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First-Order Spatial Gegenbauer Autoregressive (SGAR(1,1)) Model and some of its Properties

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Abstract. In this paper, we extend the idea of Gegenbauer process in the spatial domain by introducing a more general parameter and call this model as Spatial Gegenbauer Autoregressive (SGAR(1,1)) model. The spectral density and autocovariance functions of the model are introduced. The Yajima estimators of the Gegenbauer parameters, the log-periodogram regression estimators of the memory parameters and the Whittle's estimators of all parameters are discussed. The performance of these estimators are evaluated through a simulation study.

Keywords. Spatial Gegenbauer Autoregressive Model, SGAR(1,1), Long Memory, Log-Periodogram Estimation, Whittle Estimation.

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1 Introduction

It is common that time series data sometimes display long memory dependence, in the sense that it has a slow decaying autocorrelation function (ACF). These type of time series are usually modelled as Autoregressive Fractionally Integrated Moving Average (ARFIMA) processes as follows

$$\phi(B)(1-B)^d Y_t = \theta(B)Z_t. \tag{1.1}$$

where $\phi(B) = 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p$ is the stationary AR operator, $\theta(B) = 1-\theta_1 B-\theta_2 B^2-\cdots-\theta_p B^p$ is the invertible MA operator, B is the backward shift operator, i.e, $B^j X_i = X_{i-j}$, d is the long memory parameter that $d \in (0, 0.5)$ and Y_t and Z_t are the stationary time series and the white noise process, respectively, at time $t \in \{0, \pm1, \pm2, \cdots\}$. There is a whole host of literature covering these type of models (see for example, Granger and Joyeux (1980), Hosking (1981, 1984), Geweke and Porter-Hudak (1983), Sowell (1992a, b), Beran et. al. (2009), Boissy et. al. (2005), Gray et. al. (1989), etc.).

A generalized class of long memory time series known as GARMA models, using the theory of Gegenbauer polynomials has also been introduced in the literature (See Gray *et. al* (1989) and Chung (1996a,b)). The GARMA models are defined as

$$\phi(B)(1-2uB+B^2)^d Y_t = \theta(B)Z_t, \tag{1.2}$$

where u is called Gegenbaure parameter and $|u| \leq 1$. Distinctive properties of this generalized class are long range dependence and quasi-periodic behaviour, and they provide a model for cyclical or seasonal persistent processes, whose autocorrelation function is an hyperbolically damped oscillating sequence. As a result, the autocorrelation function is not absolutely summable and the spectrum possesses a pole at the cyclical or seasonal frequencies. The GARMA models can be also used to represent long memory depicting multiple unbounded spectral peaks away from the origin, unlike in the standard long memory ARFIMA case of Hosking (1981), which can only show unbounded spectral density peak at the zero frequency (Dissanayake, et al. (2018)).

In 2018, Dissanayake, et al. reviewed and discussed the usefulness of generalized fractionally differenced Gegenbauer processes in time series and econometric research endeavours. The k-factor Gegenbauer process has found its application in the urban transport traffic in the Paris subway (Ferrara and Guegan (2001)). Estimation of the parameters of a stationary Gegenbauer process using a wavelet methodology is presented by Boubaker, 2014. In 2018, Wu and Peiris introduced a new class of time series models generated by the vector Gegenbauer Autoregressive Moving Average structure.

Espejo, et al. (2014) extended the idea of Gegenbauer process to the spatial domain. They introduced the Gegenbauer random fields as,

$$(1 - 2u_1B_1 + B_1^2)^{d_1}(1 - 2u_2B_2 + B_2^2)^{d_2}Y_{ij} = Z_{ij}, \tag{1.3}$$

and autoregressive Gegenbauer random fields as below,

$$\phi(B_1, B_2)(1 - 2u_1B_1 + B_1^2)^{d_1}(1 - 2u_2B_2 + B_2^2)^{d_2}Y_{ij} = Z_{ij}, \tag{1.4}$$

where $\{Y_{ij} : i, j \in \mathbb{Z}\}$ is a spatial process defined on a two-dimensional regular lattice and $\{Z_{ij}\}$ is a two-dimensional white noise process, $|u_1| \leq 1$, $|u_2| \leq 1$, $|d_1| < 0.5$, $|d_2| < 0.5$, B_1 and B_2 denoting backward-shift operators for each spatial coordinates, i.e, $B_1^kB_2^lY_{ij} = Y_{i-k,j-l}$.

