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Stochastic Comparisons on the Residual Lifetimes of Series Systems with Arbitrary Components using Copulas

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Abstract. In this paper, we consider series systems consisting of arbitrary dependent components. We study the residual lifetimes of such systems based on copulas family from a new point of view. First, we extract a new explicit expression for the reliability functions of residual lifetimes of the systems. Moreover, we give some stochastic ordering properties for the residual lifetimes of series systems based on the dependence structure of the components and the corresponding mean functions. The results are expanded for series systems having used components of age $t > 0$. Subsequently, the problem of the stochastic comparison of a series system having used components and a used series system has been considered. To show the application of results, we provide some numerical examples. Finally, we present some dependence properties of the residual lifetimes of series system based on the properties of the lifetimes of components.

Keywords. Copula, Dependence, Mean function, Reliability, Residual lifetime, Stochastic order.

MSC: 62H05, 60E15.

1 Introduction

Historically, numerous authors have focused their attention on conducting studies on the coherent systems, especially series and parallel systems. The reliability of a system depends on the structure of the system and obviously to the reliability of its

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components. To get more information about coherent systems and details in this regard, refer to Barlow and Proschan (1981) as a comprehensive resource. The series system is a system with no redundancy which is utilized in various industries. Several researchers have tried to study this system according to various concepts and criteria. The mean residual lifetime (MRL) is a significant and useful criterion which has been seriously considered in recent decades. Suppose that the considered component or system is working, the matter of interest is to study the aging properties, especially the residual lifetime of component or system at the given time t. There are several papers with regards to the studies on mean residual lifetime of systems, among which we can refer to Bairamov et al. (2002), Asadi and Bayramoglu (2006), Li and Zhao (2006), Bairamov and Arnold (2007), Li and Zhang (2008), Zhao et al. (2008) and Navarro et al. (2008). Moreover, a large number of authors have tried to make a stochastic comparison between the residual lifetimes of systems among which Li and Zhao (2008), Kochar and Xu (2010), Zhang and Yan (2010), Gupta (2013), Sajadi et al. (2020) and Guo et al. (2020) can be referred.

It is important to be noted here that all the mentioned results were gained on the condition of system components being independent. In fact, we encounter systems which have arbitrary dependent components. Therefore, the analysis of dynamic reliability, particularly the study of the residual lifetimes of systems, is the matter of interest in this case, which is usually analyzed according to copulas theory. Copula function is a useful and efficient tool for describing the dependence structure between the components of a system. Recently, many researchers have focused their attention on this field, for instance you can see Navarro and Spizzichino (2010), Zhang (2010), Navarro and Rubio (2011), Rezapour et al. (2013), Durante and Foschi (2014), Tavangar (2014), Jia et al. (2014), Tavangar and Asadi (2015), Salehi (2016), Salehi and Hashemi-Bosra (2017), Navarro (2018), Salehi and Tavangar (2019), Amini-Seresht et al. (2020), Barmalzan et al. (2020) and Zhang and Yan (2022).

In some situations, it is possible for us to select a used system consisting of n components and a similar system made up of used components and compare them together (see for example Gupta (2013), Gupta et al. (2015)). Therefore, the attractive issue in here is to make a comparison between the residual lifetimes of these two systems from the theoretical viewpoint of dynamic reliability which is to be investigated in this paper for series systems. The present paper has been arranged as follows. Some required concepts and preliminaries for giving main results are presented in Section 2. In Section 3, at first an explicit expression is obtained for the reliability function (survival function) of the residual lifetime of series systems supposing that the components are arbitrary dependent. Consequently, necessary and sufficient conditions on dependence structure of components and corresponding mean functions have been provided for stochastic comparison of residual lifetimes for such series systems. Besides, we extend the results for series systems having used components. Then, we compare the residual lifetime of a used series system of age $t > 0$ with the lifetime of the similar series system made up of used components of age $t > 0$. Our findings are evaluated by some numerical examples. Finally, we present some dependence properties of the residual lifetime of series systems based on the survival copula of components lifetimes.

2 Preliminaries

In this section we introduce the copula function and dependence concepts, for more details see Nelsen (2007). Besides to provide the main results, we need to define the Schur-concave (Schur-convex) and the mean function.

Definition 2.1. *A* (n-variate) copula is a function $C : [0,1]^n \to [0,1]$ such that

1. for any $u_i \in [0,1]$, $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$,
2. for any $u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n \in [0,1], C(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n) = 0$,
3. C is n-increasing.

Definition 2.2. Let F be the bivariate distribution function (X,Y), $\bar{F}$ survival bivariate distribution function (X,Y) and $G_X(x), G_Y(y)$ be marginal functions.

a) F is positively (negatively) quadrant dependent ($PQD(NQD)$) if for every $(x,y) \in \mathbf{R}^2$, then $F(x,y) \geq (\leq) G_X(x)G_Y(y)$.

b) F is right tail increasing denote by $RTI(Y|X)$, if $\frac{\bar{F}(x,y)}{\bar{G}_X(x)}$ is non-decreasing in x.

c) X and Y are right corner set increasing denote by $RCSI(X,Y)$, if $P(X > x, Y > y | X > x', Y > y')$, is non-decreasing in x' and y'.

Definition 2.3. Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ be a n-dimensional random vector. Then $\mathbf{T}$ is positively upper orthant dependent ($PUOD$) if for all $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ in R^n,

$$P[\mathbf{T} > \mathbf{t}] \geq \prod_{i=1}^{n} P[T_i > t_i].$$

Definition 2.4. Consider two intervals A and B in $\mathbf{R}$, a function $K : A \times B \to \mathbf{R}$ is said to be reverse regular of order two, we write by RR_2, if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such $x_1 \leq x_2, y_1 \leq y_2$,

$$K(x_1,y_1)K(x_2,y_2) \leq K(x_1,y_2)K(x_2,y_1).$$

To present the next definitions we need to explain the majorization order. A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is said to be majorized by another vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ (written as $(u_1, u_2, \ldots, u_n) \leq_m (v_1, v_2, \ldots, v_n)$) if $\sum_{j=1}^{n} u_j = \sum_{j=1}^{n} v_j$ and $\sum_{j=1}^{i} u_{j:n} \geq \sum_{j=1}^{i} v_{j:n}$ for $i = 1, 2, \ldots, n-1$, where $u_{j:n}$ ($v_{j:n}$) is the jth smallest element of $\mathbf{u}$ ($\mathbf{v}$), $j = 1, 2, \ldots, n$.

Definition 2.5. Let $g : R^n \to R$ be a real-valued function, then g is Schur-concave (Schur-convex) if

$$g(u_1, u_2, \ldots, u_n) \leq (\geq) g(v_1, v_2, \ldots, v_n),$$

whenever $(u_1, u_2, \ldots, u_n) \geq_m (v_1, v_2, \ldots, v_n)$.

Definition 2.6. Let $g : R^n \to R$ be a real-valued function, then g is weakly Schur-concave (weakly Schur-convex) if

$$g(u_1, u_2, \ldots, u_n) \leq (\geq) g(\bar{u}, \bar{u}, \ldots, \bar{u}),$$

for all $(u_1, u_2, \ldots, u_n)$, where $\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$.

If a copula family is Schur-concave (Schur-convex), in fact it is weakly Schur-concave (weakly Schur-convex), too. In the following we give the concept of mean function that is useful to present the main results (see Hardy et al. (1934)).

Definition 2.7. Let $g : R^n \to R$ be a real-valued function, then the mean function associated with g is any function $m_g : R^n \to R$ such that

$$g(u_1, u_2, \ldots, u_n) = g(z, z, \ldots, z),$$

for all $(u_1, u_2, \ldots, u_n)$, where $z = m_g(u_1, u_2, \ldots, u_n)$.

