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# Quantile based Past Geometric Vitality Function of Order Statistics

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**Abstract.** Nair and Rajesh (2000) introduced the geometric vitality function, which explains the failure pattern of components or systems based on the component's geometric mean of the remaining lifetime. Recently quantile-based studies have found greater interest among researchers as an alternative method of measuring the uncertainty of a random variable. The quantile-based measures possess some unique properties to the distribution function approach. The present paper introduces a quantile-based past geometric vitality function of order statistics and its properties. Finally, we provide an application for the new measure based on some distributions which are useful in lifetime data analysis.

**Keywords.** Geometric Vitality, Order Statistics, Quantile Function, Vitality Function.

**MSC:** 62B10, 94A17.

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# 1 Introduction

In statistical theory and practice, usually, specification of a probability distribution can be made in terms of the distribution function or by the quantile function. The new methodology has received more attention among scientists in literature than its traditional method. The concept and methodologies based on quantile functions are mainly applicable when the traditional approach based on distribution functions is either complex or fails to give desired results for the study. Quantile functions are the best replacement for the distributions in modelling and analyzing statistical data. The two methodologies convey the same information regarding the distribution function but they differ in interpretation style which characterizes their unique behaviour. Many authors carried out a detailed study on quantile function, its properties, and its usefulness in model identification. Further, quantile functions are also considered to be more useful in situations where the distribution functions do not have tractable forms. For related works, one can refer to Ramberg and Schmeiser (1974), Gilchrist (2000), Hankin and Lee (2006), Sankaran and Nair (2009), Nair et al. (2011), Aswin et al. (2020) and Dileep and Sankaran (2000). For works related to the study of non-quantile functions in the context of information measures, one may refer to Di Crescenzo and Longobardi (2002) and Rajesh and Sunoj (2019). Recently, the study of information measures using quantile function are introduced by many authors in the literature. Some of the recent references are krishnan et al. (2020), Sunoj et al. (2018), krishnan et al. (2019) and Kayal and Tripathy (2018).

The ageing process receives special attention in reliability analysis concerning system components and devices under examination. Based on this concept, Kotz and Shanbhag (1980) introduced a new measure called the vitality function and obtained several characterizations for lifetime distributions. This measure has been considered as a helpful tool in modeling lifetime data. Nair and Rajesh (2000) introduced the notion of geometric vitality function (GVF), representing the geometric mean of lifetimes of components that have survived up to time t. One of the important applications of the geometric mean has been observed in the stock market and was discussed in detail by Cover and Thomas (2006). GVF can be considered as a useful tool in analyzing lifetime data similar to the vitality function. Accordingly, Sunoj et al. (2009) discussed GVF for the doubly (interval) truncated random variables. Later, Sathar et al. (2010) extended the definition of GVF to a bivariate setup and provided characterizations of some bivariate models using the functional form of the bivariate GVF. Further, Rajesh et al. (2014) proposed a nonparametric kernel-type estimator for the GVF both in the case of complete and censored samples. Gayathri and Sathar (2021) introduced past geometric vitality function (PGVF) in past life and studied several exciting properties. Assume Y is a non-negative random variable that has an absolutely continuous distribution function (cdf) $F$ and probability density function (pdf) $f$ with $E(\log(Y)) < \infty$, then the PGVF is defined as

$$\log\ \bar{G}(t) = E\left[\ \log Y|Y \leq t\ \right] = \frac{1}{F(t)}\int_0^t \log y\, f(y)\, dy. \tag{1.1}$$

Simplification of (1.1) gives

$$\log\left(\frac{\bar{G}(t)}{t}\right) = -\frac{1}{F(t)}\int_0^t \frac{F(y)}{y} dy.$$

The present paper introduces a quantile-based past geometric vitality function and studies its essential properties. The proposed measure has several advantages. First, this measure uniquely determines the corresponding quantile functions. Second, we derive past geometric vitality functions for certain quantile functions which do not have an explicit form for corresponding distribution functions. Finally, we provide an application for the new measure based on Pareto distribution which is useful in lifetime data analysis.

The outline of the article is described as follows. Section 2 introduces past geometric vitality function in terms of quantile function and studies some properties such as characterization, ageing classes and stochastic comparisons. Section 3 defines rth order statistics of the quantile-based PGVF and studies its properties. Section 4 discusses an application of the quantile-based PGVF as a risk measure. This section also compares this new risk measure with right tail deviation measure and variance. In Section 5, simulation studies and real life data application to investigating the performance of quantile-based PGVF are carried out. A brief conclusion of the present study is given in Section 6.

## 2 Quantile-based PGVF

In this section, we propose the quantile version of PGVF defined in (1.1). First, we recall some notations and preliminary concepts using quantile function. Let $Y$ be a random variable with cdf $F(\cdot)$, then the corresponding quantile function denoted by $Q(\cdot)$ and is defined as

$$Q(u) = F^{-1}(u) = inf\{y : F(y) \geq u\},\ 0 \leq u \leq 1. \tag{2.1}$$

If $f(\cdot)$ is the pdf of $Y$, then $f(Q(u))$ is called the density quantile function. The derivative of $Q(u)$, $q(u) = Q'(u)$ is known as the quantile density function of $Y$. From (2.1), we have $F(Q(u)) = u$ and differentiating it with respect to $u$ yield

$$q(u)f(Q(u)) = 1. \tag{2.2}$$

From (2.2) and substituting y=$Q(p)$ in (1.1), we proposed a new reliability measure for a non-negative random variable $Y$ in terms of quantile function, namely quantile-based PGVF ( QPGVF), which is denoted by $\log \bar{G}(Q(u))$ and is defined as

$$\log \bar{G}(Q(u)) = \frac{1}{F(Q(u))} \int_0^{Q(u)} \log Q(p)\, f(Q(p))\, dQ(p). \tag{2.3}$$

Equation (2.3) can be simplified as

$$\log \bar{G}(Q(u)) = \frac{1}{u} \int_0^{u} \log Q(p)\, dp. \tag{2.4}$$

In the following example, we evaluate $\log \bar{G}(Q(u))$ for the power-Pareto distribution, which does not have an explicitly known distribution function but has a closed-form quantile function.