They considered the following particular cases of $\phi(B_1, B_2)$ polynomial,

$$\phi(B_1, B_2) = (1 - \phi_{10}B_1 - \phi_{01}B_2 + \phi_{10}\phi_{01}B_1B_2),\ -1 \leq \phi_{10}, \phi_{01} \leq 1, \tag{1.5}$$

and

$$\phi(B_1, B_2) = (1 - \phi_{10}B_1 - \phi_{01}B_2),\ |\phi_{10}| + |\phi_{01}| < 1,$$

and the spectral functions of these processes were also introduced.

Espejo et al. (2015) have introduced the autocorrelation function of the Gegenbauer random fields and obtained the consistency together with its asymptotic normality for a class of minimum contrast estimators (MCE) of the long-range dependence parameters (d_1 and d_2).

In model (1.5), the two dimensional spectral density function is symmetric. In this paper, we replace $\phi_{10}\phi_{01}$ in equation (1.5) by $-\phi_{11}$. The purpose of doing this, is to generalise model (1.5) by reparametrising it in such a way that ϕ_{11} is not restricted to being the product of ϕ_{10} and ϕ_{01}. This generalisation would now allow for models in which the spectral density function is asymmetric about axes.

Hence, in this work we focus our attention to this more general class of models. That is, we consider the model in which the polynomial $\phi(B_1, B_2)$ is given by,

$$\phi(B_1, B_2) = 1 - \phi_{10}B_1 - \phi_{01}B_2 - \phi_{11}B_1B_2.$$

We shall also introduce,

- the autocorrelation function of this model,
- the new method to estimate the Gegenbauer parameters (u_1 and u_2),
- the log-periodogram regression method estimation of long memmory parameters and
- Whittle's method to estimate the whole parameters of the model.

The rest of the paper is organized as follows. The model is first defined in Section 2 while the spectral density and autocovariance functions are discussed in Section 3. In Section 4, the estimation of the parameters of the model are taken up by log-periodogram regression method, Yajima method and Whittle's method. Section 5 consists of some simulation results. Finally, the conclusions are drawn up in Section 6.

2 The SGAR(1,1) Model

A stationary and invertible first-order spatial Gegenbauer Autoregressive (SGAR(1,1)) process is a seasonal long memory process generated by the equation

$$(1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2)(1-2u_1B_1+B_1^2)^{d_1}(1-2u_2B_2+B_2^2)^{d_2}Y_{ij}=Z_{ij}, \tag{2.1}$$

where $\{Y_{ij} : i, j \in \mathbb{Z}\}$ is a spatial process defined on a two-dimensional regular lattice, $\{Z_{ij}\}$ is a two-dimensional white noise process with mean zero and variance σ_Z^2 with $|\phi_{10}| < 1$, $|\phi_{01}| < 1$, $|\phi_{10}+\phi_{01}| < 1-\phi_{11}$, $|\phi_{10}-\phi_{01}| < 1+\phi_{11}$, $0 < d_i < 0.5$ for $|u_i| < 1$ and $0 < d_i < 0.25$ for $|u_i| = 1$, $i = 1, 2$. The u_1 and u_2 called as Gegenbauer parameters. If $\phi_{11} = -\phi_{10}\phi_{01}$, then the term $1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2$ in (2.1) can be factored out as $(1-\phi_{10}B_1)(1-\phi_{01}B_2)$ and in this case we call the reduced process (2.1) as Separable SGAR(1,1) model (denoted as SSGAR(1,1), see Ghodsi and Shitan (2021)).

The SGAR(1,1) model can be equivalently represented by the following two equations

$$(1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2)Y_{ij}=W_{ij}, \tag{2.2}$$

and

$$(1-2u_1B_1+B_1^2)^{d_1}(1-2u_2B_2+B_2^2)^{d_2}W_{ij}=Z_{ij}. \tag{2.3}$$

where (2.2) is the spatial AR(1,1) model, which its innovations W_{ij}, are spatial Gegenbauer white noise (SGWN) process and (2.3) represent the SGWN process introduced by Espejo, et al. (2014 and 2015).

