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Ruin Probabilities for Two Risk Models with Asymptotically Independent and Dependent Classes

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Abstract. This paper presents two new insurance risk models for analyzing the ruin probabilities. Firstly, we restrict ourselves to the classical risk model contains heavy-tailed distribution of individual net losses and changeable premium income rates. Under certain technical assumptions, some asymptotic expansions and recursive formulas are obtained for the ruin probabilities. In the second risk model, we assume that the different classes of the portfolio business are dependent and compute the finite time ruin probability based on the discretization of the distribution function. We present some numerical examples in the portfolio of business and show that the value of ruin probability increases as dependence level increases. Moreover, the sensitivity of the results are investigated with respect to the parameters of Weibull and Exponential distributions.

Keywords. Classical risk model, Dependence structure, Fourier transform method, Heavy-tailed distribution, Ruin probability.

MSC: 62G32, 62F99, 62E20.

1 Introduction

Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claim paid and ruin probability is a main area in this field. In the risk theory, work concerning the financial surplus of insurance companies in continuous time has been proceeding for nearly a century. The evaluation of ruin

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probabilities strongly depends on the distribution of the claim amounts. In its simplest form, when certain events occur, an insurance contract will provide the policyholder the right to claim all or a portion of the loss. In exchange for this entitlement, the policyholder pays a specified amount called the premium and the insurer is obligated to honor its promises when they come due. In order to ensure that they will be able to pay its promised obligations, the insurance company sets aside amount called the reserve or surplus from which they can draw from when claims are due.

Discrete-time risk models themselves are also interesting stochastic models both in theory and in application, and some continuous time risk models can be approximated by discrete-time risk models. See, for example, Asmussen (2010) and references therein. Cai (2002) considered a dependent model for rates of interest, in which the rates are assumed to have an AR(1) structure. As for asymptotic formulas for ruin probabilities in risk model, Tang and Tsitsiashvili (2003) derived asymptotic formulas for the finite time ruin probability when the interest rates are independent and identically distributed (i.i.d.) random variables and loss distribution is heavy-tailed. Cai and Dickson (2004) computed the finite and infinite time ruin probabilities in a discrete-time model with a Markov chain interest model. Chen and Su (2004) obtained a precise asymptotic estimate for the finite time ruin probability in a discrete-time risk model, in which the risk model is assumed to be heavy-tailed distribution.

However, such an independent assumption was proposed mainly for the mathematical tractability rather than the practical relevance. Therefore, in recent years, more and more researchers have started to improve the model through introducing suitable dependence structures between the insurance risk and the financial risk. Chen (2011) computed the finite time ruin probability with dependent insurance and financial risks in a risk model. Yang et al. (2012) considered the discrete-time risk model with insurance risk and financial risk in some dependence structures to compute the ruin probabilities. Yang et al. (2014) derived a precise asymptotic formula for the ruin probabilities in an insurance risk model that both insurance risk and financial risk are taken into account. Sun and Wei (2014) considered a dependent insurance risk model in which the insurer makes both risk free and risky investments and obtained the ruin probabilities. Yang and Konstantinides (2012) derived the precise estimates for ruin probabilities in a discrete-time insurance risk model with dependent financial and insurance risks under the assumption that the distribution of insurance risk within one time period is consistently varying-tailed. Liu and Wang (2016) computed the ruin probabilities of a discrete-time risk model with dependent insurance and financial risks. Liu et al. (2018a) computed the finite time ruin probability of a discrete-time risk model with GARCH discounted factors and dependent risks when the common distribution of claim sizes is heavy-tailed distribution. Liu et al. (2018b) obtained some asymptotic estimates for the ruin probabilities of the discrete-time risk model with dependent claim sizes and dependent relation between insurance risks and financial risks.

Jig et al. (2020) considered a discrete-time risk model with dependence structures and computed the asymptotic estimates for finite time ruin probability with CMC simulations. Santana and Rincón (2020) obtained the approximations of ruin probability in a discrete-time risk model. Pachon et al. (2021) investigated the discrete-time risk model to show the relationships with respect to the continuous time case. Nakade and Karim (2022) obtained the equilibrium equations on steady-state probability in the discrete-time Markov process. Bazyari (2022a) derived a recursive expression for the finite time ruin probability in a generalized dual Binomial risk model where the periodic premium is one. Bazyari (2022b) derived the ruin probabilities in a discrete-time risk process with homogeneous markov chain and presented some numerical illustrations for the results. Bazyari (2023a) studied the discrete-time risk process with capital injections and reinsurance to compute the ruin probabilities.

In the present paper, our motivation is to find the more general recursive formulas for ruin probabilities and we do this by considering the asymptotically independent property of insurance risk random variables and their density functions. We consider two constructions of risk models and study the ruin probabilities in these risk models with assumption of asymptotically independent and dependent classes between the claim amounts of insurance risks. In the first model, we assume that the individual net losses belong to a heavy-tailed class of distributions and in the second model the discrete-time risk model with correlated classes of business is examined.

The remainder of this paper is organized as follows. Section 2 is concerned with structure of models. Moreover, we give some definitions on heavy-tailed distributions and the results on the aggregation of dependent random variables. Section 3 deals with the main theorem and some lemmas for computing the asymptotic ruin probabilities for the first risk model. In Section 4, we present the proof of main theorem. Also, two real examples are given to compute the numerical asymptotic ruin probabilities. In Section 5, we compute the finite time ruin probability and study two dependent class models in the portfolio of business. In addition, we give the numerical examples for different statistical distributions to obtain the ruin probabilities. Finally, conclusion is given in Section 6.

2 Structure of Models: Description and Notations

a) Let u be a positive real number, $\{U_n, n = 1, 2, \ldots\}$ be a sequence of random variables and $\{r_n \geq 0, n = 1, 2, \ldots\}$ be a sequence non-negative real numbers. Consider the following insurance risk process:

$$U_0 = u, \qquad U_n = U_{n-1}(r_n + 1) - X_n, \qquad n = 1, 2, \ldots \tag{2.1}$$

In the context of insurance risk modeling, U_n stands for the insurance company's surplus at the end of period n, u represents the initial capital at time 0, $r_n(\geq 0)$ denotes the premium income rate during the n th year, and X_n denotes the net loss for the n th year, which are calculated at the end of n, $n = 1, 2, \ldots$. In fact, X_n captures the insurance risk, i.e., the total claim amount minus the total premium incomes, during period n.

The risk model (2.1) is an extension of surplus process of the company which given in Tang (2004), where he considered the constant $r(\geq 0)$ as a constant interest rate, but in our paper we have considered it as a general flexible quantity which might change every year.

Formally, the ruin probability within finite time horizon $[0, n]$ is defined as

$$\psi(u, n) = P\Big(\min_{0\leq i\leq n} U_i < 0 | U_0 = u\Big), \qquad u \geq 0, \tag{2.2}$$

and the infinite ruin probability and infinite time horizon is defined as

$$\psi(u) = P\Big(\min_{0\leq i\leq \infty} U_i < 0 | U_0 = u\Big), \qquad u \geq 0. \tag{2.3}$$

We note that obviously $\psi(u, n) < \psi(u)$. However, the infinite time ruin probability may be sometimes also relevant for the finite time case (see Bazyari (2023b) for more details).

b) Another type of the risk process is presented. The individual model is a natural construct for a life insurance portfolio or a pension fund. (At a given time, the insureds of a portfolio and the pension fund's members are well known.) Their characteristics, sex, age, face amounts, etc., are also available as are good estimates of the needed biometric functions (probability of death, etc.). However, there is an implicit assumption underlying the use of an individual model in these contexts: the group is closed. Beginning with an initial surplus u, when time is measured in discrete units, the process is a discrete one and the surplus at the end of time period n is defined by

$$R_n = u + cn - S_n, \tag{2.4}$$

where $S_n = \sum_{k=1}^{n} Y_k$ is the aggregate claims constitute a compound Poisson process, c is the annual premium income constant over each period and Y_k, $k = 1, 2, \ldots, n$ represents the claim amount in period k, which is a sequence of independent and identically distributed (i.i.d.) random variables with $E(Y_k) - \mu_Y < c$. The probability distribution and density function of random variable Y_k, $k = 1, 2, \ldots, n$, are denoted by $F(y)$ and $f(y)$, respectively.

We can rewrite the process (2.4) as follows:

$$R_n = u + (c - Y_1) + (c - Y_2) + \cdots + (c - Y_n). \tag{2.5}$$

Let T be the time of ruin defined as $T = \inf\{n : R_n < 0\}$. If $R_n \geq 0$, then $T = \infty$ for all $k = 1, 2, \ldots$. Formally, the ruin probability within finite time horizon $[0, n]$ is defined as

$$\psi'(u, 1, n) = P\Big(T \leq n | R_0 = u\Big), \qquad u \geq 0,$$

where $R_0 = u$ stands for the insurance company's surplus at time $n = 0$. Also, the infinite ruin probability and infinite time horizon is defined as

$$\psi'(u) = P\Big(T < \infty | R_0 = u\Big), \qquad u \geq 0.$$

Note that, the finite and infinite time horizon non-ruin probabilities are given by $\phi'(u,1,n) = 1 - \psi'(u,1,n)$ and $\phi'(u) = 1 - \psi'(u)$, respectively. Given equation (2.5), we have

$$\begin{aligned} \phi'(u,1,n) &= P\Big(R_1 \geq 0, R_2 \geq 0, \ldots, R_n \geq 0\Big) \\ &= \Big(Y_1 \leq u + c, Y_1 + Y_2 \leq u + 2c, \ldots, Y_1 + Y_2 + \cdots + Y_n \leq u + nc\Big). \end{aligned}$$

A closed formula for computing the finite time ruin probability will be given in Section 5.

2.1 Ruin probabilities

We will obtain the asymptotic analysis results for finite and infinite ruin probabilities when it is assumed to incorporate dependence between the individual net losses. Consider the surplus process given in equation (2.1), we get that

$$U_0 = u, \qquad U_n = u\Pi_{k=1}^{n}(r_k + 1) - \sum_{k=1}^{n} X_k \Pi_{i=k+1}^{n}(r_i + 1), \qquad n = 1, 2, \ldots,$$

for all $u \geq 0$. Therefore, we can rewrite equations (2.2) and (2.3) based on the discounted values of the surplus process as

$$\psi(u,n) = P\Big(\min_{0 \leq i \leq n} \Pi_{k=0}^{i}(r_k + 1)^{-1} U_i < 0\Big) = P\Big(\max_{0 \leq i \leq n} \sum_{k=0}^{i} X_k \Pi_{j=0}^{k}(r_j + 1)^{-1} > u\Big), \tag{2.6}$$

and

$$\psi(u) = P\Big(\min_{0 \leq i \leq \infty} \Pi_{k=0}^{i}(r_k + 1)^{-1} U_i < 0\Big) = P\Big(\max_{0 \leq i \leq \infty} \sum_{k=0}^{i} X_k \Pi_{j=0}^{k}(r_j + 1)^{-1} > u\Big), \tag{2.7}$$

which we assume that $X_0 = 0$ and $r_0 = 0$. The computation of equations (2.6) and (2.7) will be given in Section 3.

