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A Bivariate Process based Maintenance Model for Two-Component Parallel Systems

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Abstract. This paper proposes a bivariate process-based model for maintenance and inspection planning of a parallel system, consisting of two components whose states evolve in one of three possible states: normal (0), satisfactory (1) and failure (2). The changes of states driven by a non-homogeneous Markov process are detected only by inspections and repair actions are determined by the observed state of the bivariate process. Outperforming maintenance strategies and other classical maintenance policies, the paper aims at minimizing the long-run average maintenance cost per unit time by deriving optimal inspection intervals and a preventive replacement threshold. A numerical example is given to illustrate the proposed model and examine the response of the optimal solutions to system parameters. The model explored here provides the framework for further developments.

Keywords. Bivariate Process, Markov Process, Hidden Failures, Inspection, Maintenance.

MSC: 90B25, 60K10.

1 Introduction

This paper presents an approach to the joint determination of optimal inspection and preventive replacement policy for two-component parallel systems. The model is

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developed under the assumptions that components state is hidden and detected only by inspections and their failure state is defined by regulation (soft failure) (Khatab et al., 2018; Newby and Barker, 2006). The latter means, a component experiences failure when its deterioration process exceeds a failure threshold. In that sense our model differs from those whose failures result from exposure to both the natural degradation and random shocks triggered by adverse environment.

Cold-standby systems (Wei et al., 2018) with the above operation and failure characteristics are typically encountered in practice. A particular case of a cold-standby system is the standby safety equipment adopted in nuclear plants. For such systems condition-based maintenance (CBM) is essential to retain them to an acceptable condition and increase availability against rising costs due to undetected failures. CBM is a two-step decision making process: step 1 involves monitoring the components at inspections to reveal their true state and step 2 takes in corrective maintenance (CM) and preventive maintenance (PM) actions in response to the observed components state. Since each repair and maintenance action (RMA) incurs cost, which is more for the higher level of RMAs, an intriguing question raised is how to make inspections and when to stop processing the system and carrying out a (preventive) replacement in order to minimize resulting repair and maintenance costs. This paper aims to answer this question.

For such systems, particularly safety systems, (non)periodic checking and inspections are essential to detect possible failures hidden within inter-inspection times. The earlier works on inspection models for non-self announcing failure systems are given by Barlow et al. (1963); Munford and Shahani (1972) and Keller (1974). As an extension of classical models, Jiang and Jardine (2005) propose two optimization models to determine the optimum sequence of inspection times. In contrast to classical optimum inspection policies, their model is more accurate and computationally tractable. Chelbi et al. (2008) studied an optimal inspection model for systems with self-announcing and non-self-announcing failures. Their approach was based on determining the age T for inspection which maximizes the stationary availability of the system. Rezaei (2017) presented a maintenance model for inspection planning. Liu et al. (2017) proposed an approach for obtaining optimal inspection scheme for multi-component systems characterized by hidden failures and dependent components. Recently, Seyedhosseini et al. (2018) proposed an imperfect inspection model to find the optimal periodic inspection interval.

In the literature, to further meet industry needs, the above modeling approaches were developed under a maintenance decision framework. In contrast to inspection models, they allow the use of condition monitoring information for PM decision making. During past decades numerous CBM models for systems with non-self announcing failures (e.g. see Ahmadi, 2017, 2019, 2020; He et al., 2015; Kijima, 1989) and self announcing failures (e.g. see Azizi and Salari, 2023; Cai et al., 2022; Chen et al., 2021; Forouzandeh Shahraki et al., 2020; Fink et al., 2022; Hu et al., 2021; Khatab et al., 2018; Liu et al., 2021; Jong Kim and Makis, 2009; Salari and Makis, 2017; Wang et al., 2022;

Zhang et al., 2023) are explored. For instance, He et al. (2015) propose a periodic inspection and preventive replacement policy for a system subject to hidden failures. Preventive replacement (PR) policy is implemented whenever the number of inspections scheduled between PRs reaches the quantity n. Ahmadi (2017) given partial information proposed a new approach for scheduling inspection and threshold-type replacement policy for parallel systems subject to hidden failures. PM is implemented as soon as the total number of failed components reach the threshold d. Recently, Ahmadi (2019) proposed a maintenance model integrating both the inspection and the imperfect repair policy for repairable parallel systems. His approach differs through the use of an age reduction model (Kijima, 1989). More recently, Ahmadi (2020) suggested an approach to the joint determination of inspection and replacement policy for a load-sharing k-out-of-n system whose components state is hidden and detected only by inspections. Preventive replacements are decided with respect to a basic process describing the true state of components and a performance metric (conditional mean residual lifetime).

