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Orderings of Extreme Order Statistics with Archimedean Copula and Powered Gompertz Random Variables

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Abstract. Bathtub shaped failure rate distributions are of special interest in reliability theory, survival analysis and many other fields. The so-called power Gompertz distribution is one of the popular lifetime distributions that possesses the bathtub shaped failure rate function. In this paper, we study some stochastic comparisons results for extreme order statistics from dependent powered Gompertz distributed random variables under Archimedean copula. The study has been carried out in the sense of the usual stochastic order and the dispersive order.

Keywords. Powered Gompertz Distribution, Order Statistics, *k*-out-of-*n* Systems, Majorization Orders, Stochastic Orders, Archimedean Copula.

MSC: 60E15, 90B25.

1 Introduction

Order statistics is one of basic tools which is frequently used in probability and statistics. A part from this, it has a large number of applications in various other domains including reliability theory, finance, risk management and many other fields. Different order statistics have different applications. For example, the minimum order statistic is used in reliability and survival analysis to measure the minimum survival time of a system/living organism, whereas the maximum order statistic is widely used to

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study different extreme meteorological phenomena. The applications of other order statistics could be found in different statistical inference problems. Further, the difference between maximum and minimum order statistics is a measure of dispersion. For more discussions on this, one may refer to Balakrishnan and Rao (1998a,b), Arnold et al. (1992), and David and Nagaraja (2003). Let $\{X_1, X_2, \ldots, X_n\}$ be a set of random variables, and let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics; here $X_{1:n}$ represents the minimum order statistic and $X_{n:n}$ stands for the maximum order statistic. In general, $X_{i:n}$ represents the i-th order statistic. One may be interested to know that there is a nice connection between an order statistic and the lifetime of a system. If $\{X_1, X_2, \ldots, X_n\}$ is a set of random variables representing the lifetimes of n components, then $X_{i:n}$ represents the lifetime of the $(n - i + 1)$-out-of-n system formed from these components. This is indeed a very useful relation because it essentially infers that the study of a coherent system (whose lifetime can be expressed, as a mixture of the lifetimes of r-out-of-n systems, by using the notion of signature (see Samaniego (2007)) is the same as the study of order statistics formed from a set of non-negative random variables.

Stochastic comparisons of different order statistics have been extensively studied in last few decades by different researchers (see Pledger and Proschan (1971), Proschan and Sethuraman (1976), Bon and Păltănea (2006), Balakrishnan and Zhao (2013), Barmalzan et al. (2017), Hazra et al. (2017), Alimohammadi et al. (2021) and Esna-Ashari et al. (2022)). However, most of the existing literature on stochastic comparisons of order statistics have dealt with the case when the underlying set of random variables are independent. This is mostly considered to avoid the mathematical complexities rather than the actual interest. For example, the lifetimes of the components of a technical system are mostly dependent because they often share many important factors (e.g., operating conditions, environmental conditions, stress factors, etc.) between themselves. Thus, we should take care of the inherent dependency structure between components, if any, when we deal with a technical system. Of course, there are many ways to model this dependency structure (see Kotz, Balakrishnan and Johnson (2000)), and the theory of copulas is one of the popular tools used for this purpose; see Nelsen (2006) for an encyclopaedic information on copula theory. Though many copulas have been studied in the literature, the Archimedean copula has been considered by many researchers due to its wider flexibility and also due to the fact that it includes many well known copulas (namely, Clayton-Oakes copula, Ali-Mikhail-Haq copula, Gumbel-Hougaard copula, etc.) as the particular cases. Moreover, it contains the independent copula as the particular case. Consequently, all results developed for order statistics with dependent samples under the Archimedean copula trivially hold for order statistics with independent samples. Stochastic comparisons of order statistics with dependent samples under the Archimedean copula have been considered in some of the recent articles (see Li and Fang (2015), Fang et al. (2016), Li and Li (2019), Barmalzan et al. (2020) and the references therein).

The failure rate function is a very useful measure in reliability theory. It possesses different shapes, namely, increasing, decreasing, bathtub shaped, upside down bathtub shaped, roller coaster, etc. Among all these shapes, the bathtub shaped failure rate function is of special interest in reliability and life testing experiments (see Lai and Xie (2006), Finkelstein (2008) and the references therein). Lifetimes of many electronic devices (namely, light bulbs, switches, different types of circuits, etc.) are often described by the distributions having bathtub shaped failure rate functions (see Block and Savits (1997)). Apart from modeling the lifetime distributions, many other applications of the bathtub shaped failure rate distribution could be found in the literature, for example, the useful period for a lifetime distribution can be determined by the bathtub shaped failure rate distribution (Bebbington et al.(2006)), the human mortality can be modeled by the bathtub shaped failure distribution (Bebbington et al. (2007)), etc.

One of popular distributions that possesses the bathtub shaped failure rate function is the so-called powered Gompertz distribution (Chen (2000)). A random variable X is said to have the powered Gompertz distribution, denoted by $X \sim PG(\beta, \lambda)$, if its distribution function is given by

$$F_X(x) = 1 - e^{-\lambda(e^{x^\beta}-1)}, \quad x > 0, \beta > 0, \lambda > 0.$$

Keeping the importance of the powered Gompertz distribution in mind, researchers have studied its various stochastic properties. One of the important problems in this context is the study of stochastic comparisons of orders statistics. This problem for independent samples has been considered in Bhattacharyya et al. (2020). However, the study of this problem with dependent samples has not yet been considered. As we discussed that samples are most often dependent in nature, the study of stochastic comparisons for extreme order statistics from dependent powered Gompertz random variables under Archimedean copula may be considered as a very good research problem. Thus, in this paper, we focus on studying this problem.

Throughout the paper, the increasing and the decreasing properties are not used in strict sense. Further, the notation "$c \overset{sgn}{=} d$" is used to mean that c and d possess the same sign. All random variables considered in this paper are absolutely continuous with nonnegative supports.

The orientation of the paper is as follows. In Section 2, we review some basic concepts and notions that are used throughout the paper. In Section 3, we discuss some stochastic comparison results, for the maximum and the minimum order statistics from powered Gompertz random variables under the Archimedean copula, in terms of the usual stochastic order and the dispersive order. In Section 4, we give the concluding remarks.