2.2 Preliminaries and some Definitions

The given definitions in this section will be sufficient to gain an overview of the work in this paper and to understand the motivation behind the definitions. As such we shall restrict ourselves in this section to considering non-negative random variables, with distribution function F supported on the positive real axis $[0, \infty)$. In this paper, we suppose that $X_1, X_2, \ldots$ form a sequence of identically distributed random variables, upper tail independent with generic random variable X and heavy-tailed common distribution $F(x)$, for which

$$\lim_{x \to \infty} \frac{F(-x)}{\bar{F}(x)} = 0, \tag{2.8}$$

where $\bar{F}(x) = 1 - F(x)$. For random variable X, the distribution function satisfying $\bar{F}(x) > 0$ for all $x \in (-\infty, \infty)$ is called heavy-tailed to the right, or simply heavy-tailed, if $E\Big(e^{\tau X}\Big) = \int_0^\infty e^{\tau x} F(dx) = \infty$ for all $\tau > 0$. Hereafter, all limit relationships hold for x tending to ∞ unless otherwise stated. For two positive functions $A(.)$ and $B(.)$, if $\limsup_{x\to\infty} \frac{A(x)}{B(x)} \leq 1$, we write $A(x) <^{\sim} B(x)$, if $\liminf_{x\to\infty} \frac{A(x)}{B(x)} \geq 1$, we write $A(x) >^{\sim} B(x)$, if both, $A(x) \sim B(x)$ and $\liminf_{x\to\infty} \frac{A(x)}{B(x)} \leq \limsup_{x\to\infty} \frac{A(x)}{B(x)} < \infty$, we write $A(x) \sim B(x)$.

In the risk theory, heavy-tailed distributions are often used to model large claim amounts and they play a key role in insurance and finance. We will restrict the insurance risk distribution to some classes of heavy-tailed distributions.

Definition 2.1. A distribution $F(x)$ on $(0, \infty)$ belongs to the long-tailed class $\mathcal{L}$ if for some $y > 0$,

$$\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = 1.$$

One easily sees that, for every distribution $F(x) \in \mathcal{L}$, there is a function $a(.)$ such that $a(.) : [0, \infty) \to [0, \infty)$ and the followings hold simultaneously

$$a(x) \to \infty, \qquad a(x) = o(x), \qquad \bar{F}(x \mp a(x)) \sim \bar{F}(x), \tag{2.9}$$

where $o(x)$ is a symbol for a function of x that grows slowly than x, $x \to \infty$, i.e. $\lim_{x\to\infty} \frac{o(x)}{x} = 0$.

Definition 2.2. The distribution function $F(x)$ on $(0, \infty)$ belongs to the subexponential class $\mathcal{S}$ if and only if for some $n = 2, 3, , \ldots,$

$$\lim_{x\to\infty} \frac{\bar{F}^{*n}(x)}{\bar{F}(x)} = n.$$

where F^{*n} denotes the $n-$fold convolution of function F. Subexponential distribution functions are of interest in the theory of branching processes, and in queueing theory; see for example, Pakes (1975) and Teugels (1975). Also, a distribution function K on $(-\infty, \infty)$ is still said to be subexponential if $K(x)I_{(0\leq x<\infty)}$ is subexponential, where I_A denotes the indicator function of A.

Definition 2.3. (Embrechts et al. (1997)). The distribution function $F(x)$ on the real number belongs to the dominatedly varying-tailed class V if $\bar{F}(x) = 1 - F(x)$ and for some $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} < \infty.$$

Clearly, if $F(x) \in V$, then for every $y > 0$, we have

$$0 < \liminf_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \leq \limsup_{x\to\infty} \frac{\bar{F}(xy)}{\bar{F}(x)} < \infty. \tag{2.10}$$

Definition 2.4. (Embrechts et al. (1997)). The distribution function $F(x)$ belongs to the consistently varying-tailed class C if and only if

$$\lim_{y\to 1}\liminf_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1,\qquad \textit{or equivalently}\qquad \lim_{y\to 1}\limsup_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=1.$$

Definition 2.5. (Embrechts et al. (1997)). The distribution function $F(x)$ on the real number belongs to the regularly varying-tailed class $\mathcal{R}_\alpha$ if for all x, $\bar{F}(x)>0$, and for all $y>0$,

$$\lim_{x\to\infty}\frac{\bar{F}(xy)}{\bar{F}(x)}=y^{-\alpha},$$

holds for some $0\le\alpha<\infty$.

Remark 1. If the distribution of random variable X belongs to class $V\cap\mathcal{L}$, then for any constant K, the distribution of random variable KX belongs to class $V\cap\mathcal{L}$.

Let the distribution $F(x)$ be concentrated on $(-\infty,\infty)$, for any $\eta>0$, we set

$$\bar{F}_1(\eta)=\lim\inf_{x\to\infty}\frac{\bar{F}(x\eta)}{\bar{F}(x)},\qquad \bar{F}_2(\eta)=\lim\sup_{x\to\infty}\frac{\bar{F}(x\eta)}{\bar{F}(x)},$$

and, in the terminology of Bingham et al. (1987), we define two notations J_F^+ and J_F^- as the upper and lower index of the non-negative and nondecreasing function $f(x)=\big(F(x)\big)^{-1}$, $x\ge 0$,

$$J_F^+=-\lim_{\eta\to\infty}\frac{\log\bar{F}_1(\eta)}{\log(\eta)},\qquad J_F^-=-\lim_{\eta\to\infty}\frac{\log\bar{F}_2(\eta)}{\log(\eta)}.$$

Moreover, we call the notation $J_F^\pm$ as the upper/lower index of $F(x)$. From Tang and Tsitsiashvili (2004), if for some α, such that $0\le\alpha<\infty$, $F\in\mathcal{R}_\alpha$, then $J_F^\pm=\alpha$; if for some α and β with $0\le\alpha\le\beta<\infty$, then $\alpha\le J_F^+\le J_F^-\le\beta$, and if $F\in V$, then $0\le J_F^+\le J_F^-<\infty$. For any distribution $F(x)$ with $0\le J_F^-\le\infty$ and $0<h\le J_F^-$, from Proposition 2.2.1 given in Bingham et al. (1987), there are positive constants r and x_0 such that

$$\frac{\bar{F}(xy)}{\bar{F}(x)}\le ry^{-h}, \tag{2.11}$$

holds for all $xy\ge x\ge x_0$.

2.3 Dependence Structure Between the Insurance Risks

We next introduce the dependence between the insurance risks via some restrictions on their copula function.

Theorem 2.1. *(Sklar's theorem). Let $F_{X_1,X_2}(x_1,x_2)$ be the joint distribution function of two random variables X_1 and X_2 with marginal distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$. Then the dependence structure of two random variables X_1 and X_2 is determined by a bivariate copula function C, such that*

$$F_{X_1,X_2}(x_1,x_2)=C\big(F_{X_1}(x_1),F_{X_2}(x_2)\big).$$

(See Nelsen (2006) to prove this theorem). The copula function links the marginal distributions together to form the joint distribution. In fact, the Sklar's theorem allows us to separate the modelling of the marginal distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ from the dependence structure, which expressed in C.

Definition 2.6. (Tail dependence). Let $X = (X_1, X_2)$ be a two dimensional continuous random vector. We say X is bivariate upper tail dependent if:

$$\lambda_U = \lim_{v \to 1^{-1}} P\Big(X_1 > F_{X_1}^{-1}(v) | X_2 > F_{X_2}^{-1}(v)\Big) > 0,$$

in case the limit exists. The notatons F_1^{-1} and F_2^{-1} denote the generalized inverse distribution functions of X_1 and X_2, respectively. Consequently, we say X is bivariate upper tail independent if λ_U equals to 0.

The following representation shows that tail dependence is a copula property. Thus, many copula features transfer to the tail dependence coefficient such as the invariance under strictly increasing transformations of the margins. For the continuous random variable $X = (X_1, X_2)$, straightforward calculation yields:

$$\lambda_U = \lim_{v \to 1^{-1}} \frac{1 - 2v + C(v, v)}{1 - v},$$

where C is a bivariate copula function (see Joe (1997) for more information).

Based on Resnick (2002), the asymptotic independence of two random variables X_1 and X_2 is equivalent to the following

$$\lim_{x \to \infty} \frac{P\Big(X_1 > x, X_2 > x\Big)}{P(X_1 > x)} = 0. \tag{2.12}$$

In general, for the identically distributed random variables $\{X_i, i \geq 1\}$, and for any $i \neq j$ and $j \geq 1$, if

$$\lim_{x \to \infty} \frac{P\Big(X_i > x, X_j > x\Big)}{P(X_i > x)} = 0, \tag{2.13}$$

then we say $\{X_i, i \geq 1\}$ is asymptotically independent.

Assumption 2.1. Let $X_1, X_2, \ldots, X_n$ be n dependent random variables with distributions $F_1, F_2, \ldots, F_n$, respectively. We assume that

$$\lim_{x_i \wedge x_j \to \infty} \frac{P\Big(X_i > x_i, X_j > x_j\Big)}{P(X_j > x_j)} = 0,$$

where $x_i \wedge x_j = \min\{X_i, X_j\}$.

2.4 Some Results on the Aggregation of Dependent Random Variables

For a random variable Y, $P_Y(t) = E(Y^t)$, $M_Y(t) = E(e^{tY})$ and $\varphi(t) = E(e^{itY})$ denote the probability generating function, moment generating function and characteristic function, respectively. Then, for random variables $Y_1, Y_2, \dots, Y_k$, we have

$$\begin{aligned} P_{Y_1,Y_2,\dots,Y_k}(t_1,t_2,\dots,t_k) &= E\Big(t_1^{Y_1}t_2^{Y_2}\dots t_k^{Y_k}\Big), \\ M_{Y_1,Y_2,\dots,Y_k}(t_1,t_2,\dots,t_k) &= E\Big(e^{t_1Y_1+t_2Y_2+\cdots+t_kY_k}\Big) = P_{Y_1,Y_2,\dots,Y_k}(e^{t_1},e^{t_2},\dots,e^{t_k}), \\ \varphi_{Y_1,Y_2,\dots,Y_k}(t_1,t_2,\dots,t_k) &= E\Big(e^{i(t_1Y_1+t_2Y_2+\cdots+t_kY_k)}\Big) = P_{Y_1,Y_2,\dots,Y_k}(e^{it_1},e^{it_2},\dots,e^{it_k}). \end{aligned}$$

Now, for k random variables $Y_1, Y_2, \dots, Y_k$, if $S = Y_1 + Y_2 + \cdots + Y_k$, then $P_S(t)$, $M_S(t)$ and $\varphi_S(t)$ are given as follow:

$$\begin{aligned} P_S(t) = E(t^S) = E\Big(t_1^{Y_1}t_2^{Y_2}\dots t_k^{Y_k}\Big) &= P_{Y_1,Y_2,\dots,Y_k}(t,t,\dots,t), \\ M_S(t) = E(e^{tS}) = E\Big(e^{t(Y_1+Y_2+\cdots+Y_k)}\Big) &= M_{Y_1,Y_2,\dots,Y_k}(t,t,\dots,t), \\ \varphi_S(t) = E(e^{itS}) = E\Big(e^{it(Y_1+Y_2+\cdots+Y_k)}\Big) &= \varphi_{Y_1,Y_2,\dots,Y_k}(t,t,\dots,t). \end{aligned}$$

By inverting $\varphi(t)$ using the Fast Fourier Transform method, the probability distribution function of S can be obtained. (see Embrechts et al. (1993) for the application of Fast Fourier Transform method in insurance mathematics).

