materials used, the expertise of the human resources etc. It is essential to test the mean lifetime or, in general, the quantile of any one of the brands in order to launch the new product.

The current research problem is interesting and also challenging in the sense that the information available for inferring the 'common location parameter' μ through the sufficient statistics of $(\mu, \sigma_1, \ldots, \sigma_k)$ can be utilized effectively to infer the 'quantile' $\xi = \mu + \eta\sigma_1$. Note that when $\eta = 0$, the problem reduces to the problem of inference on the 'common location parameter' μ only and has been considered by several researchers in the past from classical and decision-theoretic viewpoints. Below we give a brief review on the problem of inference on μ when the samples from two or more shifted exponential populations are available. The problem of point estimation of μ was first considered by Ghosh and Razmpour (1984) from a classical point of view. They derived various point estimators, such as the MLE, a modified version of the MLE (known as modified MLE) and the UMVUE when the scale parameters are unknown. They have also numerically compared the performances of these three estimators in terms of mean squared error. For some more results on point estimation of μ, we refer readers to Jin and Crouse (1998), Tripathy et al. (2014) and the references cited therein.

Researchers have also done some studies on hypothesis testing and interval estimation of the common location parameter μ under the current model assumption. Probably, Gunasekera (2009) was the first to consider the problem of testing μ when several shifted exponential populations are available with unknown scale parameters. The author derived certain test procedures using the generalized p-value approach, which was introduced by Tsui and Weerahandi (1989). Later on, Chang et al. (2013) showed that there does not exist any exact test procedure to test the common location parameter when the scale parameters are unknown. The authors proposed three test statistics using the likelihood function and the point estimators of the common location parameter μ. Moreover, they used the parametric bootstrap method along with the likelihood function to obtain the cut-off point for their proposed tests. Malekzadeh and Kharrati-Kopaei (2017) proposed an exact test procedure to test the common location parameter using complete as well as censored samples when the scale parameters are unknown.

The major contribution of the current article can be presented in the following manner. In Section 2, we discuss several test procedures for testing the null hypothesis $H_0 : \xi = \xi_0$ against the alternative hypothesis $H_1 : \xi \neq \xi_0$. Particularly, in Subsection 2.1, we introduce the concept of generalized variable approach and using it, several test statistics have been proposed. In Subsection 2.2, we derive test statistics using the popular parametric bootstrap method and the likelihood ratio. In Subsection 2.3, the computational approach test (CAT) and its modified version have been used to derive certain test statistics. Besides, a comprehensive simulation study has been carried out Section 2.4 to compare the performances of all the test statistics in terms of powers and sizes. Section 3 is dedicated to confidence interval of the quantile $\xi = \mu + \eta\sigma_1$. Unlike the case of the common location parameter, here, we do not have the luxury to obtain any exact confidence intervals. In Subsection 3.1, we propose some generalized confidence intervals using the generalized variable method and some of the popular point estimators of the common location μ and scale parameters σ_i; $i = 1, 2, \ldots, k$. Subsection 3.2 is devoted to the parametric bootstrap approach for deriving bootstrap confidence intervals. In Subsection 3.3, we consider a Bayesian approach, such as the Markov chain Monte Carlo (MCMC) procedure, to derive the confidence interval of the quantile. In Subsection 3.4, we discuss the performances of all the interval estimators in terms of their coverage probabilities (CPs) and average lengths (ALs). In Section 4, we discuss the application of the current model problem with the help of a real-life example and conclude the remarks.

Remark 1. It may be noted that, in the current setup of several populations, with a common location with unknown and possibly unequal scales, the quantile inference is quite different from the one for a single population. In the case of a single population, a typical quantile estimator involves an estimator of location parameter μ and an estimator of the scale parameter σ which are statistically independent, but this is not true in the case of a common location set up, where the estimator of μ depends on the estimator of $\sigma_i; i = 1, 2, \ldots, k$, as the former includes the latter (except the MLE). Further, the estimator of σ_1 involves the estimators of $\sigma_i; i = 2, 3, \ldots, k$ which are not statistically independent. Moreover, the sufficient statistics for the two models are quite different, which has a major impact on the inference procedures. However, the techniques we have used (generalized variable, CAT, boot-p, boot-t, etc.) are surely extensions of their one population counterparts.

2 Testing of Hypothesis Regarding the Quantile $\xi = \mu + \eta\sigma_1$

In this section, we discuss various test procedures in order to test the null hypothesis H_0 against the alternative H_a regarding the quantile $\xi = \mu + \eta\sigma_1$. We note that all the test procedures are not exact and have been obtained computationally.

2.1 Test Using Generalized Variable Approach

In this subsection, we introduce the generalized variable for testing the hypothesis (1.2) which was proposed by Tsui and Weerahandi (1989). This method has been successfully used by Malekzadeh and Jafari (2018) for comparing the quantiles of several exponential populations.

In order to derive the generalized test statistics for testing the hypothesis (1.2), one needs to obtain the generalized test variable, and then the generalized p-value. The following definitions will be useful in order to construct the generalized test variable as well as the generalized p-value, which we have taken from Tsui and Weerahandi (1989). Let us consider the hypothesis testing problem, say test the null hypothesis

$$H_0 : \gamma \leq \gamma_0 \quad \text{against} \quad H_a : \gamma > \gamma_0, \tag{2.1}$$

where γ_0 is a known constant.

Definition 2.1. Suppose Y is a random variable whose distribution depends only on (γ, ζ), where γ is the parameter of interest and ζ is the nuisance parameter involved in the distribution. A variable $H = H(Y; y, \gamma, \zeta)$ is said to be a generalized test variable for testing the hypothesis (2.1), if the following conditions are satisfied.

(i) The distribution of $H = H(Y; y, \gamma, \zeta)$ is free from the nuisance parameter ζ for a given value of Y.

(ii) The value of $H = H(Y; y, \gamma, \zeta)$ is free from any unknown parameters when $Y = y$ is fixed.

(iii) For fixed Y and ζ, the distribution of $H = H(Y; y, \gamma, \zeta)$ is either stochastically increasing or decreasing as a function of γ. That is, $P(H \geq h : \gamma)$ is an increasing or decreasing function of γ, for any $h \in \mathbb{R}$ and for fixed Y and ζ.

The following definition gives the concept of generalized p-value for a given generalized test variable.

Definition 2.2. Let $h = H(y; y, \gamma, \zeta)$ be the value of H for fixed $Y = y$. If H is stochastically increasing in γ, then the generalized p-value for testing the hypothesis (2.1) is obtained as

$$\sup_{H_0} P\{H(X; x, \gamma, \zeta) \geq h\} = P\{H(Y; y, \gamma_0, \zeta) \geq h\}, \tag{2.2}$$

and if $H(Y; y, \gamma, \zeta)$ is stochastically decreasing in γ, then the generalized p-value for testing the hypothesis (2.1) is given by

$$\sup_{H_0} P\{H(Y; y, \gamma, \zeta) \leq h\} = P\{H(Y; y, \gamma_0, \zeta) \leq h\}. \tag{2.3}$$

In the next subsection, we will construct certain generalized test variables in order to test the hypothesis (1.2) regarding the quantile $\xi = \mu + \eta\sigma_1$ and obtain the corresponding generalized p-values.

2.1.1 Generalized Test Variable for Testing the Quantile $\xi = \mu + \eta\sigma_1$

Let $(X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2, \ldots, k$ be the random sample of size n_i, taken from the i-th population $\text{Exp}(\mu, \sigma_i)$. Note that a complete and sufficient statistic for $(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k)$ in the current model problem is given by $(Z, T_1, T_2, \ldots, T_k)$ where the random variables Z and $T_i, i = 1, \ldots, k$ are defined as

$$X_i = \min_{1 \leq j \leq n_i}(X_{ij}), Z = \min_{1 \leq i \leq k}(X_i),\ T_i = \sum_{j=1}^{n_i}(X_{ij} - Z).$$

It is also noted that the statistics Z and $\underline{T} = (T_1, T_2, \ldots, T_k)$ are stochastically independent. The probability density functions of Z and $(T_1, T_2, \ldots, T_k)$ are given respectively by

$$f_Z(z) = a\exp\Big\{-a(z - \mu)\Big\},\ z > \mu,\ -\infty < \mu < \infty, \tag{2.4}$$

and

$$f_{\underline{T}}(\underline{t}) = \frac{1}{a}\Big[\sum_{i=1}^{k}\frac{n_i(n_i - 1)}{t_i}\Big]\Big[\prod_{i=1}^{k}\frac{t_i^{n_i - 1}e^{-t_i/\sigma_i}}{\Gamma(n_i)\sigma_i^{n_i}}\Big],\ t_i > 0, \tag{2.5}$$

where $a = \sum_{i=1}^{k}\frac{n_i}{\sigma_i}$.

In order to construct the generalized test variables for testing the quantile $\xi = \mu + \eta\sigma_1$, we define the following random variables which are slight variations of the complete and sufficient statistics. Let us define

$$X_i = \min_{1 \leq j \leq n_i} X_{ij},\ \text{and}\ S_i = \sum_{j=1}^{n_i}(X_{ij} - X_i) = n_i(\bar{X}_i - X_i),\ \text{where}\ \bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i}X_{ij}.$$

The random variable $Q_i = 2S_i/\sigma_i \sim \chi^2_{2n_i - 2}$, a chi-square distribution with degrees of freedom $2n_i - 2$. Further the random variable $V_i = (2n_i(X_i - \mu))/\sigma_i \sim \chi^2_2$ where

$n_i(X_i - \mu) \sim \text{Exp}(0, \sigma_i)$; $i = 1, 2, \ldots, k$ (see Hsieh (1986)). Utilizing these results it can be seen that the random variable $P_i = V_i/Q_i$ follows an F-distribution with degrees of freedom 2 and $2n_i - 2$. Moreover, the random variable $a(Z - \mu)$ follows an exponential distribution with scale parameter 1, so that the random variable $U = 2a(Z - \mu)$ follows a chi-square distribution with degrees of freedom 2.