Lemma 2.1. *If m_g is the mean function associated with an increasing function g, then g is weakly Schur-concave (weakly Schur-convex) if and only if*

$$m_g(u_1, u_2, \ldots, u_n) \leq (\geq) \frac{1}{n}\sum_{i=1}^{n} u_i.$$

Definition 2.8. Let X and Y be two nonnegative random variables with survival functions $\bar{F}$ and $\bar{G}$, respectively. X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{\text{st}} Y$, if for all t, $\bar{F}(t) \leq \bar{G}(t)$.

3 Results of Stochastic Comparisons

Let $T_1, T_2, \ldots, T_n$ be the arbitrarily lifetime variables of n components. Denote by $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ the ordered lifetimes of the components. The survival function of a n-dependence- components system is denoted by

$$\bar{F}(t_1, t_2, \ldots, t_n) = P(T_1 > t_1, T_2 > t_2, \ldots, T_n > t_n), \tag{3.1}$$

with marginal survival functions $\bar{F}_i(t) = P(T_i > t)$, $i = 1, 2, \ldots, n$. From Sklar's theorem (see Nelsen (2007)), the joint reliability function in (3.1) can be written as

$$\bar{F}(t_1, t_2, \ldots, t_n) = \hat{C}(\bar{F}_1(t_1), \bar{F}_2(t_2), \ldots, \bar{F}_n(t_n)),$$

where $\hat{C}$ is the survival copula. For more details about survival copulas see Nelsen (2007) (p. 32). We also suppose that $\bar{F}_i(t)$, $i = 1, 2, \ldots, n$, is strictly decreasing; therefore, these functions are invertible and their inverse will have the same behavior. Moreover, $\bar{F}(t_1, \ldots, t_n)$ will be strictly decreasing in each of its elements. In this section we focus on the series system with dependent components. In following, we obtain the reliability function of the residual lifetime of a series system using the family of copulas $\{\hat{C}_t\}_{t\geq 0}$, (see Foschi (2010)).

Let $\hat{C}$ be the survival copula of the lifetime vector $\mathbf{T} = (T_1, T_2, \ldots, T_n)$. It is known that the lifetime of the series system is $T_{1:n} = \min\{T_1, T_2, \ldots, T_n\}$. The conditional random variable $(T_{1:n})_t = \{T_{1:n} - t | T_{1:n} > t\}$ defines the residual lifetime of the series system under the condition that all of components of the system are working at time t. The survival function of $(T_{1:n})_t$ denoted by $\psi_{1:n}(x|t)$, is equal to

$$\begin{aligned}\psi_{1:n}(x|t) &= \bar{F}_t(x, \ldots, x) = P((T_{1:n})_t > x) \\ &= P(T_{1:n} > x + t | T_{1:n} > t) = \frac{\bar{F}(x+t, \ldots, x+t)}{\bar{F}(t, \ldots, t)}. \end{aligned} \tag{3.2}$$

Foschi (2010) showed that the evolution of the dependence among $((T_1 - t, \ldots, T_n - t)|T_1 > t, \ldots, T_n > t)$, can be studied in terms of the upper threshold copulas $(\hat{C}_t)$, as follows:

$$\hat{C}_t(u, \ldots, u) = \bar{F}_t\left((\bar{F}_t^{(1)})^{-1}(u), \ldots, (\bar{F}_t^{(n)})^{-1}(u)\right), \tag{3.3}$$

where $\bar{F}_t^{(i)}$ is the univariate marginal survival function defined as

$$\bar{F}_t^{(i)}(x) = P((T_i)_t > x) = P(T_i > t + x | T_1 > t, \ldots, T_n > t), \qquad i = 1, 2, \ldots, n. \tag{3.4}$$

In fact, $\hat{C}_t$ is the survival copula of the random variables $T_1 - t, \ldots, T_n - t$, provided that $T_1 > t, \ldots, T_n > t$.

From continuity and monotonicity of $\bar{F}_t$ and $\bar{F}_i(.)$, $i = 1, 2, \ldots, n$, we can write $\hat{C}_t$ in terms of $\hat{C}$ as follows:

$$\hat{C}_t(u, \ldots, u) = \frac{\hat{C}\left(\bar{F}_1((\bar{F}_t^{(1)})^{-1}(u) + t), \ldots, \bar{F}_n((\bar{F}_t^{(n)})^{-1}(u) + t)\right)}{\hat{C}(\bar{F}_1(t), \ldots, \bar{F}_n(t))}. \tag{3.5}$$

Using (3.5), the mean residual lifetime of the series system composing of n arbitrary dependent components that denote by $\Psi_{1:n}^{\hat{C}}(t)$, is equal to:

$$\Psi_{1:n}^{\hat{C}}(t) = \frac{1}{\hat{C}(\bar{F}_1(t), \ldots, \bar{F}_n(t))} \int_0^1 \hat{C}\left(\bar{F}_1((\bar{F}_t^{(1)})^{-1}(u) + t), \ldots, \bar{F}_n((\bar{F}_t^{(n)})^{-1}(u) + t)\right) du.$$

On the other, using (3.2) and (3.3), we have

$$\begin{aligned}\psi_{1:n}(x|t) &= P((T_{1:n})_t > x) \\ &= \hat{C}_t(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)) \\ &= \hat{C}_t(\varphi(x,t), \ldots, \varphi(x,t)), \end{aligned} \tag{3.6}$$

where $\varphi(x,t) = m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right)$ is the mean function of $\hat{C}_t$.

Remark 1. If the joint survival function $\bar{F}$ is symmetric (see Foschi (2010)), for $i = 1, 2, \ldots, n$, $\bar{F}_i(.) = \bar{F}(.)$ and

$$\bar{F}_t^{(i)}(x) = \bar{F}_t(x) = \frac{\bar{F}(x+t, t, t, \ldots, t)}{\bar{F}(t, t, \ldots, t)}, \qquad x, t > 0,$$

then the survival copula is symmetric and it can be written as

$$\hat{C}_t(u, \ldots, u) = \frac{\hat{C}\left(\bar{F}(\bar{F}_t^{-1}(u) + t), \ldots, \bar{F}(\bar{F}_t^{-1}(u) + t)\right)}{\hat{C}(\bar{F}(t), \ldots, \bar{F}(t))}. \tag{3.7}$$

Now we can prove the following theorems.

Theorem 3.1. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$*, denote the arbitrary dependent lifetimes of the components of series system. Assume that* T_i*'s,* $i = 1, 2, \ldots, n$*, have the survival copula* $\hat{C}$ *with mean function* $m_{\hat{C}}$*. Then* $(T_{1:n})_t \leq_{st} (T_{1:n-1})_t$ *holds if and only if for all* $t > 0$*,*

$$m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) \leq m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n-1)}(x)\right),$$

where $\hat{C}_t$ *is defined in (3.3) with mean function* $m_{\hat{C}_t}$*, and* $\bar{F}_t^{(i)}(x)$ *is the univariate marginal survival function of* $(T_i)_t$ *that is defined in (3.4), for* $i = 1, 2, \ldots, n$*.*

Proof. Using (3.6), we have for all $x, t > 0$,

$$\begin{aligned} \psi_{1:n}(x|t) &= \hat{C}_t\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) \\ &= \hat{C}_t(\varphi_1(x,t), \ldots, \varphi_1(x,t)), \end{aligned}$$

where $\varphi_1(x,t) = m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right)$. Also, the reliability function of $(T_{1:n-1})_t$ can be written as follow

$$\begin{aligned} \psi_{1:n-1}(x|t) &= \hat{C}_t\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n-1)}(x)\right) \\ &= \hat{C}_t(\varphi_2(x,t), \ldots, \varphi_2(x,t)) \end{aligned}$$

where $\varphi_2(x,t) = m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n-1)}(x)\right)$. Since $\hat{C}_t$ is an increasing function, $\psi_{1:n}(x|t) \leq \psi_{1:n-1}(x|t)$ if and only if $\varphi_1(x,t) \leq \varphi_2(x,t)$ for all x and t. □