2 Preliminaries

In this section, we discuss some well-known concepts, namely, stochastic orders, majorization orders, and the notion of copula. Before that, we introduce some notations and give the definition of the superadditive function, which will be used in the main results of this paper.

For an absolutely continuous random variable W, we denote the distribution function by $F_W(\cdot)$, the survival/reliability function by $\bar{F}_W(\cdot)$, the failure/hazard rate function by $r_W(\cdot)$; here $\bar{F}_W(\cdot) \equiv 1 - F_W(\cdot)$ and $r_W(\cdot) \equiv f_W(\cdot)/\bar{F}_W(\cdot)$. Further, by $F_W^{-1}(\cdot)$, we mean the right-continuous inverse (quantile function) of $F_W(\cdot)$. Moreover, we denote the set of real numbers by $\mathbb{R}$.

Definition 2.1. Let $\mathbb{B} \subseteq \mathbb{R}$. A function $g : \mathbb{B} \to \mathbb{R}$ is superadditive if $g(u+v) \geq g(u)+g(v)$ for all $u, v \in \mathbb{B}$.

2.1 Stochastic Orders

We often compare two or more random variables using different stochastic orders. There is a large variety of stochastic orders, namely, usual stochastic order, convex order, mrl order, etc. A thorough discussion on this topic could be found in Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

Definition 2.2. A random variable V is larger than another random variable U in the sense of the usual stochastic order, denoted by $V \geq_{st} U$, if $\bar{F}_V(x) \geq \bar{F}_U(x)$ for all $x > 0$.

Definition 2.3. A random variable V is more dispersed than another random variable U, denoted by $V \geq_{disp} U$, if $F_V^{-1}(t) - F_U^{-1}(t)$ is increasing in $t \in (0,1)$. A necessary and sufficient condition to hold $V \geq_{disp} U$ is $f_U(F_U^{-1}(F_V(x))) \leq f_V(x)$, for all $x > 0$.

2.2 Majorization Orders

Unlike the stochastic orders, majorization orders compare two vectors of real numbers. These are frequently used to establish different inequalities in mathematics, probability and related fields. To know more on this topic, one may recommend the book written by Marshall et al. (2011).

Definition 2.4. Let $\mathbf{d} = (d_1, \cdots, d_n)$ and $\mathbf{e} = (e_1, \cdots, e_n)$ be two vectors of real numbers. Further, let $d_{(1)} \leq \cdots \leq d_{(n)}$ and $e_{(1)} \leq \cdots \leq e_{(n)}$ be the representations in ascending order of the components of $\mathbf{d}$ and $\mathbf{e}$, respectively. Then,

(i) $\mathbf{d}$ is majorized by $\mathbf{e}$, denoted by $\mathbf{d} \overset{m}{\preceq} \mathbf{e}$, if $\sum_{k=1}^{i} d_{(k)} \geq \sum_{k=1}^{i} e_{(k)}$ for $i = 1, \cdots, n-1$, and $\sum_{k=1}^{n} d_{(k)} = \sum_{k=1}^{n} e_{(k)}$;

(ii) $\mathbf{d}$ is weakly supermajorized by $\mathbf{e}$, denoted by $\mathbf{d} \overset{w}{\preceq} \mathbf{e}$, if $\sum_{k=1}^{i} d_{(k)} \geq \sum_{k=1}^{i} e_{(k)}$ for $i = 1, \cdots, n$.

Definition 2.5. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then a function $\xi : \mathbb{D} \to \mathbb{R}$ is Schur-convex (Schur-concave) if $\mathbf{d} \overset{m}{\preceq} \mathbf{e} \implies \xi(\mathbf{d}) \leq (\geq)\ \xi(\mathbf{e})$ for any $\mathbf{d}, \mathbf{e} \in \mathbb{D}$. □

Below we give two important lemmas that characterize the Schur-convex/Schur-concave function (see Marshall et al. (2011)).

Lemma 2.1. *If $\mathbb{E} \subset \mathbb{R}$ is an open interval and $\xi : \mathbb{E}^n \to \mathbb{R}$ is a continuously differentiable function from $\mathbb{E}^n$ to $\mathbb{R}$, then ξ is Schur-convex (Schur-concave) on $\mathbb{E}^n$ if and only if*

(i) ξ is symmetric on $\mathbb{E}^n$;

(ii) for all $i \neq j$ and for all $\mathbf{d} \in \mathbb{E}^n$,

$$(d_i - d_j)\left(\frac{\partial \xi(\mathbf{d})}{\partial d_i} - \frac{\partial \xi(\mathbf{d})}{\partial d_j}\right) \geq 0 \, (\leq 0).$$

Lemma 2.2. *For a continuous real-valued function ξ, defined on $\mathbb{E}^n \subseteq \mathbb{R}^n$, $\boldsymbol{u} \overset{w}{\geq} \boldsymbol{v} \implies \xi(\boldsymbol{u}) \geq \xi(\boldsymbol{v})$ holds if and only if ξ is decreasing and Schur-convex on $\mathbb{E}$.*

2.3 Archimedean Copulas

Copula is one of the widely used techniques that describes the dependency structure between a set of random variables. A number of different copulas have been introduced in the literature (e.g., Archimedean copula, FGM copula, etc.). Among all existing copulas, the family of Archimedean copula is the popular one due to its wide spectrum of capturing the dependency structures (Nelsen (2006), and McNeil and N*ě*slehov*á* (2009)). Let $\psi : [0, \infty) \longrightarrow [0, 1]$ be a decreasing and continuous function with $\psi(0) = 1$ and $\psi(+\infty) = 0$, and let $\phi = \psi^{-1}$ be the pseudo-inverse of ψ. Then, the Archimedean copula is defined as

$$K_\psi(y_1, \cdots, y_n) = \psi(\phi(y_1) + \cdots + \phi(y_n)), \quad y_i \in [0, 1], \;\; i = 1, \cdots, n,$$

where ψ is the generator satisfying the conditions: $(-1)^k \psi^{[k]}(x) \geq 0$, for $k = 0, \cdots, n-2$, and $(-1)^{n-2} \psi^{[n-2]}(x)$ is decreasing and convex. This family contains many important special cases, namely, Clayton-Oakes copula, Ali-Mikhail-Haq (AMH) copula, independent copula, etc.

