JIRSS (2022)
Vol. 21, No. 02, pp 217-231
DOI: 10.22034/jirss.2022.707183

Stochastic Comparison of Hariss Family Distributions with Fixed and Randomized tilt Parameter

Somayeh Abbasi [1], **Mohammad Hossein Alamatsaz** [2]

[1] Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran.
[2] Department of Statistics, University of Isfahan, Isfahan, Iran

Received: 09/03/2022, Accepted: 22/08/2023, Published online: 14/01/2024

Abstract. In this paper, we stochastically compare Harris family distributions having random tilt parameter with Harris family distributions having fixed tilt parameter. We also study certain preservation properties of mixtures of Harris family of distributions with regards to their baseline distributions. Comparison tools are various types of orderings, such as the usual, shifted, proportional and shifted proportional stochastic orderings. Several previous findings, regarding Marshall-Olkin family, follow as special cases of our results. We shall also fit a new Harris model to a real data set to illustrate the usefulness of our comparisons.

Keywords. Marshall-Olkin Distribution, Proportional Stochastic Ordering, Shifted Stochastic Ordering, Shifted Proportional Stochastic Ordering, Usual Stochastic Ordering, Reliability Function.

MSC: 60K10, 90B25.

1 Introduction

Two large classes of general and flexible distributions were introduced in Aly (2011) and Marshall (1997). They are called Harris and Marshall-Olkin family of distributions, respectively. Both classes of such distributions are, in particular, useful in reliability

Corresponding Author: Somayeh Abbasi (s_abbasi_29@yahoo.com)
Mohammasd Hossein Alamatsaz(mh_alamatsaz@yahoo.com)

theory. In such distributions, to cover a wide range of data such as those with high degrees of skewness and kurtosis, a tilt parameter is added to the model.

The methods of generating Marshall-Olkin and Harris family distributions are as follows: Let $Y_1, Y_2, ...$ be a sequence of independently identically distributed (iid) random variables (rv's) with a common distribution function (df) F and survival function (sf) $\bar{F} = 1 - F$. Let $X = \min\{Y_1, Y_2, ..., Y_N\}$, where N is a positive integer valued rv independent of the Y_i 's with probability generating function (pgf)

$$P_N(t) = E(t^N) = \sum_{n=0}^{\infty} t^n P(N = n), \quad t \in [0, 1].$$

The random variable X can be viewed as the lifetime of a series system with iid component lifetimes $Y_1, Y_2, ..., Y_N$ and a random number N of components. The sf $\bar{H}$ of X has the representation

$$\bar{H}(x) = \sum_{n=0}^{\infty} [\bar{F}(x)]^n P(N = n), \tag{1.1}$$

so that

$$\bar{H}(x) = P_N(\bar{F}(x)). \tag{1.2}$$

Assuming N is a geometric rv, Marshall (1997) introduced the so-called Marshall-Olkin distribution with sf

$$\bar{H}(x; \theta) = \frac{\theta \bar{F}(x)}{1 - \bar{\theta}\bar{F}(x)}, \qquad 0 < \theta < \infty, \qquad \bar{\theta} = 1 - \theta. \tag{1.3}$$

Harris (1948) introduced the Harris pgf as below:

$$P_N(s; \theta, k) = \left\{\frac{\theta s^k}{1 - \bar{\theta} s^k}\right\}^{1/k}, \qquad k > 0, \qquad 0 < \theta < 1, \qquad \bar{\theta} = 1 - \theta, \tag{1.4}$$

so,

$$P_N^{-1}(s; \theta, k) = P_N(s; \frac{1}{\theta}, k). \tag{1.5}$$

By applying the Harris pgf, in Eq (1.2), the Harris family with sf

$$\bar{H}(x; \theta, k) = \left(\frac{\theta \bar{F}^k(x)}{1 - \bar{\theta}\bar{F}^k(x)}\right)^{1/k}, \qquad k > 0, \qquad 0 < \theta < \infty, \qquad \bar{\theta} = 1 - \theta. \tag{1.6}$$

was generated by Aly (2011). The df F in Eq (1.6) is called the baseline df and θ is called the tilt parameter. It is easy to see that hazard rates corresponding to F and $H(\cdot; \theta, k)$, namely, $r_F = f/\bar{F}$ and $r_H(.; \theta, k) = h(.; \theta, k)/\bar{H}(.; \theta, k)$, are related by

$$r_H(x; \theta, k) = \frac{r_F(x)}{1 - \bar{\theta}\bar{F}^k(x)}, \qquad -\infty < x < \infty, \qquad 0 < \theta < \infty, \qquad k > 0. \tag{1.7}$$

Clearly, $r_H(x;\theta,k)$ exceeds $r_F(x)$ when $0 < \theta \leq 1$. It is smaller than $r_F(x)$ when $\theta \geq 1$. They coincide when $\theta = 1$. Clearly, for $k = 1$, a Harris family distribution reduces to the Marshall-Olkin distribution.

Recently, another method of constructing Harris family was given by Batsidis and Lemonte (2015). They exhibit that Harris family is a proportional failure rate model which is obtained from a modified Marshall-Olkin family. More recently, Abbasi (2016) and Abbasi (2018) stochastically compared two Harris family distributions having different and same tilt parameters.