For i.i.d random variables $Y_1, Y_2, \dots, Y_N$, let $S = Y_1 + Y_2 + \cdots + Y_N$, then

$$\varphi_S(t) = E(e^{itS}) = E\Big(E\Big(e^{it(Y_1+Y_2+\cdots+Y_k)}\Big)|N\Big) = E\Big(\big(\varphi_S(t)\big)^N\Big) = P_N\Big(\varphi_Y(t)\Big). \tag{2.14}$$

This result can be extended to the multivariate case. Let $S = S^{(1)} + S^{(2)} + \cdots + S^{(m)}$ and $S^{(j)}$, $j = 1, 2, \dots, m$, be the random sum of $N^{(j)}$ i.id. random variables $Y_k^{(j)}$, $k = 1, 2, \dots,$ i.e. $S^{(j)} = \sum_{k=1}^{N^{(j)}} Y_k^{(j)}$. If the random variables $Y_k^{(j)}$, $j = 1, 2, \dots, m$, be dependent, then

$$\varphi_{S^{(1)},S^{(2)},\dots,S^{(m)}}(t,t,\dots,t) = P_{N^{(1)},N^{(2)},\dots,N^{(m)}}(\varphi_{Y^{(1)}}(t_1), \varphi_{Y^{(2)}}(t_2), \dots, \varphi_{Y^{(m}}(t_m)), \tag{2.15}$$

and

$$\varphi_S(t) = \varphi_{S^{(1)},S^{(2)},\dots,S^{(m)}}(t,t,\dots,t). \tag{2.16}$$

3 Results for First Model

In this section, we present the main theorem and some lemmas associated with computing the asymptotic ruin probabilities with asymptotic independence of individual net losses. The proof of theorem will be given in Section 4.

Theorem 3.1. *Let the individual net losses X_i, $i = 1, 2, \dots,$ be identically distributed as generic distribution function $F(x)$ belongs to class $V \cap \mathcal{L}$ such that $J_F^- > 0$. Using equations (2.8) and*

(2.13), we have

i) The asymptotic finite time ruin probability is given by

$$\begin{aligned}\psi(u,n) &\sim P\Big(\sum_{k=1}^{n} X_k^{+}\Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big)\\ &\sim P\Big(\sum_{k=1}^{n} X_k\Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big)\\ &\sim \sum_{k=1}^{n} P\Big(X_k\Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big), \end{aligned} \tag{3.1}$$

where $X^+ = XI_{(x>0)}$.

ii) If for $r_k, k = 1,2,\dots, s_k = (r_k+1)^{-k}$ and for some ζ, $0 < \zeta < \frac{J_F^-}{1+J_F^-}$, the conditions

$$\sum_{k=1}^{\infty} \Pi_{i=1}^{k}(r_i+1)^{-\zeta} < \infty, \qquad \sum_{k=1}^{\infty} s_k^{\zeta} < \infty, \tag{3.2}$$

hold, then the asymptotic infinite time ruin probability is given by

$$\begin{aligned}\psi(u) &\sim P\Big(\sum_{k=1}^{\infty} X_k^{+}\Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big)\\ &\sim \sum_{k=1}^{n} P\Big(X_k\Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big). \end{aligned} \tag{3.3}$$

Proof. To prove this theorem, we first give the following lemmas.

Lemma 3.1. *Suppose that $F_1(x)$ and $F_2(x)$ are two distribution functions concentrated on $(-\infty,\infty)$. Consider the convolution of these distribution functions on $(-\infty,\infty)$ and let $F(x) = F_1 * F_2(x)$. If $F_1(x) \in \mathcal{S}$ and $\bar{F}_2(x) <^{\sim} c\bar{F}_1(x)$ for some $c \geq 0$, then $\bar{F}(x) <^{\sim} (1+c)\bar{F}_1(x)$.*

Proof. See Lemma 4.4 of Tang (2004). □

Lemma 3.2. *Let $\{X_i, i \geq 1\}$ be a sequence of non-negative random variables with distribution functions $F_i(x) \in V \cap \mathcal{L}$, $i = 1,2,\dots$. If Assumption 2.1 holds, then*

$$F_1 * \cdots * F_n(x) \in V \cap \mathcal{L},$$

and for any fixed integer $n \geq 1$, we have

$$P\Big(\sum_{k=1}^{n} X_i > u\Big) \sim \sum_{i=1}^{n} P\Big(X_i > u\Big). \tag{3.4}$$

Proof. To prove the first part, combine the Lemma 4.2 of Ng et al. (2002) and Lemma 1 of Geluk (2004). First, we prove equation (3.4) for two random variables X_1 and X_2 with distribution functions $F_1(x)$ and $F_2(x)$, respectively. By Assumption 2.1, we have

$$P\Big(X_1 + X_2 > x\Big) \geq \bar{F}_1(x) + \bar{F}_2(x) - P\Big(X_1 > x, X_2 > x\Big) \sim \bar{F}_1(x) + \bar{F}_2(x).$$

On the other hand, recalling the function $a(.)$ in Section 2, by using equations (2.9) and (2.10) and Assumption 2.1, we have

$$\begin{aligned} P\Big(X_1 + X_2 > x\Big) \leq \bar{F}_1(x - a(x)) + \bar{F}_2(x - a(x)) &+ P\Big(X_1 > a(x), X_2 > \frac{x}{2}\Big) \\ &+ P\Big(X_1 > \frac{x}{2}, X_2 > a(x)\Big) \\ &\sim \bar{F}_1(x) + \bar{F}_2(x), \end{aligned}$$

therefore, equation (3.4) is proved for two random variables. Now, to prove Equation (3.4) for a sequence of non-negative random variables $\{X_i, i \geq 1\}$, we can apply the same method. □

Lemma 3.3. *Let $\{X_i, i \geq 1\}$ be identically distributed as a generic distribution function $F \in V \cap \mathcal{L}$, such that $J_F^- > 0$. If equation (4.7) of Tang (2004) holds, then we have*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{P\Big(\sum_{k=n}^{\infty} X_k^+ \Pi_{i=1}^k (r_i + 1)^{-1} > u\Big)}{P\Big(X_1 (r_1 + 1)^{-1} > u\Big)} = 0, \tag{3.5}$$

and

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{\sum_{k=n}^{\infty} P\Big(X_k \Pi_{i=1}^k (r_i + 1)^{-1} > u\Big)}{P\Big(X_1 (r_1 + 1)^{-1} > u\Big)} = 0. \tag{3.6}$$

Proof. To prove this lemma, choose h_1 such that $h_1 \in \Big(\zeta, 1 - \frac{\zeta}{J_F^-}\Big)$. Hence by condition (3.2), it holds that $\sum_{k=1}^{\infty} s_k^{h_1} < \infty$, for all large n such that

$$\sum_{k=n}^{\infty} s_k^{h_1} < 1, \qquad \textit{and} \qquad s_1 s_k^{h_1 - 1} > 1 \quad \textit{for all } k \geq n.$$

Then using the above inequalities and well-known Boole's inequality, we have

$$\begin{aligned} P\Big(\sum_{k=n}^{\infty} s_k X_k^+ > u\Big) &\leq P\Big(\sum_{k=n}^{\infty} s_k X_k^+ > \sum_{k=n}^{\infty} s_k^{h_1} u\Big) \\ &\leq P\Big(\cup_{k=n}^{\infty} \Big(s_k X_k^+ > s_k^{h_1} u\Big)\Big) \\ &\leq \sum_{k=n}^{\infty} P\Big(s_k X_k^+ > s_1 s_k^{h_1 - 1} u\Big), \end{aligned} \tag{3.7}$$

since $\sum_{k=n}^{\infty} s_k^{h_1} < 1$, so that by multiplying u on both sides of the inequality, we get $\sum_{k=n}^{\infty} s_k^{h_1} u < u$, then for the first inequality $P\Big(\sum_{k=n}^{\infty} s_k X_k^+ > u\Big) \le P\Big(\sum_{k=n}^{\infty} s_k X_k^+ > \sum_{k=n}^{\infty} s_k^{h_1} u\Big)$.

Also we choose $h_1 > 0$ such that $h_2 \in \Big(\frac{\zeta}{1-h_1}, J_F^-\Big)$. Using equation (2.11) with $h = h_2$ to the right hand side of equation (3.7), for all large $x > 0$, we have

$$\sum_{k=n}^{\infty} P\Big(s_k X_k^+ > s_1 s_k^{h_1-1} u\Big) \le A_1 \sum_{k=n}^{\infty} \Big(s_1 s_k^{h_1-1}\Big)^{h_2} P\Big(s_1 X_1 > u\Big). \tag{3.8}$$

Since the inequality $(1-h_1)h_2 > \zeta$ holds, then by condition (3.2), $\sum_{k=1}^{\infty} \Big(s_1 s_k^{h_1-1}\Big)^{-h_2} < \infty$. Therefore, the inequalities (3.7) and (3.8) give the relation (3.5).

On the other hand, it is clear that for all $x > 0$, $0 < s_k < 1$ and all n such that $k \ge n$, we have the inequality:

$$P\Big(\sum_{k=n}^{\infty} s_k X_k > u\Big) \le \sum_{k=n}^{\infty} P\Big(s_1 X_k^+ > s_1 s_k^{h_1-1} u\Big).$$

The relation (3.6) is a consequence of inequality (3.8) and the convergence of the series $\sum_{k=1}^{\infty} \Big(s_1 s_k^{h_1-1}\Big)^{-h_2}$ and this completes the proof. □

4 Proof of Theorem 3.1.

i) We know that $\{r_n \ge 0, n = 1, 2, \dots\}$, then for $n \ge 1$, the inequality

$$\sum_{k=1}^{n} X_k \Pi_{i=1}^{k} (r_i+1)^{-1} \le \max_{1\le k\le n} \sum_{i=1}^{k} X_i \Pi_{j=1}^{i} (r_j+1)^{-1} \le \sum_{k=1}^{n} X_j^+ \Pi_{i=1}^{k} (r_i+1)^{-1},$$

holds. Thus, to prove the given asymptotic finite time ruin probability in equation (3.1), we should prove that, for $n \ge 1$, the two following relations

$$P\Big(\sum_{k=1}^{n} X_k^+ \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big) \sim \sum_{k=1}^{n} P\Big(X_k \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big), \tag{4.1}$$

and

$$P\Big(\sum_{k=1}^{n} X_k \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big) \gtrsim \sum_{k=1}^{n} P\Big(X_k \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big), \tag{4.2}$$

hold. It is clear that for $u \ge 0$ and $k \ge 1$, the equality

$$P\Big(X_k^+ \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big) = P\Big(X_k \Pi_{i=1}^{k} (r_i+1)^{-1} > u\Big), \tag{4.3}$$

holds. Using equation (4.3) and Lemma 3.2, the approximation (4.1) holds.