Theorem 3.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *be the arbitrary vectors of component lifetimes of two series systems. Assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$ *with mean function* $m_{\hat{C}}$*. Then* $(T_{1:n})_t \leq_{st} (Z_{1:n})_t$ *if and only if for all* $x, t > 0$*,*

$$m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) \leq m_{\hat{C}_t}\left(\bar{G}_t^{(1)}(x), \ldots, \bar{G}_t^{(n)}(x)\right),$$

where $\bar{F}_t^{(i)}(x)$ *and* $\bar{G}_t^{(i)}(x)$ *are the univariate marginal survival functions of* $(T_i)_t$ *and* $(Z_i)_t$ *for* $i = 1, 2, \ldots, n$*, respectively.*

Proof. Let $\psi^T_{1:n}(x|t)$ and $\psi^Z_{1:n}(x|t)$ denote the survival functions of $(T_{1:n})_t$ and $(Z_{1:n})_t$, respectively. Using (3.6), we have

$$\psi^T_{1:n}(x|t) = \hat{C}_t\left(\bar{F}^{(1)}_t(x), \ldots, \bar{F}^{(n)}_t(x)\right) = \hat{C}_t(\varphi_1(x,t), \ldots, \varphi_1(x,t)),$$

and

$$\psi^Z_{1:n}(x|t) = \hat{C}_t\left(\bar{G}^{(1)}_t(x), \ldots, \bar{G}^{(n)}_t(x)\right) = \hat{C}_t(\varphi_2(x,t), \ldots, \varphi_2(x,t)),$$

where $\varphi_1(x,t) = m_{\hat{C}_t}\left(\bar{F}^{(1)}_t(x), \ldots, \bar{F}^{(n)}_t(x)\right)$ and $\varphi_2(x,t) = m_{\hat{C}_t}\left(\bar{G}^{(1)}_t(x), \ldots, \bar{G}^{(n)}_t(x)\right)$. Since for fixed $t > 0$, $\hat{C}_t$ is an increasing function, we can conclude that $\psi^T_{1:n}(x|t) \leq \psi^Z_{1:n}(x|t)$ if and only if $\varphi_1(x,t) \leq \varphi_2(x,t)$ for all x. □

Remark 2. In Theorem 3.2, if the component lifetimes of two series systems are independent then the survival copula $\hat{C}$ is the independent survival copula (denoted $\hat{C}_I(u_1, \ldots, u_n) = \prod_{i=1}^n u_i$). It is known that the mean function of $\hat{C}_I$ is equal to

$$m_{\hat{C}_I}(u_1, u_2, \ldots, u_n) = \left(\prod_{i=1}^n u_i\right)^{\frac{1}{n}},$$

(See Navarro and Spizzichino (2010)). Therefore, $(T_{1:n})_t \leq_{st} (Z_{1:n})_t$ if and only if

$$\left(\prod_{i=1}^n \frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^n \frac{\bar{G}_i(x+t)}{\bar{G}_i(t)}\right)^{\frac{1}{n}}.$$

It is notable that, $\{T_i - t|T_i > t\} \leq_{st} \{Z_i - t|Z_i > t\}$, $i = 1, 2, \ldots, n$, is one of the cases to establish the above inequality.

Examples 3.1. Let $\mathbf{T} = (T_1, T_2)$ and $\mathbf{Z} = (Z_1, Z_2)$ be the vectors of component lifetimes of two series systems. Assume that $\mathbf{T}$ and $\mathbf{Z}$ have a Ali-Mikhail-Haq (AMH) bivariate survival copula $\hat{C}$. Also, let T_i has exponential distribution with mean $\frac{1}{\lambda_i}$ and Z_i has Weibull distribution with marginal distribution function $G_i(x) = 1 - e^{-(\beta_i x)^{\alpha_i}}$, $i = 1, 2$. Using (3.2), we have

$$\psi^T_{1:2}(x|t) = \frac{\bar{F}_1(x+t)\bar{F}_2(x+t)\Big(1 - \theta F_1(t)F_2(t)\Big)}{\bar{F}_1(t)\bar{F}_2(t)\Big(1 - \theta F_1(x+t)F_2(x+t)\Big)},$$

$$\psi^Z_{1:2}(x|t) = \frac{\bar{G}_1(x+t)\bar{G}_2(x+t)\Big(1 - \theta G_1(t)G_2(t)\Big)}{\bar{G}_1(t)\bar{G}_2(t)\Big(1 - \theta G_1(x+t)G_2(x+t)\Big)},$$

where F_i and G_i are the marginal distribution functions of T_i and Z_i (i=1,2), respectively.

Since, the mean function of Archimedean copulas is equal to $\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i)\right)$ (See Navarro and Spizzichino (2010)), after some simplifications, the mean function of AMH bivariate survival copula is obtained as follows:

$$m_{\hat{C}}(u,v)=\frac{1-\theta}{\sqrt{\left(\frac{1-\theta(1-u)}{u}\right)\left(\frac{1-\theta(1-v)}{v}\right)}-\theta},\qquad \theta\in[-1,1]. \tag{3.8}$$

By replacing $u=\bar{F}_t^{(1)}(x)$ and $v=\bar{F}_t^{(2)}(x)$ in equation (3.8) and simplifications, we can obtain $m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x),\bar{F}_t^{(2)}(x)\right)$, where

$$\bar{F}_t^{(1)}(x)=\frac{\bar{F}_1(x+t)\Big(1-\theta F_1(t)F_2(t)\Big)}{\bar{F}_1(t)\Big(1-\theta F_1(x+t)F_2(t)\Big)}\ ,\qquad \bar{F}_t^{(2)}(x)=\frac{\bar{F}_2(x+t)\Big(1-\theta F_1(t)F_2(t)\Big)}{\bar{F}_2(t)\Big(1-\theta F_1(t)F_2(x+t)\Big)}.$$

In the same way we can obtain $m_{\hat{C}_t}\left(\bar{G}_t^{(1)}(x),\bar{G}_t^{(2)}(x)\right)$.

The graphs of the survival functions $\psi_{1:2}^{T}(x|t)$ and $\psi_{1:2}^{Z}(x|t)$ are given in part (a) of Fig. 1, for $\lambda_1=0.8$, $\lambda_2=2$, $\alpha_1=2$, $\beta_1=1$, $\alpha_2=1.5$ and $\beta_2=0.5$ at the fixed point $t=3$ for $\theta=0.4$. Also, we plot the graphs of mean functions $m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x),\bar{F}_t^{(2)}(x)\right)$ and $m_{\hat{C}_t}\left(\bar{G}_t^{(1)}(x),\bar{G}_t^{(2)}(x)\right)$ for the listed values of the parameters in Fig. 1 part (b). According to Fig. 1, it can be seen that $m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x),\bar{F}_t^{(2)}(x)\right)\le m_{\hat{C}_t}\left(\bar{G}_t^{(1)}(x),\bar{G}_t^{(2)}(x)\right)$ and $(T_{1:2})_t\le_{st}(Z_{1:2})_t$, for all $x,t>0$.