The process Y_{ij} defined in equation (2.2) can be written as

$$Y_{ij} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty} \frac{(k+\ell+r)!}{k!\ell!r!}\phi_{10}^{k}\phi_{01}^{\ell}\phi_{11}^{r} W_{i-k-r,j-\ell-r}, \tag{2.4}$$

and the process W_{ij} can be written in a Gegenbauer polynomial series (Gradshteyn and Ryzhik (1980), section 8.93) as follows

$$W_{ij} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} C_k^{(d_1)}(u_1)C_\ell^{(d_2)}(u_2)Z_{i-k,j-\ell}, \tag{2.5}$$

where

$$C_k^{(d_i)}(u_i) = \sum_{r=0}^{[\frac{k}{2}]}(-1)^r \frac{\Gamma(k-r+d_i)}{\Gamma(d_i)\Gamma(k+1)\Gamma(k-2r+1)}(2u_i)^{k-2r},$$

when $|u_i| < 1$ and $0 < d_i < 0.5$, for $i = 1, 2$, and

$$C_k^{(d_i)}(u_i) = \frac{\Gamma(k+2d_i)}{\Gamma(2d_i)\Gamma(k+1))}(-u_i)^k,$$

when $|u_i| = 1$ and $0 < d_i < 0.5$, for $i = 1, 2$.

In the following section we discuss the spectral and autocovariance functions of the SGAR(1,1) model.

3 Spectral Density and Autocovariance Functions

The spectral function of the SGAR(1,1) model defined in (2.2), by definition, is given as

$$f_Y(\omega_1,\omega_2) = |1-\phi_{10}e^{-i\omega_1}-\phi_{01}e^{-i\omega_2}-\phi_{11}e^{-i(\omega_1+\omega_2)}|^{-2} f_W(\omega_1,\omega_2), \tag{3.1}$$

where $|\omega_i| <, i$ for $i = 1, 2$, and $f_W(\omega_1, \omega_2)$ is the spectral function of the SGWN process defined in (2.3), which is given as follows (see Espejo, et al. (2014)),

$$\begin{aligned} f_W(\omega_1,\omega_2) &= \frac{\sigma_Z^2}{4\pi^2}|1-2u_1e^{-i\omega_1}+e^{-2i\omega_1}|^{-2d_1}|1-2u_2e^{-i\omega_2}+e^{-2i\omega_2}|^{-2d_2} \\ &= \frac{\sigma_Z^2}{4\pi^2}\left|2(\cos\omega_1-u_1)\right|^{-2d_1}|2(\cos\omega_2-u_2)\right|^{-2d_2}, \end{aligned} \tag{3.2}$$

where $(\omega_1, \omega_2) \in [-\pi, \pi]^2 - \{(\omega_1, \omega_2) | \cos(\omega_1) = u_1 \text{ or } \cos(\omega_2) = u_2\}$. It is clear that $f_W(\omega_1, \omega_2)$ and $f_Y(\omega_1, \omega_2)$ have unbounded peaks at $\nu_1 = cos^{-1}(u_1)$ and $\nu_2 = cos^{-1}(u_2)$ termed as Gegenbauer frequencies.

Using equation (3.2), it can be easily seen that $f_W(-\omega_1, -\omega_2) = f_W(\omega_1, \omega_2)$, $f_W(-\omega_1, \omega_2) = f_W(\omega_1, \omega_2)$ and $f_W(\omega_1, -\omega_2) = f_W(\omega_1, \omega_2)$. Besides since $|1 - \phi e^{-i\omega}|^2 = (1 - \phi \cos \omega)^2 + (\phi \sin \omega)^2$, then the spectral function of SSGAR(1,1) model is symmetric. Howevere, since

$$f(\omega_1, \omega_2) = |1 - \phi_{10} e^{-i\omega_1} - \phi_{01} e^{-i\omega_2} - \phi_{11} e^{-i(\omega_1+\omega_2)}|^2 = (1 - \phi_{10} \cos \omega_1 - \phi_{01} \cos \omega_2 - \phi_{11} \cos(\omega_1 + \omega_2))^2 + (\phi_{10} \sin \omega_1 - \phi_{01} \sin \omega_2 - \phi_{11} \sin(\omega_1 + \omega_2))^2$$

we have $f(-\omega_1, -\omega_2) = f(\omega_1, \omega_2)$ but $f(-\omega_1, \omega_2) \neq f(\omega_1, \omega_2)$ and $f(\omega_1, -\omega_2) \neq f(\omega_1, \omega_2)$,then the spectral function of the SGAR(1,1) model is symmetric about the origin and asymmetric about the axes.