The PM problem for systems with self announcing failures is widely addressed in literature. For example, Jong Kim and Makis (2009) proposed a CBM model for a multi-state deteriorating system with major and minor failures. The maintenance decision rule is based on two sets corresponding to the failure of minor and major components. Salari and Makis (2017) presented two condition-based maintenance policies for a production system consisting of N components whose deterioration is modelled by a three-state homogeneous Markov process (two working states and a failure state). The maintenance decision rule is based on an index integrating both the production rate and components state. Under periodic inspection and imperfect maintenance, Khatab et al. (2018) addressed the selective maintenance optimization problem for a system consisting of repairable binary components degrading according to a gamma process. Hu et al. (2021) considered a threshold-type policy with respect to the degradation of a multi-state system governed by an environmental condition-varying homogeneous Markov process. Liu et al. (2021) presented a condition-based maintenance (CBM) model for a system consisting of two heterogeneous components whose degradation conforms to a bivariate gamma process. Preventive and corrective maintenance actions are carried out based on the degradation state revealed at inspection times. Recently, Cai et al. (2022), with the same approach as Ahmadi (2020) modelled CBM based on remaining useful life for multi-component systems. Using a Markov-decision process, Fink et al. (2022) formulated both the time-based maintenance and condition-based maintenance for a k-out-of-n system whose components deteriorate according to a multivariate gamma process with Lévy copula dependence. Wang et al. (2022) developed a CBM model for a multi-component deteriorating system with stochastic and economic dependencies. Maintenance decision-making integrated both a diagnostic condition index at system level and a prognostic condition index. More recently, Azizi and Salari (2023) addressed the condition based maintenance problem for a multi-component system in which the deterioration process of each component is modelled by a three-state homogeneous Markov process (healthy state, unhealthy state and failure state). The maintenance decision rule was based on a bivariate birth/birth–death process tracking the number of units in the failure state, and an unhealthy state. Zhang et al. (2023) presented a CBM model under the Markov-decision process framework. In that sense, their approach resemble the ones suggested by Fink et al. (2022), but the Zhang et al. (2023)'s model differ in that they relaxed the homogeneity assumption of components (Liu et al., 2021).

Although the current model shares some features with previous works cited above, but, in a unifying model, it includes some characteristics which have not been studied or presented in isolation. More specifically, the common and distinctive features are as follows:

- The degradation process of each component conforms to a non-homogeneous Markov process $X_i(t)$ $(i = 1, 2)$ with three possible states: normal (0), satisfactory (1) and failure (2). This makes superiority to those models restricting their analysis to two states (Khatab et al., 2018), or transition rate remains constant as long as the component sojourns within each state (Azizi and Salari, 2023; Chen et al., 2021; Hu et al., 2021).

- Maintenance policies developed in our model resemble the ones proposed by He et al. (2015) and Ahmadi (2017, 2020); we consider both the inspection planning and the threshold-type replacement policy for the maintenance problem. The latter means a preventive replacement is implemented whenever a performance metric reaches a threshold (e.g. see Ahmadi, 2019; Cai et al., 2022; Hu et al., 2021).

- Unlike most maintenance models, the preventive replacement cost is modulated by a performance metric $\mathbf{X}(t)$. This approach allows the cost model responds to the components state in a sensible way.

Furthermore, our model mainly differs in decision rule and maintenance decision mechanism. Unlike most CBM models (e.g. see Cai et al., 2022; Chen et al., 2021; Forouzandeh Shahraki et al., 2020; Fink et al., 2022; Hu et al., 2021; Khatab et al., 2018; Liu et al., 2021; Jong Kim and Makis, 2009; Salari and Makis, 2017; Wang et al., 2022; Zhang et al., 2023), herein the action taken after an inspection are completely determined by partitioning a two-dimensional state space associated with a bivariate (damage) process $\mathbf{X}(t) = (X_1(t), X_2(t))$, in which the i^{th} element describes the component state at age t. In that sense, our model partly resembles the one suggested by Azizi and Salari (2023), but the two models differ in the assumed transition rate pattern. The transition rate in the model of Azizi and Salari (2023) remains constant as long as a component sojourns within each state, while we apply a continuous-time Markov models with time-dependent transition rates. In addition, Azizi and Salari (2023) adopted two populations (the number of units in the failure state, and the number of units in an unhealthy state) to characterize the system state, while our approach allows the consideration of an additional population (the number of units in the normal state).

Before proceeding to the model development, the main features of the model are as follows. The system maintained is a parallel system consisting of two components whose states are detected only by inspections. It is assumed that the state of the i^{th} component is described by a non-homogeneous Markov process $X_i(t)$ $(i = 1, 2)$ with three possible states: normal (0), satisfactory (1) and failure (2). The system is inspected according to a periodic policy $\Pi = \{n\tau : n = 1, 2, \cdots\}$ to reveal the true state of components $\mathbf{X}(t) = (X_1(t), X_2(t))$ defined on a state space $\Omega = \{(r, s) : r, s = 0, 1, 2\}$. To maintain a minimum level of performance, repair actions defined on an action space $\Omega_a = \{a_0, a_1, a^*\}$ are taken on the basis of (i) the bivariate process $\mathbf{X}(t)$ observed at the inspection instants and (ii) the exclusive subsets $A_0(\kappa)$, $A_1(\kappa)$ and A_2 partitioning the state space Ω. More specifically, if on inspection, the decision maker finds the bivariate process $\mathbf{X}(t)$ in the subset $A_0(\kappa) = \{(r, s) : 0 \leq r + s \leq \kappa - 1\}$, no repair action is taken (the action is denoted by the doubleton $\left\langle a, \mathbf{X} \right\rangle = a_0$); otherwise either the system is preventively replaced by new one ($\left\langle a, \mathbf{X} \right\rangle = a_1$) if $\mathbf{X}(t) \in A_1(\kappa) = \{(r, s) : \kappa \leq r + s < n\}$, or it undergoes a corrective replacement ($\left\langle a, \mathbf{X} \right\rangle = a^*$ if both components are found in the failure state, i.e. $\mathbf{X}(t) \in A_2 = \{(2, 2)\}$. Since two maintenance parameters (τ, κ) induce changes in both the level and the costs of RMAs, our aim is to determine optimal inspection interval τ^* and optimal replacement threshold κ^* which truly balance these two factors. The resulting optimization problem is solved by a renewal-reward argument to formulate a long-run average cost used as a measure of policy.