Recall that a function f is said to be convex (concave) if

$$f(\alpha x + (1-\alpha)y) \leq (\geq) \alpha f(x) + (1-\alpha) f(y),$$

for all x and y in the domain of f and $0 \leq \alpha \leq 1$.

Also, a function f is said to be logconvex (logconcave) if $\log f$ be convex (concave).

3 Results based on the Archimedean Copula

In this section, we discuss some stochastic comparison results for extreme order statistics with Archimedean copula and powered Gompertz random variables. For the proposed study, we use two different stochastic orders, namely, the usual stochastic order and the dispersive order.

3.1 Usual Stochastic Order

This subsection is devoted to study various stochastic comparison results, for maximum and minimum order statistics in terms of the usual stochastic order.

In the following theorem, we compare two maximum order statistics that are formed by powered Gompertz distributed random variables. We show that if the set of shape parameters (β_i's) of one set of random variables is majorized by that of another set, then the maximum order statistic from the first set is dominated by that of the other set with respect to the usual stochastic order.

Theorem 3.1. *Let $X_i \sim PG(\beta_i, \lambda)$ and $Y_i \sim PG(\gamma_i, \lambda)$ $(i = 1, \cdots, n)$ and the associated Archimedean copulas are with generator ψ_1 and ψ_2, respectively. Further, let $\phi_2 \circ \psi_1$ be superadditive, and ψ_1 or ψ_2 be log-convex. Then, for $\lambda \geq 1$, we have*

$$(\gamma_1, \cdots, \gamma_n) \overset{m}{\preceq} (\beta_1, \cdots, \beta_n) \Longrightarrow Y_{n:n} \leq_{st} X_{n:n}.$$

Proof: We only prove the result for the case when ψ_1 is log-convex. The result can be proved in the same line for the other case. Note that

$$F_{X_{n:n}}(x) = \psi_1\left[\sum_{i=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right], \quad x > 0,$$

and

$$F_{Y_{n:n}}(x) = \psi_2\left[\sum_{i=1}^{n} \phi_2\left(1 - e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right], \quad x > 0.$$

Then, the assumed superadditivity of $\phi_2 \circ \psi_1$ implies that

$$\psi_1\left[\sum_{i=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right] \leq \psi_2\left[\sum_{i=1}^{n} \phi_2\left(1 - e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right].$$

Thus, it suffices to show that

$$\psi_1\left[\sum_{i=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right] \leq \psi_1\left[\sum_{k=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right].$$

Let $\Delta(\beta_1, \cdots, \beta_n) = \psi_1\left[\sum_{i=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right]$. Note that $\Delta(\beta_1, \cdots, \beta_n)$ is symmetric with respect to its arguments. Thus, based on Lemma 2.1, we only need to show that $\Delta(\beta_1, \cdots, \beta_n)$ is Schur-concave in $(\beta_1, \cdots, \beta_n)$. The derivative of $\Delta(\beta_1, \cdots, \beta_n)$ with respect to β_i is

$$\frac{\partial \Delta(\beta_1, \cdots, \beta_n)}{\partial \beta_i} = \lambda \ln(x) x^{\beta_i} e^{x^{\beta_i}} e^{-\lambda(e^{x^{\beta_i}}-1)} \frac{\psi_1'\left[\sum\limits_{i=1}^{n} \phi_1\left(1 - e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right]}{\psi_1'\left[\phi_1\left(1 - e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right]}.$$

Then, for any $i \neq j$, we have

$$
\begin{aligned}
& \frac{\partial \Delta(\beta_1, \cdots, \beta_n)}{\partial \beta_i} - \frac{\partial \Delta(\beta_1, \cdots, \beta_n)}{\partial \beta_j} \\
= \ & \lambda \ln(x) \psi_1' \left[\sum_{i=1}^{n} \phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_i}} - 1)} \right) \right] \\
& \times \left[\frac{x^{\beta_i} e^{x^{\beta_i}} e^{-\lambda(e^{x^{\beta_i}} - 1)}}{\psi_1' \left[\phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_i}} - 1)} \right) \right]} - \frac{x^{\beta_j} e^{x^{\beta_j}} e^{-\lambda(e^{x^{\beta_j}} - 1)}}{\psi_1' \left[\phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_j}} - 1)} \right) \right]} \right] \\
\overset{sgn}{=} \ & \ln(x) \left[\frac{x^{\beta_j} e^{x^{\beta_j}} e^{-\lambda(e^{x^{\beta_j}} - 1)}}{\psi_1' \left[\phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_j}} - 1)} \right) \right]} - \frac{x^{\beta_i} e^{x^{\beta_i}} e^{-\lambda(e^{x^{\beta_i}} - 1)}}{\psi_1' \left[\phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_i}} - 1)} \right) \right]} \right] \\
= \ & \ln(x) \left[\left\{ \frac{x^{\beta_j} e^{x^{\beta_j}}}{e^{\lambda(e^{x^{\beta_j}} - 1)} - 1} \right\} \left\{ \frac{\psi_1(a)}{\psi_1'(a)} \right\} - \left\{ \frac{x^{\beta_i} e^{x^{\beta_i}}}{e^{\lambda(e^{x^{\beta_i}} - 1)} - 1} \right\} \left\{ \frac{\psi_1(b)}{\psi_1'(b)} \right\} \right],
\end{aligned}
$$

where $a = \phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_j}} - 1)} \right)$ and $b = \phi_1 \left(1 - e^{-\lambda(e^{x^{\beta_i}} - 1)} \right)$. Now, consider the following two cases.