Utilizing the above results, and the observed sample values $(x_{i1}, x_{i2}, \ldots, x_{in_i})$ from the i-th population, we define the generalized pivot variable (generalized p-value) for the common location parameter μ as

$$W_i = x_i - \frac{s_i P_i}{n_i(n_i - 1)}, \tag{2.6}$$

where s_i is the observed value of S_i. Taking the weighted average of all these W_is, with weights $2S_i/R_i$, where $R_i \sim \chi^2_{2n_i-2}$ and is independent from Q_i, for $i = 1, 2, \ldots, k$, we propose the generalized pivot variable for the quantile $\xi = \mu + \eta\sigma_1$ as

$$T_w = \frac{\sum_{i=1}^{k} \frac{2s_i}{R_i} W_i}{\sum_{i=1}^{k} \frac{2s_i}{R_i}} + \eta \frac{2s_1}{Q_1}. \tag{2.7}$$

The generalized test variable to test the hypothesis (1.2) on quantile can be defined using the statistic T_w as $G_w = T_w - \xi$. It can be easily observed that G_w satisfies the conditions (*i*) and (*ii*) of Definition 2.1. One can verify that, for a given sample $(x_{i1}, x_{i2}, \ldots, x_{in_i})$ and consequently s_i, the probability $P(T_w \geq r) = P(G_w \geq r - \xi)$ is stochastically decreasing as a function of ξ, for fixed $r \in \mathbb{R}$. Thus the generalized p-value for testing the hypothesis $H_0 : \xi = \xi_0$ against $H_a : \xi \neq \xi_0$ can be obtained as

$$2 \min(P(T_w \geq \xi_0), P(T_w \leq \xi_0)). \tag{2.8}$$

Note that, Ghosh and Razmpour (1984) proposed the MLE, the modified MLE and the UMVUE for the common location parameter μ. Utilizing these estimators, we will construct the generalized pivot variables for the quantile. The MLE for estimating the common location parameter μ is given by $\hat{\mu}_{ml} = Z$. Using this estimator, we define the generalized test variable for quantile ξ as

$$T_{ml} = \bar{\mu}_{ml} - \Big[\sum_{i=1}^{k} \frac{n_i Q_i}{s_i} \Big]^{-1} U + \eta \frac{2s_1}{Q_1}, \tag{2.9}$$

where $\bar{\mu}_{ml}$ is the observed value of $\hat{\mu}_{ml}$. Thus the generalized test variable for testing the hypothesis (1.2) is obtained as $G_{ml} = T_{ml} - \xi$, and the generalized p-value is computed as

$$2 \min(P(T_{ml} \geq \xi_0), P(T_{ml} \leq \xi_0)). \tag{2.10}$$

The modified MLE of the common location parameter μ is given by

$$\hat{\mu}_{mm} = Z - \frac{1}{\hat{a}_{ml}}, \text{ where } \hat{a}_{ml} = \sum_{i=1}^{k} \frac{n_i^2}{T_i}.$$

Utilizing the modified MLE $\hat{\mu}_{mm}$ of μ, we construct the generalized pivot variable for the quantile ξ as

$$T_{mm} = \bar{\mu}_{mm} - \Big[\sum_{i=1}^{k} \frac{n_i Q_i}{s_i}\Big]^{-1} U + \Big[\sum_{i=1}^{k} \frac{n_i}{\bar{x}_i - \Big[\sum_{i=1}^{k} \frac{n_i Q_i}{s_i}\Big]^{-1} U - \frac{1}{k}\sum_{i=1}^{k} W_i}\Big]^{-1} + \eta \frac{2 s_1}{Q_1}, \quad (2.11)$$

where $\bar{\mu}_{mm}$ is the observed value of $\hat{\mu}_{mm}$. The generalized test variable for testing the quantile ξ is thus given by $G_{mm} = T_{mm} - \xi$, and its p-value is obtained as

$$2\min(P(T_{mm} \geq \xi_0), P(T_{mm} \leq \xi_0)). \quad (2.12)$$

Finally, utilizing the UMVUE of the common location parameter μ given by

$$\hat{\mu}_{mv} = Z - \Big[\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{T_i}\Big]^{-1},$$

we propose the generalized pivot variable for the quantile ξ as

$$T_{mv} = \bar{\mu}_{mv} - \Big[\sum_{i=1}^{k} \frac{n_i Q_i}{s_i}\Big]^{-1} U + \Big[\sum_{i=1}^{k} \frac{n_i - 1}{\bar{x}_i - \Big[\sum_{i=1}^{k} \frac{n_i Q_i}{s_i}\Big]^{-1} U - \frac{1}{k}\sum_{i=1}^{k} W_i}\Big]^{-1} + \eta \frac{2 s_1}{Q_1}, \quad (2.13)$$

where $\bar{\mu}_{mv}$ denotes the observed value of $\hat{\mu}_{mv}$. Thus the generalized test variable for testing the hypothesis (1.2) is given by $G_{mv} = T_{mv} - \xi$. Thus the generalized p-value for this test variable can be obtained as

$$2\min(P(T_{mv} \geq \xi_0), P(T_{mv} \leq \xi_0)). \quad (2.14)$$

In all the above four generalized test variables, we reject the null hypothesis H_0 given in (1.2), if the p-values are less than the significance level α.

Remark 2. In order to make the presentation clean and also to make readers convenient, we discuss the corresponding generalized confidence intervals for the quantile ξ using the generalized pivot statistics in the Section 3.1 separately.

2.2 Parametric Bootstrap Method

In this section, we propose a parametric bootstrap method suggested by Chang et al. (2010), which has certain advantages over the usual likelihood ratio test statistics. In order to apply this method, we will first derive the likelihood function in our model to obtain the test statistics.

Note that the sufficient statistics for the underlying model is $(Z, T_1, T_2, \ldots, T_k)$. Thus the likelihood function using the sufficient statistics is given by

$$L((\mu, \sigma_1, \sigma_2, \ldots, \sigma_k)|(Z, T_1, T_2, \ldots, T_k)) = f_Z(z) h_{\underline{T}}(\underline{t}), \tag{2.15}$$

where $f_Z(z)$ and $h_{\underline{T}}(\underline{t})$ denote the probability density functions of Z and $\underline{T} = (T_1, T_2, \ldots, T_k)$ respectively, given in equations (2.4) and (2.5). In order to obtain the test statistics using likelihood ratio approach, one needs to maximize the likelihood function L under the null hypothesis H_0 and under the whole parameter space $\Theta = \{(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k), \mu \in R, \sigma_i > 0; i = 1, 2, \ldots, k\}$, then take the ratio. Thus the LRT statistics in our model is given by

$$\lambda = \frac{\sup_{H_0} L}{\sup_{\Theta} L}, \tag{2.16}$$

and after some simplification we get

$$\lambda = \Big(\frac{\hat{\sigma}_{1ml}}{\sigma_{10}}\Big)^{n_1} \prod_{i=2}^{k} \Big(\frac{\hat{\sigma}_{iml}}{\hat{\sigma}_{irm}}\Big)^{n_i} \exp\Big[-\hat{a}_{rm} A + T_1\Big(\frac{1}{\hat{\sigma}_{1ml}} - \frac{1}{\sigma_{10}}\Big) + \sum_{i=2}^{k} T_i\Big(\frac{1}{\hat{\sigma}_{iml}} - \frac{1}{\hat{\sigma}_{irm}}\Big)\Big], \tag{2.17}$$

where $\hat{\sigma}_{irm}$ and $\hat{a}_{rm}$ are the MLEs of $\sigma_i; i = 2, 3, \ldots, k$ and a, respectively, under the null hypothesis H_0 and $A = Z - \mu_0$. The MLEs under the null hypothesis are given by

$$\hat{\sigma}_{irm} = \frac{\sum_{j=1}^{n_i} (X_{ij} - \mu_0)}{n_i} = \frac{T_i}{n_i} + A, \text{ and } \hat{a}_{rm} = \frac{n_1}{\sigma_{10}} + \sum_{i=2}^{k} \frac{n_i}{\sigma_{irm}} = \frac{n_1}{\sigma_{10}} + \sum_{i=2}^{k} \frac{n_i^2}{T_i + A n_i}.$$

After substituting the MLEs of the parameters, the likelihood ratio statistic λ simplifies to

$$\lambda = \Big(\frac{T_1}{n_1 \sigma_{10}}\Big)^{n_1} \Big[\prod_{i=2}^{k} \Big(\frac{T_i}{T_i + A n_i}\Big)^{n_i}\Big] \exp\Big[-\Big(\frac{n_1}{\sigma_{10}} + \sum_{i=2}^{k} \frac{n_i^2}{T_i + A n_i}\Big) A + \sum_{i=1}^{k} n_i - \Big(\frac{T_1}{\sigma_{10}} + \sum_{i=2}^{k} \frac{n_i T_i}{T_i + A n_i}\Big)\Big]. \tag{2.18}$$

Taking logarithm on both sides of (2.18), we get the likelihood ratio statistics as

$$\lambda_1 = \log \lambda = n_1\Big[1 + \log\Big(\frac{T_1}{n_1 \sigma_{10}}\Big) - \frac{A}{\sigma_{10}}\Big] + \sum_{i=2}^{k} n_i \log\Big(\frac{T_i}{T_i + A n_i}\Big) - \frac{T_1}{\sigma_{10}}. \tag{2.19}$$

Note that, the exact distribution of the likelihood ratio statistic λ_1 under the null hypothesis is quite difficult to derive, due to its complicated structure. Moreover, we can not use the Chi-square distribution for $(-2\lambda_1)$ under H_0 (see Chang et al. (2013)). Motivated by the results of Chang et al. (2010), we employ the parametric bootstrap method to test the quantile $\xi = \mu + \eta \sigma_1$ here. In this method, one can find the suitable cut-off point numerically using simulation technique. The details of the parametric bootstrap method to test the quantile $\xi = \xi_0$ against $\xi \neq \xi_0$ consists of some algorithmic steps which we describe below.

Step-1: Using the original samples $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ from the exponential populations $\text{Exp}(\mu, \sigma_i)$; $i = 1, 2, \ldots, k$ compute the statistic λ_1 as given in (2.19).

Step-2: Under the null hypothesis H_0, generate $(X^*_{11}, X^*_{12}, \ldots, X^*_{1n_1})$ from $\text{Exp}(\mu_0, \sigma_{10})$ and $(X^*_{i1}, X^*_{i2}, \ldots, X^*_{in_i})$ from $\text{Exp}(\mu_0, \hat{\sigma}_{irm})$; $i = 2, 3, \ldots, k$, a large number of times, say B times, where $\hat{\sigma}_{irm}$ is the MLE of σ_i under H_0. Then compute λ_1 based on these sample values and denote it as λ^*_1.

Step-3: Arrange the values of λ^*_1 in the increasing order as $\lambda^*_{1(1)} \leq \lambda^*_{1(2)} \leq \ldots \leq \lambda^*_{1(B)}$. Then define the lower and upper cut-off points as $\lambda^*_{1L} = \lambda^*_{1(\frac{\alpha}{2}B)}$ and $\lambda^*_{1U} = \lambda^*_{1(1-\frac{\alpha}{2}B)}$, where α is the level of significance.

Step-4: Accept the null hypothesis H_0, if λ_1 falls between the lower and upper cut-off points, that is, if $\lambda^*_{1L} < \lambda_1 < \lambda^*_{1U}$, otherwise reject H_0. The power of this test is

$$\beta_{B_{ml}} = P(\lambda_1 < \lambda^*_{1L} \bigcup \lambda_1 > \lambda^*_{1U}). \tag{2.20}$$

In a very similar manner, one can obtain the test statistics using the modified MLE and the UMVUE of the common location parameter μ in place of the MLE. Then utilizing those test statistics, the cut-off points and the powers can be easily obtained numerically for testing H_0 against H_a. The test statistics are obtained as

$$\lambda_2 = \lambda_1 + \hat{a}_{ml}(Z - \mu_{mm}), \text{ and } \lambda_3 = \lambda_1 + \hat{a}_{ml}(Z - \mu_{mv}), \tag{2.21}$$

where λ_2 and λ_3 denote the likelihood ratio test statistics obtained by using the modified MLE and the UMVUE of μ. The power of these two tests are respectively given by

$$\beta_{B_{mm}} = P(\lambda_2 < \lambda^*_{2L} \bigcup \lambda_2 > \lambda^*_{2U}), \text{ and } \beta_{B_{mv}} = P(\lambda_3 < \lambda^*_{3L} \bigcup \lambda_3 > \lambda^*_{3U}). \tag{2.22}$$

The upper and lower cut-off points λ^*_{2L}, λ^*_{3L} and λ^*_{2U}, λ^*_{3U} are obtained numerically in a very similar manner, using the parametric bootstrap method as described above.