This paper is organized as follows. The features of the model including assumptions, modeling degradation and maintenance are given by section 2. Section 3 formulates the long-run average cost as a function of the expected cost per cycle and the expected cycle length. It is used as a measure of policy to optimize the model with respect to maintenance parameters. Section 4 includes an example for illustration purpose. Section 5 concludes the paper and gives some suggestions for further developments.

2 Features of the Model

2.1 Assumptions

- The system maintained is a parallel system consisting of two identical and independent components whose states are detected only by inspection.

- The state of the i^{th} component is described by a NHMP, $X_i(t)$, with three states: normal (0), satisfactory (1) and failure (2).

- The failure of components in normal state is not possible. This feature is common to items whose failures are defined by regulation (Khatab et al., 2018; Noortwijk et al., 1997; Newby and Barker, 2006). On this case a failure occurs when the component's deterioration process exceeds a threshold.

- The system is monitored according to a periodic inspection policy $\Pi = \{\tau, 2\tau, \cdots\}$.

- Inspections are perfect and revels the true state of components.
- The action space Ω_a includes three kinds of actions: (i) no action $\{a_0\}$, (ii) preventive replacement $\{a_1\}$ and (iii) corrective replacement $\{a^*\}$.
- The decision maker's actions are determined by the bivariate process $\mathbf{X}(t) = (X_1(t), X_2(t))$ and partitioning the state space $\Omega = \{(r,s) : r,s = 0,1,2\}$.

2.2 Modeling System

Consider a parallel system consisting of two independent and identical components whose state are detected only by inspections. The state of the component i ($i = 1,2$) is described by a NHMP, $X_i(t)$, taking its values in the state space $\Omega = \{0,1,2\}$. Specifically, the components experience three possible states: normal (0), satisfactory (1) and failure (2) with corresponding transition probabilities matrix $\mathbb{P}(t) = [P_{kl}(t)]$ and transition rate matrix $\mathbb{A}(t) = [a_{kl}(t)]$:

$$\mathbb{P}(t) = \begin{pmatrix} P_{00}(t) & P_{01}(t) & P_{02}(t) \\ 0 & P_{11}(t) & P_{12}(t) \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{A}(t) = \begin{pmatrix} -\lambda t & \lambda t & 0 \\ 0 & -\gamma t & \gamma t \\ 0 & 0 & 0 \end{pmatrix},$$

$P_{02}(t) = 1 - P_{00}(t) - P_{01}(t)$ and $P_{12}(t) = 1 - P_{11}(t)$. The transition rate matrix $\mathbb{A}(t)$ implies that, the process has propensity to shift from normal state (0) to the satisfactory state (1) and then from satisfactory state (1) to the absorbing (failure) state (2) with rates λ and γ respectively. Further, it indicates that the failure in normal state is not possible. To formulate the transition probabilities $P_{ij}(t)$, let $\dot{\mathbb{P}}(t) = [\dot{P}_{ij}(t)]$ be a matrix associated with elements $\dot{P}_{ij}(t)$:

$$\dot{P}_{ij}(t) = \frac{dP_{ij}(t)}{dt}.$$

Then, using the Kolmogorov forward equations, we have $\dot{\mathbb{P}}(t) = \mathbb{P}(t) \cdot \mathbb{A}(t)$, and given known initial conditions $P_{ii}(0) = 1$ and $P_{ij}(0) = 0$ ($i \neq j$) one can show that

$$\begin{cases} -\lambda t P_{00}(t) & = \dot{P}_{00}(t) \\ -\gamma t P_{11}(t) & = \dot{P}_{11}(t) \\ +\lambda t P_{00}(t) - \gamma P_{01}(t) & = \dot{P}_{01}(t). \end{cases}$$

Solving the above system of equations, for $t \geq 0$, yields

$$P_{00}(t) = \exp(-0.5\lambda t^2),$$
$$P_{11}(t) = \exp(-0.5\gamma t^2),$$
$$P_{01}(t) = \left(\frac{\lambda}{\gamma - \lambda}\right) \exp\left(-0.5\gamma t^2\right) \times \Big(-1 + \exp\left(0.5(\gamma - \lambda)t^2\right)\Big).$$

The behavior of transition probabilities $P_{ij}(t)$ as a function of t is presented graphically in Figure 1. For this, let $(\lambda, \gamma) = (0.2, 0.25)$ indicating that at time t the transition rates from normal state to the satisfactory state and from satisfactory state to the failure state are $0.2t$ and $0.25t$ respectively. The results reveal that for fixed values of the starting state i ($i = 0, 1$), the likelihood of failure of components increases with time implying that the process has propensity to shift to the absorbing state (2). Also, for a fixed value of t, the component is more prone to fail as its starting state increases from 0 (normal state) to 1 (satisfactory state).

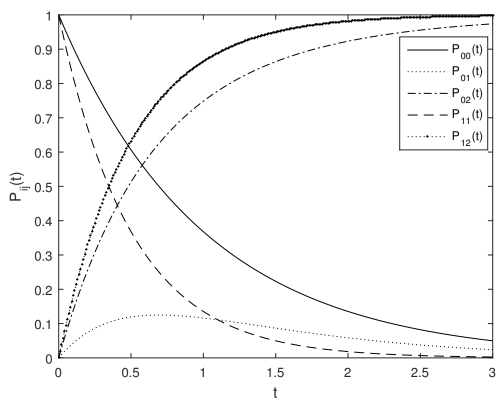

Figure 1: Transition probabilities $P_{ij}(t)$ as a function of t.