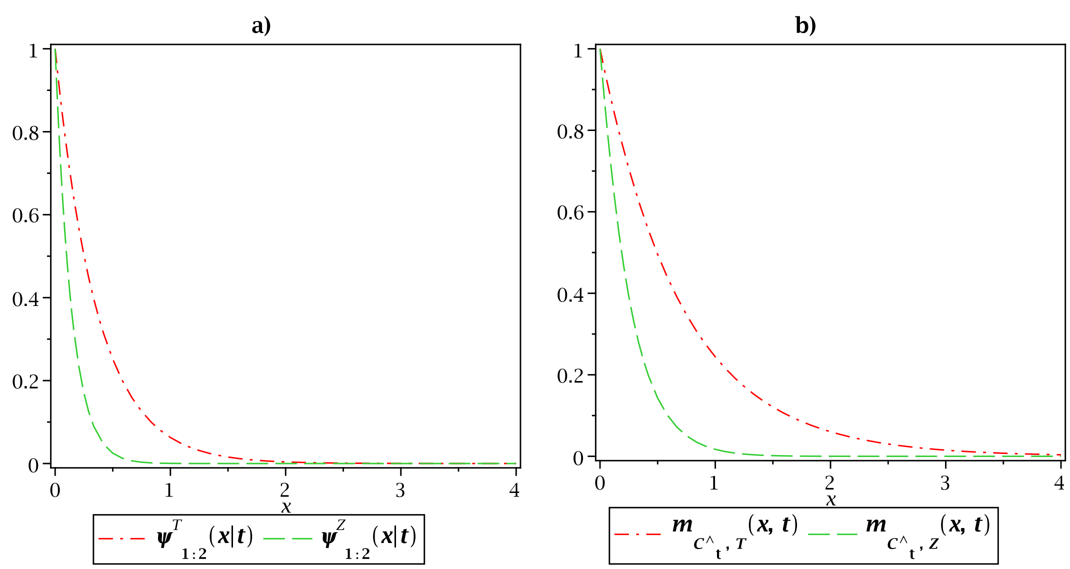

Figure 1: The curves of $\psi_{1:2}^{T}(x|t)$ and $\psi_{1:2}^{Z}(x|t)$ and related mean functions in Example 3.1.

From Theorem 3.2, we immediately deduce the following result.

Corollary 3.1. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *be the arbitrary components lifetimes of a series system and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *be the exchangeable components lifetimes of another series system. Also, assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$ *with mean function* $m_{\hat{C}}$. *Then* $(T_{1:n})_t \leq_{st} (Z_{1:n})_t$ *if and only if for all* $x, t > 0$,

$$m_{\hat{C}_t}(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)) \leq \bar{G}_t(x),$$

where $\bar{F}_t^{(i)}(x)$ *and* $\bar{G}_t(x)$ *are the univariate marginal survival functions of* $(T_i)_t$ *and* $(Z_i)_t$ *for* $i = 1, 2, \ldots, n$, *respectively.*

Proof. The proof can be obtained easily using Theorem 3.2. □

The following theorem is given a result about comparison between series systems, when the dependence structure of the component lifetimes of such systems are modeled by different copulas.

Theorem 3.3. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *denote the arbitrary dependent lifetimes of the components of two series systems. Assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the survival copula* $\hat{C}_T$ *and* $\hat{C}_Z$, *respectively, and* $\hat{C}_{t,T} \leq \hat{C}_{t,Z}$. *Then* $(T_{1:n})_t \leq_{st} (Z_{1:n})_t$, *if for all* $x, t > 0$,

$$m_{\hat{C}_{t,T}}(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)) \leq m_{\hat{C}_{t,Z}}(\bar{G}_t^{(1)}(x), \ldots, \bar{G}_t^{(n)}(x)), \tag{3.9}$$

where $\bar{F}_t^{(i)}(x)$ *and* $\bar{G}_t^{(i)}(x)$ *are the univariate marginal survival functions of* $(T_i)_t$ *and* $(Z_i)_t$ *for* $i = 1, 2, \ldots, n$, *respectively.*

Proof. Using (3.6) and after some simplifications, we have for all $x, t > 0$,

$$\begin{aligned}
\psi_{1:n}^T(x|t) &= \hat{C}_{t,T}(\varphi_1(x,t), \ldots, \varphi_1(x,t)) \\
&\leq \hat{C}_{t,T}(\varphi_2(x,t), \ldots, \varphi_2(x,t)) \\
&\leq \hat{C}_{t,Z}(\varphi_2(x,t), \ldots, \varphi_2(x,t)) = \psi_{1:n}^Z(x|t),
\end{aligned}$$

where $\varphi_1(x,t) = m_{\hat{C}_{t,T}}(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x))$ and $\varphi_2(x,t) = m_{\hat{C}_{t,Z}}(\bar{G}_t^{(1)}(x), \ldots, \bar{G}_t^{(n)}(x))$. In the above expression, the first inequality follows from (3.9) and the second inequality obtain by $\hat{C}_{t,T} \leq \hat{C}_{t,Z}$. Therefore, the proof is complete. □

The following example gives an application of Theorem 3.3.

Examples 3.2. Let $\mathbf{T} = (T_1, T_2)$ and $\mathbf{Z} = (Z_1, Z_2)$ be the vectors of component lifetimes of two series systems. Assume that $\mathbf{T}$ and $\mathbf{Z}$ have a Ali-Mikhail-Haq bivariate survival copula ($\hat{C}_T$) and Gumbel-Hougaard (GH) bivariate survival copula ($\hat{C}_Z$), respectively, i.e.

$$\begin{aligned}
\hat{C}_T(u,v) &= \frac{uv}{1-\theta_1(1-u)(1-v)}, && \theta_1 \in [-1,1], \\
\hat{C}_Z(u,v) &= \exp\left\{-\left[(-\ln u)^{\theta_2} + (-\ln v)^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}, && \theta_2 \in [1,\infty).
\end{aligned}$$

The mean function of AMH survival copula is presented in (3.8). By a similar argument as in Example 3.1, the mean function of GH survival copula is obtained as follows:

$$m_{\hat{C}_Z}(u,v) = \exp\left\{-\left[\frac{(-\ln u)^{\theta_2} + (-\ln v)^{\theta_2}}{2}\right]^{\frac{1}{\theta_2}}\right\}. \tag{3.10}$$

Now, assume that T_i and Z_i are distributed, respectively, as

$$F_i(t) = 1 - e^{-\lambda_i t} \quad , \qquad G_i(t) = 1 - \left(\frac{\beta_i}{t}\right)^{\alpha_i} \quad , \qquad i = 1, 2.$$

$\psi^T_{1:2}(x|t)$ and $m_{\hat{C}_t,T}$ are obtained in Example 3.1. However,

$$\psi^Z_{1:2}(x|t) \quad = \quad \frac{\exp\left\{-\left[(-\ln \bar{G}_1(x+t))^{\theta_2} + (-\ln \bar{G}_2(x+t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}}{\exp\left\{-\left[(-\ln \bar{G}_1(t))^{\theta_2} + (-\ln \bar{G}_2(t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}}.$$

By replacing $u = \bar{G}^{(1)}_t(x)$ and $v = \bar{G}^{(2)}_t(x)$ in equation (3.10) and simplifying the equations, we can obtain $m_{\hat{C}_t,Z}$, where

$$\bar{G}^{(1)}_t(x) \quad = \quad \frac{\exp\left\{-\left[(-\ln \bar{G}_1(x+t))^{\theta_2} + (-\ln \bar{G}_2(t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}}{\exp\left\{-\left[(-\ln \bar{G}_1(t))^{\theta_2} + (-\ln \bar{G}_2(t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}},$$

$$\bar{G}^{(2)}_t(x) \quad = \quad \frac{\exp\left\{-\left[(-\ln \bar{G}_1(t))^{\theta_2} + (-\ln \bar{G}_2(x+t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}}{\exp\left\{-\left[(-\ln \bar{G}_1(t))^{\theta_2} + (-\ln \bar{G}_2(t))^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}}.$$

For $\lambda_1 = 0.5$, $\lambda_2 = 2$, $(\alpha_1 = 2, \beta_1 = 1)$, $(\alpha_2 = 2, \beta_2 = 2)$, $\theta_1 = 0.3$ and $\theta_2 = 4$, the graphs of the survival functions $\psi^T_{1:2}(x|t)$ and $\psi^Z_{1:2}(x|t)$ and corresponding mean functions at the fixed point $t = 3$ are given in Fig. 2, that confirms the result of Theorem 3.3.