Case I: Let $x \geq 1$. Then, for $\beta_i \geq \beta_j$, we have $a \geq b$. Again, the log-convexity of ψ_1 implies that

$$0 \leq -\frac{\psi_1(b)}{\psi_1'(b)} \leq -\frac{\psi_1(a)}{\psi_1'(a)}. \tag{3.1}$$

Further, using Lemma 3.1 of Bhattacharyya et al. (2020), for $\beta_i \geq \beta_j$ and $\lambda \geq 1$, we have

$$\frac{x^{\beta_i} e^{x^{\beta_i}}}{e^{\lambda(e^{x^{\beta_i}} - 1)} - 1} \leq \frac{x^{\beta_j} e^{x^{\beta_j}}}{e^{\lambda(e^{x^{\beta_j}} - 1)} - 1}, \qquad \text{for all } x \geq 1. \tag{3.2}$$

By combining (3.1) and (3.2), we get

$$\left\{ \frac{x^{\beta_j} e^{x^{\beta_j}}}{e^{\lambda(e^{x^{\beta_j}} - 1)} - 1} \right\} \left\{ \frac{\psi_1(a)}{\psi_1'(a)} \right\} \leq \left\{ \frac{x^{\beta_i} e^{x^{\beta_i}}}{e^{\lambda(e^{x^{\beta_i}} - 1)} - 1} \right\} \left\{ \frac{\psi_1(b)}{\psi_1'(b)} \right\}. \tag{3.3}$$

Finally, using the above inequality and the fact "$\ln(x) \geq 0$ for $x \geq 1$", we get

$$\left(\beta_i - \beta_j \right) \left(\frac{\partial \Delta(\beta_1, \cdots, \beta_n)}{\partial \beta_i} - \frac{\partial \Delta(\beta_1, \cdots, \beta_n)}{\partial \beta_j} \right) \leq 0.$$

Case II: Let $0 < x \leq 1$. Then, for $\beta_i \geq \beta_j$, we have $a \leq b$. Again, the log-convexity of ψ_1 implies that

$$0 \leq -\frac{\psi_1(a)}{\psi_1'(a)} \leq -\frac{\psi_1(b)}{\psi_1'(b)}. \tag{3.4}$$

Further, using Lemma 3.1 of Bhattacharyya et al. (2020), for $\beta_i \geq \beta_j$ and $\lambda \geq 1$, we have

$$\frac{x^{\beta_i} e^{x^{\beta_i}}}{e^{\lambda(e^{x^{\beta_i}}-1)} - 1} \geq \frac{x^{\beta_j} e^{x^{\beta_j}}}{e^{\lambda(e^{x^{\beta_j}}-1)} - 1}, \qquad \text{for } 0 < x \leq 1. \tag{3.5}$$

Now, by combining (3.4) and (3.5), we get

$$\left\{\frac{x^{\beta_j} e^{x^{\beta_j}}}{e^{\lambda(e^{x^{\beta_j}}-1)} - 1}\right\}\left\{\frac{\psi_1(a)}{\psi_1'(a)}\right\} \geq \left\{\frac{x^{\beta_i} e^{x^{\beta_i}}}{e^{\lambda(e^{x^{\beta_i}}-1)} - 1}\right\}\left\{\frac{\psi_1(b)}{\psi_1'(b)}\right\}. \tag{3.6}$$

Finally, using the above inequality and the fact "$\ln x \leq 0$ for $0 < x \leq 1$", we get

$$\left(\beta_i - \beta_j\right)\left(\frac{\partial\Delta(\beta_1, \cdots, \beta_n)}{\partial\beta_i} - \frac{\partial\Delta(\beta_1, \cdots, \beta_n)}{\partial\beta_j}\right) \leq 0.$$

By combining both cases, the desired result follows from the Lemma 2.1. □

The following counterexample shows that the result of Theorem 3.1 may not hold for the case when $\lambda \ngeq 1$.

Examples 3.1. Suppose X_1, X_2, X_3 are independent random variables with $X_i \sim PG(\beta_i, 0.5)$, $i = 1, 2, 3$, where $(\beta_1, \beta_2, \beta_3) = (2, 3, 7)$, and Y_1, Y_2, Y_3 are independent random variables with $Y_i \sim PG(\gamma_i, 0.5)$, $i = 1, 2, 3$, where $(\gamma_1, \gamma_2, \gamma_3) = (4, 4, 4)$. Now, consider the Clayton copula with generators $\psi_1(t) = (\theta_1 t + 1)^{-1/\theta_1}$ and $\psi_2(t) = (\theta_2 t + 1)^{-1/\theta_2}$ with $\theta_1 = 5$ and $\theta_2 = 2$, where $\psi(t) = (\theta t + 1)^{-1/\theta}$ for $\theta \geq 1$, is log-convex function and $\phi_2 o \psi_1(t) = 0.5\left((5t+1)^{2/5} - 1\right)$ is superadditive. Clearly, $(\gamma_1, \gamma_2, \gamma_3) \overset{m}{\preceq} (\beta_1, \beta_2, \beta_3)$. The survival functions of $X_{3:3}$ and $Y_{3:3}$ under the Clayton copulas are satisfied the following inequalities:

$$\bar{F}_{X_{3:3}}(1) \approx 0.5332 < 0.6227 \approx \bar{F}_{Y_{3:3}}(1),$$

$$\bar{F}_{Y_{3:3}}(1.2) \approx 0.0850 < 0.2314 \approx \bar{F}_{X_{3:3}}(1.2),$$

which means that these survival functions cross each other and then $X_{3:3}$ and $Y_{3:3}$ are not comparable in the sense of usual stochastic order. □

In the following theorem, we compare two minimum order statistics with respect to the usual stochastic order. Here, we assume that two sets of random variables have the same λ parameter but different β parameters.

Theorem 3.2. *Let $X_i \sim PG(\beta_i, \lambda)$ and $Y_i \sim PG(\gamma_i, \lambda)$ ($i = 1, \cdots, n$) and the associated Archimedean copulas are with generator ψ_1 and ψ_2, respectively. Further, let $\phi_2 \circ \psi_1$ be superadditive, and ψ_1 or ψ_2 be log-convex. Then, we have*

$$(\gamma_1, \cdots, \gamma_n) \overset{m}{\preceq} (\beta_1, \cdots, \beta_n) \Longrightarrow X_{1:n} \leq_{st} Y_{1:n}.$$

Proof: We only prove the result for the case when ψ_1 is log-convex. The result can be proved in the same line for the other case. We have

$$F_{X_{1:n}}(x) = 1 - \psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right], \qquad x > 0,$$

and

$$F_{Y_{1:n}}(x) = 1 - \psi_2\left[\sum_{i=1}^{n}\phi_2\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right], \qquad x > 0.$$

Then, the assumed superadditivity of $\phi_2 \circ \psi_1$ implies that

$$\psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right] \leq \psi_2\left[\sum_{i=1}^{n}\phi_2\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right],$$

or equivalently

$$1 - \psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right] \geq 1 - \psi_2\left[\sum_{i=1}^{n}\phi_2\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right].$$