Therfore, the difference between SGAR(1,1) model and SSGAR(1,1) model is that, the spectral function of SSGAR(1,1) model is symmetric (because $f_W(-\omega_1, -\omega_2) = f_W(-\omega_1, \omega_2) = f_W(\omega_1, -\omega_2) = f_W(\omega_1, \omega_2)$) but the spectral function of the SGAR(1,1) model is symmetric about the origin(i.e., $f_W(-\omega_1, -\omega_2) = f_W(\omega_1, \omega_2)$ and asymmetric about the axes (i.e., $f_W(-\omega_1, \omega_2) \neq f_W(\omega_1, \omega_2)$ or/and $f_W(\omega_1, -\omega_2) \neq f_W(\omega_1, \omega_2)$) .

Figure 1 compares the spectral functions of the SGAR(1,1) and SSGAR(1,1) models for some selected parameter values. The parameters in SGAR(1,1) model were considered as $\phi_{10} = 0.1, \phi_{01} = 0.3, \phi_{11} = 0.2, d_1 = 0.1, d_2 = 0.3, u_1 = 0.3$ and $u_2 = 0.1$, and for SSGAR(1,1) were considered as $\phi_{10} = 0.1, \phi_{01} = 0.3, \phi_{11} = -0.03, d_1 = 0.1, d_2 = 0.3, u_1 = 0.3$ and $u_2 = 0.1$.

From Figure 1, it can be seen that the spectral function of the SSGAR(1,1) model is symmetric, but the spectral function of the SGAR(1,1) model is asymmetric about the axes. furthermore, both of them have some peeks away from the origin.

Given the spectral function, we can compute the autocovariance function (ACVF) $\gamma_W(h_1, h_2)$ for the SGWN process. For the case $|u_i| < 1$, for $i = 1, 2$, then the ACVF of the

SGWN process is given by (see Espejo, et al. (2015) and Dissanayake et al. (2018)),

$$\begin{aligned}\gamma_W(h_1,h_2) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(h_1\omega_1+h_2\omega_2)} f_W(\omega_1,\omega_2)d\omega_1 d\omega_2 \\ &= \frac{\sigma_Z^2}{4\pi}\prod_{i=1}^{2}\Gamma(1-2d_i)\,[2\sin(\nu_i)]^{0.5-2d_i} \\ &\quad \times[P_{h_i-0.5}^{2d_i-0.5}(u_i)+(-1)^{h_i}P_{h_i-0.5}^{2d_i-0.5}(-u_i)]. \end{aligned} \quad (3.3)$$

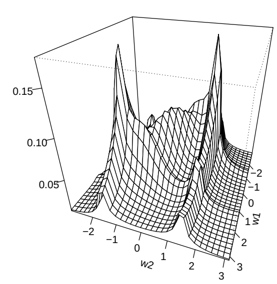
where $P_a^b(x)$ is the associated Legendre function of the first kind given as follows (see Gradshteyn and Ryzhik (2014), page 1015),

$$P_a^b(x) = \frac{1}{\Gamma(1-b)}\left(\frac{1+x}{1-x}\right)^{b/2} F(-a, a+1; 1-b; \frac{1-x}{2}),$$

in which

$$F(a,b;c;w) = \sum_{n=1}^{\infty}\frac{\Gamma(c)\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}w^n,$$

is the hypergeometric function.



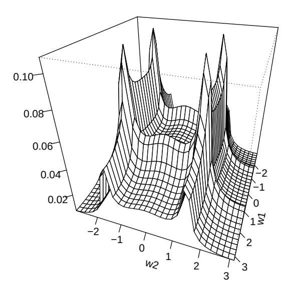


Figure 1: Spectral function of (a) SGAR(1,1) and (b) SSGAR(1,1) models for selected parameter values.