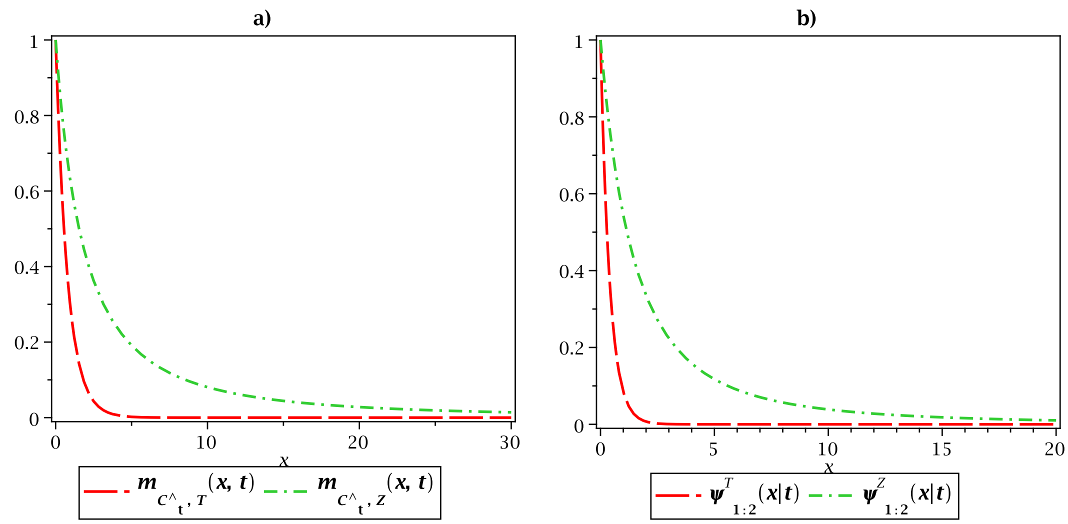

Figure 2: The curves of $\psi^{T}_{1:2}(x|t)$ and $\psi^{Z}_{1:2}(x|t)$ and related mean functions in Example 3.2.

Remark 3. It is worth noting that in Theorem 3.2, whenever$\hat{C}_T = \hat{C}_I$ (i.e.

$$T_1, T_2, \ldots, T_n$$

are independent), if

$$\frac{\bar{F}_1(x+t)\bar{F}_2(x+t)}{\bar{F}_1(t)\bar{F}_1(t)} \le \left[m_{\hat{C}_t}\left(\bar{G}_t^{(1)}(x), \bar{G}_t^{(2)}(x)\right)\right]^2,$$

then $(T_{1:2})_t \le_{st} (Z_{1:2})_t$.

In the following, we present a numerical example as an application of Remark 3, in which the residual lifetime of series system consisting of dependent components is compared with the residual lifetime of another series system consisting of independent components with the same marginal lifetime distributions.

Examples 3.3. Consider two components with lifetimes T_1 and T_2 that have the exponential distribution with mean 2 and Pareto distribution ($\alpha = 2, \beta = 2$) with $F_2(t) = 1 - \left(\frac{\beta}{t}\right)^{\alpha}$, respectively. First, assuming that these two components are dependent, we connect them as a series system. In the next case, assuming independence between T_1 and T_2, we connect the second series system. Suppose that in the first system $\mathbf{T} = (T_1, T_2)$ has AMH survival copula. The copula of $\mathbf{T}$ in the second system is the same as the independent copula, that is, C_I. For $\theta = 0.6$ and point $t = 2$, we plotted graphs of the corresponding mean functions of each copula as well as graphs of reliability functions of residual lifetime for two series systems in Fig. 3. As you can see from Fig. 3 and based on the conditions of Theorem 3.2 for the AMH copula, the reliability function of residual lifetime for first series system (dependent case, $\psi^D_{1:2}(x|t)$) is greater than the reliability function of residual lifetime for second series system (independent case, $\psi^I_{1:2}(x|t)$).

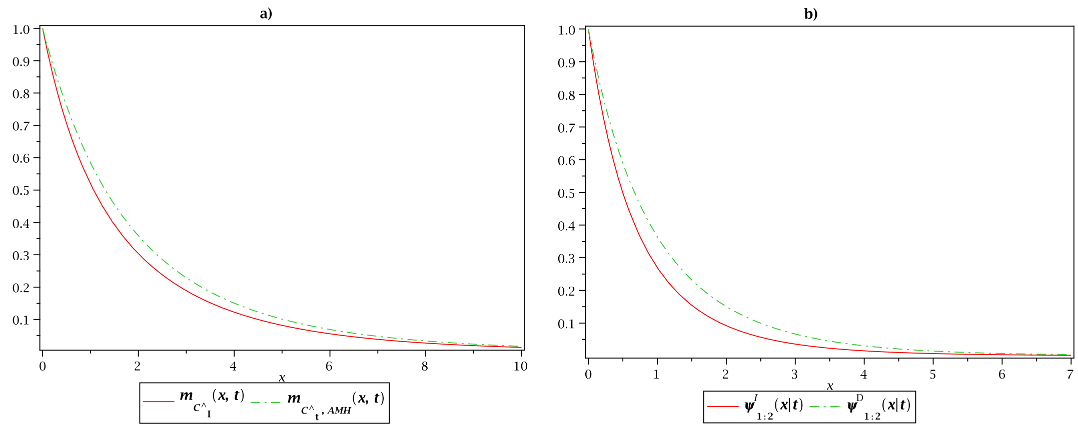

Figure 3: The curves of reliability functions $\psi^I_{1:2}(x|t)$ and $\psi^D_{1:2}(x|t)$ and related mean functions in Example 3.3.

When the components lifetimes of series systems are distributed identically, the following theorem indicates that the reliability function of residual lifetimes of such systems improves whenever the survival copula is weakly Schur-concave.

Theorem 3.4. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *denote the arbitrary dependent lifetimes of the components of two series systems. Assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$ *and* $\hat{C}_t$ *is weakly Schur-concave (weakly Schur-convex). If* $\bar{F}^{(i)}_t(x)$ *is the univariate marginal survival function of* $(T_i)_t$ *and* $\bar{G}_t(x) = \frac{1}{n}\sum_{i=1}^n \bar{F}^{(i)}_t(x)$ *is the univariate survival function of* $(Z_i)_t$ *for* $i = 1, 2, \ldots, n$. *Then for* $t > 0$,

$$(T_{1:n})_t \leq_{st} (\geq_{st})(Z_{1:n})_t.$$

Proof. From (3.6), we have

$$\begin{aligned}
\psi^T_{1:n}(x|t) &= \hat{C}_t(\bar{F}^{(1)}_t(x), \ldots, \bar{F}^{(n)}_t(x)) \\
&= \hat{C}_t(\varphi_1(x,t), \ldots, \varphi_1(x,t)) \\
&\leq (\geq)\hat{C}_t(\bar{G}_t(x), \ldots, \bar{G}_t(x)) \\
&= \psi^Z_{1:n}(x|t),
\end{aligned}$$

where $\varphi_1(x,t) = m_{\hat{C}_t}(\bar{F}^{(1)}_t(x), \ldots, \bar{F}^{(n)}_t(x))$ and $m_{\hat{C}_t}$ is the mean function associated with $\hat{C}_t$ and the last inequality follows from Lemma 2.1 and hence the proof is complete. □