Now, we prove the relation (4.2). By equation (4.9) of Tang (2004), it is clear that $F \in \mathcal{L}$. Thus there exists a function g, such that $g(u) \to \infty$ and the following relation holds

$$P\Big(X_k \Pi_{i=1}^k (r_i+1)^{-1} > u + g(u)\Big) \sim P\Big(X_k \Pi_{i=1}^k (r_i+1)^{-1} > u\Big).$$

If $D_k = \Pi_{i=1}^k (r_i+1)^{-1}$, then we have the following inequality

$$\begin{aligned} P\Big(\sum_{k=1}^n X_k \Pi_{i=1}^k (r_i+1)^{-1} > u\Big) \geq \sum_{k=1}^n P\Big(\sum_{k=1}^n X_k \Pi_{i=1}^k (r_i+1)^{-1} > u, X_k D_k > u + g(u)\Big) \\ - \sum_{1 \leq i \leq j \leq n} P\Big(X_i D_i > u + g(u), X_j D_j > u + g(u)\Big). \end{aligned}$$

From equations (2.8) and (2.13) we have

$$\sum_{1 \leq i \leq j \leq n} P\Big(X_i D_i > u + g(u), X_j D_j > u + g(u)\Big) = o\Big(\sum_{k=1}^n \bar{F}\Big(\frac{u}{D_k}\Big)\Big),$$

and

$$\begin{aligned} \sum_{k=1}^n P\Big(\sum_{k=1}^n X_k \Pi_{i=1}^k (r_i+1)^{-1} > u, X_k D_k > u + g(u)\Big) &\geq \sum_{k=1}^n P\Big(X_k D_k > u + g(u)\Big) \\ &\quad - o\Big(\sum_{k=1}^n \bar{F}\Big(\frac{u+g(u)}{D_k}\Big)\Big) \\ &\sim \sum_{k=1}^n P\Big(X_k \Pi_{i=1}^k (r_i+1)^{-1} > u\Big). \end{aligned}$$

Therefore, this completes the proof of the relation (4.2). □

ii) We will prove the given asymptotic infinite time ruin probability in equation (3.3). To prove this part, it is enough to show that

$$\psi(u) \sim \sum_{k=1}^\infty P\Big(X_k \Pi_{i=1}^k (r_i+1)^{-1} > u\Big),$$

and

$$P\Big(\sum_{k=1}^\infty X_k^+ \Pi_{i=1}^k (r_i+1)^{-1} > u\Big) \sim \sum_{k=1}^\infty P\Big(X_k \Pi_{i=1}^k (r_i+1)^{-1} > u\Big).$$

By Lemma 3.3, for any $0 < \epsilon < 1$, there are some $u_0 = u_0(\epsilon)$, and some $t = t(\epsilon) = 1, 2, \ldots,$ such that for all $u > u_0(1 + r_1)$ and all $n \geq t$, the two following inequalities hold:

$$P\Big(\sum_{k=n}^\infty X_k^+ \Pi_{i=1}^k (r_i+1)^{-1} > u\Big) \leq \epsilon P\Big(X_1 (r_1+1)^{-1} > u\Big),$$

and

$$P\Big(\sum_{k=n}^{\infty} X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big) \leq \epsilon P\Big(X_1(r_1+1)^{-1} > u\Big).$$

Applying the first part of Theorem 3.1 to $\psi(u,n)$ with $t = t(\epsilon)$ and $u > u_0$, then with considering attention to the inequality

$$\max_{1\leq n<\infty} \sum_{k=1}^{n} X_k \Pi_{i=1}^{k}(r_i+1)^{-1} \geq \max_{1\leq n<t} \sum_{k=1}^{n} X_k \Pi_{i=1}^{k}(r_i+1)^{-1},$$

we have

$$\begin{aligned}
\psi(u) \geq \psi(u,n) &\geq (1-\epsilon)\Big(\sum_{k=1}^{\infty} P\Big(X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big) - \sum_{k=t+1}^{\infty} P\Big(X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big)\Big) \\
&\geq (1-\epsilon)^2\Big(\sum_{k=1}^{\infty} P\Big(X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big). \qquad (4.4)
\end{aligned}$$

Also, using Lemma 3.1, we have the inequality

$$\begin{aligned}
\psi(u) &\leq P\Big(\sum_{k=1}^{\infty} X_k^{+} \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big) \\
&\leq (1+\epsilon) P\Big(\sum_{k=1}^{n} X_k^{+} \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big) \\
&\leq (1+\epsilon)^2 \sum_{k=1}^{n} P\Big(X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big) \\
&\leq (1+\epsilon)^2 \sum_{k=1}^{\infty} P\Big(X_k \Pi_{i=1}^{k}(r_i+1)^{-1} > u\Big), \qquad (4.5)
\end{aligned}$$

therefore, by equations (4.4) and (4.5), the relation (3.3) holds for any arbitrary $\epsilon > 0$.

4.1 Examples

In this subsection, we give two real examples for the Danish fire insurance and Swedish fire insurance and compute the value of asymptotic ruin probabilities for different values of the premium income rates and initial capitals (in Euro currency).

Examples 4.1. (Embrechts et al. (1997)). The sample consists of 500 large claims from 1st January 1980 till 31st December 1990. The unit is millions of Danish Kroner (1985 prices). The goodness of fit test is done on the data at significance level $\alpha = 0.05$. The $p - value = 0.2891$ for the goodness of fit test is reported. The result indicates evidence for the null hypothesis that the date comes from Pareto distribution. Moreover, the histogram (some claims are far out) and the mean residual life function (which is clearly increasing) in Figure 1 indicate that the distribution function is heavy-tailed. Most likely the larger claims follow a Pareto distribution. The empirical mean residual life function is first decreasing and after that linearly increasing. We consider the linear trend $r_i = 299 + 10i, i = 1, 2, 3, \ldots$, for the premium income rate for computing the ruin probabilities using (3.1) and (3.3). After estimating the parameters of Pareto distribution with maximum likelihood method, the asymptotic ruin probabilities are computed and the results are given in Tables 1 and 2, respectively.

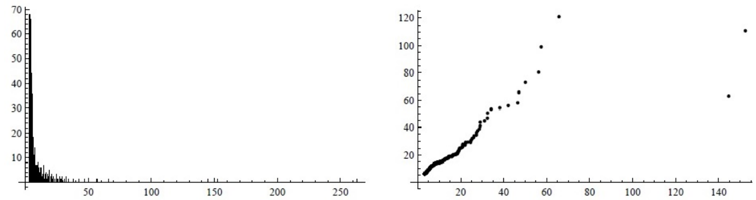

Figure 1: The histogram and the mean residual life function

Table 1: The asymptotic finite time ruin probabilities $\psi(u, n)$

n / u	3	5	7	10	15	18	20	25	30
0	0.375	0.426	0.495	0.528	0.580	0.614	0.643	0.677	0.694
20000	0.346	0.382	0.435	0.476	0.505	0.530	0.560	0.588	0.607
30000	0.283	0.323	0.377	0.407	0.445	0.481	0.502	0.521	0.538
50000	0.262	0.306	0.354	0.380	0.422	0.446	0.463	0.480	0.495
70000	0.244	0.275	0.311	0.358	0.386	0.404	0.425	0.439	0.448
100000	0.229	0.267	0.294	0.325	0.351	0.371	0.391	0.418	0.429
150000	0.218	0.234	0.270	0.310	0.336	0.357	0.369	0.383	0.395

Table 2: The asymptotic infinite time ruin probabilities with $d = 1000$

u	0	$20d$	$30d$	$50d$	$70d$	$100d$	$150d$	$200d$	$300d$
$\psi(u)$	0.720	0.631	0.576	0.533	0.470	0.456	0.427	0.402	0.388

From these two tables, we result that with decreasing the initial surplus, the ruin probability will be increased. Also, for each value of initial surplus with increasing the time, the ruin probability will be increased. We conclude that the insurance company should be more careful when the heavy-tailed claims accrue in the risk model.

Examples 4.2. (Asmussen (2010)). The sample consists of 218 claims for 1982 in units of Millions of Swedish kroner. The goodness of fit test is done on the data at significance level $\alpha = 0.05$. The $p - value = 0.6325$ for the goodness of fit test is reported. The result indicates evidence for the null hypothesis that the date comes from Lognormal distribution. Also, the histogram (some claims are far out) and the mean residual life function (which is clearly increasing) in Figure 2 indicate that the distribution function is heavy-tailed. The empirical mean residual life function is first decreasing and after that linearly increasing and this indicates the larger claims follow a Lognormal distribution. Similar to Example 1, we consider the linear trend $r_i = 299 + 10i, i = 1, 2, 3, \ldots,$ for the premium income rate for computing the ruin probabilities. After estimating the parameters of Lognormal distribution with maximum likelihood method, the asymptotic ruin probabilities are computed and the results are given in Tables 3 and 4, respectively.

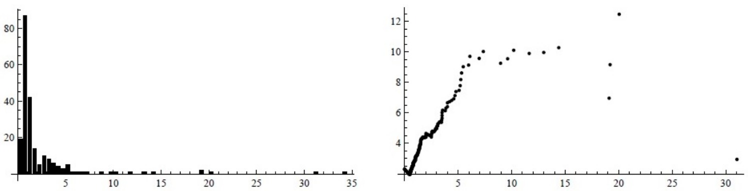

Figure 2: The histogram and the mean residual life function

Table 3: The asymptotic finite time ruin probabilities $\psi(u, n)$

n / u	3	5	7	10	15	18	20	25	30
0	0.415	0.455	0.491	0.538	0.596	0.621	0.633	0.751	0.768
20000	0.355	0.374	0.422	0.461	0.493	0.525	0.541	0.570	0.595
30000	0.310	0.343	0.381	0.412	0.457	0.474	0.497	0.542	0.578
50000	0.276	0.305	0.345	0.378	0.406	0.427	0.443	0.471	0.499
70000	0.242	0.296	0.318	0.341	0.370	0.395	0.418	0.442	0.490
100000	0.215	0.250	0.234	0.257	0.291	0.330	0.352	0.383	0.416
150000	0.184	0.216	0.239	0.258	0.281	0.307	0.319	0.341	0.372

Table 4: The asymptotic infinite time ruin probabilities with $d = 1000$

u	0	20 d	30d	50d	70d	100d	150d	200d	300d
$\psi(u)$	0.794	0.624	0.593	0.530	0.524	0.448	0.407	0.387	0.364

Similar to Example 1, since the Lognormal distribution is a subclass of heavy-tailed distributions, therefore these claims are also dangerous for the insurance company.

5 Ruin probability for Second Insurance Risk Model

In this section, we give a formula for computing the finite time ruin probability of second risk model and present two dependent class models in the portfolio of business.