**Examples 2.1.** Let $Y$ be a random variable of the power-Pareto distribution with corresponding quantile function

$$Q_Y(u) = \frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}},\ c > 0,\ \lambda_1,\ \lambda_2 > 0. \tag{2.5}$$

QPGVF corresponds to (2.5) is

$$\log \bar{G}(Q(u)) = \log c - \lambda_1 + \lambda_2 + \lambda_1 \log(u) + \lambda_2 \frac{1-u}{u} \log(1-u). \tag{2.6}$$

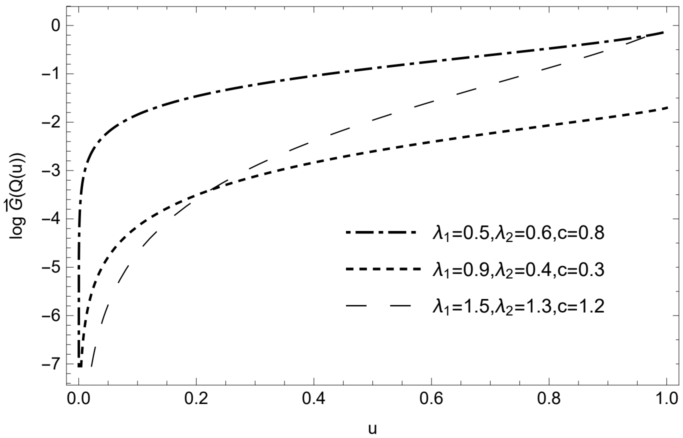


Figure 1: QPGVF for power-Pareto distribution with various values of parameters.

Figure 1 provides the plot of $\log \bar{G}(Q(u))$ for (2.5) for different values of $\lambda_1$ , $\lambda_2$ and c. From Figure 1, it is clear that $\log \bar{G}(Q(u))$ is an increasing function in terms of various values of $\lambda_1$ , $\lambda_2$ and c. By integrating by parts in (2.4), we get

$$\log \bar{G}(Q(u)) = \log Q(u) - \frac{1}{u}\int_0^u \frac{p\ \ q(p)}{Q(p)}dp. \tag{2.7}$$

Differentiating (2.7) with respect to $u$, we get the following relationship

$$A(u) = \frac{\frac{d}{du}\log \bar{G}(Q(u))}{q(u)\log Q(u)} + \frac{\log \bar{G}(Q(u))}{uq(u)\log Q(u)}, \tag{2.8}$$

where $A(u) = \frac{1}{u\ q(u)}$, is the reversed hazard quantile function. From (2.8), we can write

$$\log \bar{G}(Q(u)) = \int_0^u A(p)\log Q(p)q(p)dp + K,$$

where $K$ denotes the constant of integration. In the following theorem, we discuss the uniqueness property of QPGVF .

**Theorem 2.1.** *Quantile-based past geometric vitality function uniquely determines the corresponding quantile function.*

*Proof.* Differentiating (2.4) with respect to $u$ yields

$$Q(u) = \exp\left[\log \bar{G}(Q(u)) + u\log \bar{G}'(Q(u))\right]. \tag{2.9}$$

Using (2.9), one can easily derive the corresponding quantile function if we know the QPGVF and hence the proof is complete. □

The following example illustrates Theorem 2.1.

**Examples 2.2.** Suppose $Y$ follows log-logistic distribution with quantile function

$$Q(u) = \frac{1}{\alpha}\left(\frac{u}{1-u}\right)^{\frac{1}{\beta}}, \; \alpha, \; \beta > 0. \tag{2.10}$$

QPGVF corresponds to (2.10) simplifies to

$$\log \bar{G}(Q(u)) = -\log\alpha + \frac{1}{\beta}\left[\log u + \frac{1-u}{u}\log(1-u)\right]. \tag{2.11}$$

Conversly, assume (2.11) holds. Then using (2.9), we obtain

$$Q(u) = \exp\left[-\log\alpha + \frac{1}{\beta}\log u + \frac{1}{\beta}\frac{1-u}{u}\log(1-u) - \frac{u}{\beta}\left(\frac{\log(1-u)}{u} + \frac{1-u}{u^2}\log(1-u)\right)\right],$$

and, on simplification, we get (2.10).

In the following we define a class of distributions using QPGVF.

**Definition 2.1.** A non-negative random variable $Y$ is said to have increasing (decreasing) QPGVF denoted as IQPGVF (DQPGVF) if $\log \bar{G}\,(Q(u))$ is increasing (decreasing) in $u \geq 0$.

The next theorem gives the necessary and sufficient condition for $\log \bar{G}\,(Q(u))$ to be an increasing (decreasing) function of $u$.

**Theorem 2.2.** *Let $Y$ be a non-negative random variable. $Y$ has IQPGVF (DQPGVF) if and only if* $\log \bar{G}\,(Q(u)) \leq (\geq) \log Q(u)$.