Examples 3.4. Let the arbitrary lifetimes of two components T_1 and T_2 have a FGM bivariate survival copula, $\hat{C}$. Assume that T_i has a continuous distribution $\bar{F}_i$, $i = 1, 2$. Also, let $T_{1:2}$ ($Z_{1:2}$) be a series system lifetime consisting of two arbitrary (exchangeable) components. In this case $(T_1)_t$ and $(T_2)_t$ have the marginal survival function $\bar{F}_t^{(1)}(x)$ and $\bar{F}_t^{(2)}(x)$, and consider that $(Z_1)_t$ and $(Z_2)_t$ are identical with common marginal survival function $\bar{G}_t(x) = \frac{1}{2}\left[\bar{F}_t^{(1)}(x) + \bar{F}_t^{(2)}(x)\right]$. Therefore, we have

$$\psi_{1:2}^{T}(x|t) = \hat{C}_t(\bar{F}_t^{(1)}(x), \bar{F}_t^{(2)}(x)) = \frac{\bar{F}_1(x+t)\bar{F}_2(x+t)(1+\theta F_1(x+t)F_2(x+t))}{\bar{F}_1(t)\bar{F}_2(t)(1+\theta F_1(t)F_2(t))},$$

$$\psi_{1:2}^{Z}(x|t) = \hat{C}_t(\bar{G}_t(x), \bar{G}_t(x)) = \frac{\left(\bar{F}_1(x+t) + \bar{F}_2(x+t)\right)^2\left[1 + \theta\left(1 - \frac{\bar{F}_1(x+t)+\bar{F}_2(x+t)}{2}\right)^2\right]}{\left(\bar{F}_1(t) + \bar{F}_2(t)\right)^2\left[1 + \theta\left(1 - \frac{\bar{F}_1(t)+\bar{F}_2(t)}{2}\right)^2\right]}.$$

Durante and Sempi (2003) showed that FGM copula is Schur-concave, in fact, it is weakly Schur-concave. Hence, from Theorem 3.4, it is trivial that for all $x > 0$,

$$\psi_{1:2}^{T}(x|t) \leq \psi_{1:2}^{Z}(x|t).$$

The graphs of the survival functions of $(T_{1:2})_t$ and $(Z_{1:2})_t$, at a fixed point $t = 3$, are plotted in Fig. 4, when T_1 and T_2 have the FGM bivariate exponential distribution with $\lambda_1 = 2$, $\lambda_2 = 0.5$ and $\theta = 0.2$.

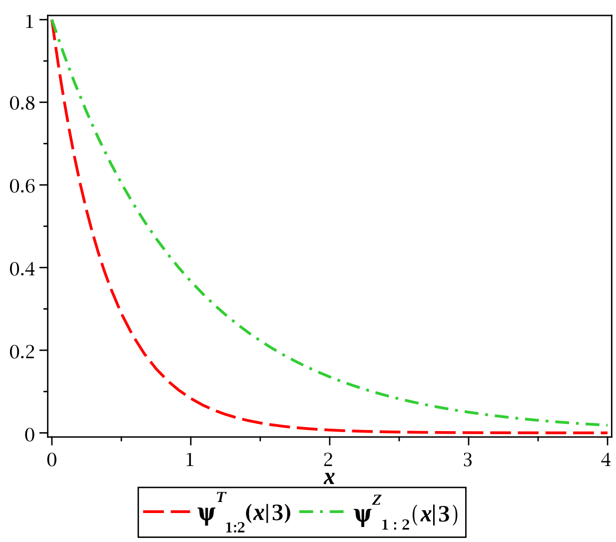

Figure 4: The curves of survival functions of $(T_{1:2})_t$ and $(Z_{1:2})_t$, in Example 3.4.

The reliability analysis and stochastic ordering properties of systems consisting of used components of age $t > 0$ studied by Zhang and Li (2003), Gupta (2013) and

Gupta et al. (2015). Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ have the survival copula $\hat{C}$. Also, let $(T_t)_i = (T_i - t | T_i > t)$, $i = 1, 2, \ldots, n$, are the residual lifetimes of used components, where $(T_t)_i$ has the marginal survival function $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$, $i = 1, 2, \ldots, n$. The survival function of the lifetime of the series system including used components denoted by $(\psi_{1:n}(x))_t$, is equal to

$$\begin{aligned} (\psi_{1:n}(x))_t &= P((T_t)_{1:n} > x) \\ &= \hat{C}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \\ &= \hat{C}(\kappa(x,t), \ldots, \kappa(x,t)), \end{aligned} \tag{3.11}$$

where $\kappa(x,t) = m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right)$.

The following theorems can be proved similarly to Theorems 3.1-3.4, respectively, and hence their proofs are omitted.

Theorem 3.5. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ denoted the arbitrary dependent lifetimes of the components of a series system. Let T_i's, $i = 1, 2, \ldots, n$, have the survival copula $\hat{C}$ with mean function $m_{\hat{C}}$. Then $(T_t)_{1:n} \leq_{st} (T_t)_{1:n-1}$ if and only if for all $x, t > 0$,*

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_{n-1}(x+t)}{\bar{F}_{n-1}(t)}\right),$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ is the marginal survival function of $(T_t)_i$, $i = 1, 2, \ldots, n$.

Theorem 3.6. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ be the arbitrary vectors of components lifetimes of two series systems. Also assume that $\mathbf{T}$ and $\mathbf{Z}$ have the same survival copula $\hat{C}$ with mean function $m_{\hat{C}}$. Then $(T_t)_{1:n} \leq_{st} (Z_t)_{1:n}$ if and only if for all $x, t > 0$,*

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq m_{\hat{C}}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \ldots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right),$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ and $\frac{\bar{G}_i(x+t)}{\bar{G}_i(t)}$ are the marginal survival functions of $(T_t)_i$ and $(Z_t)_i$, $i = 1, 2, \ldots, n$, respectively.

Theorem 3.7. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ be the arbitrary components lifetimes of a series system and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ be the exchangeable components lifetimes of another series system. Also, assume that $\mathbf{T}$ and $\mathbf{Z}$ have the same survival copula $\hat{C}$ with mean function $m_{\hat{C}}$, then $(T_t)_{1:n} \leq_{st} (Z_t)_{1:n}$ if and only if for all $x, t > 0$,*

$$m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \leq \frac{\bar{G}(x+t)}{\bar{G}(t)},$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ and $\frac{\bar{G}(x+t)}{\bar{G}(t)}$ are the marginal survival functions of $(T_t)_i$ and $(Z_t)_i$, $i = 1, 2, \ldots, n$, respectively.

Theorem 3.8. *Let* $\mathbf{T} = (T_1, T_2, \dots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ *denote the arbitrary dependent lifetimes of the components of two series systems. Also, assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the survival copulas* $\hat{C}_T$ *and* $\hat{C}_Z$*, respectively, such that* $\hat{C}_T \le \hat{C}_Z$*. Then* $(T_t)_{1:n} \le_{st} (Z_t)_{1:n}$*, if for all* $x, t > 0$,

$$m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \dots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \le m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \dots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right), \tag{3.12}$$

where $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *and* $\frac{\bar{G}_i(x+t)}{\bar{G}_i(t)}$ *are the marginal survival functions of* $(T_t)_i$ *and* $(Z_t)_i$*,* $i = 1, 2, \dots, n$*, respectively.*

Proof. From (3.11) and after some simplifications, we have for all $x, t > 0$,

$$\begin{aligned}(\psi^T_{1:n}(x))_t &= \hat{C}_T(\kappa_1(x,t), \dots, \kappa_1(x,t))\\ &\le \hat{C}_T(\kappa_2(x,t), \dots, \kappa_2(x,t))\\ &\le \hat{C}_Z(\kappa_2(x,t), \dots, \kappa_2(x,t)) = (\psi^Z_{1:n}(x))_t,\end{aligned}$$

where $\kappa_1(x,t) = m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \dots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right)$ and $\kappa_2(x,t) = m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \dots, \frac{\bar{G}_n(x+t)}{\bar{G}_n(t)}\right)$. In the above expression, the first inequality follows from (3.12) and the second inequality obtain by $\hat{C}_T \le \hat{C}_Z$. Therefore, the proof is complete. □

Examples 3.5. Let us consider two random vectors **T** and **Z** in Example 3.2 that have AMH and GH survival copulas, respectively. Then we have

$$\begin{aligned}(\psi^T_{1:2}(x))_t &= \frac{\bar{F}_1(x+t)\bar{F}_2(x+t)}{\bar{F}_1(t)\bar{F}_2(t) - \theta_1(\bar{F}_2(t) - \bar{F}_2(x+t))(\bar{F}_2(t) - \bar{F}_2(x+t))},\\ (\psi^Z_{1:2}(x))_t &= \exp\left\{-\left[\left(-\ln\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}\right)^{\theta_2} + \left(-\ln\frac{\bar{G}_2(x+t)}{\bar{G}_2(t)}\right)^{\theta_2}\right]^{\frac{1}{\theta_2}}\right\}.\end{aligned}$$

From (3.8) and (3.10), we can obtain easily $m_{\hat{C}_T}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \frac{\bar{F}_2(x+t)}{\bar{F}_2(t)}\right)$ and $m_{\hat{C}_Z}\left(\frac{\bar{G}_1(x+t)}{\bar{G}_1(t)}, \frac{\bar{G}_2(x+t)}{\bar{G}_2(t)}\right)$.