Theorem 5.1. *Suppose that $Y_k, k = 1, 2, \ldots, n$, be a sequence of i.i.d. random variables. Then the finite time non-ruin probability is*

$$\varphi'(u, 1, n) = \int_0^{u+c} \varphi'\Big(u + c - y, 1, n - 1\Big) dF_Y(y). \tag{5.1}$$

Proof. From the probability theory, we have

$$\varphi'(u,1,n) = \int_0^{u+c} \varphi'\Big(u+c-y,2,n\Big)dF_Y(y),$$

where

$$\varphi'(q,l,n) = P\Big(Y_l \le q+c, Y_l + Y_{l+1} \le q+2c, \dots, Y_l + Y_{l+1} + \cdots + Y_{n-l} \le q + c(n-l)\Big).$$

Since $Y_k, k = 1,2,\dots,n$, are a sequence of i.i.d. random variables, then $\varphi'(q,2,n) = \varphi'(q,1,n-1)$ and this completes the proof. □

From equation (5.1) the computation of non-ruin is not easy. To compute the approximation of $\varphi'(q,1,n)$, an algorithm in Theorem 5.2 is presented. In this algorithm the discretization of the distribution function F_Y is used. Discretization methods are given in Toth and Houtte (1992), Klugman et al. (1998) and Campos (2014).

Let $F_{\tilde{Y}}$ be the discretization distribution function derived by one of these methods and $\tilde{Y}$ be the discrete random variable. If $P(\tilde{Y} = d) = f_d$, $d = 0,1,\dots,M$, then

$$F_{\tilde{Y}}(d) = P(\tilde{Y} = d) = \sum_{j=0}^{d} f_j,$$

where f_j is the mass probability.

Let the constant integers p and k denote the premium income and initial surplus respectively, and the surplus process takes only integer values. Also the notations $\psi'_{k,1,n}$ and $\varphi'_{k,1,n}$ denote the finite time ruin and finite time non-ruin probabilities computed using the discretization of the distribution function F_Y over the interval $[1,n]$.

Theorem 5.2. *For constant integers p, j and k, the finite time non-ruin probability can be calculated from the following recursive formula*

$$\varphi'_{k,1,n} = \sum_{j=0}^{\min(k+p,M)} \varphi'_{k+p-j,1,n-1} f_j, \qquad n = 2,3,\dots, \tag{5.2}$$

where

$$\varphi'_{k,1,1} = F_{\min(k+p,M)} = \sum_{j=0}^{\min(k+p,M)} f_j, \qquad n = 2,3,\dots, \tag{5.3}$$

Proof. The proof of this theorem comes from Theorem 5.1. □

Using the Fast Fourier Transform method, an approximation for the discretization distribution function F_Y will be obtained. The computed function $F_{\tilde{Y}}$ is used in equations (5.2) and (5.3) for the estimation of the non-ruin probability.

5.1 Two Dependent Class Models in the Portfolio of Business

In this section, the Poisson model and Negative Binomial model with common component and dependent class in the portfolio of business are considered. In these models, it is assumed that the structure of the models is consisted of m dependent classes of business and the total claim amount of business in period $i, i = 1, 2, \ldots,$ is given by

$$Y_i = Y_{i,1} + Y_{i,2} + \cdots + Y_{i,m},$$

where $Y_{i,j}$ denotes the total claim amounts for the jth class of business in the period i and for any $i \neq v$, Y_i and Y_v are i.i.d. For a fixed period $i, i = 1, 2, \ldots,$ different classes of business are assumed to be dependent. For the class of business $j, j = 1, 2, \ldots, m$, in the period i, the notations $Y_{i,j,k}$ and $N_{i,j}$ denote the kth individual claim and number of claims, respectively. Then $Y_{i,j} = \sum_{k=1}^{N_{i,j}} Y_{i,j,k}$. For j fixed, with $F_{Y^{(j)}}(0) = 0$, denotes the common distribution function of i.i.d. random variables $Y_{i,j,k}$, $i, k = 1, 2, \ldots, N_{i,j}$. We denote by $Y^{(j)}$ a random variable with distribution function $F_{Y^{(j)}}(y)$. The nth moment of $F_{Y^{(j)}}(y)$ is denoted by $\mu_n^{(j)}$ with $\mu_1^{(j)} = \mu^{(j)}$.

Also for j fixed, $N_{i,j}, i = 1, 2, \ldots,$ are identically distributed random variables and denoted by $N^{(j)}$ a random variable with their distribution function. Similarly, for j fixed, $Y_{i,j}, i = 1, 2, \ldots,$ are identically distributed random variables and denotes by $Y^{0(j)}$ a random variable with this distribution function. It is supposed that $N^{(j)}$ and $Y^{(j)}$ are independent random variables. For the class of business, for j fixed with positive risk margin γ_j and any period $i, i = 1, 2, \ldots,$ the premium income is

$$c_j = E\left(Y^{0(j)}\right)(1 + \gamma_j) = \mu^{(j)} E\left(N^{(j)}\right)(1 + \gamma_j), \qquad j = 1, 2, \ldots, m.$$

Therefore, the premium income for the business in the period $i, i = 1, 2, \ldots,$ is $c = c_1 + c_2 + \cdots + c_m$.

5.2 Poisson Distribution with Dependent Classes

In this subsection, Poisson distribution with m dependent classes of business is considered. Suppose that for any fixed period $i, i = 1, 2, \ldots,$ the random variables $N_{i,1}$, $N_{i,2}$ and $N_{i,3}$ are identically distributed and define $N^{(j)}$, $j = 1, 2, 3$, as follows:

$$\begin{aligned} N^{(1)} &= N^{(11)} + N^{(12)} + N^{(13)} + N^{(123)}, \\ N^{(2)} &= N^{(22)} + N^{(12)} + N^{(23)} + N^{(123)}, \\ N^{(3)} &= N^{(33)} + N^{(13)} + N^{(23)} + N^{(123)}, \end{aligned}$$

where for $a, b = 1, 2, 3$, the random variables $N^{(ab)}$ and $N^{(123)}$ are distributed as $Poisson(\lambda_{ab})$ and $Poisson(\lambda_{123})$, respectively. Now, since the distribution of sum of n independent Poisson random variables $Y_1, Y_2, \ldots, Y_n$ with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, is distributed as Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i$, then we get that the random variable $N^{(p)}, p = 1, 2, 3$, is distributed as Poisson distribution with parameter λ_p, where for any fixed p, the parameters λ_1, λ_2 and λ_3 are defined by

$$\begin{aligned}\lambda_1 &= \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{123},\\ \lambda_2 &= \lambda_{22} + \lambda_{12} + \lambda_{23} + \lambda_{123},\\ \lambda_3 &= \lambda_{33} + \lambda_{13} + \lambda_{23} + \lambda_{123}.\end{aligned}$$

Also, for any $a \neq b$ the covariance between two random variables $N^{(a)}$ and $N^{(b)}$ is defined by

$$Cov\Big(N^{(a)}, N^{(b)}\Big) = Var\Big(N^{(ab)}\Big) + Var\Big(N^{(123)}\Big).$$

The joint probability generating function of random variables $\Big(N^{(1)}, N^{(2)}, N^{(3)}\Big)$ is

$$\begin{aligned}P_{N^{(1)},N^{(2)},N^{(3)}}(t_1, t_2, t_3) &= E\Big(t_1^{\left(N^{(11)}+N^{(12)}+N^{(13)}+N^{(123)}\right)} t_2^{\left(N^{(22)}+N^{(12)}+N^{(23)}+N^{(123)}\right)} t_3^{\left(N^{(33)}+N^{(13)}+N^{(23)}+N^{(123)}\right)}\Big)\\ &= \Big\{\Pi_{j=1}^{3} E\Big(t_j^{N^{(jj)}}\Big)\Big\} E\Big((t_1t_2)^{N^{(12)}}\Big) E\Big((t_1t_3)^{N^{(13)}}\Big) E\Big((t_2t_3)^{N^{(23)}}\Big) E\Big((t_1t_2t_3)^{N^{(123)}}\Big)\\ &= \exp\Big(\sum_{j=1}^{3}(\lambda_{jj}t_j - 1)\Big) + \exp\Big(\sum_{j<k}(t_jt_k - 1)\Big) + \exp\Big(\lambda_{123}(t_1t_2t_3 - 1)\Big).\end{aligned}$$

Given equation (2.15), the characteristic function of random variables $\Big(Y^{(1)}, Y^{(2)}, Y^{(3)}\Big)$ is

$$\begin{aligned}\varphi_{Y_1,Y_2,Y_3}(t_1, t_2, t_3) &= P_{N^{(1)},N^{(2)},N^{(3)}}\Big(\varphi_{Y^{(1)}}(t_1), \varphi_{Y^{(2)}}(t_2), \varphi_{Y^{(3)}}(t_3)\Big)\\ &= \exp\Big(A_1 + A_2 + A_3\Big),\end{aligned}$$

where

$$A_1 = \sum_{j=1}^{3} \lambda_{jj}(\varphi_{Y^{(j)}}(t_j) - 1), \qquad A_2 = \sum_{j<k} \lambda_{jk}\Big(\varphi_{Y^{(j)}}(t_j)\varphi_{Y^{(k)}}(t_k) - 1\Big),$$

and

$$A_3 = \lambda_{123}\Big(\varphi_{Y^{(1)}}(t_1)\varphi_{Y^{(2)}}(t_2)\varphi_{Y^{(3)}}(t_3) - 1\Big).$$

Using equations (2.14) and (5.4), we have

$$\varphi_Y(t) = \varphi_{Y^{(1)},Y^{(2)},Y^{(3)}}(t, t, t) = \exp\Big(\lambda\varphi_Y(t) - 1\Big), \tag{5.4}$$

where $\lambda_1 = \lambda_{11} + \lambda_{22} + \lambda_{33} + \lambda_{12} + \lambda_{13} + \lambda_{23} + \lambda_{123}$, and

$$\begin{aligned}\varphi_Y(t) &= \frac{\lambda_{11}}{\lambda}\varphi_{Y^{(1)}}(t) + \frac{\lambda_{22}}{\lambda}\varphi_{Y^{(2)}}(t) + \frac{\lambda_{33}}{\lambda}\varphi_{Y^{(3)}}(t) + \frac{\lambda_{12}}{\lambda}\varphi_{Y^{(1)}+Y^{(2)}}(t)\\ &\quad + \frac{\lambda_{13}}{\lambda}\varphi_{Y^{(1)}+Y^{(3)}}(t) + \frac{\lambda_{23}}{\lambda}\varphi_{Y^{(2)}+Y^{(3)}}(t) + \frac{\lambda_{123}}{\lambda}\varphi_{Y^{(1)}+Y^{(2)}+Y^{(3)}}(t),\end{aligned}$$

where $\varphi_{Y+Z}(t) = \varphi_Y(t)\varphi_Z(t)$.