2.3 The Computational Approach Test (CAT)

In this subsection, we apply the CAT for testing the hypothesis H_0 against H_a which was proposed by Pal et al. (2007). This method works as good as other exact methods, and is easy to apply. In this method, it is not required to obtain the exact distribution of the test statistics, however one needs to handle the computation carefully. Exploiting the superior computational facilities available in hand, we try to test the hypothesis (1.2) and compute the size and power of the test statistics numerically.

This method is applied by using both the MLE and the Modified MLE of the common location parameter μ. Using the Monte- Carlo simulation procedure, we test the hypothesis $H_0 : \xi = \xi_0$ against $H_a : \xi \neq \xi_0$. The details of the numerical algorithm is given below for computing the size and power of the test.

Step-1: Given the sample values $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ from $\text{Exp}(\mu, \sigma_i), i = 1, 2, \ldots, k$, compute the MLEs of the parameters μ and σ_i. Utilizing the MLEs of the parameters μ and σ_1, compute the MLE of the quantile $\xi = \mu + \eta\sigma_1$ as

$$\hat{\xi}_{ml} = \hat{\mu}_{ml} + \eta\hat{\sigma}_{1ml}.$$

Step-2: Generate the artificial sample values, say $(Y_{11}, Y_{12}, \ldots, Y_{1n_1})$ under the null hypothesis H_0, that is from $\text{Exp}(\mu_0, \sigma_{10})$. The artificial samples $(Y_{i1}, Y_{i2}, \ldots, Y_{in_i})$ are generated from the populations $\text{Exp}(\mu_0, \hat{\sigma}_{irm}); i = 2, 3, \ldots, k$, where $\hat{\sigma}_{irm}$ is the MLE of σ_i under the null hypothesis H_0.

Step-3: Using these artificial samples, compute the MLE of $\xi = \mu + \eta\sigma_1$ and denote it as $\hat{\xi}_{0ml} = \hat{\mu}_{0ml} + \eta\hat{\sigma}_{10ml}$.

Step-4: Repeat the Steps 2 and 3, a large number of times, say A times, and get the estimates as $\hat{\xi}_{01ml}, \hat{\xi}_{02ml}, \ldots, \hat{\xi}_{0Aml}$. Arrange these estimates in the increasing order as $\hat{\xi}_{0(1)ml} \le \hat{\xi}_{0(2)ml} \le \ldots \hat{\xi}_{0(A)ml}$.

Step-5: The lower and upper cut-off points are obtained as $\hat{\xi}_L = \hat{\xi}_{0(\frac{\alpha}{2}A)ml}$ and $\hat{\xi}_U = \hat{\xi}_{0((1-\frac{\alpha}{2})A)ml}$ respectively.

Step-6: Reject the null hypothesis H_0 if $\hat{\xi}_{ml} > \hat{\xi}_U$ or $\hat{\xi}_{ml} < \hat{\xi}_L$, otherwise accept it.

Step-7: The power of the test, denoted by $\beta_{C_{ml}}$, is computed as,

$$\beta_{C_{ml}} = P(\hat{\xi}_{ml} > \hat{\xi}_U \bigcup \hat{\xi}_{ml} < \hat{\xi}_L). \tag{2.23}$$

The above procedure can be slightly modified by considering the hypothesis

$$H_0^* : h(\xi) = (\xi - \xi_0)^2 = 0 \; vs \; H_a^* : h(\xi) > 0.$$

Let us call the modification of the test as modified computational approach test (MCAT). The details of the algorithm for applying it, can be described as follows.

Step-1: For given random samples, generate the artificial samples as discussed above and similarly compute $\hat{h}_0 = (\hat{\xi}_{0ml} - \xi_0)^2$ for a large number of times, say A times, to obtain $\hat{h}_{01}, \hat{h}_{02}, \ldots, \hat{h}_{0A}$ and then arrange these values in increasing order as $\hat{h}_{0(1)} \le \hat{h}_{0(2)} \le \ldots \le \hat{h}_{0(A)}$.

Step-2: Compute the statistic $\delta = (\hat{\xi}_{ml} - \xi_0)^2$. Then reject the null hypothesis H_0^* if $\delta > \hat{h}_{0((1-\alpha)A)}$, otherwise accept it.

Step-3: The power of the test MCAT, say $\beta_{MC_{ml}}$, can be computed as,

$$\beta_{MC_{ml}} = P(\delta > \hat{h}_{0((1-\alpha)A)}). \tag{2.24}$$

Further, using the MLE and the modified MLE of the common location parameter μ, we can construct the statistics for CAT and MCAT. The modified MLE of ξ is given by $\hat{\xi}_{mm} = \mu_{mm} + \eta\sigma_{1ml}$. Utilizing this we obtain the power of the test CAT as

$$\beta_{C_{mml}} = P(\hat{\xi}_{mm} > \hat{\xi}_{MU} \bigcup \hat{\xi}_{mm} < \hat{\xi}_{ML}), \tag{2.25}$$

where $\hat{\xi}_{MU}$ and $\hat{\xi}_{ML}$ are the upper and lower cut-off points of the test CAT. The power of the modified CAT, when the MLE of μ is replaced by the modified MLE of μ is computed as

$$\beta_{MC_{mml}} = P(\delta_M > \hat{h}_{M0((1-\alpha)A)}), \tag{2.26}$$

where $\delta_M = (\hat{\xi}_{mm} - \xi_0)^2$ is obtained using the original sample and $\hat{h}_{M0} = (\hat{\xi}_{0mm} - \xi_0)^2$ is obtained using the artificial sample.

Remark 3. All the four tests, discussed in this section are compared in terms of their sizes and powers numerically in Section 2.4.

2.4 Computational Results: Power and Size Comparison of Test Procedures

In this section a comprehensive simulation study has been carried out in order to compare the performances of all the proposed test procedures which had been derived in the previous subsections (Subsections 2.1-2.3) numerically using Monte-Carlo simulation method in terms of their sizes and powers.

It can be easily seen that, all the proposed test procedures for testing the hypothesis $H_0 : \xi = \xi_0$ against $H_a : \xi \neq \xi_0$, could not be obtained in closed forms, in the sense that the powers/sizes can not be obtained analytically. However, from an application point of view, it is quite necessary to compare their performances. In view of this and taking advantages of modern computational facilities, we try to compare the performances of all the proposed test procedures numerically in terms of power and size.

In order to evaluate and compare the performances of test procedures, 20,000 random samples each from the $k(\geq 2)$ exponential populations $\text{Exp}(\mu, \sigma_i)$ of sample sizes n_i; $i = 1, 2, \ldots, k$, have been generated using the 'R-software' (version 3.6.2). In our simulation study, we have considered $\mu_0 = 1$, $\sigma_{10} = 1$ and $\eta = -\log(1 - 0.95)$, (95-th quantile) so that $\xi_0 = 3.999$ for convenience. Note that all the test statistics are location invariant. The nominal level for testing the hypothesis is taken as $\alpha = 0.05$. The power of all the tests depends on sample sizes as well as $\rho_i = \sigma_i/\sigma_1; i = 2, 3, \ldots, k$. The effect of ρ_i on size has been seen by fixing σ_1 and varying $\sigma_i; i = 2, 3, \ldots, k$, from small to large, so that ρ_i varies from small to large.

The simulation study was conducted using $k = 2, 3, 4, 5$ populations and various combinations of sample sizes (n_i) and parameter ranges. However, for presentation purposes, we have reported the simulation results only for the case $k = 2$. The other simulation results will be available for readers on request to authors. While presenting the simulation results for $k = 2$ populations, we used the notation $\rho_2 = \rho$ in tables.

In the case of computational approach test procedures (that is CAT and MCAT) and parametric bootstrap test procedure, the number of replications have been taken as 2500, that is the values $A = B = 2500$. Also in the case of generalized variable approach test, the same number of replications for the inner loop has been considered. A high level of accuracy has been achieved in computing the size/power in the sense that the standard error of the simulation is approximately bounded above by 0.005.

The size and power of all the proposed tests have been computed for various combinations of sample sizes and parameter ranges. The criterion for choosing the tests in terms of their size values has been discussed in Xu and Tian (2017), Malekzadeh and Kharrati-Kopaei (2020), and Malekzadeh and Mahmoudi (2020). Accordingly, a lower bound of the 95% confidence interval for the estimated sizes is $0.05 - 1.96\sqrt{\frac{0.0475}{20000}}$. If the estimated size of a method is lower or upper than it, we conclude that the method is liberal or conservative, respectively. For illustration purposes, we have presented the sizes for some selected sample sizes and parameters in Table 1. The powers of some selected test procedures have been presented in Tables 2-3. In tables, corresponding to one choice of ρ or σ_2 there correspond eight values which present the sizes/powers of the test for eight combinations of sample sizes. Below we discuss the outcomes of our comprehensive simulation study, which we write in the forms of some remarks.

Remark 4. The 95% lower bound for estimated size of all the tests is 0.047. From our computational results as well as Table 1, it has been observed that the test G_w is neither liberal nor conservative. The generalized tests G_{mm} and G_{mv} are liberal, whereas the other tests are conservative. It is also observed that these two liberal tests, G_{mm} and G_{mv}, do not attain the size within 20% of the nominal level $\alpha = 0.05$, and hence have been excluded for comparing the performances in terms of powers.

In Tables 2-3 we have presented the powers of all the tests except G_{mm} and G_{mv} for some selected combinations of sample sizes and parameters. In Table 2, the power comparison has been done by varying the common location parameter μ and fixing σ_1, where as in Table 3, the same has been done by fixing μ and varying σ_1.

Remark 5. (a) All the three parametric bootstrap tests, B_{ml}, B_{mm}, and B_{mv}, have very similar performance in terms of power. Further, it is seen that these three tests come just after the generalized tests G_w and G_{ml}, when we vary μ and fix σ_1. However, when we fix μ and vary over σ_1 in order to vary ξ, the performance of the parametric bootstrap tests worsens in terms of power.

(b) The MCAT always perform better than the CAT. When we analyze further, we see that the MCAT as well as the CAT based on the MLE (M_{ml} and C_{ml}) and the modified MLE (M_{mm} and C_{mm}) have similar performances.

(c) The generalized test based on the MLE, that is G_{ml}, performs better than all the other tests in terms of powers for all combinations of samples sizes and parameters.

(d) A very similar type of pattern in terms of power/size has been seen for other combinations of sample sizes and parameters.

Remark 6. The simulation study also has been carried out for other choices of ξ_0 such as 99-th and 90-th quantile, however a similar pattern in terms of performances has been noticed, and thus the thus the computational results have not been presented here.

Table 1: Sizes of All the Proposed Tests for Various Combinations of Sample Sizes and $\alpha = 0.05$.