Now we can obtain the ACVF of the SGAR(1,1) process. Since $E(W_{ij}) = 0$, we have $E(Y_{ij}) = 0$, and therefore the autocovariance function of the SGAR(1,1) process is given as,

$$\begin{aligned}\gamma_Y(h_1,h_2) &= E(Y_{i+h_1,j+h_2}Y_{ij})\\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(k+\ell+r)!}{k!\ell!r!}.\frac{(m+n+p)!}{m!n!p!}\phi_{10}^{k+m}\phi_{01}^{\ell+n}\phi_{11}^{r+p}\\ &\times\ \gamma_W(h_1+k+r-m-p,h_2+l+r-n-p),\end{aligned} \tag{3.4}$$

where $h_1, h_2 \in \mathbb{Z}$ and $\gamma_W(.,.)$ is given as in (3.3).

For the SSGAR(1,1) process, (i.e. when $\phi_{11} = -\phi_{10}\phi_{01}$), Y_{ij} defined in (3) can be written as

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\phi_{10}^{k}\phi_{01}^{\ell}W_{i-k,j-\ell}, \tag{3.5}$$

and its ACVF is given by,

$$\gamma_Y(h_1,h_2) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\phi_{10}^{k+m}\phi_{01}^{\ell+n}\gamma_W(h_1+k-m,h_2+\ell-n) \tag{3.6}$$

where $\gamma_W(.,.)$ is given as in (3.3).

Figure 2 gives an example of the autocorrelation function of the SGAR(1,1) and SSGAR(1,1) models for selected parameter values. It can be seen that the autocorrelation functions of both models are hyperbolically damped which is a properties of long memory processes.

In the next section the estimation of parameters of the SGAR(1,1) process is briefly discussed.

4 Parameters Estimation

To obtain preliminary estimates for the long memory parameters (d_1, d_2) and the coefficients (u_1, u_2), log-periodogram regression method and Yajima (1996) estimators can be used, respectively. All the parameters could be estimated using Whittle's method. These methods will be described in the following.

4.1 Yajima Estimation of u_1 and u_2 Parameters

In practice, u_1 and u_2 are unknown and have to be estimated. We extend the idea of Yajima (1996) for estimating the frequency of an unbounded spectral function in

Figure 2: Autocorrelation function of (a) SGAR(1,1) and (b) SSGAR(1,1) models for selected parameter values.

one dimension to estimate the Gegenbauer parameters of the SGAR(1,1) model in two dimension as follows:

$$(\hat{u}_1, \hat{u}_2) = (\cos(\omega_1), \cos(\omega_2)) \tag{4.1}$$

where

$$(\omega_1, \omega_2) = arg \max I_{j_1, j_2}; \qquad j_i = -\left[\frac{n_i - 1}{2}\right], \ldots, \left[\frac{n_i}{2}\right] \text{for} \quad i = 1, 2,$$

where I_{j_1,j_2} will be introduced in the next section.

4.2 Log-Periodogram Regression Estimation of the Long Memory Parameters

Assume that $\{Y_{ij} : i, j \in \mathbb{Z}\}$ is a SGAR(1,1) process as defined in (2.1). Let $U_{ij} = (1 - 2u_1 B_1 + B_1^2)^{d_1}(1 - 2u_2 B_2 + B_2^2)^{d_2} Y_{ij}$. The spectral function of Y_{ij} can be written as

$$f_Y(\omega_1, \omega_2) = |1 - 2u_1 e^{-i\omega_1} + e^{-2i\omega_1}|^{-2d_1} |1 - 2u_2 e^{-i\omega_2} + e^{-2i\omega_2}|^{-2d_2} f_U(\omega_1, \omega_2), \tag{4.2}$$

where

$$f_U(\omega_1, \omega_2) = \frac{\sigma^2}{4\pi^2} |1 - \phi_{10} e^{-i\omega_1} - \phi_{01} e^{-i\omega_2} - \phi_{11} e^{-i(\omega_1 + \omega_2)}|^{-2},$$

and $\omega_1, \omega_2 \in [-\pi, \pi]$.