To reveal the true state of components and take an appropriate repair action the system is monitored according to a periodic inspection policy $\Pi = \{\tau, 2\tau, \cdots\}$. The repair action after inspection, denoted by the doubleton $\langle a, \cdot \rangle$, is determined by the bivariate underlying process $\mathbf{X}(t) = \big(X_1(t), X_2(t)\big)$ and partitioning state space $\Omega = \{(r,s) : r, s = 0, 1, 2\}$: if on inspection $\mathbf{X}(t)$ is found in state $A_0(\kappa) = \{(r,s) : 0 \leq r + s \leq \kappa - 1\}$, the system is not repaired and it is left to continue ($\langle a, \mathbf{X} \rangle = a_0$); if $\mathbf{X}(t)$ falls in state $A_1(\kappa) = \{(r,s) : \kappa \leq r + s < n\}$, the system is preventively replaced by new one ($\langle a, \mathbf{X} \rangle = a_1$); otherwise the system undergoes a corrective replacement ($\langle a, \mathbf{X} \rangle = a^*$. In other words,

$$\langle a, \mathbf{X} \rangle = \begin{cases} a_0, & \mathbf{X} \in A_0(\kappa); \\ a_1, & \mathbf{X} \in A_1(\kappa); \\ a^*, & \mathbf{X} \in A_2. \end{cases}$$

where r, s are non-negative integer values, $n = 4$ and κ is a decision variable taking its values in $\{1, 2\}$. As noted the decision process is determined by the excursions of the bivariate process $(X_1, X_2) \in \Omega$ and the decision maker's actions are then carried out by partitioning the state space Ω:

$$\Omega = A_0(\kappa) \cup A_1(\kappa) \cup A_2.$$

3 The Long-run Average Cost

3.1 Expected Cost per Cycle

A cycle consists of a sequence of inspections and maintenance actions that ultimately ends with replacement. The possible maintenance costs incurred in each cycle include an inspection cost c and a corrective replacement cost C_F. Also, a penalty cost C_R is incurred due to undetected failures within inter-inspection times. Further, an inspection and subsequent preventive replacement action for i^{th} component at age t induces the random cost $C_{X_i(t)}$ modulated by the bivariate process $X_i(t)$ ($i = 1, 2$). More specifically,

$$C_{X_i(\tau)} = \begin{cases} C_0, & X_i(\tau) = 0; \\ C_1, & X_i(\tau) = 1; \\ C_2, & X_i(\tau) = 2. \end{cases}$$

where C_j $(j = 0, 1, 2)$ $(C_0 < C_1 < C_2)$ is the cost of inspection and preventive replacement of a component if it is found in state j $(j = 0, 1, 2)$. As noted the piecewise function $C_{X_i(\tau)}$ in terms of the Heaviside step function $H(X_i(\tau) - j)$ can be expressed as

$$C_{X_i(\tau)} = C_0 + (C_1 - C_0)\Big(H(X_i(\tau) - 1)\Big) + (C_2 - C_1)\Big(H(X_i(\tau) - 2)\Big),$$

where

$$H(x) = \begin{cases} 1, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

The above modelling approach helps to formulate the preventive maintenance cost of a two-component parallel system through using a random cost function modulated by the bivariate process $\mathbf{X}(t) = \Big(X_1(t), X_2(t)\Big)$:

$$\begin{aligned} C_{\mathbf{X}(\tau)} = 2C_0 &+ (C_1 - C_0)\Big[\Big(H(X_1(\tau) - 1)\Big) + \Big(H(X_2(\tau) - 1)\Big)\Big] \\ &+ (C_2 - C_1)\Big[\Big(H(X_1(\tau) - 2)\Big) + \Big(H(X_2(\tau) - 2)\Big)\Big]. \end{aligned} \tag{3.1}$$

Let $C_\tau(\kappa; \mathbf{i})$ denote the cost per cycle starting with $\mathbf{i} = (i_1, i_2)$: when the system state is found in $A_0(\kappa)$, $\mathbf{X}(\tau) \in A_0(\kappa)$, the system restarts from the current state $\mathbf{X}(\tau)$ and the cyclic cost makes up from the planned inspection cost c and the future costs induced by taking no action $C_\tau\Big(\kappa; \mathbf{X}(\tau)\Big)$. On finding the system in state $A_1(\kappa)$, the maintenance cost includes the random preventive replacement cost $C_{\mathbf{X}(\tau)}$ (3.1) and the future cost starting from zero $C_\tau(\kappa; \mathbf{0})$. Also, the cyclic cost consists of the penalty cost of the system being unavailable due to an undetected failure and the corrective replacement cost on failure. In other words,

$$\begin{aligned} C_\tau(\kappa; \mathbf{i}) = &\Big(c + C_\tau\Big(\kappa; \mathbf{X}(\tau)\Big)\Big)\mathbf{1}_{A_0(\kappa)}(\mathbf{X}(\tau)) \\ &+ \Big(C_{\mathbf{X}(\tau)} + C_\tau(\kappa; \mathbf{0})\Big)\mathbf{1}_{A_1(\kappa)}(\mathbf{X}(\tau)) + \Big(C_F + C_R(\tau - T_{\mathbf{i}:n})\Big)\mathbf{1}_{A_2}(\mathbf{X}(\tau)), \end{aligned} \tag{3.2}$$

where

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A, \end{cases}$$

and $T_{\mathbf{i}:n}$ denotes the system's lifetime given the starting state $\mathbf{i} = (i_1, i_2)$ and $A_i(\kappa)$ $(i = 1, 2)$ are the subsets of the state space $\Omega = \{(r, s) : r, s = 0, 1, 2\}$:

$$A_0(\kappa) = \{(r, s) : 0 \le r + s \le \kappa - 1\}, \quad A_1(\kappa) = \{(r, s) : \kappa \le r + s < n\}.$$