Many financial and economical events have distributions with wide tails and sharp peaks, which indicate the existence of pessimism and excessive optimism that investors have shown in the market. Due to the fact that Many well-knowns distributions always underestimate the values that occur rarely, they are not suitable for cases where such observations are of great importance and may have serious consequences for asset allocation and risk management or lead to a deficit in payment or even bankruptcy. In such cases, it will be very important to develop the model so that the new model has more flexibility on real data. Therefore, one of the ways to develop the model is to use the Harris distribution family. Now, in the selection of the tilt parameter, stochastic orderings can play a very important role in order to enable us to choose the best tilt parameter space and power factor in order to achieve the desired utility. Of course, one of the other merits of this distribution is the control of the hazard rate by adding the tilt parameter and the power coefficient parameter in its hazard rate function.

Due to various reasons, in many practical situations the tilt parameter may not be constant and the occurrence of heterogeneity is sometimes unpredictable and cannot be explained. But often it may not be possible to ignore this type of heterogeneity. For instance, survival analysis is mainly concerned with investigating the hazard of death at any time when an individual patient is involved in a clinical trial or other medical study. Due to the difference between individuals in their susceptibility to causes of death or disease, response to treatment, and influence of various risk factors, the observed covariate, such as demographic, physiological, or lifestyle characteristics, are taken into account. Nevertheless, heterogeneity unexplained by observed covariate usually plays an important role because it sometimes leads to a misleading conclusion (cf. Li (2011)). In the analysis of survival of a patient with recently metastasized cancer, the number of involved organs is not fixed but it randomly changes from patient to patient. It is possible that in one patient only one organ is involved, in another person two organs are involved, and in another person n organs are involved in cancer cells.

Therefore, it is important to inspect the unobserved random factors' influence on the random variable. Considering this fact, we need to study the mixture of the family of distributions. Mixture distributions are often used in mixture models, which are used to express probabilities of sub-populations within a larger population. A mixture model can accommodate the historically observed data in that sense and offers a flexible solution for different distributional forms. Recently, Aghababaei (2010), Aghababaei (2011) and Alamatsaz (2008) were concerned with stochastic comparison of certain distributions with their mixtures. Also, Abbasi (2019) provided some bounds related to mixture of Harris family of distributions.

Stochastic orderings have phenomenal performance in comparing probability distributions. They have an important role in reliability, survival snalysis, economic and insurance. For instance, recently Payandeh Najafabadi (2016) used stochastic orderings to compare Series and Parallel Systems with Heterogeneous Extended Generalized Exponential Components. Batsidis and Lemonte (2015) were concerned with the behavior of the failure rate function and some stochastic order relations in the Harris family. Here, we are dealing with stochastic comparison of mixtures of Harris family distributions when the tilt parameter is a rv. Our tools in comparisons are various types of stochastic orderings such as usual, shifted, proportional and shifted proportional stochastic orderings.

In Harris family distribution, there is no theoretical basis for choosing the baseline distribution and the distribution of its tilt parameter; when tilt parameter is a rv. Therefore, it is important to see how a Harris family rv responds to the change of its baseline distribution and tilt parameter. This paper, mainly investigates how stochastic orders between tilt parameters affect the corresponding Harris family distributions with fixed and randomized tilt parameters. Considering the utility desired, we are able to select a fix or rv tilt parameter. We shall also fit a new Harris model (denoted by HXTG) to a real data set and illustrate the usefulness of our comparisons.

Our results enfold all findings on stochastic orderings of Nanda (2012), as special cases, who stochastically compared members of the Marshall-Olkin family. We shall use the terms increasing in place of non-decreasing and decreasing in place of non-increasing. In Section 2, we state acronyms and useful relations among stochastic orderings to be used in the sequel. In Section 3, we discuss stochastic comparisons of Harris family distributions with their tilt-mixtures. At first, without any restriction on the baseline distribution, we compare Harris family distributions with their mixtures using usual stochastic orderings and aging concepts such as shifted, proportional and shifted proportional orderings. In Section 4, we fit the proposed HXTG distribution to a real data set and compare it with its baseline distribution and also Marshal-Olkin distribution (denoted by MOXTG). Clearly, one can use our results to one's benefit by choosing its tilt parameter to be fixed or rv, i.e., HXTG model or unconditional HXTG model.

2 Relations among different Types of Stochastic Orderings

First we present acronyms which are used in this paper. For more details, we refer readers to Abbasi (2016), Abbasi (2018), Abbasi (2019), Lillo (2001), Marshall (2007) and Shaked (2007).

Eq.	equation	DHR	decreasing hazard rate
pdf	probability density function	IRHR	increasing reversed hazard rate
st	simple stochastic	DRHR	decreasing reversed hazard rate
lr	likelihood ratio	plr	proportional likelihood ratio
hr	hazard rate	phr	proportional hazard rate
rh	reversed hazard rate	prh	proportional reversed hazard rate
E	expectation	IPLR	increasing proportional likelihood ratio
AI	ageing intensity	IPHR	increasing proportional hazard rate
lr↑	up likelihood ratio	IPRH	increasing proportional reversed hazard rate
lr↓	down likelihood ratio	plr↑	up proportional likelihood ratio
hr↑	up hazard rate	plr↓	down proportional likelihood ratio
hr↓	down hazard rate	phr↑	up proportional hazard rate
rh↑	up reversed hazard rate	phr↓	down proportional hazard rate
rh↓	down reversed hazard rate	UIPLR	up increasing proportional likelihood ratio
ILR	increasing likelihood ratio	UIPHR	up increasing proportional hazard rate
DLR	decreasing likelihood ratio	DIPLR	down increasing proportional likelihood ratio
IHR	increasing hazard rate	DIPHR	down increasing proportional hazard rate

Table 1, due to Abbasi (2018), summarizes several useful relationships among the stochastic orderings used in the sequel.