Thus, to show that

$$1 - \psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right] \geq 1 - \psi_2\left[\sum_{i=1}^{n}\phi_2\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right],$$

it suffices to prove that

$$1 - \psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right] \geq 1 - \psi_1\left[\sum_{k=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\gamma_i}}-1)}\right)\right].$$

Let $\xi(\beta_1, \cdots, \beta_n) = 1 - \psi_1\left[\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right]$. Note that $\xi(\beta_1, \cdots, \beta_n)$ is symmetric with respect to its arguments. Thus, based on Lemma 2.1, we must establish that $\xi(\beta_1, \cdots, \beta_n)$ is Schur-concave in $(\beta_1, \cdots, \beta_n)$, for any fixed $x > 0$. The derivative of $\xi(\beta_1, \cdots, \beta_n)$ with respect to β_i is

$$\frac{\partial\xi(\beta_1, \cdots, \beta_n)}{\partial\beta_i} = -\lambda\ln(x)x^{\beta_i}e^{x^{\beta_i}}e^{-\lambda(e^{x^{\beta_i}}-1)}\psi_1'\left(\sum_{i=1}^{n}\phi_1\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right)\right)\phi_1'\left(e^{-\lambda(e^{x^{\beta_i}}-1)}\right).$$

Then, for any $i \neq j$, we have

$$
\begin{aligned}
& \frac{\partial \xi(\beta_1, \cdots, \beta_n)}{\partial \beta_i} - \frac{\partial \xi(\beta_1, \cdots, \beta_n)}{\partial \beta_j} \\
= \ & -\lambda \ln(x) \psi_1' \left(\sum_{i=1}^{n} \phi_1 \left(e^{-\lambda (e^{x^{\beta_i}} - 1)} \right) \right) \\
& \times \left[\frac{x^{\beta_i} e^{x^{\beta_i}} e^{-\lambda (e^{x^{\beta_i}} - 1)}}{\psi_1' \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_i}} - 1)} \right) \right]} - \frac{x^{\beta_j} e^{x^{\beta_j}} e^{-\lambda (e^{x^{\beta_j}} - 1)}}{\psi_1' \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_j}} - 1)} \right) \right]} \right] \\
\overset{sgn}{=} \ & \ln(x) \left[\frac{x^{\beta_i} e^{x^{\beta_i}} \psi_1 \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_i}} - 1)} \right) \right]}{\psi_1' \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_i}} - 1)} \right) \right]} - \frac{x^{\beta_j} e^{x^{\beta_j}} \psi_1 \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_j}} - 1)} \right) \right]}{\psi_1' \left[\phi_1 \left(e^{-\lambda (e^{x^{\beta_j}} - 1)} \right) \right]} \right] \\
= \ & \ln(x) \left[\frac{x^{\beta_i} e^{x^{\beta_i}} \psi_1(u)}{\psi_1'(u)} - \frac{x^{\beta_j} e^{x^{\beta_j}} \psi_1(v)}{\psi_1'(v)} \right], \qquad (3.7)
\end{aligned}
$$

where $u = \phi_1 \left(e^{-\lambda (e^{x^{\beta_i}} - 1)} \right)$ and $v = \phi_1 \left(e^{-\lambda (e^{x^{\beta_j}} - 1)} \right)$. Now, consider the following two cases.

Case I: Let $x \geq 1$. Then, for $\beta_i \geq \beta_j$, we have $u \geq v$. Further, the log-convexity of ψ_1 implies

$$
0 \leq -\frac{\psi_1(v)}{\psi_1'(v)} \leq -\frac{\psi_1(u)}{\psi_1'(u)}. \qquad (3.8)
$$

Since $x^{\beta} e^{x^{\beta}}$ is increasing in $\beta > 0$, for any fixed $x \geq 1$, we have

$$
x^{\beta_i} e^{x^{\beta_i}} \quad \geq \quad x^{\beta_j} e^{x^{\beta_j}}, \text{ for } \beta_i \geq \beta_j. \qquad (3.9)
$$

On combining (3.8) and (3.9), we get

$$
x^{\beta_i} e^{x^{\beta_i}} \frac{\psi_1(u)}{\psi_1'(u)} \quad \leq \quad x^{\beta_j} e^{x^{\beta_j}} \frac{\psi_1(v)}{\psi_1'(v)}.
$$

On using the above inequality together with the fact "$\ln(x) \geq 0$ for $x \geq 1$" in (3.7), we get

$$
\left(\beta_i - \beta_j \right) \left(\frac{\partial \xi(\beta_1, \cdots, \beta_n)}{\partial \beta_i} - \frac{\partial \xi(\beta_1, \cdots, \beta_n)}{\partial \beta_j} \right) \leq 0.
$$

Case II: Let $0 < x \leq 1$. Then, for $\beta_i \geq \beta_j$, we have $u \leq v$. Again, the log-convexity of ψ_1 implies

$$
0 \leq -\frac{\psi_1(u)}{\psi_1'(u)} \leq -\frac{\psi_1(v)}{\psi_1'(v)}. \qquad (3.10)
$$

Since $x^{\beta}e^{x^{\beta}}$ is decreasing in $\beta > 0$, for any fixed $0 < x \leq 1$, we have

$$x^{\beta_i}e^{x^{\beta_i}} \quad \leq \quad x^{\beta_j}e^{x^{\beta_j}}, \qquad \text{for } \beta_i \geq \beta_j. \tag{3.11}$$

On combining (3.10) and (3.11), we get

$$x^{\beta_i}e^{x^{\beta_i}}\frac{\psi_1(u)}{\psi_1'(u)} \quad \geq \quad x^{\beta_j}e^{x^{\beta_j}}\frac{\psi_1(v)}{\psi_1'(v)}.$$

Using the above inequality together with the fact "$\ln(x) \leq 0$ for $0 < x \leq 1$" in (3.7), we get

$$\left(\beta_i - \beta_j\right)\left(\frac{\partial\xi(\beta_1, \cdots, \beta_n)}{\partial\beta_i} - \frac{\partial\xi(\beta_1, \cdots, \beta_n)}{\partial\beta_j}\right) \leq 0.$$

On combining Cases I and II, we get that $\xi(\beta_1, \cdots, \beta_n)$ is Schur-concave in $(\beta_1, \cdots, \beta_n)$ and hence, the desired result is obtained. □

Remark 1. It is to be noted that the sufficient conditions "$\phi_2 \circ \psi_1$ is superadditive" and "ψ_1 or ψ_2 is log-convex", given in Theorems 3.1 and 3.2, are hold true for many copulas. We can easily verify that these conditions indeed hold for the Clayton copula with generator $\psi(t) = (\theta t + 1)^{-1/\theta}$ for $\theta \geq 0$, and the Ali-Mikhail-Haq (AMH) copula with generator $\psi(t) = (1 - \theta)/(e^t - \theta)$ for $\theta \in [0, 1)$.