Taking expectation of both sides of the equation (3.2) gives the expected cost per cycle $\mathcal{C}_\tau(\kappa; \mathbf{i}) = \mathbb{E}(C_\tau(\kappa; \mathbf{i}))$ as

$$\begin{aligned} \mathcal{C}_\tau(\kappa; \mathbf{i}) = & \sum_{\mathbf{j} \in A_0(\kappa)} (c + \mathcal{C}_\tau(\kappa; \mathbf{j})) \times P_{\mathbf{ij}}(\tau) \\ & + \Big(C_F + C_R \mu(\tau; \mathbf{i})\Big) \times P_{\mathbf{i2}}(\tau) + \mathcal{C}_\tau(\kappa; \mathbf{0}) \sum_{\mathbf{j} \in A_1(\kappa)} P_{\mathbf{ij}}(\tau) + \sum_{\mathbf{j} \in A_1(\kappa)} H_c(\mathbf{j}) \times P_{\mathbf{ij}}(\tau), \end{aligned} \tag{3.3}$$

where $P_{\mathbf{ij}}(\tau) = P\Big(\mathbf{X}(\tau) = (j_1, j_2) | \mathbf{X}(0) = (i_1, i_2)\Big) = P_{i_1 j_1}(\tau) \times P_{i_2 j_2}(\tau)$,

$$H_c(\mathbf{j}) = 2C_0 + (C_1 - C_0)\Big(H(j_1 - 1) + H(j_2 - 1)\Big) + (C_2 - C_1)\Big(H(j_1 - 2) + H(j_2 - 2)\Big),$$

and $\mu(\tau; \mathbf{i})$ implies the mean past lifetime of the system with the lifetime $T_{\mathbf{i}:n}$ such that $\mu(\tau; \mathbf{i}) = \int_0^\tau P_{\mathbf{i2}}(t) dt / P_{\mathbf{i2}}(\tau)$.

3.2 Expected Cycle Length

The cycle length starting from $\mathbf{i} = (i_1, i_2)$ is defined as $L_\tau(\kappa; \mathbf{i})$: The expected length of a cycle $\mathcal{L}_\tau(\kappa; \mathbf{i}) = \mathbb{E}\Big(L_\tau(\kappa; \mathbf{i})\Big)$ is obtained as the expected cost per cycle: on failure, a full period τ is completed. On observing the system in $A_0(\kappa)$, the random time until replacement is made up of a completed inspection time and an additional cycle length starting in state $\mathbf{X}(\tau)$. On finding the system state in the critical region $A_1(\kappa)$ the cycle length comprises a completed inspection time as well as the remaining cycle length $L_\tau(\kappa; \mathbf{0})$ starting from zero. In other words,

$$L_\tau(\kappa; \mathbf{i}) = \Big(\tau + L_\tau(\kappa; \mathbf{X}(\tau))\Big) \mathbf{1}_{A_0(\kappa)}(\mathbf{X}(\tau)) + \Big(\tau + L_\tau(\kappa; \mathbf{0})\Big) \mathbf{1}_{A_1(\kappa)}(\mathbf{X}(\tau)) + \tau \mathbf{1}_{A_2}(\mathbf{X}(\tau))$$

The average length of a cycle $\mathcal{L}_\tau(\kappa; \mathbf{i}) = \mathbb{E}\Big(L_\tau(\kappa; \mathbf{i})\Big)$ becomes

$$\mathcal{L}_\tau(\kappa; \mathbf{i}) = \tau + \sum_{\mathbf{j} \in A_0(\kappa)} \mathcal{L}_\tau(\kappa; \mathbf{j}) \times P_{\mathbf{ij}}(\tau) + \mathcal{L}_\tau(\kappa; \mathbf{0}) \sum_{\mathbf{j} \in A_1(\kappa)} P_{\mathbf{ij}}(\tau). \tag{3.4}$$

3.3 The Long-run Average Cost Rate

Since the perfect repair instants are regeneration points and the sequences of regeneration points consists a renewal process, this allows the use of the renewal reward theorem

(Ross, 1970) for formulating the average cost rate $\mathbb{C}_\tau(\kappa;\mathbf{i})$ in terms of the expected cost per cycle (3.3) and the expected cycle length (3.4):

$$\mathbb{C}_\tau(\kappa;\mathbf{i}) = \frac{\mathcal{C}_\tau(\kappa;\mathbf{i})}{\mathcal{L}_\tau(\kappa;\mathbf{i})}. \tag{3.5}$$

The average cost rate (3.5) is used as a measure of policy to determine the optimal inspection policy $\Pi^* = \{n\tau^* : n = 1, 2, \cdots\}$ and the preventive replacement policy implemented by the optimal replacement threshold κ^*:

$$(\tau^*, \kappa^*) = \underset{(\tau,\kappa)\in(0,\infty)\times A^c}{\operatorname{argmin}} \mathbb{C}_\tau(\kappa;\mathbf{i}), \tag{3.6}$$

where $A^c = \{1, 2, 3\}$.