Table 1: Some useful relations among various types of stochastic orderings

$\leq_{lr}$	$\Rightarrow$	$\leq_{hr}$				$\leq_{hr\uparrow}$	$\Rightarrow$	$\leq_{hr}$	$\Rightarrow$	$\leq_{st}$	$\Rightarrow$	$\leq_{E}$
$\Uparrow$						$\Uparrow$						
$\leq_{plr\uparrow}$	$\Rightarrow$	$\leq_{prh\uparrow}$	$\Rightarrow$	$\leq_{rh\uparrow}$	$\Leftarrow$	$\leq_{lr\uparrow}$	$\Leftarrow$	$\leq_{plr\uparrow}$	$\Rightarrow$	$\leq_{phr\uparrow}$	$\Rightarrow$	$\leq_{hr\uparrow}$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\leq_{plr}$	$\Rightarrow$	$\leq_{prh}$	$\Rightarrow$	$\leq_{rh}$	$\Leftarrow$	$\leq_{lr}$	$\Leftarrow$	$\leq_{plr}$	$\Rightarrow$	$\leq_{phr}$	$\Rightarrow$	$\leq_{hr}$
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$
$\leq_{plr\downarrow}$		$\leq_{prh\downarrow}$	$\Rightarrow$	$\leq_{rh\downarrow}$	$\Leftarrow$	$\leq_{lr\downarrow}$	$\Leftarrow$	$\leq_{plr\downarrow}$	$\Rightarrow$	$\leq_{phr\downarrow}$	$\Rightarrow$	$\leq_{hr\downarrow}$
				$\Downarrow$		$\Downarrow$						
		$\leq_{st}$	$\Leftarrow$	$\leq_{rh}$		$\leq_{hr\downarrow}$						

3 Stochastic Comparison under Mixtures

In Eq (1.6), let the parameter Θ be an absolutely continuous rv with df $G(.)$ and pdf $g(.)$. Then, its corresponding unconditional Harris sf is given by

$$\begin{aligned} \bar{H}(x;k) &= \int_0^\infty \bar{H}(x;\theta,k)g(\theta)d\theta, \\ &= E[\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)}]^{\frac{1}{k}}\bar{F}(x). \end{aligned} \tag{3.1}$$

We denote the corresponding rv by X^*. Clearly, pdf of $X^*$is given by

$$\begin{aligned} h(x;k) &= \int_0^\infty h(x;\theta,k)g(\theta)d\theta, \\ &= f(x)E[\frac{\Theta}{(1-\bar{\Theta}\bar{F}^k(x))^{k+1}}]^{\frac{1}{k}}. \end{aligned} \tag{3.2}$$

Marshall (2007) have compared $\bar{H}(.;k)$ for the case $k=1$; i.e., the mixture of Marshall-Olkin distribution, with its baseline distribution F under several stochastic orderings. In what follows, we shall compare, more generally, Harris distributions; i.e., when $k>0$ is arbitrary, with its mixtures. Our results enfold Nanda (2012)'s findings in this connection.

Theorem 3.1. *Let X and X^* be two rv's with sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Then, $X \leq_{lr} (\geq_{lr}) X^*$ if $P(\Theta \geq \nu) = 1$ ($P(0 < \Theta \leq \nu) = 1$).*

Proof. Using Eq (3.2), we have

$$\begin{aligned} \frac{h(x;k)}{h(x;\nu,k)} &= [\frac{(1-\bar{\nu}\bar{F}^k(x))^{k+1}}{\nu}]^{\frac{1}{k}} E[\frac{\Theta}{(1-\bar{\Theta}\bar{F}^k(x))^{k+1}}]^{\frac{1}{k}} \\ &= E[\frac{\Theta(1-\bar{\nu}\bar{F}^k(x))^{k+1}}{\nu(1-\bar{\Theta}\bar{F}^k(x))^{k+1}}]^{\frac{1}{k}}. \end{aligned}$$

So

$$\begin{aligned} \frac{d}{dx}(\frac{h(x;k)}{h(x;\nu,k)}) &= \frac{d}{dx}(E[\frac{\Theta(1-\bar{\nu}\bar{F}^k(x))^{k+1}}{\nu(1-\bar{\Theta}\bar{F}^k(x))^{k+1}}]^{\frac{1}{k}}), \\ &= E[\frac{(k+1)\Theta^{\frac{1}{k}}\bar{F}^{k-1}(x)f(x)(1-\bar{\nu}\bar{F}^k(x))^{\frac{1}{k}}(\Theta-\nu)}{\nu^{\frac{1}{k}}(1-\bar{\Theta}\bar{F}^k(x))^{\frac{1}{k}+2}}]. \end{aligned} \tag{3.3}$$

Clearly, Eq (3.3) is non-negative provided that $P(\Theta \geq \nu) = 1$ and is non-positive provided that $P(0 < \Theta \leq \nu) = 1$. Thus, the proof is completed. □

Remark 1. Taking Θ as a degenerate rv, the above results coincide with those of Theorem 3.2 in Abbasi (2018) when $k_1 = k_2 = k$

Observing Table 1, we obtain the following corollary.