Remark 2. The sufficient conditions, as mentioned in Remark 1, can nicely be interpreted as follows.

(i) Let $K_{\psi_1}(u)$ and $K_{\psi_2}(u)$ be two n-dimensional Archimedean copulas with generators ψ_1 and ψ_2, respectively, and let ϕ_1 and ϕ_2 be their respective pseudo-inverses. Then the fact "$\phi_2 \circ \psi_1$ is supperadditive" implies $C_{\psi_1}(u) \leq C_{\psi_2}(u)$, for all $u \in [0, 1]^n$. Moreover, for some sub-families of Archimedean copulas, the superadditivity of $\phi_2 \circ \psi_1$ roughly means that the Kendall's τ for the copula with generator ψ_2 is greater than that with generator ψ_1. Consequently, it represents more positive dependent (Li and Fang (2015));

(ii) Let the joint distribution of (X_1, X_2) be described by the Archimedean copula $K_\psi(u, v) = \psi[\phi(u) + \phi(v)]$. Then, the log-convexity (log-concavity) of ψ implies that (X_1, X_2) satisfies the TP_2 (RR_2) property (Karlin (1968)).

Nadeb et al. (2021) provided some general results for the usual stochastic ordering of the extreme order statistics arising from two sets of random variables with different marginal distributions and different underlying Archimedean copulas structure. It is important to note that the following results are different from the results given by Nadeb et al. (2021), because $F_X(x; \beta) = 1 - e^{-\lambda(e^{x^\beta} - 1)}$ and also $\bar{F}_X(x; \beta) = e^{-\lambda(e^{x^\beta} - 1)}$ are not log-concave with respect to β.

Torrado (2021) obtained distribution-free results to compare coherent systems, in the usual stochastic order under some majorization conditions, with heterogeneous and dependent components where the dependency structure can be defined by any copula. One of the assumptions in the Torrado (2021) results is that $F_X(x;\beta)$ be decreasing (increasing) with respect to β. It should be noted that our results are different from the results given by Torrado (2021), because $F_X(x;\beta) = 1 - e^{-\lambda(e^{x^\beta}-1)}$ is not decreasing (increasing) with respect to β.

Das et al. (2022) also provided some general results for the usual stochastic order between extreme order statistics when the parameter vectors verify the p-larger order or the reciprocally majorization order. Besides, extreme order statistics arising from the dependent MPHRS and MPRHRS models are compared in the sense of the reversed hazard rate order and the hazard rate order as well. Their botained results are based on increasing (decreasing) property of $F_X(x;e^\beta)$ and $F_X(x;\frac{1}{\beta})$ with respect to β. It should be noted that our results are different from the results given by Das et al. (2022), because $F_X(x;e^\beta) = 1 - e^{-\lambda(e^{x^{e^\beta}}-1)}$ and $F_X(x;\frac{1}{\beta}) = 1 - e^{-\lambda(e^{x^{1/\beta}}-1)}$ is not decreasing (increasing) with respect to β.

3.2 Dispersive Order

In this subsection, all results are studied in terms of the dispersive order under Archimedean copula for the joint distribution of random variables.

In the following theorem, we compare two maximum order statistics with respect to the dispersive order. Here, we assume that one of them is formed from a heterogeneous sample with heterogeneity with respect to the λ parameter, and the other one is formed from a homogeneous sample.

Theorem 3.3. *Let $X_i \sim PG(\beta,\lambda_i)$ and $Y_i \sim PG(\beta,\mu)$, $i = 1,\cdots,n$, and let both samples have the same Archimedean copula with generator ψ. Suppose that $\left(\ln(1-\psi)^{-\frac{1}{\mu}}+1\right)\frac{1-\psi}{\psi'}$ is concave. Then,*

(i) for $\beta \geq 1$ and $\mu = \frac{1}{n}\sum_{i=1}^{n}\lambda_i$, we have $X_{n:n} \geq_{disp} Y_{n:n}$;

(ii) for $\beta = 1$ and $\mu \leq \frac{1}{n}\sum_{i=1}^{n}\lambda_i$, we have $X_{n:n} \geq_{disp} Y_{n:n}$.

Proof: Note that

$$F_{X_{n:n}}(x) = \psi\left[\sum_{i=1}^{n}\phi\left(1-e^{-\lambda_i(e^{x^\beta}-1)}\right)\right], \qquad x > 0,$$

and

$$F_{Y_{n:n}}(x) = \psi\left[n\phi\left(1-e^{-\mu(e^{x^\beta}-1)}\right)\right], \qquad x > 0.$$

From these, we have

$$f_{X_{n:n}}(x) = \beta x^{\beta-1} e^{x^\beta} \psi' \left[\sum_{i=1}^{n} \phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right) \right] \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i (e^{x^\beta}-1)}}{\psi' \left[\phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right) \right]},$$

and

$$f_{Y_{n:n}}(x) = \beta x^{\beta-1} e^{x^\beta} \psi' \left[n\phi\left(1 - e^{-\mu (e^{x^\beta}-1)}\right) \right] \frac{n\mu e^{-\mu (e^{x^\beta}-1)}}{\psi' \left[\phi\left(1 - e^{-\mu (e^{x^\beta}-1)}\right) \right]}.$$

Further, note that, for $x > 0$,

$$F^{-1}_{Y_{n:n}}(x) = \left[\ln\left(\ln\left(1 - \psi\left[\frac{1}{n}\phi(x)\right]\right)^{-\frac{1}{\mu}} + 1 \right) \right]^{\frac{1}{\beta}},$$

and hence,

$$F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x)) = \left[\ln\left(\ln\left(1 - \psi\left[\frac{1}{n}\sum_{i=1}^{n} \phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right)\right]\right)^{-\frac{1}{\mu}} + 1 \right) \right]^{\frac{1}{\beta}}.$$