*Proof.* For a DQPGVF, we have $\frac{d}{du}\log \bar{G}\,(Q(u)) \leq 0$. By using (2.8), we get the result as $\log \bar{G}\,(Q(u)) \geq \log Q(u)$. The proof of the converse part is easy by retracing the above-given steps. In a similar manner for an IQPGVF, we can obtain the result $\log \bar{G}\,(Q(u)) \leq \log Q(u)$. Hence the theorem is proved. □

In Table 1, the QPGVF of some distributions are derived and the monotonicity of the QPGVF is established.

Table 1: Quantile functions, QPGVF and monotone nature.

| Distribution | Quantile functions | $\log \bar{G}(Q(u))$ | Monotone nature |
|---|---|---|---|
| Pareto | $\sigma(1-u)^{\frac{-1}{\alpha}}$ | $\log\sigma + \frac{1}{\alpha} + \frac{1}{\alpha}\frac{(1-u)}{u}\log(1-u)$ | IQPGVF<br>DRHR |
| Log-logistic | $\frac{1}{\alpha}(\frac{u}{1-u})^{\frac{1}{\beta}}$ | $-\log\alpha + \frac{1}{\beta}[\log(u) + \frac{1-u}{u}\log(1-u)]$ | IQPGVF<br>DRHR |
| Power | $\alpha u^{\frac{1}{\beta}}$ | $\log\alpha - \frac{1}{\beta}[1-\log u]$ | IQPGVF<br>DRHR |
| Exponential | $\frac{1}{\lambda}[-\log(1-u)]$ | $-\log\lambda - \frac{1}{u}\int_0^u \log(\log(1-p))dp$ | IQPGVF<br>BT |

*Remark* 1. Since the monotonicity of the reversed hazard rate function and the reversed hazard quantile function are the same, we say that $Y$ has an increasing (decreasing) reversed hazard rate [IRHR (DRHR)] if reversed hazard quantile of $Y$, denoted as $A_Y(u)$ is increasing (decreasing) in $u$. In Table 1, we studied the monotonic nature of QPGVF using some distributions which belong to IRHR (DRHR) classes. From Table 1, we can find that IRHR (DRHR) property does not imply IQPGVF (DQPGVF) property.

Let us recall some definitions of stochastic orderings from Shaked and Shanthikumar (2007). For quantile-based stochastic orderings, one may refer to Nair et al. (2013). In the same way, the order based on QPGVF for past lifetime is given through the following definition.

**Definition 2.2.** Let $W$ and $Y$ be two non-negative random variables then $W \leq_{QPGVF} Y$, if $\log \bar{G}^W(Q_W(u)) \leq \log \bar{G}^Y(Q_Y(u))$ for all $0 < u < 1$.

The following example illustrates Definition 2.2.

**Examples 2.3.** Suppose W$\sim U(0,a)$ and Y$\sim U(0,b)$ (if $a < b$) then the corresponding quantile functions are $Q^W(u) = au$ and $Q^Y(u) = bu$, respectively.

Using (2.4), we get

$$\log \bar{G}^W(Q_W(u)) - \log \bar{G}^Y(Q_Y(u)) = \log a + (-1 + \log(u)) - (\log b + (-1 + \log(u))) < 0. \tag{2.12}$$

From (2.12), we have $W \leq_{QPGVF} Y$.

**Definition 2.3.** Let $W$ and $Y$ be two non-negative random variables then $W \leq_{st} Y$, if $Q_W(u) \leq Q_Y(u)$ for all $0 < u < 1$.

**Theorem 2.3.** *If $W$ and $Y$ are two random variables such that $W \leq_{st} Y$, then $W \leq_{QPGVF} Y$.*

*Proof.* if $W \leq_{st} Y$ then $Q_W(u) \leq Q_Y(u)$ and consequently

$$\int_0^u \log Q_W(p)dp \leq \int_0^u \log Q_Y(p)dp,$$

or equivalently

$$\frac{1}{u}\int_0^u \log Q_W(p)dp \leq \frac{1}{u}\int_0^u \log Q_Y(p)dp.$$

Thus $\log \bar{G}^W(Q_W(u)) \leq \log \bar{G}^Y(Q_Y(u))$. □

**Definition 2.4.** Let $W$ and $Y$ be two non-negative random variables then $W \leq_{disp} Y$, if $Q_Y(u) - Q_W(u)$ is increasing in $u \in (0, 1)$.

**Theorem 2.4.** *If $W$ and $Y$ are two random variables such that $W \leq_{disp} Y$, then $W \leq_{QPGVF} Y$.*

*Proof.* if $W \leq_{disp} Y$ then $Q_Y(u) - Q_W(u)$ is increasing in $u$. Using (2.4),

$$\log \bar{G}^W(Q_W(u)) = \frac{1}{u}\int_0^u \log Q_W(p)dp \leq \frac{1}{u}\int_0^u \log Q_Y(p)dp = \log \bar{G}^Y(Q_Y(u)).$$

Thus $\log \bar{G}^W(Q_W(u)) \leq \log \bar{G}^Y(Q_Y(u))$. □

**Definition 2.5.** Let $W$ and $Y$ be two non-negative random variables such that $A_W(u) \leq A_Y(u)$ for all $u \in (0, 1)$. Then $W$ is said to be smaller than $Y$ in reversed hazard quantile order (RHQ), denoted by $W \leq_{RHQ} Y$.

In the following theorem, we consider reversed hazard quantile order to compare two random variables based on QPGVF.

**Theorem 2.5.** *If $W$ and $Y$ are two random variables such that $W \leq_{RHQ} Y$, then $W \leq_{QPGVF} Y$.*

*Proof.* Let $W \leq_{RHQ} Y$. So, $A_W(u) \leq A_Y(u)$ implies

$$\frac{1}{u}\int_0^u A_W(p)\log Q_W(p)q_W(p)dp \leq \frac{1}{u}\int_0^u A_Y(p)\log Q_Y(p)q_Y(p)dp.$$

Thus, $\log \bar{G}^W(Q_W(u)) \leq \log \bar{G}^Y(Q_Y(u))$. □

In the following example, we verify that the inequality obtained in the case of QPGVF need not be the same in the case of PGVF and vice versa.