Corollary 3.1. *i) $X \leq_{hr} (\leq_{rh}, \leq_{st}, \leq_E) X^*$ if $\Theta \geq \nu$ with probability 1.*
ii) $X^* \leq_{hr} (\leq_{rh}, \leq_{st}, \leq_E,) X$ if $0 < \Theta \leq \nu$ with probability 1.

In the following theorem, we compare the ageing intensity ordering between a Harris family and its mixture. First, we give the following lemma.

Lemma 3.1. *Let X be a rv with sf $\bar{H}(.;\nu,k)$ and X^* be a rv with sf $\bar{H}(.;k)$ and hazard rate $r_H(.;k)$. Then, if $P(0 < \Theta \leq 1) = 1$, $\frac{r_H(x;k)}{r_H(x;\nu,k)}$ is decreasing in x.*

Proof. We have

$$\begin{aligned}
\frac{r_H(x;k)}{r_H(x;\nu,k)} &= \frac{1-\bar{\nu}\bar{F}^k(x)}{r_F(x)}\frac{h(x;k)}{\bar{H}(x;k)} \\
&= \frac{1-\bar{\nu}\bar{F}^k(x)}{r_F(x)}\frac{\int_0^1 f(x)\frac{u^{\frac{1}{k}}g(u)}{(1-(1-u)\bar{F}^k(x))^{\frac{1}{k}+1}}du}{\int_0^1 \bar{F}(x)\frac{u^{\frac{1}{k}}g(u)}{(1-(1-u)\bar{F}^k(x))^{\frac{1}{k}}}du} \\
&= \frac{(1-\bar{\nu}\bar{F}^k(x))\int_0^1 \frac{u^{\frac{1}{k}}g(u)}{(1-(1-u)\bar{F}^k(x))^{\frac{1}{k}+1}}du}{\int_0^1 \frac{u^{\frac{1}{k}}g(u)}{(1-(1-u)\bar{F}^k(x))^{\frac{1}{k}}}du} \\
&= E[\frac{1-\bar{\nu}\bar{F}^k(x)}{1-(1-Z)\bar{F}^k(x)} \mid x], \qquad (3.4)
\end{aligned}$$

where $Z \mid x$ is a rv having pdf

$$h_{Z|x}(z;k) = a(x;k)\frac{z^{\frac{1}{k}}g(z)}{(1-(1-z)\bar{F}^k(x))^{\frac{1}{k}}}, \qquad 0 < z \leq 1,$$

with $a(x;k)$ as the normalizing constant. It is easy to verify that

$$\frac{h_{Z|x_2}(z;k)}{h_{Z|x_1}(z;k)} = \frac{a(x_2;k)}{a(x_1;k)}[\frac{1-(1-z)\bar{F}^k(x_1)}{1-(1-z)\bar{F}^k(x_2)}]^{\frac{1}{k}},$$

is increasing in $0 < z \leq 1$ for any $0 \leq x_1 \leq x_2$. This implies that $Z \mid x_1 \leq_{lr} Z \mid x_2$, which yields $Z \mid x_1 \leq_{st} Z \mid x_2$. By relation (1.A.7) of [16] this in turn implies that for any increasing function $Q(z)$, $E(Q(Z) \mid x_1) \leq E(Q(Z) \mid x_2)$. If we put $Q(z) = \frac{-(1-\bar{\nu}\bar{F}^k(x))}{1-(1-z)\bar{F}^k(x)}$ which is increasing in $0 < z < 1$, for any x and $k > 0$, we obtain

$$E[\frac{-(1-\bar{\nu}\bar{F}^k(x_1))}{1-(1-Z)\bar{F}^k(x_1)} \mid x_1] \leq E[\frac{-(1-\bar{\nu}\bar{F}^k(x_2))}{1-(1-Z)\bar{F}^k(x_2)} \mid x_2],$$

or, equivalently,

$$E[\frac{1-\bar{\nu}\bar{F}^k(x_2)}{1-(1-Z)\bar{F}^k(x_2)} \mid x_2] \leq E[\frac{1-\bar{\nu}\bar{F}^k(x_1)}{1-(1-Z)\bar{F}^k(x_1)} \mid x_1]. \qquad (3.5)$$

This completes the proof. □

Theorem 3.2. *Let X and X^* be two rv's with sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Then, if $P(0 < \Theta \leq 1) = 1$, we have $X \leq_{AI} X^*$.*

Proof. $X \leq_{AI} X^*$ if and only if,

$$\frac{1}{r_H(x;\nu,k)}\int_0^x r_H(u;\nu,k)du \leq \frac{1}{r_H(x;k)}\int_0^x r_H(u;k)du,$$

or, equivalently, by Eq (1.7) and Eq (3.4), if and only if,

$$\frac{1-\bar{\nu}\bar{F}^k(x)}{r_F(x)}\int_0^x \frac{r_F(u)}{1-\bar{\nu}\bar{F}^k(u)}du \quad \leq \quad \frac{1}{r_F(x)E[\frac{1}{1-(1-Z)\bar{F}^k(x)} \mid x]} \int_0^x r_F(u)E[\frac{1}{1-(1-Z)\bar{F}^k(u)} \mid u]du.$$