ρ	(n_1, n_2)	G_w	G_{ml}	G_{mm}	G_{mv}	B_{ml}	B_{mm}	B_{mv}	C_{ml}	C_{mm}	M_{ml}	M_{mm}
0.25	(5,5)	0.0470	0.0533	0.0425	0.0410	0.0475	0.0475	0.0475	0.0482	0.0484	0.0502	0.0496
	(15,15)	0.0491	0.0547	0.0431	0.0430	0.0529	0.0529	0.0529	0.0519	0.0522	0.0504	0.0511
	(25,25)	0.0463	0.0507	0.0396	0.0395	0.0506	0.0506	0.0506	0.0501	0.0501	0.0507	0.0515
	(40,40)	0.0516	0.0542	0.0447	0.0448	0.0492	0.0492	0.0492	0.0530	0.0530	0.0536	0.0529
	(5,10)	0.0488	0.0542	0.0425	0.0419	0.0496	0.0496	0.0499	0.0490	0.0490	0.0510	0.0515
	(15,25)	0.0492	0.0579	0.0500	0.0497	0.0502	0.0502	0.0505	0.0510	0.0510	0.0485	0.0481
	(10,5)	0.0468	0.0565	0.0365	0.0353	0.0490	0.0490	0.0482	0.0474	0.0473	0.0479	0.0478
	(25,15)	0.0501	0.0511	0.0364	0.0364	0.0548	0.0548	0.0548	0.0486	0.0486	0.0494	0.0492
0.50	(5,5)	0.0432	0.0533	0.0343	0.0334	0.0519	0.0519	0.0519	0.0534	0.0533	0.0516	0.0520
	(15,15)	0.0472	0.0489	0.0352	0.0345	0.0532	0.0532	0.0532	0.0526	0.0525	0.0523	0.0528
	(25,25)	0.0486	0.0487	0.0347	0.0344	0.0497	0.0497	0.0497	0.0530	0.0530	0.0528	0.0528
	(40,40)	0.0485	0.0476	0.0363	0.0359	0.0514	0.0514	0.0514	0.0535	0.0534	0.0529	0.0536
	(5,10)	0.0463	0.0518	0.0416	0.0407	0.0523	0.0523	0.0525	0.0493	0.0495	0.0519	0.0517
	(15,25)	0.0483	0.0511	0.0409	0.0406	0.0507	0.0507	0.0509	0.0489	0.0489	0.0457	0.0456
	(10,5)	0.0426	0.0538	0.0321	0.0313	0.0470	0.0470	0.0459	0.0518	0.0513	0.0508	0.0505
	(25,15)	0.0480	0.0510	0.0341	0.0339	0.0536	0.0536	0.0537	0.0508	0.0508	0.0536	0.0519
1.00	(5,5)	0.0443	0.0553	0.0342	0.0337	0.0501	0.0501	0.0501	0.0530	0.0531	0.0526	0.0532
	(15,15)	0.0470	0.0501	0.0350	0.0347	0.0477	0.0477	0.0477	0.0487	0.0489	0.0492	0.0482
	(25,25)	0.0450	0.0472	0.0306	0.0306	0.0523	0.0523	0.0523	0.0508	0.0505	0.0521	0.0522
	(40,40)	0.0503	0.0526	0.0389	0.0384	0.0528	0.0528	0.0528	0.0478	0.0478	0.0478	0.0476
	(5,10)	0.0464	0.0520	0.0376	0.0371	0.0513	0.0513	0.0522	0.0483	0.0481	0.0485	0.0481
	(15,25)	0.0469	0.0501	0.0329	0.0330	0.0488	0.0488	0.0490	0.0486	0.0489	0.0489	0.0492
	(10,5)	0.0436	0.0523	0.0395	0.0389	0.0514	0.0514	0.0505	0.0469	0.0468	0.0455	0.0453
	(25,15)	0.0460	0.0545	0.0368	0.0369	0.0510	0.0510	0.0509	0.0503	0.0503	0.0506	0.0507
2.00	(5,5)	0.0452	0.0563	0.0384	0.0366	0.0488	0.0488	0.0488	0.0521	0.0516	0.0524	0.0536
	(15,15)	0.0463	0.0490	0.0321	0.0323	0.0474	0.0474	0.0474	0.0492	0.0495	0.0503	0.0504
	(25,25)	0.0519	0.0499	0.0332	0.0333	0.0517	0.0517	0.0517	0.0474	0.0472	0.0475	0.0472
	(40,40)	0.0520	0.0470	0.0371	0.0367	0.0493	0.0493	0.0493	0.0541	0.0543	0.0517	0.0524
	(5,10)	0.0465	0.0540	0.0349	0.0341	0.0497	0.0497	0.0497	0.0529	0.0529	0.0520	0.0515
	(15,25)	0.0479	0.0471	0.0323	0.0320	0.0501	0.0501	0.0503	0.0509	0.0509	0.0516	0.0517
	(10,5)	0.0482	0.0522	0.0412	0.0404	0.0545	0.0545	0.0540	0.0511	0.0508	0.0491	0.0499
	(25,15)	0.0459	0.0474	0.0373	0.0373	0.0503	0.0503	0.0501	0.0500	0.0500	0.0484	0.0495
4.00	(5,5)	0.0492	0.0545	0.0410	0.0404	0.0489	0.0489	0.0489	0.0495	0.0492	0.0497	0.0485
	(15,15)	0.0508	0.0525	0.0429	0.0424	0.0520	0.0520	0.0520	0.0477	0.0477	0.0507	0.0491
	(25,25)	0.0536	0.0504	0.0413	0.0409	0.0494	0.0494	0.0494	0.0509	0.0510	0.0506	0.0512
	(40,40)	0.0517	0.0537	0.0429	0.0429	0.0567	0.0567	0.0567	0.0547	0.0547	0.0570	0.0564
	(5,10)	0.0457	0.0560	0.0346	0.0330	0.0484	0.0484	0.0491	0.0491	0.0489	0.0493	0.0495
	(15,25)	0.0523	0.0515	0.0347	0.0345	0.0544	0.0544	0.0546	0.0520	0.0523	0.0535	0.0523
	(10,5)	0.0471	0.0541	0.0434	0.0435	0.0514	0.0514	0.0505	0.0490	0.0490	0.0489	0.0485
	(25,15)	0.0520	0.0495	0.0443	0.0442	0.0527	0.0527	0.0526	0.0527	0.0527	0.0528	0.0525
5.00	(5,5)	0.0439	0.0493	0.0362	0.0356	0.0550	0.0550	0.0550	0.0507	0.0511	0.0515	0.0505
	(15,15)	0.0515	0.0496	0.0375	0.0377	0.0539	0.0539	0.0539	0.0518	0.0512	0.0509	0.0523
	(25,25)	0.0493	0.0482	0.0381	0.0381	0.0529	0.0529	0.0529	0.0570	0.0574	0.0567	0.0558
	(40,40)	0.0525	0.0498	0.0397	0.0397	0.0502	0.0502	0.0502	0.0509	0.0509	0.0523	0.0519
	(5,10)	0.0510	0.0554	0.0363	0.0358	0.0578	0.0578	0.0582	0.0503	0.0500	0.0509	0.0509
	(15,25)	0.0490	0.0461	0.0341	0.0343	0.0530	0.0530	0.0533	0.0506	0.0505	0.0517	0.0516
	(10,5)	0.0494	0.0484	0.0422	0.0415	0.0523	0.0523	0.0518	0.0506	0.0504	0.0509	0.0502
	(25,15)	0.0496	0.0514	0.0448	0.0451	0.0556	0.0556	0.0557	0.0523	0.0524	0.0522	0.0513

Remark 7. Though we have obtained the theoretical results for a general $k(\geq 2)$ population, it is not possible to provide simulation results for a general k. We have conducted the simulation study considering the number of populations up to $k = 5$ only. The conclusions regarding the performances of test statistics in terms of their power and size remain very similar even if we increase the number of populations from $k = 2$ to 5.

Table 2: Powers of Some Selected Tests for Various Combinations of Sample Sizes and $\alpha = 0.05, \sigma_1 = 1$

μ	(n_1, n_2)	G_w	G_{ml}	B_{ml}	B_{mm}	B_{mv}	C_{ml}	C_{mm}	M_{ml}	M_{mm}
1.05	(5,5)	0.0401	0.0457	0.0404	0.0404	0.0404	0.0552	0.0554	0.0560	0.0560
	(15,15)	0.0559	0.1111	0.0884	0.0884	0.0884	0.0519	0.0520	0.0541	0.0548
	(25,25)	0.1324	0.3226	0.2135	0.2135	0.2135	0.0482	0.0484	0.0516	0.0507
	(40,40)	0.4191	0.9425	0.6493	0.6493	0.6493	0.0498	0.0499	0.0515	0.0515
	(5,10)	0.0392	0.0564	0.0475	0.0475	0.0480	0.0518	0.0518	0.0544	0.0547
	(15,25)	0.0826	0.2010	0.1375	0.1375	0.1377	0.0506	0.0504	0.0520	0.0519
	(10,5)	0.0394	0.0586	0.0483	0.0483	0.0480	0.0512	0.0508	0.0535	0.0529
	(25,15)	0.0787	0.1972	0.1387	0.1387	0.1387	0.0548	0.0544	0.0562	0.0569
1.10	(5,5)	0.0428	0.0764	0.0526	0.0526	0.0526	0.0434	0.0432	0.0532	0.0511
	(15,15)	0.1586	0.5323	0.3017	0.3017	0.3017	0.0491	0.0490	0.0527	0.0518
	(25,25)	0.6240	0.9962	0.9251	0.9251	0.9251	0.0528	0.0528	0.0562	0.0548
	(40,40)	0.9992	1.0000	1.0000	1.0000	1.0000	0.0539	0.0540	0.0549	0.0536
	(5,10)	0.0354	0.1263	0.0860	0.0860	0.0863	0.0444	0.0447	0.0516	0.0496
	(15,25)	0.2829	0.8994	0.6083	0.6083	0.6084	0.0506	0.0502	0.0554	0.0545
	(10,5)	0.0368	0.1324	0.0900	0.0900	0.0895	0.0509	0.0508	0.0535	0.0508
	(25,15)	0.2785	0.8978	0.6239	0.6239	0.6239	0.0535	0.0536	0.0566	0.0560
1.50	(5,5)	0.1648	0.8791	0.6679	0.6679	0.6679	0.0523	0.0531	0.0751	0.0704
	(15,15)	1.0000	1.0000	1.0000	1.0000	1.0000	0.0829	0.0852	0.0989	0.0949
	(25,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.1120	0.1136	0.1263	0.1219
	(40,40)	1.0000	1.0000	1.0000	1.0000	1.0000	0.1626	0.1650	0.1746	0.1700
	(5,10)	0.4755	0.9991	0.9860	0.9860	0.9871	0.0560	0.0569	0.0822	0.0790
	(15,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.0827	0.0844	0.1044	0.1005
	(10,5)	0.4353	0.9983	1.0000	1.0000	1.0000	0.0748	0.0764	0.0900	0.0829
	(25,15)	0.9999	1.0000	1.0000	1.0000	1.0000	0.1159	0.1191	0.1278	0.1210
2.00	(5,5)	0.6244	0.9993	1.0000	1.0000	1.0000	0.0894	0.0956	0.1276	0.1191
	(15,15)	1.0000	1.0000	1.0000	1.0000	1.0000	0.2012	0.2078	0.2316	0.2231
	(25,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.3201	0.3278	0.3480	0.3391
	(40,40)	1.0000	1.0000	1.0000	1.0000	1.0000	0.5056	0.5119	0.5293	0.5209
	(5,10)	0.8610	1.0000	1.0000	1.0000	1.0000	0.0821	0.0866	0.1227	0.1191
	(15,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.2019	0.2076	0.2360	0.2280
	(10,5)	0.8170	1.0000	1.0000	1.0000	1.0000	0.1422	0.1481	0.1649	0.1536
	(25,15)	1.0000	1.0000	1.0000	1.0000	1.0000	0.3322	0.3399	0.3525	0.3409
3.00	(5,5)	0.9744	1.0000	1.0000	1.0000	1.0000	0.2353	0.2574	0.3025	0.2895
	(15,15)	1.0000	1.0000	1.0000	1.0000	1.0000	0.6722	0.6902	0.7147	0.7072
	(25,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.9107	0.9178	0.9259	0.9225
	(40,40)	1.0000	1.0000	1.0000	1.0000	1.0000	0.9938	0.9939	0.9948	0.9942
	(5,10)	0.9916	1.0000	1.0000	1.0000	1.0000	0.2292	0.2463	0.3124	0.3055
	(15,25)	1.0000	1.0000	1.0000	1.0000	1.0000	0.6776	0.6925	0.7261	0.7188
	(10,5)	0.9843	1.0000	1.0000	1.0000	1.0000	0.4763	0.5022	0.5202	0.5057
	(25,15)	1.0000	1.0000	1.0000	1.0000	1.0000	0.9148	0.9223	0.9299	0.9254