Define the two-dimensional periodogram ordinates of the process as follows,

$$I(\omega_{1,j_1}, \omega_{2,j_2}) = \frac{1}{n_1 n_2} \left| \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} y_{s,t} e^{i(s\omega_{1,j_1} + t\omega_{2,j_2})} \right|^2,$$

where $(y_{11}, y_{12}, \ldots, y_{1n_2}, y_{21}, y_{22}, \ldots, y_{2n_2}, \ldots, y_{n_1 1}, \ldots, y_{n_1 n_2})$ is an observed dataset of grid size $n_1 \times n_2$ of the SGAR(1,1) process and $\omega_{i,j_i} = 2\pi j_i / n_i$, $j_i = -\left[\frac{n_i-1}{2}\right], \cdots, \left[\frac{n_i}{2}\right]$ for $i = 1, 2$.

By taking logarithm of (4.2) and adding $\ln I_{j_1,j_2}$ to both sides, we obtain the multiple regression equation as follows (see Shitan (2008))

$$\ln I_{j_1,j_2} = \beta_0 + \beta_1 x_{1,j_1} + \beta_2 x_{2,j_2} + \varepsilon_{j_1,j_2}, \tag{4.3}$$

where $I_{j_1,j_2} = I(\omega_{1,j_1}, \omega_{2,j_2})$, $\beta_0 = \ln f_U(0,0) - C, \beta_1 = d_1, \beta_2 = d_2$, $x_{1,j_1} = x(\omega_{1,j_1}) = -2\ln|1 - 2u_1 e^{-i\omega_{1,j_1}} + e^{-2i\omega_{1,j_1}}|$, $x_{2,j_2} = x(\omega_{2,j_2}) = -2\ln|1 - 2u_2 e^{-i\omega_{2,j_2}} + e^{-2i\omega_{2,j_2}}|$, $\varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2}/f_{j_1,j_2}) + C$, $f_{j_1,j_2} = f_Y(\omega_{1,j_1}, \omega_{2,j_2})$ and $C = 0.5772$ is Euler's constant. Note that, according (3.2) and (4.3), if there exists some j_1 and j_2 such that $\cos(\omega_{1,j_1}) = u_1$ or $\cos(\omega_{2,j_2}) = u_2$ or $I_{j_1,j_2} = 0$, the corresponding elements in $\ln I_{j_1,j_2}$, x_{1,j_1} and x_{2,j_2} should be excluded.

Suppose u_1 and u_2 have been estimated by Yajima method, presented in section 4.1, and d_1 and d_2 can be estimated by ordinary least square (OLS) method. After estimating d_1 and d_2, we can obtain estimates of U_{ij} as $\hat{U}_{ij} = (1 - 2u_1 B_1 + B_1^2)^{\hat{d}_1}(1 - 2u_2 B_2 + B_2^2)^{\hat{d}_2} Y_{ij}$, and from which the usual methods of estimation like maximum likelihood (Basu and Reinsel, 1993) or whittle can be employed to estimate out the parameters ϕ_{10}, ϕ_{01}, ϕ_{11} and σ_Z^2.

4.3 Whittle's Method

The Whittle's estimator $\hat{\boldsymbol{\eta}}$ is the value of $\boldsymbol{\eta} = (\phi_{10}, \phi_{01}, \phi_{11}, d_1, d_2, u_1, u_2)'$ that minimizes the Whittle's likelihood function as follows

$$L_W(\boldsymbol{\eta}) = \frac{1}{n_1 n_2} \sum_{j_1} \sum_{j_2} \frac{I(\omega_{1,j_1}, \omega_{2,j_2})}{g(\omega_{1,j_1}, \omega_{2,j_2}; \boldsymbol{\eta})}, \tag{4.4}$$

where $\omega_{i,j_i} = \frac{2\pi j_i}{n_i}$, $j_i = -\left[\frac{n_i-1}{2}\right], \cdots, \left[\frac{n_i}{2}\right]$, for $i = 1, 2$, and $g(\omega_{1j_1}, \omega_{2j_2}; \boldsymbol{\eta}) = \frac{4\pi^2}{\sigma^2} f(\omega_{1,j_1}, \omega_{2,j_2})$. An estimator for σ_Z^2 is $L_W(\hat{\boldsymbol{\eta}})$.