This, in turn, is equivalent to

$$\int_0^x r_F(u)\frac{E[\frac{1-\bar{\nu}\bar{F}^k(u)}{1-(1-Z)\bar{F}^k(u)} \mid u] - E[\frac{1-\bar{\nu}\bar{F}^k(x)}{1-(1-Z)\bar{F}^k(x)} \mid x]}{(1-\bar{\nu}\bar{F}^k(u))(1-\bar{\nu}\bar{F}^k(x))}du \geq 0,$$

for all $x > 0$, which is true by the decreasing property of inequality (3.5). Thus, we have the result. □

In the following theorems, we shall only give the proofs for the case when $\Theta \geq 1$ with probability 1. Proofs of the case $0 < \Theta \leq 1$ with probability 1 are similar and thus omitted.

Theorem 3.3. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ ($0 < \Theta \leq 1$) with probability 1 and $0 < \nu \leq 1$ ($\nu \geq 1$). Then*
i) $X \leq_{plr\uparrow} (\geq_{plr\uparrow}) X^*$ if $F \in UIPLR$,
ii) $X \leq_{plr} (\geq_{plr}) X^*$ if $F \in IPLR$,
iii) $X \leq_{lr\uparrow} (\geq_{lr\uparrow}) X^*$ if $F \in ILR$.

Proof. i) $X \leq_{plr\uparrow} X^*$ if and only if,

$$\frac{h(\lambda x;k)}{h(x+t;\nu,k)} = \frac{f(\lambda x)}{f(x+t)}E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x+t)}{1-\bar{\Theta}\bar{F}^k(\lambda x)})^{\frac{1}{k}+1}],$$

is increasing in x for all $t \geq 0$, $0 < \lambda \leq 1$ and $k > 0$. But, $F \in UIPLR$ is equivalent to $\frac{f(\lambda x)}{f(x+t)}$ being increasing in x for all $t \geq 0$ and $0 < \lambda \leq 1$. Also,

$$\frac{d}{dx}E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x+t)}{1-\bar{\Theta}\bar{F}^k(\lambda x)})^{\frac{1}{k}+1}] = E[(k+1)(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x+t)}{1-\bar{\Theta}\bar{F}^k(\lambda x)})^{\frac{1}{k}} (\frac{\bar{\nu}\bar{F}^{k-1}(x+t)f(x+t)(1-\bar{\Theta}\bar{F}^k(\lambda x))}{(1-\bar{\Theta}\bar{F}^k(\lambda x))^2} - \frac{\bar{\Theta}\lambda\bar{F}^{k-1}(\lambda x)f(\lambda x)(1-\bar{\nu}\bar{F}^k(x+t))}{(1-\bar{\Theta}\bar{F}^k(\lambda x))^2})], \tag{3.6}$$

is non-negative if $0 < \nu \leq 1$ and $\Theta > 1$ with probability 1. Thus, since both factors are non-negative, this completes the assertion.
With proper choices of $t(=0)$ or $\lambda(=1)$, proofs of parts (ii) and (iii) are immediate. □

Theorem 3.4. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ ($0 < \Theta \leq 1$) with probability 1 and $0 < \nu \leq 1$ ($\nu \geq 1$). Then*
i) $X \leq_{plr\downarrow} (\geq_{plr\downarrow}) X^*$ if $F \in DIPLR$,
ii) $X \leq_{lr\downarrow} (\geq_{lr\downarrow}) X^*$ if $F \in DLR$.

Proof. i) $X \leq_{plr\downarrow} X^*$ if and only if,

$$\frac{h(\lambda x+t;k)}{h(x;\nu,k)} = \frac{f(\lambda x+t)}{f(x)} E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x)}{1-\bar{\Theta}\bar{F}^k(\lambda x+t)})^{\frac{1}{k}+1}],$$

is increasing in x for all $t \geq 0$, $0 < \lambda \leq 1$ and $k > 0$. But, $F \in DIPLR$ is equivalent to $\frac{f(\lambda x+t)}{f(x)}$ being increasing in x for all $t \geq 0$ and $0 < \lambda \leq 1$. Also,

$$\begin{aligned} \frac{d}{dx} E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x)}{1-\bar{\Theta}\bar{F}^k(\lambda x+t)})^{\frac{1}{k}+1}] = E[(k+1)(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x)}{1-\bar{\Theta}\bar{F}^k(\lambda x+t)})^{\frac{1}{k}} \\ (\frac{\bar{\nu}\bar{F}^{k-1}(x)f(x)(1-\bar{\Theta}\bar{F}^k(\lambda x+t))}{(1-\bar{\Theta}\bar{F}^k(\lambda x+t))^2} \\ -\frac{\bar{\Theta}\lambda\bar{F}^{k-1}(\lambda x+t)f(\lambda x)(1-\bar{\nu}\bar{F}^k(x))}{(1-\bar{\Theta}\bar{F}^k(\lambda x+t))^2})], \end{aligned} \tag{3.7}$$

is non-negative if $0 < \nu \leq 1$ and $\Theta > 1$ with probability 1. Thus, since both factors are non-negative, this proves the assertion.
With choosing $\lambda(=1)$, proof of part (ii) is immediate. □

Theorem 3.5. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ ($0 < \Theta \leq 1$) with probability 1 and $0 < \nu \leq 1$ ($\nu \geq 1$). Then*
i) $X \leq_{phr\uparrow} (\geq_{phr\uparrow}) X^*$ if $F \in UIPHR$,
ii) $X \leq_{phr} (\geq_{phr}) X^*$ if $F \in IPHR$,
iii) $X \leq_{hr\uparrow} (\geq_{hr\uparrow}) X^*$ if $F \in IHR$.