Let $L_1(\lambda; x) = \psi\left[\frac{1}{n}\sum_{i=1}^{n} \phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right)\right]$ and $l_1(\lambda; x) = \psi'\left[\frac{1}{n}\sum_{i=1}^{n} \phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right)\right]$. It is easy to observe that

$$\begin{aligned} f_{Y_{n:n}}(F^{-1}_{Y_{n:n}}(F_{X_{n:n}})) &= n\mu\beta \left[\ln\left(\ln(1 - L_1(\lambda; x))^{-\frac{1}{\mu}} + 1 \right) \right]^{\frac{\beta-1}{\beta}} \left(\ln(1 - L_1(\lambda; x))^{-\frac{1}{\mu}} + 1 \right) \\ &\quad \times \frac{1 - L_1(\lambda; x)}{l_1(\lambda; x)} \psi' \left[\sum_{i=1}^{n} \phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right) \right]. \end{aligned}$$

(i) Since $\left(\ln(1-\psi)^{-\frac{1}{\mu}} + 1\right) \frac{1-\psi}{\psi'}$ is concave and $\mu \le \frac{1}{n}\sum_{i=1}^{n} \lambda_i$, we get

$$\begin{aligned} \left(\ln(1 - L_1(\lambda; x))^{-\frac{1}{\mu}} + 1\right) \frac{1 - L_1(\lambda; x)}{l_1(\lambda; x)} &\ge \frac{1}{n\mu} \sum_{i=1}^{n} \frac{\left(\lambda_i \left(e^{x^\beta} - 1\right) + \mu\right) e^{-\lambda_i (e^{x^\beta}-1)}}{\psi'\left(\phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right)\right)} \\ &\ge \frac{1}{n\mu} \sum_{i=1}^{n} \frac{\lambda_i e^{x^\beta} e^{-\lambda_i (e^{x^\beta}-1)}}{\psi'\left(\phi\left(1 - e^{-\lambda_i (e^{x^\beta}-1)}\right)\right)}. \qquad (3.12) \end{aligned}$$

Let us set $\mu = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$. Because e^x is convex, we have

$$e^{-\mu\left(e^{x^\beta}-1\right)} = e^{-\frac{1}{n}\sum_{i=1}^{n} \lambda_i\left(e^{x^\beta}-1\right)} \le \frac{1}{n}\sum_{i=1}^{n} e^{-\lambda_i\left(e^{x^\beta}-1\right)},$$

and then

$$1-e^{-\mu\left(e^{x^\beta}-1\right)} \;\; \geq \;\; 1-\frac{1}{n}\sum_{i=1}^{n} e^{-\lambda_i\left(e^{x^\beta}-1\right)} = \frac{1}{n}\sum_{i=1}^{n}\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right).$$

Since ϕ is a convex function, we have

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n}\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right)\right) \leq \frac{1}{n}\sum_{i=1}^{n}\phi\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right).$$

Also, since $\psi'(x) \leq 0$, we get

$$\begin{aligned} L_1(\lambda; x) &= \psi\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right)\right] \\ &\leq \psi\left[\phi\left(\frac{1}{n}\sum_{i=1}^{n}\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right)\right)\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(1-e^{-\lambda_i\left(e^{x^\beta}-1\right)}\right) \leq 1-e^{-\mu\left(e^{x^\beta}-1\right)}. \end{aligned} \tag{3.13}$$

From (3.13), we conclude that

$$x \;\; \geq \;\; \left[\ln\left(\ln\left(1-L_1(\lambda; x)\right)^{-\frac{1}{\mu}}+1\right)\right]^{\frac{1}{\beta}}.$$

For $\beta \geq 1$, we have

$$x^{\beta-1} \;\; \geq \;\; \left[\ln\left(\ln\left(1-L_1(\lambda; x)\right)^{-\frac{1}{\mu}}+1\right)\right]^{\frac{\beta-1}{\beta}}. \tag{3.14}$$

In view of (3.12) and (3.14), we get

$$f_{X_{n:n}}(x) \;\; \geq \;\; f_{Y_{n:n}}(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))),$$

which completes the proof of Part (i).

(ii) According to (3.12), it can be seen that $\left(\ln\left(1-\psi\right)^{-\frac{1}{\mu}}+1\right)\frac{1-\psi}{\psi'}$ is concave, $\mu \leq \frac{1}{n}\sum\limits_{i=1}^{n}\lambda_i$ and $\beta = 1$. Then

$$f_{X_{n:n}}(x) \;\; \geq \;\; f_{Y_{n:n}}(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x))),$$

which completes the proof of Part (ii). □

In the following theorem, we establish the dispersive order between minimum order statistics from dependent powerd Gompertz random variables.

Theorem 3.4. *Let $X_i \sim PG(\beta, \lambda_i)$ and $Y_i \sim PG(\beta, \mu)$, $i = 1, \cdots, n$, and let both samples have the same Archimedean copula with generator ψ.*

(i) If $\left(\ln(\psi)^{-\frac{1}{\mu}} + 1\right)\frac{\psi}{\psi'}$ is convex (concave) and $t\phi'(t)$ is increasing (decreasing), then, for $\beta = 1$ and $\mu \geq (\leq\ )\frac{1}{n}\sum_{i=1}^{n}\lambda_i$, we have $X_{1:n} \leq_{disp} (\geq_{disp}) Y_{1:n}$;

(ii) If $\left(\ln(\psi)^{-\frac{1}{\mu}} + 1\right)\frac{\psi}{\psi'}$ is concave, $\ln\psi$ is concave and $t\phi'(t)$ is decreasing, then, for $\beta \geq 1$ and $\mu = \frac{1}{n}\sum_{i=1}^{n}\lambda_i$, we have $X_{1:n} \geq_{disp} Y_{1:n}$.