Table 3: Powers of Some Selected Tests for Various Combinations of Sample Sizes and $\alpha = 0.05$, $\mu = 1$

σ_1	(n_1, n_2)	G_w	G_{ml}	B_{ml}	B_{mm}	B_{mv}	C_{ml}	C_{mm}	M_{ml}	M_{mm}
1.05	(5,5)	0.0527	0.1336	0.0526	0.0526	0.0526	0.0547	0.0542	0.0626	0.0593
	(15,15)	0.3720	0.9435	0.0539	0.0539	0.0539	0.0620	0.0620	0.0692	0.0672
	(25,25)	0.9779	1.0000	0.0527	0.0527	0.0527	0.0652	0.0651	0.0708	0.0688
	(40,40)	1.0000	1.0000	0.0595	0.0595	0.0595	0.0733	0.0733	0.0772	0.0754
	(5,10)	0.0529	0.2728	0.0538	0.0538	0.0546	0.0586	0.0586	0.0648	0.0636
	(15,25)	0.6378	0.9996	0.0514	0.0514	0.0512	0.0611	0.0611	0.0680	0.0659
	(10,5)	0.0436	0.2591	0.0519	0.0519	0.0505	0.0593	0.0592	0.0640	0.0619
	(25,15)	0.6622	0.9998	0.0559	0.0559	0.0559	0.0612	0.0609	0.0658	0.0636
1.10	(5,5)	0.0561	0.4422	0.0492	0.0492	0.0492	0.0663	0.0661	0.0782	0.0751
	(15,15)	0.9683	1.0000	0.0535	0.0535	0.0535	0.0773	0.0773	0.0865	0.0840
	(25,25)	1.0000	1.0000	0.0610	0.0610	0.0610	0.0914	0.0915	0.1009	0.0988
	(40,40)	1.0000	1.0000	0.0618	0.0618	0.0618	0.1117	0.1118	0.1206	0.1168
	(5,10)	0.1531	0.8591	0.0484	0.0484	0.0486	0.0667	0.0664	0.0807	0.0795
	(15,25)	0.9832	1.0000	0.0553	0.0553	0.0555	0.0792	0.0792	0.0924	0.0898
	(10,5)	0.1473	0.8348	0.0541	0.0541	0.0529	0.0701	0.0702	0.0810	0.0757
	(25,15)	0.9950	1.0000	0.0608	0.0608	0.0606	0.0915	0.0915	0.0996	0.0947
1.50	(5,5)	0.7457	0.9999	0.0750	0.0750	0.0750	0.2026	0.2018	0.2508	0.2424
	(15,15)	1.0000	1.0000	0.1721	0.1721	0.1721	0.4012	0.4009	0.4374	0.4262
	(25,25)	1.0000	1.0000	0.3009	0.3009	0.3009	0.5818	0.5815	0.6140	0.6026
	(40,40)	1.0000	1.0000	0.4819	0.4819	0.4819	0.7565	0.7565	0.7758	0.7704
	(5,10)	0.8239	1.0000	0.0820	0.0820	0.0822	0.2073	0.2068	0.2606	0.2559
	(15,25)	1.0000	1.0000	0.1772	0.1772	0.1775	0.4156	0.4152	0.4587	0.4515
	(10,5)	0.9715	1.0000	0.1228	0.1228	0.1214	0.3182	0.3175	0.3533	0.3397
	(25,15)	1.0000	1.0000	0.2977	0.2977	0.2977	0.5705	0.5702	0.5945	0.5828
2.00	(5,5)	0.9374	1.0000	0.1766	0.1766	0.1766	0.4229	0.4221	0.4848	0.4722
	(15,15)	1.0000	1.0000	0.5413	0.5413	0.5413	0.8016	0.8012	0.8290	0.8212
	(25,25)	1.0000	1.0000	0.7908	0.7908	0.7908	0.9398	0.9397	0.9476	0.9454
	(40,40)	1.0000	1.0000	0.9491	0.9491	0.9491	0.9913	0.9913	0.9924	0.9921
	(5,10)	0.9372	1.0000	0.1829	0.1829	0.1846	0.4318	0.4320	0.5043	0.4978
	(15,25)	1.0000	1.0000	0.5497	0.5497	0.5503	0.8028	0.8027	0.8302	0.8247
	(10,5)	0.9996	1.0000	0.3535	0.3535	0.3514	0.6437	0.6439	0.6782	0.6636
	(25,15)	1.0000	1.0000	0.7931	0.7931	0.7929	0.9403	0.9403	0.9464	0.9442
3.00	(5,5)	0.9750	1.0000	0.4645	0.4645	0.4645	0.7451	0.7448	0.7905	0.7816
	(15,15)	1.0000	1.0000	0.9334	0.9334	0.9334	0.9850	0.9851	0.9887	0.9877
	(25,25)	1.0000	1.0000	0.9948	0.9948	0.9948	0.9997	0.9997	0.9997	0.9996
	(40,40)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	(5,10)	0.9735	1.0000	0.4801	0.4801	0.4830	0.7509	0.7504	0.7997	0.7958
	(15,25)	1.0000	1.0000	0.9338	0.9338	0.9341	0.9872	0.9874	0.9903	0.9894
	(10,5)	1.0000	1.0000	0.7900	0.7900	0.7881	0.9332	0.9326	0.9439	0.9398
	(25,15)	1.0000	1.0000	0.9932	0.9932	0.9932	0.9992	0.9992	0.9995	0.9995

3 Interval Estimation of Quantile $\xi = \mu + \eta\sigma_1$

In this section, we derive certain confidence intervals for the quantile $\xi = \mu + \eta\sigma_1$. Note that, under the current model set up it is not possible to obtain the asymptotic confidence interval, since the regularity condition does not hold. Further, it is not possible to derive the intervals in closed forms. In this section, we utilize the generalized variable approach, bootstrap approach and the MCMC approach to obtain confidence intervals and hence evaluate their performances numerically.

3.1 Interval Estimation Using Generalized Variable Method

In this subsection, we discuss the generalized p-value method proposed by Tsui and Weerahandi (1989) and Weerahandi (1993) for constructing confidence interval as well as testing the hypothesis of a function of parameter(s). This method has been successfully employed by Krishnamoorthy and Lu (2003) for testing hypothesis and finding confidence interval of the common mean μ of several normal populations.

The following definition will be useful in order to construct a confidence interval of a parameter or any function of parameters in a given model problem set up.

Definition 3.1. Let X be any random variable and the distribution of X depends on (γ, ζ), where γ is the parameter of interest and ζ is the nuisance parameter. A random variable $T = T(X; x, \gamma, \zeta)$ is said to be a generalized pivot variable for obtaining the generalized confidence interval of γ, if it satisfies the following two conditions:

(i) The distribution of $T(X; x, \gamma, \zeta)$ is free from all the unknown parameters for a fixed $X = x$.

(ii) The value of $T(X; x, \gamma, \zeta)$ at $X = x$, is γ, that is, $T(X; x, \gamma, \zeta) = \gamma$ the parameter of interest.

Utilizing the Definition 3.1, and the generalized pivot variable T_w, we obtain the $(1-\alpha)100\%$ confidence interval for the quantile $\xi = \mu + \eta\sigma_1$ as

$$(T_w(\alpha/2), T_w(1-\alpha/2)). \tag{3.1}$$

In a similar manner, utilizing the generalized pivot variable T_{ml}, T_{mm} and T_{mv}, we can obtain the generalized confidence intervals respectively as

$$(T_{ml}(\alpha/2), T_{ml}(1-\alpha/2)), \tag{3.2}$$

$$(T_{mm}(\alpha/2), T_{mm}(1-\alpha/2)), \tag{3.3}$$

and

$$(T_{mv}(\alpha/2), T_{mv}(1-\alpha/2)). \tag{3.4}$$

3.2 Interval Estimation Using Parametric Bootstrap Method

In this section, we obtain two approximate confidence intervals, such as bootstrap percentile (boot-p) and bootstrap-t (boot-t) for the quantile ξ by using the bootstrap sampling method. The boot-p and boot-t intervals were proposed by Efron (1982) and Hall and Martin (1988), respectively. The algorithms to obtain the boot-p and boot-t confidence intervals are presented in the following subsection.

3.2.1 Bootstrap-p Confidence Interval

The details of the computational steps to construct the boot-p confidence interval for the quantile ξ can be written as follows.

Step-1: For given sample values $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ from the $k(\geq 2)$ exponential populations $\text{Exp}(\mu, \sigma_i)$, $i = 1, 2, \ldots, k$, compute the MLEs of the parameters μ, and σ_i as given in Section 2. Using these estimators, obtain the MLE of the quantile $\xi = \mu + \eta\sigma_1$ as $\hat{\xi}_{ml} = \hat{\mu}_{ml} + \eta\hat{\sigma}_{1ml}$.

Step-2: Generate bootstrap samples $(X^*_{i1}, X^*_{i2}, \ldots, X^*_{in_i})$ from $\text{Exp}(\hat{\mu}_{ml}, \hat{\sigma}_{iml})$; $i = 1, 2, \ldots, k$,, Compute the bootstrap MLEs for the parameters as $\hat{\mu}^*_{ml}$, $\hat{\sigma}^*_{1ml}$, $\hat{\sigma}^*_{2ml}$, $\ldots$, $\hat{\sigma}^*_{kml}$. Then, utilizing these MLEs of the parameters, compute the bootstrap MLE of the quantile ξ as

$$\hat{\xi}^*_{ml} = \hat{\mu}^*_{ml} + \eta\hat{\sigma}^*_{1ml}.$$

Step-3: Repeat Step-2 a large number times, say B times, and obtain the bootstrap estimates of ξ as $\hat{\xi}^*_{ml1}, \hat{\xi}^*_{ml2}, \ldots, \hat{\xi}^*_{mlB}$.