Proof. i) $X \leq_{phr\uparrow} X^*$ if and only if,

$$\frac{\bar{H}(\lambda x;k)}{\bar{H}(x+t;\nu,k)} = \frac{\bar{F}(\lambda x)}{\bar{F}(x+t)} E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x+t)}{1-\bar{\Theta}\bar{F}^k(\lambda x)})^{\frac{1}{k}}],$$

is increasing in x for all $t \geq 0$, $0 < \lambda \leq 1$ and $k > 0$. But, $F \in UIPHR$ is equivalent to $\frac{\bar{F}(\lambda x)}{\bar{F}(x+t)}$ being increasing in x for all $t \geq 0$ and $0 < \lambda \leq 1$. Using Eq (3.6), the second factor is also increasing in x if $0 < \nu \leq 1$ and $\Theta \geq 1$ with probability 1. Thus, since both factors are non-negative, thus the assertion follows.
With proper choices of $t(=0)$ or $\lambda(=1)$, proofs of parts (ii) and (iii) are immediate. □

Theorem 3.6. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ ($0 < \Theta \leq 1$) with probability 1 and $0 < \nu \leq 1$ ($\nu \geq 1$). Then*
i) $X \leq_{phr\downarrow} (\geq_{phr\downarrow}) X^*$ if $F \in DIPHR$,
ii) $X \leq_{hr\downarrow} (\geq_{hr\downarrow}) X^*$ if $F \in DHR$.

Proof. i) $X \leq_{phr\downarrow} X^*$ if and only if,

$$\frac{\bar{H}(\lambda x+t;k)}{\bar{H}(x;\nu,k)} = \frac{\bar{H}(\lambda x+t)}{\bar{H}(x)} E[(\frac{\Theta}{\nu})^{\frac{1}{k}}(\frac{1-\bar{\nu}\bar{F}^k(x)}{1-\bar{\Theta}\bar{F}^k(\lambda x+t)})^{\frac{1}{k}}],$$

is increasing in x for all $t \geq 0$, $0 < \lambda \leq 1$ and $k > 0$. But, $F \in DIPHR$ is equivalent to $\frac{\bar{F}(\lambda x+t)}{\bar{F}(x)}$ being increasing in x for all $t \geq 0$ and $0 < \lambda \leq 1$. Using Eq (3.7), the second factor is also increasing in x if $0 < \nu \leq 1$ and $\Theta \geq 1$ with probability 1. Thus, since both factors are non-negative, this proves the assertion.

By choosing $\lambda = 1$, proof of part (ii) is immediate. □

Remark 2. Taking Θ as a degenerate rv, some of the results in Theorem 2.3, 2.4, 2.5 and 2.6 enfold Theorem 3.1 in Abbasi (2018) as a special case when $k_1 = k_2 = k$

Remark 3. Taking Θ as a degenerate rv and $\nu = 1$, our results coincide with Batsidis and Lemonte (2015)'s findings in this connection. Also, by Remark 1 of Batsidis and Lemonte (2015) or, taking $\nu = 1$ and $k = 1$, our results contain Nanda (2012)'s findings in this connection. Some results of Nanda (2012) and consequently the results of proposition 2 in Batsidis and Lemonte (2015) are not correct as the conventional definitions of down and up orderings but, they are correct according to their own definitions.

In the following theorem, we shall investigate preservation of decreasing hazard rate average (DHRA) and new worse than used (NWU) characteristics by mixtures of Harris family of distributions. F has DHRA property, if $\bar{F}(ct) \leq (\bar{F}(t))^c$, for all $0 < c < 1$. This means that the system improving, as time goes by, is less intuitive.
F has NWU property, if $\bar{F}(t+u) \leq \bar{F}(t)\bar{F}(u)$, for $t > 0$ and $u > 0$. This means that a device of any particular age has a stochastically bigger remaining lifetime than dose a new device. For more details see Barlow (1981).

Theorem 3.7. *Let $P(0 < \Theta < 1) = 1$.*

i) DHRA characteristic is preserved by transformation to Mixture of Harris family.
ii) NWU characteristic is preserved by transformation to Mixture of Harris family.