For $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha_1 = 3$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 2$, $\theta_1 = 0.4$ and $\theta_2 = 3$, the graphs of the survival functions $(\psi^T_{1:2}(x))_t$ and $(\psi^Z_{1:2}(x))_t$ and their corresponding mean functions are given in Fig. 5 at the fixed point $t = 2$. Hence, this shows that the result of Theorem 3.8 holds.

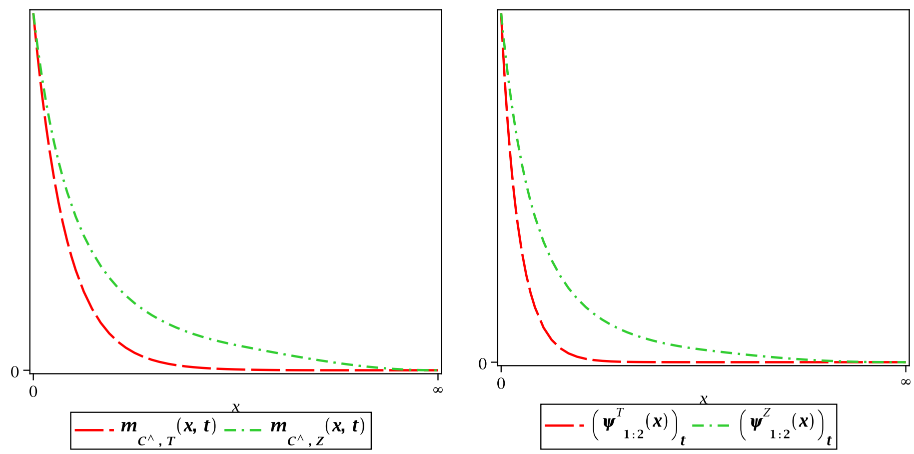

Figure 5: The curves of the survival functions of $(T_t)_{1:2}$ and $(Z_t)_{1:2}$ and related mean functions in Example 3.5.

Theorem 3.9. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *and* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ *denote the arbitrary dependent lifetimes of the components of two series systems. Also, assume that* $\mathbf{T}$ *and* $\mathbf{Z}$ *have the same survival copula* $\hat{C}$, *such that* $\hat{C}$ *is weakly Schur-concave (weakly Schur-convex). If* $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *is the marginal survival function of* $(T_t)_i$ *and* $\frac{\bar{G}(x+t)}{\bar{G}(t)} = \frac{1}{n}\sum_{i=1}^{n} \frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *is the marginal survival function of* $(Z_t)_i$ *for* $i = 1, 2, \ldots, n$, *then for* $t > 0$,

$$(T_t)_{1:n} \leq_{st} (\geq_{st})(Z_t)_{1:n}.$$

In the following theorem, under some conditions, the residual lifetime of a used series system of age t with the lifetime of similar series system made up of used components of age t, are stochastically compared.

Theorem 3.10. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ *denote the arbitrary dependent lifetimes of the components of series system. Also, assume that* $\mathbf{T}$ *has the survival copula function* $\hat{C}$, *such that for all* $t > 0$, $\hat{C}_t \leq (\geq)\hat{C}$. *Then* $(T_{1:n})_t \leq_{st} (\geq_{st})(T_t)_{1:n}$ *holds if for all* $x, t > 0$,

$$m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) \leq (\geq) m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right),$$

where $\bar{F}_t^{(i)}(x)$ *and* $\frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$ *are the univariate survival functions of* $(T_i)_t$ *and* $(T_t)_i$, $i = 1, 2, \ldots, n$, *respectively.*

Proof. Let $\psi_{1:n}(x|t)$ and $(\psi_{1:n}(x))_t$ denote the survival functions of $(T_{1:n})_t$ and $(T_t)_{1:n}$, respectively. Then using (3.6), we have, for all $x, t > 0$,

$$\begin{aligned} \psi_{1:n}(x|t) &= \hat{C}_t\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) \\ &= \hat{C}_t(\varphi(x,t), \ldots, \varphi(x,t)), \end{aligned}$$

where $\varphi(x,t) = m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right)$. However, the reliability function of $(T_t)_{1:n}$ can be written as follow

$$
\begin{aligned}
(\psi_{1:n}(x))_t &= \hat{C}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right) \\
&= \hat{C}(\kappa(x,t), \ldots, \kappa(x,t)),
\end{aligned}
$$

where $\kappa(x,t) = m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right)$. Since $\hat{C}$ is an increasing function, $\psi_{1:n}(x|t) \leq (\geq)$ $(\psi_{1:n}(x))_t$ if and only if $\varphi(x,t) \leq (\geq) \kappa(x,t)$ for all x, t. □

To show the application of Theorem 3.10, we provide the following numerical example.

Examples 3.6. Let $\mathbf{T} = (T_1, T_2)$ be a random vector of lifetimes of components with the GH bivariate survival copula. Assume that the marginal distribution of T_i is Pareto distribution with parameters (α_i, β_i), $i = 1, 2$. From (3.10), we can compute $m_{\hat{C}_t}\left(\bar{F}_t^{(1)}(x), \bar{F}_t^{(2)}(x)\right)$ and $m_{\hat{C}}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \frac{\bar{F}_2(x+t)}{\bar{F}_2(t)}\right)$. Let us consider the used series system of age t and the other series system including the used components of age t. From (3.2) and (3.11) we obtain $\psi_{1:n}(x|t)$ and $(\psi_{1:n}(x))_t$. For $\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 2, \beta_2 = 2$ and $\theta = 4$, the graphs of the survival functions $\psi^{T}_{1:2}(x|t)$ and $(\psi^{Z}_{1:2}(x))_t$ and their corresponding mean functions, i.e. $m_{\hat{C}_t}$ and $m_{\hat{C}}$, at the fixed point $t = 2$ are given in Fig. 6.

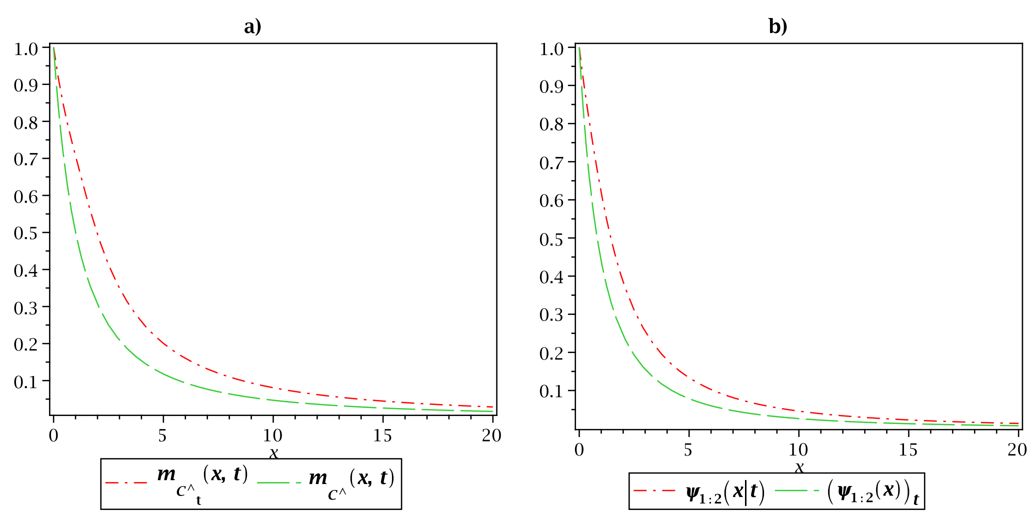

Figure 6: The curves of mean functions $m_{\hat{C}_t}$ and $m_{\hat{C}}$ and the curves of the survival functions of $(T_{1:2})_t$ and $(T_t)_{1:2}$ in Example 3.6 .