Proof: We have

$$F_{X_{1:n}}(x) = 1 - \psi\left[\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right], \qquad x > 0,$$

and

$$F_{Y_{1:n}}(x) = 1 - \psi\left[n\phi\left(e^{-\mu(e^{x^\beta}-1)}\right)\right], \qquad x > 0.$$

From these, we get

$$f_{X_{1:n}}(x) = \beta x^{\beta-1}e^{x^\beta}\psi'\left[\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]\sum_{i=1}^{n}\frac{\lambda_i e^{-\lambda_i(e^{x^\beta}-1)}}{\psi'\left[\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]},$$

and

$$f_{Y_{1:n}}(x) = \beta x^{\beta-1}e^{x^\beta}\psi'\left[n\phi\left(e^{-\mu(e^{x^\beta}-1)}\right)\right]\frac{n\mu e^{-\mu(e^{x^\beta}-1)}}{\psi'\left[\phi\left(e^{-\mu(e^{x^\beta}-1)}\right)\right]}.$$

It is easy to observe that

$$F^{-1}_{Y_{1:n}}(x) = \left[\ln\left(\ln\left(\psi\left[\frac{1}{n}\phi(1-x)\right]\right)^{-\frac{1}{\mu}} + 1\right)\right]^{\frac{1}{\beta}}.$$

Therefore, we have

$$F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x)) \quad = \quad \left[\ln\left(\ln\left(\psi\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]\right)^{-\frac{1}{\mu}} + 1\right)\right]^{\frac{1}{\beta}}.$$

Set $L_2(\lambda; x) = \psi\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]$ and $l_2(\lambda; x) = \psi'\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]$. It can be seen that

$$\begin{aligned} f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}})) \quad &= \quad n\mu\beta\left[\ln\left(\ln(L_2(\lambda; x))^{-\frac{1}{\mu}} + 1\right)\right]^{\frac{\beta-1}{\beta}}\left(\ln(L_2(\lambda; x))^{-\frac{1}{\mu}} + 1\right) \\ &\quad \times\frac{L_2(\lambda; x)}{l_2(\lambda; x)}\psi'\left[\sum_{i=1}^{n}\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]. \end{aligned}$$

(i) Since $\left(\ln(\psi)^{-\frac{1}{\mu}}+1\right)\frac{\psi}{\psi'}$ is a convex (concave) function, $t\phi'(t)$ is increasing (decreasing) in $t\in[0,1]$ and $\mu\geq(\leq\ )\frac{1}{n}\sum\limits_{i=1}^{n}\lambda_i$, then

$$\begin{aligned}\left(\ln(L(\lambda;x))^{-\frac{1}{\mu}}+1\right)\frac{L_2(\lambda;x)}{l_2(\lambda;x)} &\leq (\geq)\frac{1}{n\mu}\sum_{i=1}^{n}\frac{\left(\lambda_i\left(e^{x^\beta}-1\right)+\mu\right)e^{-\lambda_i(e^{x^\beta}-1)}}{\psi'\left[\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]}\\ &\leq (\geq)\frac{1}{n\mu}\sum_{i=1}^{n}\frac{\lambda_i e^{x^\beta}e^{-\lambda_i(e^{x^\beta}-1)}}{\psi'\left[\phi\left(e^{-\lambda_i(e^{x^\beta}-1)}\right)\right]}. \end{aligned}\tag{3.15}$$

Also, according to (3.15), it can be see that, if $\beta=1$, then

$$f_{X_{1:n}}(x)\quad\leq(\geq)\quad f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))),$$

which completes the proof of Part (i).

(ii) The log-concavity of ψ and "$\mu=\frac{1}{n}\sum\limits_{i=1}^{n}\lambda_i$" together imply

$$\ln(L_2(\lambda;x))^{-\frac{1}{\mu}}=-\frac{1}{\mu}\ln(L_2(\lambda;x))\quad\leq\quad\frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_i}{\mu}(e^{x^\beta}-1)=(e^{x^\beta}-1).\tag{3.16}$$

Thus, we have

$$\left[\ln\left(\ln(L_2(\lambda;x))^{-\frac{1}{\mu}}+1\right)\right]^{\frac{1}{\beta}}\quad\leq\quad x.$$

Therefore, for $\beta\geq1$, we have

$$\left[\ln\left(\ln(L_2(\lambda;x))^{-\frac{1}{\mu}}+1\right)\right]^{\frac{\beta-1}{\beta}}\quad\leq\quad x^{\beta-1}.\tag{3.17}$$

In view of (3.16) and (3.17), we get

$$f_{X_{1:n}}(x)\quad\geq\quad f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))),$$

which completes the proof of Part (ii).

Remark 3. It is useful to note that the stated conditions in the Theorems 3.3 and 3.4 can be verified for some Archimedean copulas. Consider the following cases.

1. If $\psi(x)=e^{-x^\theta}$, for $\theta\in(0,1]$, then both $ln(\psi)$ and $\left(-\frac{1}{\mu}\ln(\psi)+1\right)\frac{\psi}{\psi'}$ are convex, and $t\phi'(t)$ is increasing in $t\in[0,1]$;
2. If $\psi(x)=\frac{1}{2e^x-1}$, then both $ln(\psi)$ and $\left(-\frac{1}{\mu}\ln(\psi)+1\right)\frac{\psi}{\psi'}$ are convex, and $t\phi'(t)$ is increasing in $t\in[0,1]$;
3. If $\psi(x)=(0.5(e^x+1))^{-2}$, then both $ln(\psi)$ and $\left(-\frac{1}{\mu}\ln(\psi)+1\right)\frac{\psi}{\psi'}$ are concave, and $t\phi'(t)$ is decreasing in $t\in[0,1]$;
4. If $\psi(x)=1-(1-e^{-x})^\theta$, for $\theta\geq1$, then both $ln(\psi)$ and $\left(-\frac{1}{\mu}\ln(\psi)+1\right)\frac{\psi}{\psi'}$ are concave and $t\phi'(t)$ is decreasing in $t\in[0,1]$.

4 Concluding Remarks

In this paper, some stochastic comparisons results for extreme order statistics are discussed in terms of the usual stochastic order and the dispersive order. We assume that the underlying sets of random variables are dependent under Archimedean copulas, and follow the powered Gompertz distribution.

The powered Gompertz distribution is one of the popular lifetime distributions that has the bathtub shaped failure rate function. Note that the bathtub shaped failure rate distribution has a wide range of applications in reliability and related fields. Furthermore, in this study, we consider the Archimedean copulas which are not only capable of describing a wide spectrum of dependency structures but also computationally convenient. Thus, the study conducted here may have wider applications compared to those existing in the literature.

We conclude our discussion by mentioning some open problems. In our study, we only consider the usual stochastic order and the dispersive order. The study of the same problem, as done here, with respect to other stochastic orders (namely, hazard rate order, reversed hazard rate order, likelihood ratio order and etc.) may be explored in future.

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