Since this maximization procedure cannot be done analytically, we use numerical approach to estimate the parameters.

5 Numerical Study

5.1 Simulation Results

In order to provide the finite sample performance of the proposed estimators in section 4, some Monte-Carlo simulations of the SGAR(1,1) process defined in (2.1) are carried out. To simulate the SGAR(1,1) model, we first generate W_{ij} using (2.5), in which $Z_{ij} \overset{iid}{\sim} N(0,1)$, and then we generate Y_{ij} using (2.4). Since the sums in (2.5) and (2.4) are infinite, the calculation continued untill the sums converge.

We consider 500 replications with different sample sizes $(n_1, n_2) = (20\times20), (50\times50)$ and (80×80) and two sets of parameters $\boldsymbol{\theta} = (\phi_{10}, \phi_{01}, \phi_{11}, d_1, d_2, u_1, u_2, \sigma_Z^2)$ = (1): (0.1, 0.3, 0.2, 0.1, 0.3, 0.3, 0.1, 1) and (2): (0.3, 0.4, 0.2, 0.3, 0.2, 0.3, 0.4, 1).

We investigate the performance of the Yajima estimators (YE) of the Gegenbauer parameters defined in (4.1), the Log-Periodogram Regression (LPR) estimators of the memory parameters of the model defined in (4.3) and the Whittle (WITL) estimators of all parameters discussed in section 4 through a simulation study. All the simulations were carried out using R software.

Table 1 and 2 shows the estimated bias and RMSE (root mean squared error) values of introduced estimators. It appears from these tables that the bias and RMSE values from WITL method are often smaller than the other methods for all grid sizes. It can also be seen that the bias and RMSE values decrease when the grid size increases.

5.2 A Real Data Example

In this subsection, we applied the SGAR(1,1) model to the dataset of the yield of barley (kg), from an agricultural uniformity trial experiment (on a regular grid of 28 $\times$ 7) presented by Kempton and Howes (1981) at Plant Breeding Institute, Cambridge, England.

Figure 3 shows the sample spatial correlation and peridogram functions of the data. From Figure 3(a), it is clear that the data are highly correlated. As such, we fitted the SGAR(1,1) model to the mean corrected data using Whittle's estimation and the result is as follows,

$$
\begin{aligned}
&(1 - 0.365B_1 - 0.176B_2 + 0.145B_1B_2)(1 - 2(0.99999)B_1 + B_1^2)^{0.143} \\
&(1 - 2(0.621)B_2 + B_2^2)^{0.000001} Y_{ij} = Z_{ij},
\end{aligned}
$$

with $\hat{\sigma}_Z^2 = 1.002$ and AIC=6.653.

Furthermore, as a comparison we also fitted the SSGAR(1,1) model and the fitted model is,

$$(1 - 0.755B_1)(1 + 0.187B_2)(1 - 2(0.49999)B_1 + B_1^2)^{0.058}$$
$$(1 - 2(0.650)B_2 + B_2^2)^{0.000001}Y_{ij} = Z_{ij},$$

with $\hat{\sigma}_Z^2 = 1.042$ and AIC=4.732.

It was found that the AIC value for the SSGAR(1,1) model is smaller. Therefore, we can conclude that the SSGAR(1,1) model is the better model compare to the SGAR(1,1) model for the data, because the correlation function is not cyclicall.

Figure 3: The sample spatial autocorrelations (a) and the periodogram (b) of the yield of barley data.

6 Conclusion

The main objective of this research is to extend the idea of Gegenbauer process to the spatial domain and introduce a model known as Spatial Gegenbauer Autoregressive (SGAR(1,1)) model.

The Spectral Density and Autocovariance functions were discussed and the estimation of the parameters of the model by two different methods namely, log-periodogram regression method and Whittle's method were provided. The numerical simulation results indicated that the bias and RMSE values of the Whittle estimators were uniformly smaller than the other methods for all grid sizes. The SGAR(1,1) model was also applied to a real data example.