Step-4: Let $F(x) = P(\hat{\xi}^*_{ml} \leq x)$ be the cumulative distribution function of $\hat{\xi}^*$. Then, the approximate $(1-\alpha)100\%$ boot-p confidence interval of the quantile ξ is obtained as

$$BP_{ml} = (\hat{\xi}_{Boot-p}(\alpha/2), \hat{\xi}_{Boot-p}(1-\alpha/2)),$$

where $\hat{\xi}_{Boot-p}(x) = F^{-1}(x)$.

In a similar manner, we get the Bootstrap confidence interval using the modified MLE of the common location parameter μ as

$$BP_{mm} = (\hat{\xi}_{Boot-p}(\alpha/2), \hat{\xi}_{Boot-p}(1-\alpha/2)).$$

3.2.2 Bootstrap-t Confidence Interval

The details of the computational steps to construct the boot-t confidence interval for the quantile ξ can be written as follows.

Step-1: For the given sample values $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ from $k(\geq 2)$ exponential populations $\text{Exp}(\mu, \sigma_i)$; $i = 2, 3, \ldots, k$, compute the MLEs of the parameters μ, σ_1, $\sigma_2, \ldots, \sigma_k$ as given in Section 2. Using these MLEs, obtain the MLE of the quantile ξ as

$$\hat{\xi}_{ml} = \hat{\mu}_{ml} + \eta\hat{\sigma}_{1ml}.$$

Step-2: Generate the bootstrap samples $(X^*_{i1}, X^*_{i2}, \ldots, X^*_{in_i})$ from $\text{Exp}(\hat{\mu}_{ml}, \hat{\sigma}_{iml})$, $i = 2, 3, \ldots, k$, and using these sample values, compute the bootstrap MLEs of the parameters as $\hat{\mu}^*_{ml}$, $\hat{\sigma}^*_{1ml}$, $\hat{\sigma}^*_{2ml}, \ldots, \hat{\sigma}^*_{kml}$. Using the bootstrap MLEs of the parameters, compute the bootstrap MLE of the quantile ξ as

$$\hat{\xi}^*_{ml} = \hat{\mu}^*_{ml} + \eta\hat{\sigma}^*_{1ml}.$$

Step-3: Using the bootstrap MLE of the quantile ξ, compute the statistic $T^* = \frac{\hat{\xi}^*_{ml} - \hat{\xi}_{ml}}{\sqrt{Var(\hat{\xi}^*_{ml})}}$.

Step-4: Repeat Steps 2 and 3, a large number of times, say B times. Then construct the approximate $(1-\alpha)100\%$ confidence interval of the quantile ξ as

$$BT_{ml} = (\hat{\xi}_{Boot-t}(\alpha/2), \hat{\xi}_{Boot-t}(1-\alpha/2)),$$

where $\hat{\xi}_{Boot-t} = \hat{\xi}_{ml} + F^{-1}(x)\sqrt{Var(\hat{\xi}^*_{ml})}$ and $F(x) = P(T^* \leq x)$ is the cumulative distribution function of T^*.

In a similar manner, we get the confidence interval of the quantile ξ using the modified MLE of the common location parameter μ as

$$BT_{mm} = (\hat{\xi}_{Boot-t}(\alpha/2), \hat{\xi}_{Boot-t}(1-\alpha/2)).$$

3.3 Bayesian Confidence Interval Using MCMC Approach

In this subsection, we derive the confidence interval for the quantile $\xi = \mu + \eta\sigma_1$, assuming certain prior probability for the parameters μ, σ_1 and σ_2. Taking the advantages of superior computational facilities, we use the MCMC method along with the Metropolis-Hastings algorithm and Gibbs sampling technique and obtain the highest posterior density (HPD) interval for the quantile ξ.

In order to apply the method, we need to assume certain prior probabilities for the parameters μ, σ_1 and σ_2. We consider the prior probability density functions of the scale parameters and the common location parameter as suggested by Jana et al. (2016), which are respectively given by

$$p_i(\sigma_i) = \frac{c_i^{d_i}}{\Gamma(d_i)} \exp^{-\frac{c_i}{\sigma_i}} \sigma_i^{1-d_i},\ \sigma_i > 0, c_i, d_i > 0,\ i = 1, 2, \ldots, k, \tag{3.5}$$

and

$$p(\mu|(\sigma_1, \sigma_2, \ldots, \sigma_k)) = \Big(\sum_{i=1}^{k} \frac{1}{\sigma_i}\Big)^2 \exp -\Big(\sum_{i=1}^{k} \frac{1}{\sigma_i}\Big)(c - r\mu), -\infty < \mu < \frac{c}{r}. \tag{3.6}$$

The joint posterior density function of $(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k)$ is obtained as

$$\pi(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k) \propto L((\mu, \sigma_1, \sigma_2, \ldots, \sigma_k)|(Z, \underline{T})) \prod_{i=1}^{k} p_i(\sigma_i) p(\mu|(\sigma_1, \sigma_2, \ldots, \sigma_k)), \tag{3.7}$$

where $L(\mu, \sigma_1, \sigma_2, \ldots, \sigma_k|Z, \underline{T})$ denotes the likelihood function as given in Section 2. The conditional posterior probability density functions of the parameters μ, and σ_i, can be obtained as

$$\mu|(\sigma_1, \sigma_2, \ldots, \sigma_k, Z, \underline{T}) \propto \exp\Big(-\sum_{i=1}^{k} \frac{n_i Z - D_i\mu}{\sigma_i}\Big),$$

and

$$\sigma_i|(\mu,\sigma_1,\sigma_2,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_k Z,\underline{T}) \propto \Big(\sum_{i=1}^{k}\frac{1}{\sigma_i}\Big)^2 \sigma_i^{1-d_i-n_i}\exp\Big\{-\Big(\frac{N_i-D_i\mu}{\sigma_i}\Big)\Big\},\ i=1,2,\ldots,k.$$

respectively, where $D_i = n_i + r$, and $N_i = T_i + c_i + n_i Z + c$.

It is easy to observe that, the posterior densities of all the $(k+1)$ parameters μ, σ_1, $\ldots$, σ_k do not have closed form expressions. In this situation, some computational approaches like MCMC procedure will be very much helpful to obtain the confidence interval approximately. Here, we will use the MCMC method that uses the well known Random Walk Metropolis-Hastings (RWMH) algorithm, to generate the samples from the posterior densities of μ, σ_1, $\ldots$, σ_k. The details of the computational steps for the RWMH procedure can be elaborately described to generate the samples from the posterior distribution of $\mu, \sigma_1, \ldots, \sigma_k$ in the following manner.

Step-1: Suppose the K-th iteration of the Markov chain consists of $(\mu^{(K-1)}$, $\sigma_1^{(K-1)}$, $\sigma_2^{(K-1)}$, $\ldots, \sigma_k^{K-1})$.

Step-2: Using the RWMH algorithm, generate $\epsilon \sim N(0,\sigma_\mu^2)$, $\epsilon_i \sim N(0,\sigma_{\sigma_i}^2)$; $i=1,2,\ldots,k$. Let $\mu^{(*)} = \mu^{(K-1)} + \epsilon$, $\sigma_i^{(*)} = \sigma_i^{(K-1)} + \epsilon_i$.

Step-3: Next, compute the term

$$\Pi(\mu^{(*)},\underline{\sigma}^{(*)}) = \frac{L((\mu^{(*)},\underline{\sigma}^{(*)})|(Z,\underline{T}))p(\mu^{(*)})\prod_{i=1}^{k}p_i(\sigma_i^{(*)})}{L((\mu^{(K-1)},\underline{\sigma}^{(K-1)})|(Z,\underline{T}))p(\mu^{(K-1)})\prod_{i=1}^{k}p_i(\sigma_i^{(K-1)})},$$

where we denote $\underline{\sigma}^{(*)} = (\sigma_1^{(*)},\sigma_2^{(*)},\ldots,\sigma_k^{(*)})$, and $\underline{\sigma}^{(K-1)} = (\sigma_1^{(K-1)},\sigma_2^{(K-1)},\ldots,\sigma_k^{(K-1)})$.

Step-4: Define the quantity $V = \min(1,\Pi(\mu^{(*)},\underline{\sigma}^{(*)})$. Next, generate a random number $u \sim U(0,1)$. If $u \le V$, accept $(\mu^{(*)},\underline{\sigma}^{(*)})$ and update the parameters as $\mu^{(k)} = \mu^{(*)}$, $\sigma_i^{(K)} = \sigma_i^{(*)}; i=1,2,\ldots,k$, and otherwise set $\mu^{(K)} = \mu^{(K-1)}$, $\sigma_i^{(K)} = \sigma_i^{(K-1)}; i=1,2,\ldots,k$. Then update the quantile at the K^{th} step as $\xi^{(K)} = \mu^{(K)} + \eta\sigma_1^{(K)}$. Repeat these steps for $K=1,2,\ldots,N$ where N is a suitably chosen large number.

The choice of the values of σ_μ^2 and $\sigma_{\sigma_i}^2$, is crucial in the RWMH method. Chib and Greenberg (1995) discussed these issues and pointed out that for small choices of these values, there is a chance of acceptance. Hence, we chose the values of σ_μ^2, $\sigma_{\sigma_i}^2$ in such a way that the acceptance rate is between 20% and 30%. Using these MCMC samples, we obtain the $100(1-\alpha)$% HPD interval for the quantile ξ by applying the method given in Chen and Shao (1999).

Note that the HPD credible interval is obtained as follows. Suppose $\{\xi_{(i)}; i = 1,2,\ldots,N\}$ is the corresponding ordered MCMC sample from $\{\xi^i; i=1,2,\ldots,N\}$. Then the $100(1-\alpha)$% HPD credible interval for the quantile ξ is given by

$$(\xi_{(j^*)},\xi_{(j^*+[(1-\alpha)N])}),$$

where j^* is chosen in such a way that

$$\xi_{(j^*+[(1-\alpha)N])} - \xi_{(j^*)} = \min_{1\leq j\leq N-[(1-\alpha)N]}(\xi_{(j+[(1-\alpha)N])} - \xi_{(j)}).$$

3.4 Computational Results: Comparing the Confidence Intervals

In this subsection, we carry out a detailed simulation study in order to compare the performances of all the proposed interval estimators in terms of CPs and ALs. It is easy to observe that, none of the interval estimators have closed form expressions, hence they have been compared using Monte-Carlo simulation method numerically.

In order to evaluate and compare the performances of all the proposed interval estimators, we take the same number of replications for generating the sample as in Subsection 2.4. The sample size selection and the parameter choices will also remain same as of the Subsection 2.4. It can be observed that all the interval estimators are location invariant and the CP and AL are function of $\rho_i = \sigma_i/\sigma_1$, $i = 1, 2, \ldots, k$. The 95% confidence interval have been computed for the 95^{th} quantile, that is the level of significance is taken as $\alpha = 0.05$. In the case of bootstrap confidence interval and generalized confidence intervals the number of replications in the inner loop, B, is taken as 2500. In the computation of HPD interval using the MCMC approach, the choice of hyper parameters c_i, d_i ($i = 1, 2, \ldots, k$) and the upper limit c/r have been chosen suitably. In fact, the prior for random variable μ has been generated using the condition that $\mu < c/r$.