4 Numerical Example

Numerical results are based on a bivariate process $\mathbf{X}(t) = (X_1(t), X_2(t))$ characterized by three states (normal state (0), satisfactory state (1) and failure state (2)). For the numerical illustration, we set $(\lambda, \gamma) = (0.2, 0.25)$. The choice for the cost parameters are $c = 0.25$, $C_0 = 0.5$, $\mathbf{C}_1 = (C_1, C_2, C_F) = (1.5, 2.5, 10)$. Using the above set of values, we aim at finding both the optimal inspection and replacement policy implemented by the optimal inspection interval τ^* and the optimal replacement threshold κ^*. The results are developed by examining the response of the model to costs $\mathbf{C}_2 = (4.5, 7.5, 30)$, $\mathbf{C}_3 = (9, 15, 60)$ and $C_R \in \{0.20, 1.0, 5.0, 25\}$ and the degradation parameters $\lambda \in \{0.05, 0.2, 0.8\}$ and $\gamma \in \{0.025, 0.25, 0.75, 1.25\}$.

From Figure 2 and Table 2, it is easy to see that the maintenance policy is characterized by the optimal maintenance parameters $(\tau^*, \kappa^*) = (0.95, 3.0)$. Therefore, to achieve the minimum expected cost per unit time, the model postulates that to detect the true state of components the system should be monitored according to the policy $\Pi^* = \{0.95n : n = 1, 2, \cdots\}$: if the revealed state $(X_1(t), X_2(t))$ on inspection at age $t = n\tau^*$ falls in the set $A_0(\kappa^*) = \{(r, s) : 0 \le r + s < \kappa^*\} = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (0, 2)\}$ no repair action is taken; the system undergoes a preventive replacement if the components state is found in $A_1(\kappa^*) = \{(r, s) : \kappa^* \le r + s < 4\} = \{(1, 2), (2, 1)\}$; otherwise, $(X_1(t), X_2(t)) = (2, 2)$, a corrective replacement should be undertaken. This maintenance policy driven by the optimal inspection interval τ^* and the optimal replacement threshold κ^* determines the optimal expected cost $\mathbb{C}_{\tau^*}(\kappa^*, \mathbf{0}) = 0.49$.

The optimal solutions for different combinations of maintenance cost and degradation parameters are illustrated in Table1 to Table3.

The main findings of our numerical results are outlined as follows:

- Apart from few cases outlined below, changing both the degradation and cost parameters does not impact on the optimal replacement threshold $\kappa^* = 3$:

(**i**) as the process is more prone to move up to the failure state from the satisfactory state at a constant rate $\gamma t = 1.25t$ ($\gamma/\lambda = 25$), given that maintaining the system incurs the least cost $\mathbf{C}_1$, the preventive replacement policy is implemented whenever the bivariate process $\mathbf{X}(t)$ on inspection is found in the set $A_1(\kappa^*) = \{(r,s) : \kappa^* \leq r+s < 2\}$ with $\kappa^* = 1$ (i.e. a degradation process makes a transition to the satisfactory or the failure state) (see Table 1). An intuitive implication of this result is that as the system becomes more susceptible to failure (γ/λ increases), the model ideally responds to it by the reduction of the replacement threshold $\kappa^* : 3 \mapsto 1$.

(**ii**) in the case when the maintenance cost increases uniformly and the process in contrary to the case **(i)** has propensity to sojourn in state 1 ($\lambda/\gamma = 32$), the model adapts itself via moving up the threshold value $\kappa^* = 4$ (see Tables 2 and 3). In this case the action space includes only two types of actions $\Omega_a = \{a_0, a^*\}$ and the model is simplified by dropping the set $A_1(\kappa)$ (PR region) from the model. The equations (3.3) and (3.4) can be rewritten respectively as:

$$C_\tau(\kappa;\mathbf{i}) = \sum_{\mathbf{j}\in A_0(\kappa)} (c + C_\tau(\kappa;\mathbf{j})) \times P_{\mathbf{ij}}(\tau) + \Big(C_F + C_R\mu(\tau;\mathbf{i})\Big) \times P_{\mathbf{i2}}(\tau),$$

and

$$\mathcal{L}_\tau(\kappa;\mathbf{i}) = \tau + \sum_{\mathbf{j}\in A_0(\kappa)} \mathcal{L}_\tau(\kappa;\mathbf{j}) \times P_{\mathbf{ij}}(\tau).$$

- The optimal period of inspection τ^* strongly depends on the degradation parameters (λ,γ) as other parameters remain fixed: as transition rates decrease the system is monitored less frequently (τ^* increases) implying a reduction in the amount of maintenance undertaken on the system and resulting expected costs.

- The optimal period of inspection τ^* is sensitive to the cost parameters $\mathbf{C}_i$ ($i = 1,2,3$) as other parameters remain fixed: as CM cost in contrast to PM cost increases more rapidly, the inspections are scheduled more frequently. This ideally averts increasing costs by an early detection of failures.

The optimal solutions for different combinations of the penalty cost and the maintenance cost parameters are examined as other parameters remain fixed (see Table 4).

The results summarized in Table 4 indicate that:

- in all cases the optimal replacement threshold κ^* does not respond to $\mathbf{C}_i$ and C_R.

- as maintenance cost parameters remain constant, increasing the penalty cost makes inspections slightly more frequent (see the first two rows of Table 4). This penalizes a costly strategy resulting from undetected failures within inter-inspection times.

- as maintenance costs increase more rapidly, changes in penalty cost does not induce significant changes in optimal solutions (see the last row of Table 4).

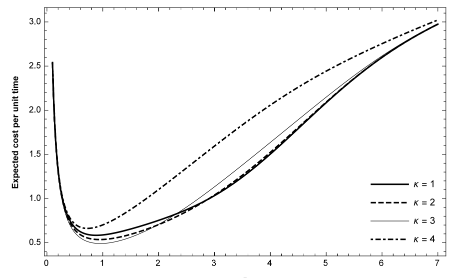

Figure 2: Expected cost per unit time for different $\kappa \in \{1, 2, 3, 4\}$ and $(\lambda, \gamma) = (0.2, 0.25)$.