The importance of this study is that by introducing the Spatial Gegenbauer Autoregressive (SGAR(1,1)) Model, it adds to the literature of spatial models and thereby extends the field of spatial modelling.

Table 1: The Bias and RMSE values estimated by YE, LPR and WITL method for true parameters $\boldsymbol{\theta}$=(0.1, 0.3, 0.2, 0.1, 0.3, 0.3, 0.1, 1). '-' means that, the estimator of the parameter has not been defined.

		ϕ_{10}		ϕ_{01}		ϕ_{11}		d_1	
Grid size	Method	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	YE	-	-	-	-	-	-	-	-
20×20	LPR	-	-	-	-	-	-	0.0042	0.0012
	WITL	-0.0048	0.0003	-0.0048	0.0002	-0.0047	0.0002	-0.0032	0.0002
	YE	-	-	-	-	-	-	-	-
50×50	LPR	-	-	-	-	-	-	0.0153	0.0006
	WITL	-0.0014	0.0000	-0.0009	0.0000	-0.0023	0.0000	-0.0007	0.0000
	YE	-	-	-	-	-	-	-	-
80×80	LPR	-	-	-	-	-	-	0.0254	0.0008
	WITL	-0.0003	0.0000	-0.0006	0.0000	-0.0008	0.0000	0.0004	0.0000

		d_2		u_1		u_2		σ_Z^2	
Grid size	Method	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
	YE	-	-	0.0901	0.1652	0.1740	0.1575	-	-
20×20	LPR	0.1257	0.0180	-	-	-	-	-	-
	WITL	-0.0045	0.0003	0.0091	0.0001	-0.0354	0.0033	0.0060	0.0087
	YE	-	-	0.0085	0.1537	0.0152	0.0221	-	-
50×50	LPR	0.0454	0.0026	-	-	-	-	-	-
	WITL	-0.0044	0.0001	0.0092	0.0001	-0.0368	0.0014	-0.0115	0.0010
	YE	-	-	-0.0355	0.1274	-0.0070	0.0050	-	-
80×80	LPR	0.0289	0.0010	-	-	-	-	-	-
	WITL	-0.0006	0.0000	0.0095	0.0001	-0.0213	0.0005	-0.0042	0.0004

Table 2: The Bias and RMSE values estimated by YE, LPR and WITL method for true parameters $\boldsymbol{\theta}$=(0.3, 0.4, 0.2, 0.3, 0.2, 0.1, 0.4, 1). '-' means that, the estimator of the parameter has not been defined.

		ϕ_{10}		ϕ_{01}		ϕ_{11}		d_1	
Grid size	Method	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
20×20	YE	-	-	-	-	-	-	-	-
	LPR	-	-	-	-	-	-	0.1056	0.0145
	WITL	-0.0011	0.0012	-0.0202	0.0015	-0.0378	0.0024	-0.0511	0.0039
50×50	YE	-	-	-	-	-	-	-	-
	LPR	-	-	-	-	-	-	0.0195	0.0005
	WITL	-0.0005	0.0000	-0.0044	0.0000	-0.0054	0.0001	-0.0054	0.0002
80×80	YE	-	-	-	-	-	-	-	-
	LPR	-	-	-	-	-	-	0.0254	0.0008
	WITL	-0.0004	0.0000	-0.0009	0.0000	-0.0018	0.0000	-0.0016	0.0000

		d_2		u_1		u_2		σ_Z^2	
Grid size	Method	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
20×20	YE	-	-	0.5717	0.4698	0.4005	0.2158	-	-
	LPR	0.1908	0.0394	-	-	-	-	-	-
	WITL	-0.0316	0.0020	-0.0926	0.0009	-0.0891	0.0083	0.0076	0.0087
50×50	YE	-	-	0.5977	0.5237	0.4082	0.2358	-	-
	LPR	0.1364	0.0188	-	-	-	-	-	-
	WITL	-0.0065	0.0002	-0.0353	0.0013	0.0258	0.0007	0.0264	0.0019
80×80	YE	-	-	0.5317	0.4628	0.3245	0.0050	-	-
	LPR	0.1112	0.0158	-	-	-	-	-	-
	WITL	-0.0019	0.0000	-0.0211	0.0004	-0.0173	0.0003	0.0183	0.0008

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