**Examples 2.4.** Let $W$ and $Y$ follow the power distributions with pdf $f_W(w) = \alpha w^{\alpha-1}, \alpha > 0$ and $f_Y(y) = \beta y^{\beta-1}, \beta > 0, \alpha \geq \beta$, respectively. Using (1.1), we have

$$\log \bar{G}(W, t) = \left(\frac{-1 + \alpha \log t}{\alpha}\right) \geq \left(\frac{-1 + \beta \log t}{\beta}\right) = \log \bar{G}(Y, t).$$

Also, from Table 1, we have

$$\log \bar{G}^W(Q_W(u)) = \frac{1}{\alpha}\left[-1 + \log(u)\right] \leq \frac{1}{\beta}\left[-1 + \log(u)\right] = \log \bar{G}^Y(Q_Y(u)),$$

for $0 \leq u \leq 1$. Hence $W \leq_{QPGVF} Y$ does not imply that $W \leq_{QPGVF} Y$. Also, interchanging the roles of W and Y implies that $W \leq_{PGVF} Y$ does not lead to $W \leq_{PGVF} Y$.

Sunoj et al. (2013) defined quantile version of past entropy function (PQE) as

$$\bar{\psi}_{Q(u)} = 1 - \frac{1}{u}\int_0^u \log A(p)dp. \tag{2.13}$$

In the following theorem, we obtain relationship between the QPGVF and PQE.

**Theorem 2.6.** *Let $Y$ be a non-negative random variable with QPGVF $\log \bar{G}(Q(u))$ and PQE $\bar{\psi}_{Q(u)}$. The relationship*

$$\bar{\psi}_{Q(u)} - \log \bar{G}(Q(u)) = k, \tag{2.14}$$

*where k is a constant holds for all $u \in (0, 1)$ if and only if $Y$ follows power distribution with quantile function given in Table 1.*

*Proof.* Let (2.14) holds, using (2.13) and (2.4), we get

$$\int_0^u \log A(p)dp + \int_0^u \log Q(p)dp = (1 - k)u. \tag{2.15}$$

Differentiating (2.15) with respect to $u$, we have

$$A(u) = K_1[Q(u)]^{-1},$$

which is the reversed hazard function of power distribution and $K_1 = e^{1-k}$.
Conversely, let $Y$ follows power with quantile function given in Table 1. By direct calculation, we get

$$\bar{\psi}_{Q(u)} = \log \frac{\alpha}{\beta} + \frac{\beta - 1}{\beta} + \frac{1}{\beta}\log u,$$

and

$$\log \bar{G}(Q(u)) = \log \alpha + \frac{1}{\beta} - \frac{1}{\beta}\log u.$$

Now $\bar{\psi}_{Q(u)} - \log \bar{G}(Q(u)) = k = 1 - \log \beta$, is a constant. □

The relative performance of QPGVF with PQE for different life distributions are shown in Table 2.

Table 2: Comaprison of QPGVF and PQE measures on specific lifetime distributions.

| Distribution | Quantile functions | parameters | $\log \bar{G}(Q(u))$ | $\bar{\psi}_{Q(u)}$ |
|---|---|---|---|---|
| Pareto | $\sigma(1-u)^{\frac{-1}{c}}, u \in (0,1)$ | $c = 0.2, \sigma = 3$ and $u = 0.2$ | 1.6357 | 1.7431 |
| | | u=0.3 | 1.937 | 2.5106 |
| | | u=0.65 | 4.0868 | 6.0708 |
| | | u=0.8 | -1.9221 | -1.0177 |
| Log logistic | $\frac{1}{\alpha}(\frac{u}{1-u})^{\frac{1}{c}}$ | $\alpha = 5, c = 2$ and $u = 0.2$ | -2.8604 | -2.4462 |
| | | u=0.3 | -2.6276 | -2.1529 |
| | | u=0.65 | -2.1075 | -1.3659 |
| | | u=0.8 | -1.9221 | -1.0177 |
| Power | $\alpha u^{\frac{1}{\sigma}}$ | $\alpha = 8, \sigma$=0.2 and $u = 0.2$ | -10.9677 | -8.3583 |
| | | u=0.3 | -8.9404 | -6.3309 |
| | | u=0.65 | -5.0745 | -2.4650 |
| | | u=0.8 | -4.03628 | -1.4268 |

On critically examining the values obtained in Table 2, it can be seen that there are certain situations where the QPGVF is lower than the quantile-based past entropy measure.

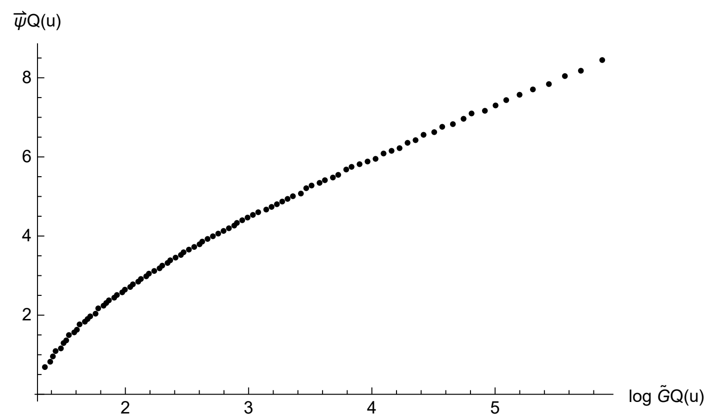


Figure 2: Graph of $\log \bar{G}(Q(u))$ against $\bar{\psi}_Q(u)$ for Pareto distribution.