Table 1: Optimal parameters $(\tau^*, \kappa^*, \mathbb{C}_{\tau^*})$ for different $\mathbf{C}_i$ and γ given $\lambda = 0.05$.

	γ											
	0.025			0.25			0.75			1.25		
$\mathbf{C}_i$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$
$\mathbf{C}_1$	2.25	3.00	0.21	1.52	3.00	0.28	1.29	3.00	0.31	1.36	1.00	0.32
$\mathbf{C}_2$	1.34	3.00	0.37	1.00	3.00	0.46	0.89	3.00	0.49	0.83	3.00	0.51
$\mathbf{C}_3$	0.94	3.00	0.52	0.75	3.00	0.64	0.68	3.00	0.66	0.64	3.00	0.68

Table 2: Optimal parameters $(\tau^*, \kappa^*, \mathbb{C}_{\tau^*})$ for different $\mathbf{C}_i$ and γ given $\lambda = 0.2$.

	γ											
	0.025			0.25			0.75			1.25		
$\mathbf{C}_i$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$
$\mathbf{C}_1$	1.80	3.00	0.27	0.95	3.00	0.49	0.81	3.00	0.54	0.75	3.00	0.56
$\mathbf{C}_2$	1.04	3.00	0.48	0.57	3.00	0.84	0.52	3.00	0.90	0.50	3.00	0.93
$\mathbf{C}_3$	0.72	4.00	0.68	0.41	3.00	1.19	0.38	3.00	1.26	0.36	3.00	1.29

Table 3: Optimal parameters $(\tau^*, \kappa^*, \mathbb{C}_{\tau^*})$ for different $\mathbf{C}_i$ and γ given $\lambda = 0.8$.

	γ											
	0.025			0.25			0.75			1.25		
$\mathbf{C}_i$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$
$\mathbf{C}_1$	1.66	3.00	0.29	0.65	3.00	0.74	0.51	3.00	0.94	0.46	3.00	1.00
$\mathbf{C}_2$	0.96	4.00	0.50	0.38	3.00	1.31	0.29	3.00	1.64	0.28	3.00	1.72
$\mathbf{C}_3$	0.70	4.00	0.70	0.27	3.00	1.87	0.21	3.00	2.32	0.20	3.00	2.43

Table 4: Optimal parameters $(\tau^*, \kappa^*, \mathbb{C}_{\tau^*})$ for different $\mathbf{C}_i$ and C_r given $(\lambda, \gamma) = (0.2, 0.25)$.

	C_r								
	0.2			1			5		
$\mathbf{C}_i$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$	τ^*	κ^*	$\mathbb{C}_{\tau^*}$
$\mathbf{C}_1$	0.97	3.00	0.487	0.97	3.00	0.488	0.95	3.00	0.49
$\mathbf{C}_2$	0.585	3.00	0.8460	0.585	3.00	0.8461	0.5729	3.00	0.8464
$\mathbf{C}_3$	0.4165	3.00	1.1964	0.4165	3.00	1.1964	0.4165	3.00	1.1964

5 Conclusion

Through the use of a bivariate process and the identification of a renewal process this paper has presented a new approach to the joint determination of optimal inspection and preventive replacement policy for a two-component parallel system whose state is detected by inspections.

The model is examined for the case when the state transitions of components are driven by a three-state non-homogeneous Markov process with known characteristics. The results of the model provide a sound inspection and preventive replacement policy and give an intuition on the behavior of the optimal solutions as system's parameters vary.

The structure explored here allows further developments via the study of complex systems (e.g. see Zarezadeh et al., 2019; Zarezadeh and Asadi, 2019) with independent or dependent multi-state components, (ii) the consideration of systems with a more general failure mechanism and (iii) the inclusion of imperfect repair. The latter can be implemented by considering age reduction models and partitioning the state space to four exclusive subsets.

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References

Azizi, F. and Salari, N. (2023), *A novel condition-based maintenance framework for parallel manufacturing systems based on bivariate birth/birth–death processes*. Reliability Engineering & System Safety, **229**, 108798.

Ahmadi, R. (2020), *Reliability and maintenance modeling for a load-sharing k-out-of-n system subject to hidden failures*. Computers and Industrial Engineering, **150**, 1-13.

Ahmadi, R. (2019), *A new approach to maintenance optimization of repairable parallel systems subject to hidden failures*. Journal of Operational Research Society, **71**(9), 1448-1465.

Ahmadi, R. and Wu, S.(2017), *A novel data-driven approach to optimizing replacement policy*. Reliability Engineering and System Safety, **167**, 506-516.

Barlow, R., Hunter, L.C. and Proschan, F.(1963), *Optimum checking procedures*. Journal off the Society for Industrial and Applied Mathematics, **11**, 1078-1095.

Chelbi, A., Ait-Kadi, D. and Aloui, H.(2008), *Optimal inspection and preventive maintenance policy for systems with self-announcing and non-self-announcing failures*. Journal of Quality in Maintenance Engineering, **14**(1), 34-45.

Cai, B., Wang, Y., Zhang, Y., Liu, Y., Ge, W., Li, R., Liu, Y. and Liu, G. (2022), *Condition-based maintenance method for multi-component system based on RUL prediction: Subsea tree system as a case study*. Computers & Industrial Engineering, **173**, 108650.