For correlated compound Poisson random variables $Y^{(1)}, Y^{(2)}, Y^{(3)}$, the random variable $Y = \sum_{i=1}^{3} Y^{(i)}$ has a compound Poisson distribution, as for the independent case mentioned in the Appendix, but with different parameter λ and different claim size characteristic function $\varphi_Y(t)$, which is associated to the distribution function

$$\begin{aligned} F_Y(t) &= \frac{\lambda_{11}}{\lambda}F_{Y^{(1)}}(t) + \frac{\lambda_{22}}{\lambda}F_{Y^{(2)}}(t) + \frac{F_{33}}{\lambda}F_{Y^{(3)}}(t) + \frac{\lambda_{12}}{\lambda}F_{Y^{(1)}+Y^{(2)}}(t) \\ &\quad + \frac{\lambda_{13}}{\lambda}F_{Y^{(1)}+Y^{(3)}}(t) + \frac{\lambda_{23}}{\lambda}F_{Y^{(2)}+Y^{(3)}}(t) + \frac{\lambda_{123}}{\lambda}F_{Y^{(1)}+Y^{(2)}+Y^{(3)}}(t), \end{aligned}$$

with

$$\begin{aligned} \mu &= \frac{\lambda_{11}}{\lambda}\mu^{(1)} + \frac{\lambda_{22}}{\lambda}\mu^{(2)} + \frac{F_{33}}{\lambda}\mu^{(3)} + \frac{\lambda_{12}}{\lambda}\Big(\mu^{(1)} + \mu^{(2)}\Big) \\ &\quad + \frac{\lambda_{13}}{\lambda}\Big(\mu^{(1)} + \mu^{(3)}\Big) + \frac{\lambda_{23}}{\lambda}\Big(\mu^{(2)} + \mu^{(3)}\Big) + \frac{\lambda_{123}}{\lambda}\Big(\mu^{(1)} + \mu^{(2)} + \mu^{(3)}\Big) \\ &= \frac{1}{\lambda}\Big(\lambda_1\mu^{(1)} + \lambda_2\mu^{(2)} + \lambda_3\mu^{(3)}\Big). \end{aligned}$$

5.3 Negative Binomial Model with Common Component

For Poisson random variable, modelling the number of claims N means that the variance $Var(N)$ is equal to $E(N)$. But from Panjer and Willmot (1992), sometimes in practice may be the inequality $Var(N) > E(N)$ establishes. The Negative Binomial distribution is used to model claim numbers in such situations. For random variable N, the probability function of Negative Binomial is

$$P(N = n) = \binom{n + \alpha - 1}{\alpha - 1}\Big(\frac{1}{\alpha + \beta}\Big)^{\alpha}\Big(\frac{\beta}{\alpha + \beta}\Big)^{n}, \qquad \alpha, \beta > 0, \qquad n = 0, 1, 2, \ldots.$$

Besides, for Negative Binomial distribution, the probability generating function is $\Big(1 - \beta(t-1)\Big)^{-\alpha}$.

As similar to the Poisson distribution, we consider m dependent classes of business. Assume that for the jth, $j = 1, 2, 3$, class of business the number of claim sizes is the sum of two random variables. The notation $N^{(jj)}$ denotes for the first random variable, which is specific to each class and is independent of the specific random variables of the other classes. For any fixed i, the random variables $N^{(j)}$, $j = 1, 2, 3$, are independent. The notation $N^{(j0)}$ denotes for the second random variable. It is assumed that, there is a dependence structure between the second random variables of the different classes.

The random variables $N^{(j0)}$, $j = 1, 2, 3$, are dependent and if the random variable Θ is distributed as $Gamma(\alpha_0, 1)$, then $N^{(j0)}$, $j = 1, 2, 3$, are distributed as Poisson Gamma mixture distribution $N^{(j0)}|\Theta = \theta \sim Poisson(\theta\beta_j)$, where $N^{(j0)}|\Theta = \theta$, $j = 1, 2, 3$, are independent random variables. (See Kocherlakota and Kocherlakota, (1992), for more details).

For any fixed j, $j = 1, 2, 3$, the random variable $N^{(j)}$, is defined by

$$N^{(j)} = N^{(jj)} + N^{(j0)},$$

where the random variables $N^{(jj)}$ and $N^{(j0)}$ are distributed as Negative Binomial distribution $NB(\alpha_{jj}, \beta_j)$ and $NB(\alpha_0, \beta_j)$, respectively. Therefore, the random variable $N^{(j)}$, $j = 1, 2, 3$, is distributed as Negative Binomial distribution $NB(\alpha_j, \beta_j)$, where $\alpha_j = \alpha_{jj}+\alpha_0$.

The joint probability distribution function of random variables $\Big(N^{(10)}, N^{(20)}, N^{(30)}\Big)$ is as follows

$$\begin{aligned} P_{N^{(10)},N^{(20)},N^{(30)}}(t_1, t_2, t_3) &= E\Big(E\Big(t_1^{N^{(10)}} t_2^{N^{(20)}} t_3^{N^{(30)}}\Big)\Big) \\ &= M_{\Theta}\Big(\beta_1(t_1 - 1) + \beta_2(t_2 - 1) + \beta_3(t_3 - 1)\Big) \\ &= \Big(1 - \beta_1(t_1 - 1)\Big) - \beta_2(t_2 - 1) - \beta_3(t_3 - 1)\Big)^{-\alpha_0}. \end{aligned}$$

Therefore, the probability generating function of $\Big(N^{(1)}, N^{(2)}, N^{(3)}\Big)$ is

$$\begin{aligned} P_{N^{(1)},N^{(2)},N^{(3)}}(t_1, t_2, t_3) &= E\Big(t_1^{\left(N^{(11)}+N^{(10)}\right)} t_2^{\left(N^{(22)}+N^{(20)}\right)} t_3^{\left(N^{(33)}+N^{(30)}\right)}\Big) \\ &= E\Big(t_1^{\left(N^{(11)}\right)}\Big)E\Big(t_2^{\left(N^{(22)}\right)}\Big)E\Big(t_3^{\left(N^{(33)}\right)}\Big)E\Big(t_1^{N_{10}} t_2^{N_{20}} t_3^{N_{30}}\Big) \\ &= \Pi_{j=1}^{3}\Big(1 - \beta_j(t_j - 1)\Big)^{-\alpha_{jj}} P_{N^{(10)},N^{(20)},N^{(30)}}(t_1, t_2, t_3) \\ &= \Pi_{j=1}^{3}\Big(1 - \beta_j(t_j - 1)\Big)^{-\alpha_{jj}}\Big(1 - \sum_{k=1}^{3} \beta_j(t_j - 1)\Big)^{-\alpha_0}. \end{aligned}$$

Also, for any $a \neq b$, the covariance between two random variables $N^{(a)}$ and $N^{(b)}$ is defined by

$$\begin{aligned} Cov\Big(N^{(a)}, N^{(b)}\Big) &= Cov\Big(N^{(aa)} + N^{(a0)}, N^{(bb)} + N^{(b0)}\Big) \\ &= Cov\Big(E\Big(N^{(a0)}|\Theta\Big), E\Big(N^{(b0)}|\Theta\Big)\Big) + E\Big(Cov\Big(N^{(a0)}, N^{(b0)}\Big)|\Theta\Big) \\ &= \beta_a\beta_b Var(\Theta) = \alpha_0\beta_a\beta_b, \end{aligned}$$

where $N^{(j0)}|\Theta = \theta$, $j = 1, 2, 3$, are independent random variables. The marginal distribution of random variable $W^{(j)}$, $j = 1, 2, 3$, is a compound Negative Binomial with parameter α_j, λ_j and $F_{X^{(j)}}$. The characteristic function of $W^{(j)}$ is $\varphi_{W^{(j)}}(t)$ ans is obtained as

$$\begin{aligned} \varphi_{W^{(j)}}(t) &= \Big(1 - \beta_j(\varphi_{X^{(j)}}(t_j) - 1)\Big)^{-\alpha_{jj}}\Big(1 - \beta_j(\varphi_{X^{(j)}}(t_j) - 1)\Big)^{-\alpha_0} \\ &= \Big(1 - \beta_j(\varphi_{X^{(j)}}(t_j) - 1)\Big)^{-\alpha_j}. \end{aligned}$$

By (2.15), the characteristic function $\left(W^{(1)}, W^{(2)}, W^{(3)}\right)$ is given by

$$\begin{aligned} P_{W^{(1)},W^{(2)},W^{(3)}}(t_1,t_2,t_3) &= P_{N^{(1)},N^{(2)},N^{(3)}}\left(\varphi_{Y^{(1)}}(t_1), \varphi_{Y^{(2)}}(t_2), \varphi_{Y^{(3)}}(t_3)\right) \\ &= \Pi_{j=1}^{3}\left(1-\beta_j(\varphi_{Y^{(j)}}(t_j)-1)\right)^{-\alpha_{jj}} \\ &\quad \times\left(1-\beta_1\left(\varphi_{Y^{(1)}}(t_1)-1\right)-\beta_2\left(\varphi_{Y^{(2)}}(t_2)-1\right)-\beta_3\left(\varphi_{Y^{(3)}}(t_3)-1\right)\right)^{-\alpha_0}. \end{aligned}$$

Using equation (2.14), we have

$$\varphi_W(t) = \varphi_{W^{(1)},W^{(2)},W^{(3)}}(t,t,t). \tag{5.5}$$

Taking inverse of equation (5.5) and using the Fast Fourier Transform method, the probability distribution F_Y of Y will be obtained.

5.4 The Approximation of Function F_Y

In both models, the non-ruin probability can be computed using $F_{\tilde{Y}}$, which is obtained by the Fast Fourier Transform method applied to $\varphi_W(t)$. First discretize the function $F_Y^{(j)}$, $j = 1, 2, \ldots, m$, to take their Fourier Transform (see for example, Panjer and Willmot (1992)). These Fourier Transforms are inverted in either equation (5.4) or (5.5). The obtained results are inverted with the Fast Fourier Transform method produces the vector of mass probabilities defining the probability distribution function $F_{\tilde{Y}}$. This approximation of F_Y is used in equations (5.2) and (5.3) in the computation of $\varphi'_{k,1,n}$.

5.5 Numerical Examples

In this subsection, two examples having Poisson model and Negative Binomial model with common component distributions with two class of business are presented. In both examples, the random variables $Y^{(j)}$, $j = 1, 2$,are distributed as Weibull distribution and $N^{(j)}$, $j = 1, 2$, are distributed as Poisson and Negative Binomial distributions. For $n = 10, 20$, the finite time ruin probabilities $\psi'_{u,1,n}$ and moments of random variables are computed.

Examples 5.1. In this example, we assume that the random variables $Y^{(1)}$ and $Y^{(2)}$ are distributed with *Weibull*(0.5, 1, 1.25) and *Exponential*(2.25) distributions, respectively. Also, the random variables $N^{(j)}$, $j = 1, 2$, are distributed as *Poisson*(4). The moments of random variables $Y^{(i)}$, $N^{(i)}$ and $Y^{0(i)}$ are computed and the results are given in Table 5. The values of covariances and correlations are given in Table 6. Also, the numerical results for the finite time ruin probabilities are presented in Tables 7 and 8 with different initial surplus.