Though the simulation study has been conducted for many choices of parameters and many combination of the sample sizes, for illustration purposes we have presented the CPs and ALs of all the interval estimators for some specific values of sample sizes and parameters. This has been presented in Tables 4-5.

While comparing the interval estimators in terms of CPs and ALs, it has been seen that some of the intervals do not attain the nominal level $1 - \alpha = 0.95$, though they have the smaller ALs. It is natural to opt for confidence intervals which has higher CP with smallest AL. In our case, it is not possible to chose the intervals that satisfy this criteria. In order to have the twin effect of AL and CP, we use a third criteria known as 'probability coverage density (PCD)' which is defined as the ratio of CP to AL. This unified criterion shows which confidence interval is more dense apart from attaining the minimum desirable confidence level. This PCD criteria for selecting the interval estimators was proposed by Unhapipat et al. (2016).

The following observations were made (which we present in the forms of some remarks) during our simulation study as well as from the Tables 4-6 regarding ALs, CPs and PCDs of the interval estimators.

Remark 8. The ALs of all the confidence intervals decrease as the sample sizes increase from small to large, except the HPD whose pattern is not clear. The CPs of all the confidence intervals lie between 72% and 99%.

Remark 9. Though the AL of boot-p and boot-t confidence intervals are pretty much equal, the CPs of boot-p interval is always higher than the CP of boot-t interval. Further, if we analyze closely, the boot-p confidence intervals using the MLE and the modified MLE of common location parameter μ performs very much similar. The same behavior also has been seen in the case of boot-t confidence intervals. The HPD credible interval has the largest length among all the proposed confidence intervals, however its CP is higher than the boot-p confidence intervals.

Remark 10. (i) If we fix the nominal level at 95% (which is the actual nominal level) the generalized confidence intervals T_w and T_{ml} attain it, however the other two generalized confidence intervals T_{mm} and T_{mv} have CPs higher than the nominal level. If we rank the confidence intervals in terms of highest CPs, when the actual nominal level is 0.95, T_{mv} has the best performance followed by T_{mm}, and then T_w and T_{ml}.

(ii) If we rank the confidence intervals in terms of shortest lengths, the generalized confidence interval T_{ml} has the best performance followed by T_{mm}, T_{mv} and T_w.

(iii) If we use the twine criteria that is both CPs and ALs, the qualified confidence intervals are all the four generalized confidence intervals. However, it is not possible to rank the qualified intervals further. In order to have a clear cut winner, we apply the unified performance measure criterion- the PCD.

(iv) The PCD values of all the generalized confidence intervals such as T_w, T_{ml}, T_{mm} and T_{mv} increase as the sample sizes increase from small to large.

(v) In terms of PCD, the generalized confidence interval T_{ml} has the best performance followed by T_{mm}, T_{mv} and T_w for all combinations of sample sizes.

Remark 11. The confidence intervals are also compared considering the number of populations $k = 3, 4, 5$ in terms of AL, CP and PCD values. It has been noticed that a very similar type of trend appears when we increase the number of populations. Hence, for convenience, we have reported the tabulated values only for the case of $k = 2$. However, the other simulation results can be available to the readers on requesting upon the authors.

Remark 12. Some authors also used the lower bound of estimated CPs as a criterion for observing the nature of the intervals (for example, see Malekzadeh and Kharrati-Kopaei (2020)). Using that criterion, we have computed the 95% lower bound of the estimated CPs, and is obtained as 0.947. It has been observed that the intervals T_w, T_{mm} and T_{mv} are conservative. The intervals BP_{ml}, BP_{mm}, BT_{ml}, and BT_{mm} are liberal. The interval T_{ml} is neither liberal nor conservative. The HPD interval is liberal, except for a few ranges of the parameters. The qualified intervals for PCD comparison are the generalized confidence intervals, which are not liberal.

Table 4: The CPs of Several Interval Estimators for Various Combinations of Sample Sizes with $\alpha = 0.05$

ρ	(n_1, n_2)	T_w	T_{ml}	T_{mm}	T_{mv}	BP_{ml}	BP_{mm}	BT_{ml}	BT_{mm}	HPD
0.25	(5,5)	0.95	0.94	0.96	0.96	0.87	0.86	0.80	0.80	0.95
	(15,15)	0.95	0.95	0.96	0.96	0.92	0.92	0.89	0.89	0.97
	(25,25)	0.95	0.95	0.96	0.96	0.93	0.93	0.91	0.91	0.98
	(40,40)	0.95	0.95	0.96	0.96	0.93	0.93	0.92	0.92	0.95
	(5,10)	0.95	0.94	0.96	0.96	0.88	0.87	0.81	0.81	0.98
	(15,25)	0.95	0.95	0.95	0.95	0.92	0.92	0.89	0.89	0.99
	(10,5)	0.95	0.94	0.96	0.96	0.89	0.89	0.86	0.85	0.92
	(25,15)	0.95	0.95	0.96	0.96	0.93	0.93	0.91	0.91	0.97
0.50	(5,5)	0.95	0.94	0.96	0.96	0.85	0.84	0.79	0.79	0.95
	(15,15)	0.95	0.95	0.96	0.96	0.92	0.91	0.89	0.89	0.97
	(25,25)	0.95	0.95	0.96	0.96	0.92	0.92	0.90	0.90	0.98
	(40,40)	0.95	0.95	0.96	0.96	0.93	0.93	0.92	0.92	0.95
	(5,10)	0.95	0.95	0.96	0.96	0.87	0.86	0.81	0.81	0.98
	(15,25)	0.95	0.95	0.96	0.96	0.92	0.91	0.88	0.88	0.98
	(10,5)	0.95	0.95	0.96	0.97	0.88	0.87	0.84	0.84	0.92
	(25,15)	0.95	0.95	0.97	0.97	0.92	0.92	0.91	0.91	0.97
1.00	(5,5)	0.95	0.95	0.96	0.96	0.82	0.80	0.77	0.76	0.93
	(15,15)	0.96	0.95	0.97	0.97	0.91	0.90	0.88	0.88	0.96
	(25,25)	0.96	0.95	0.96	0.97	0.92	0.91	0.90	0.90	0.98
	(40,40)	0.95	0.95	0.96	0.96	0.93	0.93	0.92	0.92	0.95
	(5,10)	0.96	0.95	0.96	0.96	0.84	0.83	0.78	0.78	0.97
	(15,25)	0.95	0.95	0.96	0.96	0.91	0.90	0.88	0.88	0.98
	(10,5)	0.95	0.95	0.96	0.96	0.87	0.86	0.84	0.84	0.91
	(25,15)	0.96	0.95	0.97	0.97	0.92	0.91	0.90	0.90	0.96
2.00	(5,5)	0.96	0.95	0.97	0.97	0.79	0.76	0.75	0.74	0.90
	(15,15)	0.95	0.95	0.96	0.96	0.90	0.88	0.87	0.87	0.93
	(25,25)	0.95	0.95	0.97	0.97	0.92	0.91	0.90	0.90	0.96
	(40,40)	0.95	0.95	0.96	0.96	0.92	0.92	0.92	0.91	0.94
	(5,10)	0.96	0.95	0.97	0.97	0.82	0.79	0.76	0.76	0.93
	(15,25)	0.95	0.95	0.96	0.96	0.90	0.89	0.88	0.88	0.95
	(10,5)	0.95	0.95	0.96	0.96	0.87	0.84	0.84	0.83	0.89
	(25,15)	0.95	0.94	0.96	0.96	0.91	0.90	0.90	0.89	0.95
4.00	(5,5)	0.96	0.95	0.97	0.97	0.78	0.73	0.74	0.72	0.86
	(15,15)	0.95	0.95	0.96	0.96	0.89	0.87	0.87	0.86	0.90
	(25,25)	0.95	0.95	0.96	0.96	0.91	0.90	0.90	0.90	0.93
	(40,40)	0.95	0.95	0.96	0.96	0.92	0.91	0.91	0.91	0.90
	(5,10)	0.95	0.95	0.97	0.97	0.80	0.76	0.75	0.74	0.86
	(15,25)	0.95	0.95	0.97	0.97	0.89	0.88	0.87	0.87	0.90
	(10,5)	0.95	0.95	0.96	0.96	0.85	0.82	0.82	0.82	0.87
	(25,15)	0.95	0.95	0.95	0.95	0.91	0.90	0.90	0.90	0.93
5.00	(5,5)	0.95	0.95	0.96	0.96	0.77	0.72	0.73	0.72	0.80
	(15,15)	0.95	0.95	0.96	0.96	0.89	0.87	0.88	0.87	0.83
	(25,25)	0.95	0.94	0.96	0.96	0.91	0.89	0.89	0.89	0.85
	(40,40)	0.95	0.95	0.96	0.96	0.92	0.91	0.91	0.91	0.88
	(5,10)	0.95	0.95	0.97	0.97	0.78	0.74	0.74	0.72	0.78
	(15,25)	0.95	0.95	0.96	0.96	0.89	0.88	0.87	0.87	0.83
	(10,5)	0.95	0.95	0.96	0.96	0.85	0.81	0.82	0.81	0.77
	(25,15)	0.95	0.95	0.95	0.95	0.91	0.89	0.90	0.89	0.85

Table 5: The ALs of All the Interval Estimators for Various Combinations of Sample Sizes with $\alpha = 0.05$