Chen, Y., Liu, Y. and Jiang, T. (2021), *1. Optimal maintenance strategy for multi-state systems with single maintenance capacity and arbitrarily distributed maintenance time*. Reliability Engineering & System Safety, **211**, 107576.

Fink, J., Reenberg, A.A., Kulahci, A.M. and Nielsen, B.F. (2022), *A numerical study of Markov decision process algorithms for multi-component replacement problems*. Reliability Engineering & System Safety, **299**(3), 898-909.

Forouzandeh Shahraki, A., Yadav, O.P. and Vogiatzis, C. (2020), *Selective maintenance optimization for multi-state systems considering stochastically dependent components and stochastic imperfect maintenance actions*. Reliability Engineering & System Safety, **196**, 106738.

He, K., Maillart, L.M. and Prokopyev, O.A. (2015), *Scheduling preventive maintenance as a function of an imperfect inspection interval*. IEEE Transactions on Reliability, **64**(3), 983-997.

Hu, J., Xu, A., Li, B. and Liao, H. (2021), *Condition-based maintenance planning for multi-state systems under time-varying environmental conditions*. Computers & Industrial Engineering, **158**, 107380.

Jiang, K. and Jardine, A.K.S. (2005), *Two optimization models of the optimum inspection problem*. Journal of the Operational Research Society, **56**, 1176-1183.

Khatab, A., Diallo, C., Aghezzaf, E.H. and Venkatadri, U. (2018), *Condition-based selective maintenance for stochastically degrading multi-component systems under periodic inspection and imperfect maintenance*. Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, **232**(4), 447-463.

Liu, B., Pandey, M.D., Wang, X. and Zhao, X. (2021), *A finite-horizon condition-based maintenance policy for a two-unit system with dependent degradation processes*. Reliability Engineering & System Safety, **295**(2), 705-717.

Keller, J. (1974), *Optimum checking schedules for systems subject to random failures*. IEEE Transactions on Reliability, **66**, 219-232.

Noortwijk, J.V., Kok, M. and Cooke, R. (1997), *Optimum checking schedules for systems subject to random failures*. European Journal of Operational Research, **82**(2), 25-56.

Jong Kim, M. and Makis, V. (2009), *Optimal maintenance policy for a multi-state deteriorating system with two types of failures under general repair Constant transition rate*. Computers & Industrial Engineering, **57**(1), 298-303.

Kijima, M. (1989), *Some results for repairable systems with general repair*. Journal of Applied Probability, **26**, 89-102.

Liu, B., Yeh, R.H., Xie, M. and Kuo, W. (2017), *Maintenance Scheduling for Multicomponent Systems with Hidden Failures*. IEEE Transactions on Reliability, **66**(4), 1280-1292.

Mercier, S. and Castro, I.T. (2019), *Stochastic comparisons of imperfect maintenance models for a gamma deteriorating system*. European Journal of Operational Research, **273**, 237-248.

Munford, A. and Shahani, A. (1972), *A nearly optimal inspection policy*. Operational Research Quarterly, **23**, 373-379.

Newby, M.J. and Barker, C.T. (2006), *A bivariate process model for maintenance and inspection planning*. International Journal of Pressure Vessels and Piping, **83**, 270–275.

Perry, D. (2000), *Control limit policies in a replacement modelwith additive phase-type distributed damage and linear restoration*. Operational Research Letters, **27**, 127-134.

Rangan, A. and Grace, R.E. (1989), *On the Replacement of Systems Subject to Shocks and Wear-Dependent Failure*. Advances in Applied Probability, **21**(1), 238-240.

Rezaei, E. (2017), *A new model for the optimization of periodic inspection intervals with failure interation: A case study for a turbin rotor*. Case studies in Engineering Failure Analysis, **9**, 148-156.

Ross, S.M. (1970), *Applied probability models with optimization applications*. San Francisco, CA: Holden-Day.

Seyedhosseini, S.M., Moakedi, H. and Shahanaghi, K. (2018), *Imperfect inspection optimization for a two-component system subject to hidden and two-stage revealed failures over a finite time horizon*. Reliability Engineering and System Safety, **174**, 141-156.

Salari, N. and Makis, V. (2017), *Comparison of two maintenance policies for a multi-unit system considering production and demand rates*. International Journal of Production Economics, **193**, 381-391.

Wang, Y., Li, X., Chen, J. and Liu, Y. (2022), *A condition-based maintenance policy for multi-unit systems subject to deterioration*. Reliability Engineering & System Safety, **219**, 108174.

Wei, Y., Zhiying, W., Junlin, X. and Yaofeng, X. (2018), *Redundancy optimization of cold-standby systems under periodic inspection and maintenance*. Reliability Engineering & System Safety, **180**, 394-402.

Yang, L., Ye, Z.S., Lee, C.G., Yang, S.F. and Peng, R. (2019), *A two-phase preventive maintenance policy considering imperfect repair and postponed replacement*. European Journal of Operational Research, **274**(3), 966-977.

Zarezadeh, S. and Asadi, M. (2019), *Coherent systems subject to multiple shocks with applications to preventative maintenance*. Relibility ngineering & System Safety, **185**, 124-132.

Zarezadeh, S., Asadi, M. and Eftekhar, S. (2019), *Signature-based information measures of multi-state networks*. Probability in the Engineering and Informational Sciences, **33**(3), 438-459.

Zhang, N., Deng, Y., Liu, B. and Zhang, J. (2023), *Condition-based maintenance for a multi-component system in a dynamic operating environment*. Reliability Engineering & System Safety, **231**, 108988.