Proof. i) For any $0 < c < 1$ and $k > 0$, by Eq(3.1), we have

$$\bar{H}(cx;k) = E_\Theta[\bar{H}(cx;\Theta,k)].$$

Let F have DHRA and $0 < \Theta < 1$ with probability 1. By Corollary 5.1 of Abbasi (2018), DHRA is preserved by transformation to Harris family. Thus, for non-negative $\bar{H}$ function, we have

$$E_{\Theta}[\bar{H}(cx;\Theta,k)] \leq E_{\Theta}[(\bar{H}(x;\Theta,k))^c].$$

Let $U = \bar{H}(x;\Theta,k)$. Rv U is non-negative. Thus, $[E(U^c)]^{\frac{1}{c}}$ is an increasing function of $c > 0$. Hence, for any $0 < c < 1$, $E(U^c) \leq [E(U)]^c$. Consequently

$$\begin{aligned} E_{\Theta}[(\bar{H}(x;\Theta,k))^c] &\leq [E_{\Theta}(\bar{H}(x;\Theta,k))]^c, \\ &= [\bar{H}(x;k)]^c. \end{aligned} \quad (3.8)$$

By above definition of DHRA, we have the result.
ii) Let F have NWU and $0 < \Theta < 1$ with probability 1. By Abbasi (2018), NWU is preserved by transformation to Harris family. Thus, for non-negative $\bar{H}$ function, we have

$$\begin{aligned} \bar{H}(t+u;k) &= E_{\Theta}[\bar{H}(t+u;\Theta,k)] \\ &\leq E_{\Theta}[\bar{H}(t;\Theta,k)\bar{H}(u;\Theta,k)] \\ &\leq E_{\Theta}[\bar{H}(t;\Theta,k)]E_{\Theta}[\bar{H}(u;\Theta,k)] \\ &= \bar{H}(t;k)\bar{H}(u;k), \end{aligned} \quad (3.9)$$

since the random variables $\bar{H}(t;\Theta,k)$ and $\bar{H}(u;\Theta,k)$ are monotone in the same direction, the last inequality holds and we have the result. □

4 Application

In this section, to show the applicability of our model, we investigate the data set originally considered by Bjerkedal (1960). This data set has also been analyzed by Cordeiro et al. (2014). The data shows the survival times of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility to human tuberculosis. Thus, they are worth dealing with in this study. Here, we are primarily considering the animals in the same cage that were under the same regimen. The regimen value is the common logarithm of the number of bacillary units in 0.5 ml of challenge solution; i.e., regimen 6.6 corresponds to 4.0×106 bacillary units per 0.5 ml ($\log(4.0 \times 106) = 6.6$). The data are: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Cordeiro et al. (2014) considered three baseline distributions denoted by Ch, XTG and FW and their transformations to Marshall-Olkin family (i.e., $k = 1$ in Harris family) denoted by MOCh, MOXTG and MOFW. They fitted these distributions to a real data set and compared them. They concluded that MOXTG model fits the data well and then can be used to model the survival times of guinea pigs.

Now, we consider XTG, MOXTG and Harris of XTG (HXTG) distributions and fit them to above real data using a SAS Program. The pdf and sf of XTG distribution are

$$f(x;\tau_1,\tau_2,\tau_3) = \tau_1\tau_2(\frac{x}{\tau_3})^{\tau_2-1}exp((\frac{x}{\tau_3})^{\tau_2} + \tau_1\tau_3[1-exp((\frac{x}{\tau_3})^{\tau_2})]), \qquad x > 0,$$

and

$$\bar{F}(x;\tau_1,\tau_2,\tau_3) = exp(\tau_1\tau_3[1-exp((\frac{x}{\tau_3})^{\tau_2})]) \qquad x > 0,$$

respectively, where $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_3 > 0$. Table 2 lists the MLEs and standard errors, in parentheses, of the parameters of XTG, MOXTG and HXTG distributions. AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) are listed in Table 3. We note that HXTG distribution has the lowest AIC and BIC values in relation to its special XTG and MOXTG models and it could be chosen as the best model among the fitted models. HXTG model provides an adequate fit to the real data and it fits the data well and therefore can be used to model the survival times of guinea pigs. By using our results, we can chose either the HXTG or unconditional HXTG models. Let the tilt parameter follow a uniform distribution in unconditional HXTG model. Figure 1 shows that if $P(0 < \nu \leq \Theta \leq 1) = 1$ then $\frac{h(x;k)}{h(x;\nu,k)}$ is increasing. Thus, according to Theorem 1 we have that in the likelihood ratio ordering sense HXTG rv is smaller than unconditional HXTG rv. Figure 2 shows that if $P(0 < \Theta \leq \nu \leq 1) = 1$ then $\frac{h(x;k)}{h(x;\nu,k)}$ is decreasing. Hence, the unconditional HXTG rv is smaller than HXTG model in likelihood ratio ordering. Since the usual stochastic ordering is implied by likelihood ratio ordering, we can conclude that if $P(0 < \nu \leq \Theta \leq 1) = 1$, then sf of unconditional HXTG rv is larger than HXTG rv. If the utility issue is the sf, using unconditional HXTG model will be better than HXTG model. Also, if the utility is a low hazard, using unconditional HXTG model will be more favorable than HXTG model.