In Figure 2 , we plot $\log \bar{G}(Q(u))$ against $\bar{\psi}_Q(u)$ for different choices of $u$. We can observe that QPGVF shows an increasing nature for various values of $u$. Furthermore, we can conclude that QPGVF increases accordingly when the uncertainty contained in the PQE shows an increasing nature.

## 3 Quantile-based PGVF of Order Statistics

Order statistics are widely applicable in many fields like reliability and survival analysis, quality control, goodness-of-fit tests, statistical inference and in probability and statistics. In this section, we discuss the QPGVF of order statistics. Suppose $Y_1, Y_2, ..., Y_n$ are independent and identically distributed (i.i.d) observations with cdf $F(y)$ and pdf $f(y)$. If we arrange the observations in the increasing order of magnitude as $Y_{1:n} \leq Y_{2:n} \leq ... \leq Y_{n:n}$, then the pdf of rth order statistic is given by

$$f_{r:n}(y) = \frac{1}{B(r, n-r+1)} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y), 1 \leq r \leq n, \tag{3.1}$$

where $B(m,k)$ denote the beta function given by

$$B(m,k) = \int_0^1 y^{m-1}(1-y)^{k-1} dy; \quad m, k > 0.$$

Suppose that $(n-r+1)$ - out- of- $n$ system is functioning at time $t$, then $\log \bar{G}_{r:n}(t)$ represents the geometric mean of lifetimes of systems in the past period $(0,t)$. Gayathri and Sathar (2021) defined PGVF of $r$-th order statistics as follows:

$$\log \bar{G}_{r:n}(t) = \frac{1}{F_{r:n}(t)} \int_0^t \log y f_{r:n}(y) dy, \tag{3.2}$$

where $F_{r:n}(t) = \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i}$. Then, we define QPGVF of $r$-th order statistics, $\log \bar{G}_{r:n}(Q(u))$ as

$$\log \bar{G}_{r:n}(Q(u)) = \frac{1}{F_{r:n}(Q(u))} \int_0^u \log Q(p)\, f_{r:n}(Q(p))\, dQ(p). \tag{3.3}$$

Putting $F(Q(u)) = u$, (3.1) becomes

$$\begin{aligned} f_{r:n}(Q(u)) &= \frac{1}{B(r, n-r+1)} u^{r-1}(1-u)^{n-r} f(Q(u)), \quad 1 \le r \le n \\ &= \frac{1}{B(r, n-r+1)} u^{r-1}(1-u)^{n-r} \frac{1}{q(u)} \\ &= \frac{g_r(u)}{q(u)}, \end{aligned} \tag{3.4}$$

where $g_r(u) = \frac{1}{B(r,n-r+1)} u^{r-1}(1-u)^{n-r}$ is the pdf of beta distribution. Using (3.4), (3.3) can be written as

$$\log \bar{G}_{r:n}(Q(u)) = \frac{B(r, n-r+1)}{B_u(r, n-r+1)} \int_0^u \log Q(p) g_r(p) dp, \tag{3.5}$$

where $\frac{B_u(r,n-r+1)}{B(r,n-r+1)}$ is the quantile form of $F_{r:n}(t)$ (see Nair et al. (2013)) with $B_u(r, n-r+1) = \int_0^u p^{r-1}(1-p)^{n-r} dp$, is the incomplete beta function. In the following theorem, we establish that $\log \bar{G}_{r:n}(Q(u))$ determines the corresponding quantile function uniquely.

**Theorem 3.1.** *Quantile-based past geometric vitality function of order statistics uniquely determines the quantile function.*

*Proof.* Differentiating (3.5) with respect to $u$, we get

$$\log \bar{G}'_{r:n}(Q(u)) = \frac{d}{du}[\log \bar{G}_{r:n}(Q(u))] = \frac{u^{r-1}(1-u)^{n-r}}{B_u(r, n-r+1)}[\log Q(u) - \log \bar{G}_{r:n}(Q(u))]. \tag{3.6}$$

Equation (3.6) can be simplified as

$$Q(u) = \exp\left[\log \bar{G}_{r:n}(Q(u)) + \frac{B_u(r, n-r+1)}{u^{r-1}(1-u)^{n-r}} \log \bar{G}'_{r:n}(Q(u))\right], \tag{3.7}$$

and hence the proof. □

The following example illustrates Theorem 3.1.

**Examples 3.1.** Suppose $Y$ follows the power distribution with the quantile function

$$Q(u) = \alpha u^{\frac{1}{\sigma}}, \ \alpha, \ \sigma > 0. \tag{3.8}$$

Then, the nth order QPGVF simplifies to

$$\log \bar{G}_{n:n}(Q(u)) = \log \alpha + \frac{1}{n\sigma}(-1 + n \log u). \tag{3.9}$$

Conversely, assume that (3.9) holds. Then, using (3.7) , we get

$$Q(u) = \exp\left[\log \bar{G}_{n:n}(Q(u)) + \frac{B_u(n,1)}{u^{n-1}} \log \bar{G}'_{n:n}(Q(u))\right],$$

which gives

$$Q(u) = \exp\left[\log \alpha + \frac{1}{n\sigma}(-1 + n \log u) + \frac{B_u(n,1)}{u^{n-1}} \frac{1}{\sigma u}\right].$$