Table 5: The moment values of random variables $Y^{(i)}$, $N^{(i)}$ and $Y^{0(i)}$

Moment	First class of business	Second class of business
$\mathrm{E}\left(Y^{(i)}\right)$	1.125	1.125
$\mathrm{E}\left(Y^{(i)2}\right)$	2.531	2.531
$\mathrm{E}\left(N^{(i)}\right)$	4.00	4.00
$\mathrm{Var}\left(N^{(i)}\right)$	4.00	4.00
$\mathrm{E}\left(Y^{0(i)}\right)$	4.50	4.50
$\mathrm{Var}\left(Y^{0(i)}\right)$	10.125	30.375

Table 6: The values of covariance and correlation

	$\rho\left(N^{(1)}, N^{(2)}\right) = 0$	$\rho\left(N^{(1)}, N^{(2)}\right) = 0.25$	$\rho\left(N^{(1)}, N^{(2)}\right) = 0.75$
λ_0	0	1.00	3.00
$Cov\left(N^{(1)}, N^{(2)}\right)$	0	1.00	3.00
$Cov\left(Y^{0(1)}, Y^{0(2)}\right)$	0	1.265	3.796
$\rho\left(Y^{0(1)}, Y^{0(2)}\right)$	0	0.072	0.216

Table 7: The finite time ruin probability $\psi'(u, 1, 10)$

u	$\psi'(u, 1, 10, 0)$	$\psi'(u, 1, 10, 0.25)$	$\psi'(u, 1, 10, 0.75)$
0	0.7420	0.7821	0.8120
5	0.5031	0.5216	0.5633
10	0.3948	0.4250	0.4476
15	0.2851	0.2974	0.3204
20	0.2104	0.2592	0.2891
25	0.1569	0.1958	0.2570
30	0.0843	0.1435	0.2089
35	0.0472	0.0820	0.1225
40	0.0390	0.0561	0.0843
45	0.0285	0.0343	0.0460
50	0.0137	0.0205	0.0255
60	0.0085	0.0108	0.0174
70	0.0052	0.0067	0.0081
80	0.0031	0.0042	0.0065
90	0.0019	0.0023	0.0052
100	0.0007	0.0010	0.0021
110	0.0004	0.0006	0.0014
120	0.0002	0.0003	0.0008
130	0.0001	0.0001	0.0004
140	0.0000	0.0000	0.0002
150	0.0000	0.0000	0.0001

Table 8: The finite time ruin probability $\psi'(u, 1, 20)$

u	$\psi'(u, 1, 20, 0)$	$\psi'(u, 1, 20, 0.25)$	$\psi'(u, 1, 20, 0.75)$
0	0.6451	0.6791	0.7105
5	0.5620	0.5860	0.6229
10	0.5237	0.5387	0.5611
15	0.4486	0.4610	0.4945
20	0.3171	0.3400	0.4260
25	0.2709	0.2904	0.3827
30	0.2230	0.2515	0.2700
35	0.1384	0.1533	0.2267
40	0.0733	0.1206	0.1805
45	0.0305	0.0811	0.1352
50	0.0278	0.0450	0.0960
60	0.0104	0.0298	0.0437
70	0.0085	0.0100	0.0254
80	0.0025	0.0066	0.0081
90	0.0010	0.0023	0.0057
100	0.0007	0.0009	0.0021
110	0.0004	0.0006	0.0008
120	0.0002	0.0005	0.0007
130	0.0002	0.0003	0.0004
140	0.0000	0.0001	0.0002
150	0.0000	0.0000	0.0001

Examples 5.2. In this example, we assume that the random variables $Y^{(1)}$ and $Y^{(2)}$ are distributed with *Weibull*$(0.5, 1, 1.25)$ and *Exponential*(2.25) distributions, respectively, and the random variables $N^{(j)}$, $j = 1, 2$, are distributed as *Negative Binomial*$(1, 4)$. The moments of random variables $Y^{(i)}$, $N^{(i)}$ and $Y^{0(i)}$ are computed and the results given in Table 9. The values of covariances and correlations are given in Table 10. Also, the numerical results for the finite time ruin probabilities are presented in Tables 11 and 12 with different initial surplus. It is clear that, in both models the value of ruin probability increases as dependence level increases. This result is similar for $\psi'(u, 1, 10)$ and $\psi'(u, 1, 20)$. Also, the ruin probability decreases as the initial surplus increase.

Table 9: The moment values of random variables $Y^{(i)}$, $N^{(i)}$ and $Y^{0(i)}$

Moment	First class of business	Second class of business
$E\left(Y^{(i)}\right)$	1.125	1.125
$E\left(Y^{(i)2}\right)$	2.531	2.531
$E\left(N^{(i)}\right)$	4.00	4.00
$Var\left(N^{(i)}\right)$	20.00	20.00
$E\left(Y^{0(i)}\right)$	4.50	4.50
$Var\left(Y^{0(i)}\right)$	28.125	48.375

Table 10: The values of covariance and correlation

	$\rho\left(N^{(1)}, N^{(2)}\right) = 0$	$\rho\left(N^{(1)}, N^{(2)}\right) = 0.25$	$\rho\left(N^{(1)}, N^{(2)}\right) = 0.75$
α_0	0	0.3125	0.9375
$Cov\left(N^{(1)}, N^{(2)}\right)$	0	5.000	15.000
$Cov\left(Y^{0(1)}, Y^{0(2)}\right)$	0	6.328	18.984
$\rho\left(Y^{0(1)}, Y^{0(2)}\right)$	0	0.1716	0.5147

Table 11: The finite time ruin probability $\psi'(u,1,10)$

u	$\psi'(u,1,10,0)$	$\psi'(u,1,10,0.25)$	$\psi'(u,1,10,0.75)$
0	0.7803	0.7948	0.8125
5	0.6245	0.6502	0.6840
10	0.5902	0.6031	0.6192
15	0.5531	0.5619	0.5955
20	0.4694	0.4820	0.5303
25	0.3900	0.4106	0.4211
30	0.2821	0.3492	0.3509
35	0.2305	0.2701	0.3155
40	0.1558	0.2353	0.2908
45	0.0736	0.1944	0.2330
50	0.0354	0.1200	0.2041
60	0.0267	0.0871	0.1725
70	0.0113	0.0329	0.1288
80	0.0067	0.0176	0.0650
90	0.0033	0.0094	0.0361
100	0.0020	0.0053	0.0097
110	0.0008	0.0022	0.0063
120	0.0006	0.0010	0.0045
130	0.0004	0.0008	0.0021
140	0.0002	0.0005	0.0007
150	0.0001	0.0002	0.0004

Table 12: The finite time ruin probability $\psi'(u,1,20)$

u	$\psi'(u,1,20,0)$	$\psi'(u,1,20,0.25)$	$\psi'(u,1,20,0.75)$
0	0.6925	0.7212	0.7302
5	0.6431	0.6504	0.6640
10	0.6042	0.5601	0.5812
15	0.5884	0.5530	0.5681
20	0.5239	0.5319	0.5490
25	0.4740	0.4900	0.5007
30	0.3115	0.4377	0.4423
35	0.2968	0.3645	0.3815
40	0.2471	0.2816	0.3130
45	0.2093	0.2017	0.2495
50	0.1745	0.1854	0.2001
60	0.0762	0.1205	0.1476
70	0.0253	0.0891	0.1254
80	0.0104	0.0575	0.0715
90	0.0082	0.0288	0.0433
100	0.0054	0.0094	0.0120
110	0.0021	0.0075	0.0082
120	0.0009	0.0021	0.0045
130	0.0006	0.0008	0.0016
140	0.0005	0.0006	0.0007
150	0.0003	0.0005	0.0006

5.6 Sensitivity of the Results with Respect to the Parameters

In this subsection, special consideration is given to the finite time ruin probability within the parameter of distributions. The sensitivity of the obtained results in Examples 3 and 4 are investigated with respect to the parameters of Weibull and Exponential distributions. Firstly, in Example 3 for Weibull distribution with the second parameter $k = 1.125$ and $Y^{(2)}$ distributed as *Exponential*(2.25), we suppose that the first parameter of

Weibull distribution, δ, takes the values $0.025, 0.05, 0.1, 0.2, 0.5, 1, 2, 3, 4, 10, 15, 20$, then the finite time ruin probabilities are computed and results are presented in Table 13. The results show that the finite time ruin probabilities increase as parameter δ, increases. The Graph of ruin probabilities are shown in Figure 3 for some of the parameters. The similar results will be held for any constant k and also we get the similar results for Example 4 when we apply the above changes.

The sensitivity of the results is quite evident with respect to the parameter of Exponential distribution. In this case, when the random variables $Y^{(1)}$ and $Y^{(2)}$ are distributed with $Weibull(0.5, 1, 1.125)$ and $Exponential(\eta)$, $\eta = 0.025, 0.5, 0.1, 1.5, 2.25, 3, 5, 10$ distributions, respectively, the values of finite time ruin probabilities are computed and results are presented in Table 14. Ruin probabilities increase as the parameter of Exponential distribution increases. The Graph of ruin probabilities are shown in Figure 4 for some of the parameters.

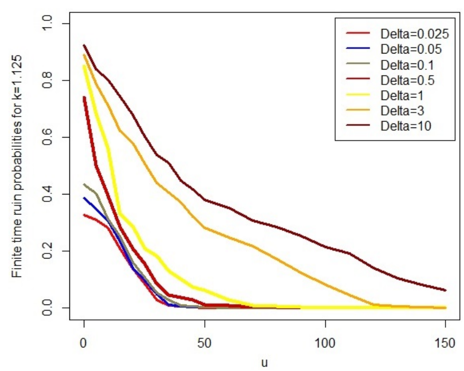

Figure 3: Finite time ruin probabilities with respect to parameter δ

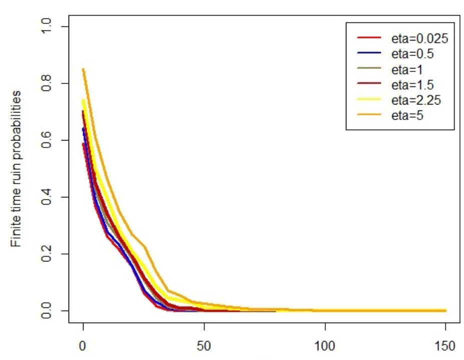

Figure 4: Finite time ruin probabilities with respect to parameter η

Table 13: The finite time ruin probability $\psi'(u, 1, 10)$ for $k = 1.125$

u	$\delta = 0.025$	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.2$	$\delta = 0.5$	$\delta = 1$
0	0.3290	0.3864	0.4352	0.5271	0.7420	0.8513
5	0.3118	0.3512	0.4027	0.4483	0.5031	0.6856
10	0.2820	0.3077	0.3149	0.3309	0.3948	0.5611
15	0.2107	0.2360	0.2504	0.2720	0.2851	0.3320
20	0.1391	0.1433	0.1630	0.1872	0.2104	0.2891
25	0.0854	0.0965	0.1095	0.1245	0.1569	0.2104
30	0.0300	0.0481	0.0554	0.0661	0.0843	0.1833
35	0.0097	0.0132	0.0296	0.0305	0.0472	0.1295
40	0.0062	0.0075	0.0094	0.0194	0.0390	0.1027
45	0.0041	0.0051	0.0071	0.0088	0.0285	0.0744
50	0.0008	0.0032	0.0042	0.0054	0.0137	0.0635
60	0.0007	0.0009	0.0014	0.0029	0.0085	0.0290
70	0.0005	0.0006	0.0008	0.0011	0.0052	0.0093
80	0.0004	0.0005	0.0007	0.0006	0.0031	0.0067
90	0.0001	0.0003	0.0004	0.0005	0.0019	0.0035
100	0.0000	0.0000	0.0001	0.0003	0.0007	0.0010
110	0.0000	0.0000	0.0000	0.0001	0.0004	0.0008
120	0.0000	0.0000	0.0000	0.0000	0.0002	0.0006
130	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004
140	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