Remark 4. When $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is the independent lifetimes variables of n components of a series system, then the survival function in (3.1) equals to

$$\bar{F}(t_1, t_2, \ldots, t_n) = \hat{C}_I(\bar{F}_1(t_1), \bar{F}_2(t_2), \ldots, \bar{F}_n(t_n)) = \prod_{i=1}^{n} \bar{F}_i(t_i),$$

As mentioned in Remark 2, the mean function of $\hat{C}_I$ is $m_{\hat{C}_I}(u_1, u_2, \ldots, u_n) = \left(\prod_{i=1}^{n} u_i\right)^{\frac{1}{n}}$. Hence,

$$\begin{aligned} m_{\hat{C}_{I_t}}\left(\bar{F}_t^{(1)}(x), \ldots, \bar{F}_t^{(n)}(x)\right) &= \left(\prod_{i=1}^{n} \bar{F}_t^{(i)}(x)\right)^{\frac{1}{n}} \\ &= \left(\prod_{i=1}^{n} \frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}\right)^{\frac{1}{n}} = m_{\hat{C}_I}\left(\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)}, \ldots, \frac{\bar{F}_n(x+t)}{\bar{F}_n(t)}\right). \end{aligned}$$

Now suppose that the component lifetimes of series system are exchangeable and the survival copula is symmetrical as well. Under these assumptions, in the following theorems we investigate some dependence properties of the residual lifetime of series system according to the properties of the lifetimes of dependent exchangeable components.

Proposition 3.1. *$\hat{C}_t$ is RTI for all $t > 0$, if and only if $\hat{C}$ is RR_2.*

Proof. From Definition 2.2 part b), $\hat{C}_t$ is RTI, if and only if $\frac{\bar{F}(x,y)}{\bar{G}(x)}$ is non-decreasing in x, for all $y \geq 0$. It means that for all $t \geq 0$, and $x' \geq x \geq 0$,

$$\frac{\bar{F}(x+t, x+t)}{\bar{F}(x+t, t)} \leq \frac{\bar{F}(x'+t, x'+t)}{\bar{F}(x'+t, t)}.$$

Therefore,

$$\bar{F}(x+t, x+t)\bar{F}(x'+t, t) \leq \bar{F}(x'+t, x'+t)\bar{F}(x+t, t).$$

After some simplifications, we have

$$\bar{F}(x+t, x+t)\bar{F}(x'+t, x'+t) \leq \bar{F}(x'+t, t)\bar{F}(x+t, t). \tag{3.13}$$

The inequality in (3.13) implies that $\bar{F}$ is RR_2, then $\hat{C}$ is RR_2. Hence, the proof is complete. □

Proposition 3.2. *If $\hat{C}_t$ is RCSI, then*

$$\Psi_{1:2}^{\hat{C}}(t) \geq \Psi_{1:2}^{\hat{C}}(t'), \quad \textit{for all } \; t > t'.$$

Proof. From Definition 2.2 part d), if $\hat{C}_t$ is RCSI, then $\frac{\bar{F}(x+t,x+t)}{\bar{F}(t,t)}$ is non-decreasing in t, i.e. for $t \geq t'$,

$$\frac{\bar{F}(x+t, x+t)}{\bar{F}(t, t)} \geq \frac{\bar{F}(x+t', x+t')}{\bar{F}(t', t')}, \tag{3.14}$$

Hence, (3.14) implies that $\hat{C}_t \geq \hat{C}_{t'}$. Taking the integral from both sides of inequality (3.14), we obtain $\Psi_{1:2}^{\hat{C}}(t) \geq \Psi_{1:2}^{\hat{C}}(t')$, for $t \geq t'$. Therefore, the proof is complete. □

Theorem 3.11. *$\hat{C}_t$ is PUOD for all $t > 0$, if and only if*

$$\left[\hat{C}(u,\ldots,u)\right]^{n-1}\hat{C}(u',\ldots,u') \geq \left[\hat{C}(u,\ldots,u,u')\right]^{n}, \quad 0 < u' < u < 1.$$

Proof. It is known that the survival copula connecting to a multivariate survival function $\bar{F}$ is *PUOD* if and only if the joint survival function $\bar{F}$ is *PUOD*, i.e.,

$$\bar{F}(x_1,\ldots,x_n) \geq \bar{F}(x_1)\times\ldots\times\bar{F}(x_n).$$

Then, $\hat{C}_t(u,\ldots,u)$ being *PUOD* is equivalent to $\bar{F}_t$ being *PUOD*, for all $t > 0$, i.e.,

$$\frac{\bar{F}(x+t,\ldots,x+t)}{\bar{F}(t,\ldots,t)} \geq \frac{\bar{F}(x+t,\ldots,t,t)}{\bar{F}(t,\ldots,t)}\times\ldots\times\frac{\bar{F}(t,\ldots,t,x+t)}{\bar{F}(t,\ldots,t)}. \tag{3.15}$$

By multiplying $\bar{F}(t,\ldots,t) > 0$, on the sides of inequality (3.15) and using Sklar's Theorem, we have

$$\begin{aligned}&\left[\hat{C}(\bar{F}(t),\ldots,\bar{F}(t))\right]^{n-1}\hat{C}\Big(\bar{F}(x+t),\ldots,\bar{F}(x+t)\Big)\\ &\qquad\geq \hat{C}\Big(\bar{F}(x+t),\bar{F}(t),\ldots,\bar{F}(t)\Big)\times\ldots\times\hat{C}\Big(\bar{F}(t),\ldots,\bar{F}(t),\bar{F}(x+t)\Big).\end{aligned}$$

By replacing $u = \bar{F}(t)$, $u' = \bar{F}(x+t)$, and using symmetric property of the copula, the proof is complete. □

Remark 5. An immediate result of Theorem 3.11 is that, $\hat{C}_t$ is *PQD*(*NQD*) for all $t \geq 0$, if and only if

$$\hat{C}(u,u)\hat{C}(\acute{u},\acute{u}) \geq (\leq)\left[\hat{C}(u,\acute{u})\right]^{2}, \quad 0 < u < \acute{u} < 1.$$

4 Discussions and Conclusions

In this article, we studied two types of series systems (the used series system and the series system composed of the used components) consisting of arbitrary dependent components based on the copulas family and a concept called mean function. First, we extracted a closed expression for the reliability functions of residual lifetime of the series system with arbitrary dependent components from the view point of Foschi (2010) in term of the upper threshold copulas. Then, the residual lifetime of two series systems consisting of different components are compared in usual stochastic order based on the dependence structure of components mean functions. As a special case of these results, the residual lifetime of series system composed of dependent components is compared with the residual lifetime of another series system composed of independent components with the same marginal lifetime distributions. Another special case is when both series systems consist of independent components. In this case, the residual lifetimes of the systems is investigated and we reached to the well-known results in the existing literatures. In order to apply the derived results, various numerical examples have been presented along with graphs.

In the following, the series systems consisting of the used components are examined and the stochastic comparative results based on the mean function and the dependence structure of components have also been obtained. Afterwards, the residual lifetime of a used series system of age t, consisting of arbitrary dependent components is compared with the lifetime of the similar series system made up of used components of age t. At the end, we presented some dependence properties for the residual lifetime of a series system based on the lifetime properties of its components.

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