On simplification, we get

$$Q(u) = \alpha u^{\frac{1}{\sigma}}.$$

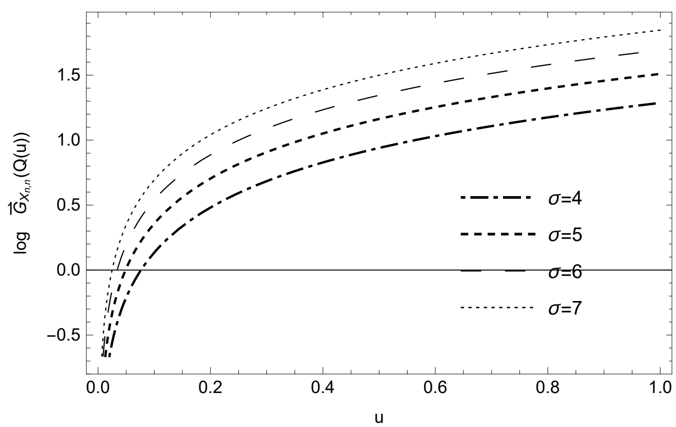


Figure 3: QPGVF of $n$-th order statistic arising from power distribution.

In Figure 3, we plot $\log \bar{G}_{n:n}(Q(u))$ against $u$ for the power distribution by fixing $\alpha = 2$ and $n = 5$ and for different values of $\sigma$ and $u$. Figure 3 shows that the QPGVF of $n$th order statistic has an increasing tendency as $u$ increases. Using $dF_{r:n}Q(p)$ instead of $f_{r:n}Q(p)$ in (3.3), we get

$$\log \bar{G}_{r:n}(Q(u)) = \log Q(u) - \frac{1}{B_u(r, n-r+1)} \int_o^u \frac{q(p)}{Q(p)} B_p(r, n-r+1) dp. \tag{3.10}$$

Differentiating (3.10) with respect to $u$ results in

$$A_{r:n}(u) = \frac{\frac{d}{du}\log \bar{G}_{r:n}(Q(u))}{q(u)\left[\log \bar{G}_{r:n}(Q(u)) - \log Q(u)\right]},$$

where $A_{r:n}(u) = \frac{f_{r:n}Q(u)}{F_{r:n}Q(u)}$, is the reversed hazard quantile based on order statistics.

In the following theorem we provide characterization of a random variable based on QPGVF in the context of order statistics.

**Theorem 3.2.** *The relationship* $\log \bar{G}_{n:n}(Q(u)) = a + bu$, *where* $a$ *and* $b$ *are real constants, holds if and only if the quantile function* $Q(u) = \theta e^{\lambda u}$, *where* $\lambda = \frac{n+1}{n} b$ *and* $\theta = e^a$.

*Proof.* The if part can be easily obatained from (3.5) and the 'only if' part can be proved using (3.7) as follows:

$$\begin{aligned} Q(u) &= \exp\left(\log \bar{G}_{n:n}(Q(u)) + \frac{B_u(n,1)}{u^{n-1}} \log \bar{G}'_{n:n}(Q(u))\right) \\ &= \exp\left(\log \bar{G}_{n:n}(Q(u)) + \frac{u}{n} \log \bar{G}'_{n:n}(Q(u))\right). \end{aligned}$$

On solving the above equation, we get

$$Q(u) = \theta e^{\lambda u}, \text{where } \theta = e^a \text{and } \lambda = \frac{n+1}{n} b.$$

□

The following theorem discusses the monotone property for the QPGVF with respect to different choices of $n$.

**Theorem 3.3.** *If* $\log Q(u)$ *is increasing in* $u$, *then* $\log \bar{G}_{n:n}(Q(u))$ *is nondecreasing in* $n \geq 1$.

*Proof.* By writing (3.3) for $r = n$, we have

$$\log \bar{G}_{n:n}(Q(u)) = \frac{1}{F_{n:n}(Q(u))} \int_0^u \log Q(p)\, f_{n:n}(Q(p))\, dQ(p).$$

Using (3.5), the above equation can be simplified to

$$\log \bar{G}_{n:n}(Q(u)) = \int_0^u \log Q(p) s^t_{n:n}(p) dQ(p),$$

where $s^t_{n:n}(p) = n\frac{(p)^{n-1}}{q(p)}\frac{B_p(n,1)}{B(n,1)}$, $p \leq u$ represents the pdf of $[Q_{Y_{n:n}}(p)|Q_{Y_{n:n}(p)} < u]$.
Hence we have

$$\log \bar{G}_{n:n}\,(Q(u)) = E[\log Q_{Y_{n:n}}(p)|Q_{Y_{n:n}}(p) < u]. \tag{3.11}$$

Similarly, we get $\log \bar{G}_{n+1:n+1}\,(Q(u)) = E[\log Q_{Y_{n+1:n+1}}(p)|Q_{Y_{n+1}:n+1}(p) < u]$.

Consider $\frac{s^t_{n:n}(p)}{s^t_{n+1:n+1}(p)}$ is decreasing on the interval $(0,u)$. We get the relation

$$[Q_{Y_{n:n}}(p)|Q_{Y_{n:n}}(p) < u] \leq_{lr} [Q_{Y_{n+1:n+1}}(p)|Q_{Y_{n+1:n+1}}(p) < u],$$

which results in that

$$[Q_{Y_{n:n}}(p)|Q_{Y_{n:n}}(p) < u] \leq_{st} [Q_{Y_{n+1:n+1}}(p)|Q_{Y_{n+1:n+1}}(p) < u].$$

Thus, we have

$$E[\log Q_{Y_{n:n}}(p)|Q_{Y_{n:n}(p)} < u] \leq E[\log Q_{Y_{n+1:n+1}}(p)|Q_{Y_{n+1:n+1}}(p) < u],$$

since $\log Q(u)$ is increasing in $u$. From (3.11), we get the desired result directly. □