u	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 10$	$\delta = 15$	$\delta = 20$
0	0.8824	0.8902	0.9112	0.9256	0.9301	0.9524
5	0.7735	0.7894	0.8145	0.8401	0.8627	0.8730
10	0.6881	0.7130	0.7720	0.8032	0.8145	0.8201
15	0.5507	0.6240	0.6934	0.7455	0.7619	0.7922
20	0.4112	0.5822	0.6523	0.6811	0.6933	0.7105
25	0.3645	0.5075	0.5725	0.6052	0.6341	0.6436
30	0.2952	0.4411	0.5103	0.5378	0.5760	0.5925
35	0.2360	0.4060	0.4760	0.5115	0.5270	0.5557
40	0.2021	0.3731	0.4118	0.4500	0.4736	0.5021
45	0.1879	0.3209	0.3670	0.4192	0.4579	0.4894
50	0.1250	0.2817	0.3209	0.3811	0.3991	0.4231
60	0.0984	0.2500	0.2911	0.3546	0.3600	0.3775
70	0.0465	0.2188	0.2282	0.3091	0.3185	0.3250
80	0.0221	0.1716	0.1930	0.2865	0.2922	0.3001
90	0.0090	0.1240	0.1477	0.2540	0.2768	0.2895
100	0.0053	0.0840	0.1161	0.2141	0.2407	0.2686
150	0.0002	0.0008	0.0084	0.0643	0.0833	0.1134

Also, in Example 3, for Weibull distribution with $\delta = 0.5$ we suppose that the second parameter, k, takes the values $0.025, 0.05, 0.5, 1, 1.125, 2, 5$, and $Y^{(2)}$ is distributed with *Exponential*(2.25), then the finite time ruin probabilities are computed and the results are presented in Table 15. The results show that the finite time ruin probabilities decrease when k increases in the interval $[0, 0.5]$ but the probabilities increase when k increases in the interval $(0.5, \infty]$. These results will be held for any constant δ, which in this case the ruin probabilities decrease when k increases in the interval $(0, \delta]$, and the probabilities increase when k increases in the interval (δ, ∞). The similar results will be held for example 4 when we consider Weibull distribution with constant δ for different values of parameter δ. Therefore, it doesn't matter if the random variables $N^{(j)}$, $j = 1, 2$, are distributed as Poisson or Negative Binomial distribution.

Table 14: The finite time ruin probability $\psi'(u,1,10)$ for different parameters of Exponential distribution

u	$\eta=0.025$	$\eta=0.5$	$\eta=1$	$\eta=1.5$	$\eta=2.25$	$\eta=3$	$\eta=5$	$\eta=10$
0	0.5908	0.6430	0.6844	0.7036	0.7420	0.7931	0.8524	0.9152
5	0.3703	0.3916	0.4297	0.4532	0.5031	0.5304	0.6117	0.7231
10	0.2625	0.2811	0.3121	0.3407	0.3948	0.4177	0.4630	0.5494
15	0.2161	0.2330	0.2505	0.2610	0.2851	0.2900	0.3508	0.4260
20	0.1570	0.1600	0.1833	0.1955	0.2104	0.2261	0.2691	0.3418
25	0.0620	0.0752	0.1080	0.1172	0.1569	0.1683	0.2280	0.2900
30	0.0174	0.0330	0.0475	0.0590	0.0843	0.0902	0.1405	0.2193
35	0.0050	0.0086	0.0163	0.0244	0.0472	0.0546	0.0715	0.1305
40	0.0010	0.0030	0.0077	0.0096	0.0390	0.0447	0.0533	0.0987
45	0.0008	0.0011	0.0041	0.0083	0.0285	0.0298	0.0314	0.0730
50	0.0002	0.0007	0.0018	0.0052	0.0137	0.0185	0.0278	0.0428
60	0.0000	0.0002	0.0008	0.0020	0.0085	0.0097	0.0151	0.0301
70	0.0000	0.0000	0.0004	0.0013	0.0052	0.0069	0.0081	0.0151
80	0.0000	0.0000	0.0001	0.0007	0.0031	0.0037	0.0055	0.0083
90	0.0000	0.0000	0.0000	0.0003	0.0019	0.0022	0.0030	0.0055
100	0.0000	0.0000	0.0000	0.0001	0.0007	0.0007	0.0009	0.0020
110	0.0000	0.0000	0.0000	0.0000	0.0004	0.0005	0.0007	0.0009
120	0.0000	0.0000	0.0000	0.0000	0.0002	0.0002	0.0004	0.0007
130	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0002	0.0005
140	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 15: The finite time ruin probability $\psi'(u,1,10)$ for $\delta=0.5$

u	$k=0.025$	$k=0.05$	$k=0.5$	$k=1$	$k=1.125$	$k=2$	$k=5$
0	0.4863	0.3714	0.2810	0.6128	0.7420	0.8160	0.9235
5	0.3230	0.2798	0.2472	0.4772	0.5031	0.7643	0.8720
10	0.2828	0.2339	0.1961	0.3516	0.3948	0.6252	0.8216
15	0.2251	0.2100	0.1557	0.2645	0.2851	0.5372	0.7630
20	0.1702	0.1124	0.0879	0.1990	0.2104	0.4600	0.6602
25	0.1135	0.0950	0.0720	0.1255	0.1769	0.3818	0.5734
30	0.0840	0.0402	0.0319	0.0761	0.0843	0.3304	0.5001
35	0.0677	0.0215	0.0127	0.0320	0.0472	0.2991	0.4638
40	0.0300	0.0090	0.0081	0.0144	0.0390	0.2550	0.4122
45	0.0110	0.0072	0.0055	0.0085	0.0285	0.2263	0.3790
50	0.0085	0.0034	0.0010	0.0061	0.0137	0.1815	0.3284
60	0.0064	0.0008	0.0006	0.0045	0.0085	0.1260	0.2915
70	0.0040	0.0003	0.0002	0.0018	0.0052	0.0757	0.2509
80	0.0012	0.0001	0.0000	0.0008	0.0031	0.0485	0.2031
90	0.0008	0.0000	0.0000	0.0006	0.0019	0.0114	0.1716
100	0.0005	0.0000	0.0000	0.0005	0.0007	0.0082	0.1200
110	0.0001	0.0000	0.0000	0.0002	0.0004	0.0039	0.0974
120	0.0000	0.0000	0.0000	0.0001	0.0002	0.0008	0.0633
130	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0351
140	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0153
150	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0009

6 Conclusion

In the present paper, we firstly considered the construction of discrete-time risk model where the long-tailed and dominatedly varying-tailed of the net losses $X_i, i=1,2,\ldots,$ are asymptotically independent random variables with distribution function $F(x)$. The asymptotic ruin probability problem in the finite and infinite horizon cases studied using convolution of distribution functions with other assumptions. For two real examples with Pareto and Lognormal distributions, which the goodness of fit tests are done on the data at significance level 0.05, the numerical asymptotic ruin probabilities estimated. It must be emphasized that the results show that when the distribution function is heavy-tailed, the insurance company is likely to go ruin. In the second dependent discrete-time risk model, the finite time ruin probability computed based on the discretization of the distribution function. We studied the impact on the ruin probability of a dependence relation between two classes of insurance business. In the first example, we considered the aggregation of the classes of business via a common shock model and in the second one, the aggregation is made via the Negative Binomial model with common component. We observed that the increase in the ruin probability $\psi'(u, 1, 10)$ with the introduction of a relation of dependence is more important in the Negative Binomial model than in the Poisson model. The similar results have been obtained for ruin $\psi'(u, 1, 20)$. We concluded that in the presence of the large values of parameters in Exponential and Weibull distributions will lead to large ruin probabilities.. Also, increasing or decreasing the ruin probabilities for constant δ depends on the value of k. In this case, for $k \leq \delta$, the ruin probability decreases with increasing k and for $k > \delta$, the ruin probability increases with increasing k.

For the future research, we propose to consider the first model with capital injections and reinsurance. Moreover, there is a limitation with the proposed approach for the second risk model, when the model contains a heavy-tailed distribution and it can be the potential directions for future research.

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Appendix

The result for independent class of a compound Poisson distribution is presented. Suppose that three random variables $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ are independent compound Poisson distribution with parameter λ. Given $N^{(j)} \sim Poisson(\lambda_j)$, where $N^{(j)}$, $j = 1, 2, 3$, are independent random variables, the joint probability generating function of $\left(N^{(1)}, N^{(2)}, N^{(3)}\right)$ is

$$P_{N^{(1)},N^{(2)},N^{(3)}}(t_1, t_2, t_3) = \Pi_{j=1}^{3} P_{N^{(j)}}(t_j).$$

With equation (2.15), the joint characteristic function of $\left(W^{(1)}, W^{(2)}, W^{(3)}\right)$ is

$$\varphi_{W^{(1)},W^{(2)},W^{(3)}}(t_1, t_2, t_3) = \Pi_{j=1}^{3} P_{N^{(j)}}\Big(\varphi_{Y^{(j)}}(t_j)\Big),$$

which by equation (2.16) leads to

$$\varphi_W(t) = \varphi_{W^{(1)},W^{(2)},W^{(3)}}(t, t, t) = \exp\Big(\lambda\varphi_Y(t) - 1\Big),$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ and

$$\varphi_Y(t) = \frac{1}{\lambda}\Big(\lambda_1\varphi_{Y^{(1)}}(t) + \lambda_2\varphi_{Y^{(2)}}(t) + \lambda_3\varphi_{Y^{(3)}}(t)\Big).$$

The random variable $W = \sum_{i=1}^{3} W^{(i)}$ has a compound Poisson distribution, with parameter λ and claim size characteristic function $\varphi_Y(t)$ which is associated to the distribution function

$$F_Y(t) = \frac{1}{\lambda}\Big(\lambda_1 F_{Y^{(1)}}(t) + \lambda_2 F_{Y^{(2)}}(t) + \lambda_3 F_{Y^{(3)}}(t)\Big),$$

where $\mu_Y(t) = \frac{1}{\lambda}\Big(\lambda_1\mu_{Y^{(1)}}(t) + \lambda_2\mu_{Y^{(2)}}(t) + \lambda_3\mu_{Y^{(3)}}(t)\Big)$.