Table 2: MLEs and standard errors in parentheses

Model	Estimates				
HXTG($\tau_1,\tau_2,\tau_3,\nu,k$)	0.004319	3.5230	389.28	0.002822	2.2756
	(-)	(0.6537)	(-)	(0.002493)	(0.6966)
MOXTG(τ_1,τ_2,τ_3,ν)	0.001701	2.4390	389.28	0.01223	
	(0.008469)	(0.2782)	(418.98)	(0.04449)	
XTG(τ_1,τ_2,τ_3)	1.851	1.3925	83522433		
	(2.7536)	(0.1125)	(-)		

Table 3: Statistics AIC and BIC

Model	Statistics AIC	Statistics BIC
HXTG	792.6	804.0
MOXTG	796.1	805.3
XTG	810.3	817.2

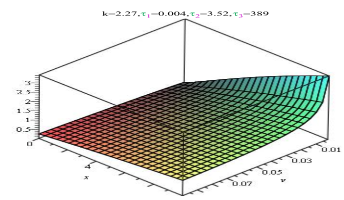

Figure 1: Illustrating $\frac{h(x;k)}{h(x;\nu,k)}$ with baseline pdf XTG and $P(\nu \leq \Theta \leq 1) = 1$, is increasing.

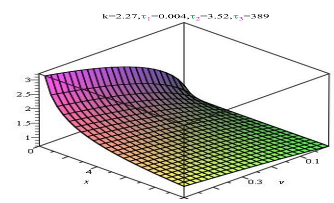

Figure 2: Illustrating $\frac{h(x;k)}{h(x;\nu,k)}$ with baseline pdf XTG and $P(\Theta \leq \nu \leq 1) = 1$, is decreasing.

5 Discussion and conclusion:

The class of Harris family of distribution, which was introduced by Aly (2011), contains a tilt parameter so that it can cover a wide range of data such as those with high degrees of skewness and kurtosis, we describe the better choice of the tilt parameter by comparing a Harris family distribution with a mixed Harris family distribution considering higher likelihood ratio, lower risk (Hazard rate order), longer lifetime (usual stochastic order), higher expectation and lower aging intensity orderings. We also investigated preservation of certain aging properties of the baseline distribution, such as the likelihood ratio and lower shifted proportional hazard rate orderings, by a mixed Harris family distribution. Finally, to show the applicability of our model, we fitted the proposed Harris model to a real data set and compared it with its mixture.

Acknowledgements
The authors are grateful to two anonymous referees whose valuable comments and suggestions led to improvement of the manuscript.

References

Abbasi, S.; Alamatsaz, M. H.and Cramer, E. (2016), *Preservation properties of stochastic orderings by transformation to Harris family with different tilt parameters*. Latin American Journal Of Probability And Mathematical Statistics (Alea). **13**, 465-479.

Abbasi, S. and Alamatsaz, M. H. (2018), *Preservation properties of stochastic orders by transformation to Harris family*. Probability and Mathematical Statistics (PMS). **38** (2), 441-458.

Abbasi, S.; Alamatsaz, M. H. (2019), *Some bounds related to Harris family of distributions*. Communications in Statistics: Theory and Methods. **48** (16), 4082-4095.

Aghababaei Jazi, M.; Alamatsaz, M. H. (2010), *Ordering comparison of logseries random variable with its mixture*. Communications in Statistics: Theory and Methods.**39**, 3252-3263.

Aghababaei Jazi, M.; Alamatsaz, M. H.; Abbasi, S. (2011), *A unified approach to ordering comparison of GPS distributions with their mixtures*. Communications in Statistics: Theory and Methods. **40**, 2591-2604.

Alamatsaz, M. H.; Abbasi, S. (2008) *Ordering comparison of negative binomial random variable with its mixture*. Statistics and Probability Letters. **78**, 2234-2239.

Aly, EAA.; Benkherouf, L. (2011) *A new family of distributions based on probability generating functions*. Sankhya B. **73**, 70-80.

Barlow, R. E.; Proschan, F. (1981), *Statistical Theory of Reliability and Life Testing. Probability Models*. To Begin With: Silver Springs, Maryland.

Batsidis, A.; Lemonte, A. J. (2015) *On the Harris extended family of distributions*. Statistics. **49**, 1400-1421.

Bjerkedal, T. (1960) *Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli.* American Journal of Epidemiology. **72**, 130–148.

Cordeiro G. M.; Lemonte A. J.; Ortega E. M. M. (2014) *The Marshall–Olkin family of distributions: mathematical properties and new Models* Journal of Statistical Theory and Practice,**8**, 343–366.

Harris, TE. (1948) *Branching processes*. Annals of Mathematical Statistics. **19**, 474-494.

Li X., Zhao P., (2011) *On the mixture of proportional odds models.* Communications in Statistics-Theory and Methods. **40**, 333-344.

Lillo, R. E.; Nanda, A. K.; Shaked, M. (2001) *Preservation of some likelihood ratio stochastic orders by order statistics*. Statistics and Probability Letters. **51**, 111-119.

Marshall, A. W.; Olkin, I. (1997) *A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*. Biometrika. **84**, 641-652.

Marshall, A. W.; Olkin, I. (2007) *Life distributions: Structure of nonparametric, semiparametric, and parametric families*. New York: Springer, Springer Series in Statistics.

Nanda, A. k.; Das, S. (2012), *Stochastic orders of the Marshall–Olkin extended distribution*. Statistics and Probability Letters. **82**, 295-302.

Payandeh Najafabadi, A. T; Barmlzan, G. (2016) *Stochastic Comparisons of Series and Parallel Systems with Het- erogeneous Extended Generalized Exponential Components*. Journal of the Iranian Statistical Society (JIRSS). **15**(1), 45-58.

Shaked, M.; Shanthikumar, J. G. (2007) *Stochastic Orders*. Academic Press, New York.