**Counterexample 3.1.** Let Y follows the Pareto distribution with the quantile function $Q(u) = \sigma(1-u)^{\frac{-1}{\alpha}}$, $\alpha \geq 0, \sigma > 0$. Using (3.5), we have

$$\log \bar{G}_{1:n}(Q(u)) = \log \sigma - \frac{1}{n\alpha(1-u)^n} + \frac{1}{n\alpha}\left(1 - n\, log(1-u)\right).$$

By fixing $\sigma = 3$, $\alpha = 7$ and $u = 0.3$ and for different values of $n$, we get

$$\log \bar{G}_{1:5}(0.3) = 1.00814 > 0.772091 = \log \bar{G}_{1:9}(0.3).$$

This implies $\log \bar{G}_{1:n}(Q(u))$ is decreasing in n, even though $\log Q(u)$ is increasing in $u$.

*Remark* 2. From the above counterexample 3.1, it can be seen that although $\log Q(u)$ is increasing in $u$, the QPGVF evaluated for the smallest order statistic violates Theorem 3.3 and hence the result in Theorem 3.3 could not be generalized to $Y_{r:n}$.

Motivated by Di Crescenzo and Longobardi (2002), the order-based QPGVF is given through the following definition.

**Definition 3.1.** Let $W$ and $Y$ be two non-negative random variables then $W_{k:n} \leq_{QPGVF} Y_{k:n}$, if $\log \bar{G}^W_{k:n}(Q_W(u)) \leq \log \bar{G}^Y_{k:n}(Q_Y(u))$.

**Examples 3.2.** Suppose $W$ and $Y$ follow the power distribution with parameters $\beta_i = 1$ and $\alpha_i$, for i=1, 2 given in Table 1. We have $\log \bar{G}_{n:n}(Q(u)) = \log \alpha - \frac{1}{n\beta} + \frac{1}{\beta}\log(u)$. Now if $\alpha_1 < \alpha_2$, we obtain

$$\log \bar{G}^{W}_{n:n}(Q_W(u)) - \log \bar{G}^{Y}_{n:n}(Q_Y(u)) \leq 0.$$

which implies $W_{k:n} \leq_{QGVF} Y_{k:n}$.

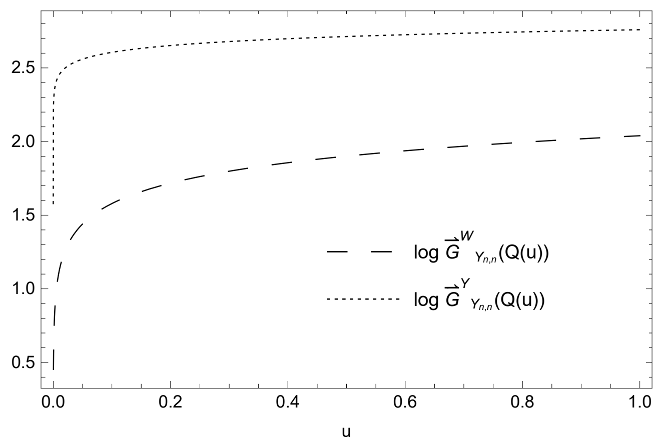


Figure 4: QPGVF of $n$-th order statistic arising from power distribution.

Figure 4 provides a plot for QPGVF of $n$-th order statistic for varying values of $u$ with respect to the parameter values $\alpha_1 = 5,\ \alpha_2 = 3\alpha_1$ and $\beta_1 = 8,\ \beta_2 = 2\beta_1$. From this figure, it is easily observed that $\log \bar{G}^{W}_{n:n}(Q_W(u))$ is less than $\log \bar{G}^{Y}_{n:n}(Q_Y(u))$.

**Theorem 3.4.** *The relationship* $\log \bar{G}_{n:n}(Q(u)) = \frac{u}{b}$, *where b is real constant, holds if and only if the quantile function is expressed as* $Q(u) = e^{\lambda u}$, *where* $\lambda = \frac{n+1}{nb}$.

*Proof.* The if part can be easilly obatained from (3.5) and the 'only if' part can be proved using (3.7), which reduces to $Q(u) = \exp\Big(\log \bar{G}_{n:n}(Q(u)) + \frac{u}{n}\log \bar{G}'_{n:n}(Q(u))\Big)$, and by solving we get the required Q(u). □

In the following theorem, we see how the monotonicity of QPGVF is affected by an increasing transformation.

**Theorem 3.5.** *If $W$ is IQPGVF and $\phi$ is a nonnegative, increasing and convex function, then $\phi(W)$ is also IQPGVF.*

*Proof.* Let $Y = \phi(W)$ be a non-negative, increasing and convex function. Then, the pdf of $Y$ is

$$g(y) = \frac{f((\phi)^{-1}(y))}{\phi'((\phi)^{-1}(y))} = \frac{1}{\phi'(Q(u))q(u)}.$$

Using (3.10), QPGVF of order $Y_{r:n}$ is given by

$$\log \bar{G}^{Y}_{r:n}(Q(u)) = \log Q_Y(u) - \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_Y(p)}{Q_Y(p)} B_p(r, n-r+1)dp, \tag{3.12}$$

which is equivalent to

$$\begin{aligned}
\log \bar{G}^{Y}_{r:n}(Q(u)) &= \log Q_W(u) - \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_W(p)\phi^{'}(Q_W(p))}{Q_W(p)} B_p(r, n-r+1)dp \\
&= \log Q_W(u) - \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_W(p)(\phi^{'}(Q_W(p)) - 1 + 1)}{Q_W(p)} B_p(r, n-r+1)dp \\
&= \log Q_W(u) - \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_Y(p)}{Q_Y(p)} B_p(r, n-r+1)dp + \\
&\quad \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_W(p)(1 - \phi^{'}(Q_W(p)))}{Q_W(p)} B_p(r, n-r+1)dp \\
&= \log \bar{G}^{W}_{r:n}(Q(u)) + \frac{1}{B_u(r, n-r+1)} \int_0^u \frac{q_W(p)(1 - \phi^{'}(Q_W(p)))}{Q_W(p)} B_p(r, n-r+1)dp.
\end{aligned}$$