ρ	(n_1, n_2)	T_w	T_{ml}	T_{mm}	T_{mv}	BP_{ml}	BP_{mm}	BT_{ml}	BT_{mm}	HPD
0.25	(5,5)	1.019	0.180	0.256	0.279	4.872	4.860	4.872	4.860	6.926
	(15,15)	0.228	0.052	0.056	0.056	2.952	2.950	2.952	2.950	6.917
	(25,25)	0.128	0.030	0.031	0.032	2.297	2.297	2.297	2.297	6.984
	(40,40)	0.078	0.019	0.019	0.019	1.819	1.820	1.819	1.820	5.267
	(5,10)	0.999	0.092	0.126	0.132	4.988	4.981	4.988	4.981	7.638
	(15,25)	0.226	0.033	0.035	0.035	2.959	2.957	2.959	2.957	7.696
	(10,5)	0.403	0.139	0.164	0.171	3.518	3.511	3.518	3.511	5.919
	(25,15)	0.130	0.045	0.047	0.047	2.286	2.283	2.286	2.283	6.221
0.50	(5,5)	1.015	0.283	0.360	0.384	4.753	4.728	4.753	4.728	6.667
	(15,15)	0.208	0.085	0.089	0.089	2.919	2.915	2.919	2.915	6.758
	(25,25)	0.114	0.050	0.051	0.051	2.279	2.273	2.279	2.273	6.814
	(40,40)	0.069	0.031	0.031	0.031	1.816	1.816	1.816	1.816	4.972
	(5,10)	0.893	0.161	0.192	0.198	4.954	4.939	4.954	4.939	7.661
	(15,25)	0.197	0.058	0.060	0.060	2.932	2.929	2.932	2.929	7.538
	(10,5)	0.494	0.199	0.226	0.231	3.417	3.397	3.417	3.397	5.819
	(25,15)	0.127	0.068	0.071	0.071	2.275	2.270	2.275	2.270	6.103
1.00	(5,5)	1.255	0.417	0.516	0.546	4.574	4.514	4.574	4.514	6.357
	(15,15)	0.232	0.126	0.131	0.131	2.862	2.846	2.862	2.846	6.372
	(25,25)	0.126	0.075	0.076	0.076	2.257	2.250	2.257	2.250	6.477
	(40,40)	0.074	0.046	0.047	0.047	1.810	1.807	1.810	1.807	4.984
	(5,10)	0.822	0.263	0.298	0.308	4.703	4.662	4.703	4.662	7.309
	(15,25)	0.182	0.094	0.096	0.096	2.908	2.896	2.908	2.896	7.138
	(10,5)	0.891	0.264	0.300	0.307	3.351	3.310	3.351	3.310	5.524
	(25,15)	0.186	0.094	0.097	0.097	2.244	2.232	2.244	2.232	5.823
2.00	(5,5)	2.158	0.576	0.746	0.798	4.265	4.169	4.265	4.169	5.617
	(15,15)	0.423	0.170	0.178	0.179	2.815	2.791	2.815	2.791	5.527
	(25,25)	0.233	0.100	0.102	0.102	2.246	2.231	2.246	2.231	5.643
	(40,40)	0.139	0.062	0.063	0.063	1.796	1.789	1.796	1.789	4.746
	(5,10)	0.971	0.401	0.455	0.467	4.511	4.446	4.511	4.446	6.376
	(15,25)	0.257	0.137	0.142	0.142	2.856	2.837	2.856	2.837	6.239
	(10,5)	1.975	0.321	0.392	0.405	3.263	3.200	3.263	3.200	5.023
	(25,15)	0.408	0.117	0.121	0.121	2.220	2.202	2.220	2.202	5.183
4.00	(5,5)	4.518	0.737	1.081	1.185	4.061	3.915	4.061	3.915	4.655
	(15,15)	0.947	0.209	0.225	0.226	2.774	2.736	2.774	2.736	4.525
	(25,25)	0.525	0.122	0.126	0.127	2.223	2.204	2.223	2.204	4.541
	(40,40)	0.315	0.075	0.076	0.076	1.784	1.774	1.784	1.774	4.557
	(5,10)	1.674	0.567	0.681	0.709	4.301	4.186	4.301	4.186	4.687
	(15,25)	0.532	0.181	0.190	0.190	2.807	2.778	2.807	2.778	4.524
	(10,5)	4.501	0.372	0.532	0.559	3.200	3.126	3.200	3.126	4.489
	(25,15)	0.937	0.134	0.142	0.143	2.208	2.186	2.208	2.186	4.539
5.00	(5,5)	5.750	0.787	1.232	1.365	4.017	3.854	4.017	3.854	4.238
	(15,15)	1.212	0.219	0.239	0.241	2.763	2.723	2.763	2.723	4.281
	(25,25)	0.677	0.128	0.133	0.134	2.215	2.192	2.215	2.192	4.372
	(40,40)	0.408	0.078	0.080	0.080	1.781	1.771	1.781	1.771	4.488
	(5,10)	2.075	0.620	0.776	0.866	4.140	4.012	4.140	4.012	4.231
	(15,25)	0.683	0.194	0.204	0.205	2.795	2.760	2.795	2.760	4.300
	(10,5)	5.748	0.382	0.589	0.622	3.171	3.090	3.171	3.090	4.227
	(25,15)	1.217	0.138	0.149	0.150	2.205	2.181	2.205	2.181	4.384

Table 6: The PCDs of Some of the Selected Interval Estimators for Various Combinations of Sample Sizes with $\alpha = 0.05$

ρ	(n_1, n_2)	T_w	T_{ml}	T_{mml}	T_{umv}	ρ	(n_1, n_2)	T_w	T_{ml}	T_{mml}	T_{umv}
0.25	(5,5)	0.933	5.207	3.737	3.434	2.00	(5,5)	0.443	1.643	1.295	1.212
	(15,15)	4.170	18.193	17.144	17.044		(15,15)	2.248	5.572	5.413	5.390
	(25,25)	7.411	31.218	30.539	30.484		(25,25)	4.087	9.519	9.441	9.429
	(40,40)	12.215	50.773	50.313	50.298		(40,40)	6.804	15.285	15.265	15.262
	(5,10)	0.953	10.258	7.560	7.228		(5,10)	0.985	2.363	2.124	2.074
	(15,25)	4.202	28.520	26.965	26.858		(15,25)	3.703	6.911	6.811	6.798
	(10,5)	2.368	6.798	5.859	5.650		(10,5)	0.483	2.951	2.453	2.375
	(25,15)	7.329	20.946	20.299	20.248		(25,15)	2.334	8.108	7.949	7.933
0.50	(5,5)	0.940	3.336	2.661	2.498	4.00	(5,5)	0.211	1.288	0.893	0.816
	(15,15)	4.578	11.122	10.808	10.766		(15,15)	1.005	4.543	4.270	4.242
	(25,25)	8.316	18.898	18.788	18.774		(25,25)	1.807	7.795	7.612	7.600
	(40,40)	13.872	30.804	30.732	30.720		(40,40)	3.013	12.632	12.536	12.530
	(5,10)	1.067	5.893	4.981	4.832		(5,10)	0.569	1.671	1.423	1.367
	(15,25)	4.847	16.307	15.954	15.920		(15,25)	1.780	5.250	5.095	5.081
	(10,5)	1.932	4.749	4.277	4.176		(10,5)	0.212	2.559	1.805	1.719
	(25,15)	7.491	13.868	13.704	13.682		(25,15)	1.012	7.075	6.706	6.678
1.00	(5,5)	0.760	2.271	1.866	1.764	5.00	(5,5)	0.165	1.201	0.780	0.706
	(15,15)	4.113	7.564	7.407	7.383		(15,15)	0.785	4.349	4.020	3.989
	(25,25)	7.596	12.685	12.646	12.642		(25,25)	1.402	7.385	7.171	7.153
	(40,40)	12.832	20.508	20.556	20.554		(40,40)	2.323	12.130	11.994	11.984
	(5,10)	1.163	3.604	3.233	3.133		(5,10)	0.459	1.527	1.247	1.120
	(15,25)	5.209	10.110	10.013	10.000		(15,25)	1.391	4.893	4.716	4.698
	(10,5)	1.066	3.600	3.204	3.128		(10,5)	0.165	2.481	1.622	1.537
	(25,15)	5.132	10.085	9.989	9.977		(25,15)	0.781	6.88	6.392	6.362

4 An Application Through Real Life Example & Concluding Remarks

In this section, we consider a real life example, where our model fits well and demonstrate the test procedures as well as interval estimation methods. The data sets are about the operational times (in hours) between successive failures of air conditioning equipments in two aircrafts; Plane 7915 and Plane 8044. Barlow et al. (1972) showed that the two parameter exponential distribution fits these two data sets well. Further, the equality of the location parameters has been tested at the level of significance 10%. The data sets are given as, Plane 7915: 359, 9, 12, 270, 603, 3, 104, 2, 438; Plane 8044: 487, 18, 100, 7, 98, 5, 85, 91, 43, 230.

In Table 7, we compute the interval length and the intervals for the quantile ξ, using these two data sets with level of significance 0.95.

Table 7: Computing the Lower Limits, Upper Limits and Lengths of Confidence Intervals with $\alpha = 0.05$

Interval	T_w	T_{ml}	T_{mm}	T_{mv}	BP_{ml}	BP_{mm}	BT_{ml}	BT_{mm}	HPD
$\hat{\xi}_L$	628.8835	675.6342	674.8448	674.6932	257.2489	263.9333	200.5226	144.7706	662.2297
$\hat{\xi}_U$	707.9302	708.7513	710.4287	710.6968	989.7874	1030.487	933.0611	911.324	989.9744
Length	79.04668	33.11711	35.58397	35.58397	732.5385	766.5534	732.5385	766.5534	327.7447

In the Table 7, it is observed that the generalized confidence interval T_{ml} has the shortest length among all the proposed confidence intervals.

Next, we consider to test the hypothesis $H_0 : \xi = 700$ against the alternative $H_a : \xi \neq 700$ at level of significance $\alpha = 0.05$ using the given data sets. The p-values for all the proposed test procedures are computed and given in Table 8.

Table 8: Computing the p-values of all the Proposed Tests with $\alpha = 0.05$

Method	G_w	G_{ml}	G_{mm}	G_{mv}	B_{ml}	B_{mm}	B_{mv}	C_{ml}	C_{mm}	M_{ml}	M_{mm}
p-value	0.5906	0.6734	0.6278	0.6234	0.0200	0.0200	0.0200	0.7600	0.7400	0.6900	0.6600

The p-values, indicate that all the three parametric bootstrap tests, such as B_{ml}, B_{mm}, and B_{mv} reject the null hypothesis and the other tests such as G_w, G_{ml}, G_{mm}, G_{mv}, C_{ml}, C_{mm}, M_{ml}, and M_{mm} accept the null hypothesis at level of significance $\alpha = 0.05$.

5 Concluding Remarks

The problem of point estimation of quantiles $\xi = \mu + \eta\sigma_1$ when samples are available from two or more shifted exponential populations with a common location and different scale parameters has been well studied by authors (see, for example, Kumar and Sharma (1996)). Surprisingly, under the same model set-up, the interval estimation and the hypothesis testing of the quantile $\xi = \mu + \eta\sigma_1$ have not been considered so far in the literature. In this article, we have considered these two problems in detail. We also note that, unlike the case of common location parameter, it is difficult to derive any exact test procedures as well as intervals in the case of quantile for the underlying model.

In this regard, we first proposed several test procedures such as tests based on generalized variable and p-value approaches, tests based on parametric bootstrap approach and the tests based on a computational approach proposed by Pal et al. (2007). All the test procedures are evaluated numerically in terms of size and power. From our computational results, we concluded that the tests based on the generalized variable approach that uses the MLE of the common location parameter (denoted as G_{ml}) have the best performance in terms of power.

Several interval estimators for the quantile, namely the intervals based on generalized variable method, parametric bootstrap method and MCMC method have been proposed. The confidence intervals are compared in terms of CPs and ALs. While comparing their performances in terms of these two criteria, it has been observed that none of the intervals outperforms others. In fact, it is not possible to decide which of the confidence intervals perform the best using these two criteria. In order to get a better picture regarding the performance of the intervals, we employ a third criteria known as 'probability coverage density (PCD)'. In terms of PCD values, it has been seen that the interval based on the generalized statistic that uses MLE of the common location parameter (denoted as T_{ml}) has the best performance, followed by T_{mm}, T_{mv}

and T_w. It is also noted that the conclusions regarding performances of tests as well as confidence intervals are purely based on our comprehensive simulation study. Finally, we discussed a real-life example for application purposes.

We note that in the case of normal distribution Khatun et al. (2020), considered the problems of hypothesis tetsing and confidence interval for the quantiles of the first population. Authors proposed several test procedures and interval estimators using some of the popular estimators of the common mean. However, there are no such results available for estimating the common location parameter of several exponential populations. Using various estimators of the common location parameter and different scale parameters, we have developed some test procedures and confidence intervals of the quantile of the first population. However, there are no such results for estimating the common location parameter of several exponential populations. Using various estimators of the common location parameter and different scale parameters, we have developed some test procedures and confidence intervals of the quantile of the first population.

Declarations

Conflicts of interests: Authors confirm that there are no relevant financial or non-financial competing interests to report.

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