Since $\phi(\cdot)$ is convex, $\phi^{'}(Q_W(p)) < \phi^{'}(Q_W(u))$, $0 < p < u$. Therefore, $\phi^{'}(Q_W(p))$ is increasing and non-negative. Moreover, by the assumption, $W$ is IQPGVF. Hence $\phi(W)$ is also IQPGVF. □

**Definition 3.2.** $W_{k:n}$ is smaller than $Y_{k:n}$ in dispersive order, denoted by $W_{k:n} \leq_{disp} Y_{k:n}$, if $Q_Y(u) - Q_W(u)$ is increasing in $u$.

In the following theorem, we discuss the connection between PQGVF and dispersive ordering.

**Theorem 3.6.** *If $W_{k:n} \leq_{disp} Y_{k:n}$, then $\log \bar{G}^{W}_{n:n}(Q_W(u)) \leq \log \bar{G}^{Y}_{n:n}(Q_Y(u))$.*

*Proof.* Assume that $W_{k:n} \leq_{disp} Y_{k:n}$. This implies that $Q_Y(u) - Q_W(u)$ is increasing in $u$. Using (3.5), we have

$$\frac{n}{u^n}\int_0^u \log Q_Y(u)p^{n-1}dp \geq \frac{n}{u^n}\int_0^u \log Q_W(u)p^{n-1}dp.$$

Hence the theorem. □

In the following theorem, we derive the expression for QPGVF under scalar transformation.

**Theorem 3.7.** *If* $Y_{r:n} = \frac{W_{r:n}}{b}$, *with* $b > 0$, *then* $\log \bar{G}^Y_{r:n}(Q_Y(u)) = \log \bar{G}^W_{r:n}(Q_W(u)) - \log b$.

*Proof.* Let $Y_{r:n} = \frac{W_{r:n}}{b}$, with b $>$ 0. Then

$$F_r(y) = P[\, Y_{r:n} \;\leq\; y] = P\left[\frac{W_{r:n}}{b} \;\leq\; y\right] = F_r\,(by)\,.$$

Thus, $Q_{Y_{r:n}}(u) = \frac{Q_{W_{r:n}}(u)}{b}$ and we have

$$\begin{aligned}\log \bar{G}^Y_{r:n}(Q_Y(u)) &= \frac{B(r,n-r+1)}{B_u(r,n-r+1)}\int_0^u \log Q_{Y_{r:n}}(p)g_r(p)dp\\ &= \frac{B(r,n-r+1)}{B_u(r,n-r+1)}\int_0^u \log \frac{Q_{W_{r:n}}(p)}{b}g_r(p)dp\\ &= \log \bar{G}^W_{r:n}(Q_W(u)) - \log b.\end{aligned}$$

□

# 4 Application

Wang (1998) used right tail deviation as a risk measure. It is well known that the quantile function $Q_Y(u)$ of a random variable $Y$ plays a significant role in comparing risks. Quantile-based measures can be used to measure risk in situations having no closed form survival function. This section uses the QPGVF ($\log \bar{G}Q(u)$) at a fixed value $u = u_0$ as a risk measure.

Wang (1998) defined the right tail deviation measure $D(Y)$ as

$$D(Y;r,t_0) = \int_{t_0}^{\infty}\left(\frac{\bar{F}(y)}{\bar{F}(t_0)}\right)^{\frac{1}{2}}dy - \int_{t_0}^{\infty}\frac{\bar{F}(y)}{\bar{F}(t_0)}dy. \tag{4.1}$$

Substituting $y = Q_Y(p)$ in (4.1), the quantile form of the right tail deviation measure reduces to

$$D(Y;r,u_0) = \int_{u_0}^{1} \left(\frac{1-p}{1-u_0}\right)^{\frac{1}{2}} q_Y(p)dp - \int_{u_0}^{1} \frac{1-p}{1-u_0} q_Y(p)dp. \tag{4.2}$$

In the following example, we consider some statistical models for comparing $\log \bar{G}Q(u_0)$, $D(Y;u_0)$ and variance $\sigma^2_{Q(u_0)}$.

**Examples 4.1.** For the Pareto distribution, we have $\log \bar{G}Q(u_0) = \log \alpha + \frac{1}{\beta} + \frac{1}{\beta}\frac{1-u_0}{u_0}\log(1-u_0)$, $D(Y;u_0) = \alpha((1-u_0)^{\frac{-1}{\beta}})(\frac{2}{\beta-2} - \frac{1}{\beta-1})$, for $\beta > 2$, and $\sigma^2_{Q(u_0)} = \frac{\alpha^2}{1-\frac{2}{\beta}} + \frac{2\alpha^2\beta}{(\beta-1)(1-\frac{1}{\beta})} + \frac{(\alpha\beta)^2}{(\beta-1)^2}$, for $\beta > 1$. Table 3 gives the numerical illustration of the risk measures for Pareto type I distribution for different choices of parameters.

Table 3: Numerical illustration of $\log \bar{G}Q(u_0)$ , $D(Y;r;u_0)$ and $(\sigma^2_{Q(u_0)})$ of Pareto type I distribution for different values of parameters

.

| $\alpha$ | $\beta$ | $\log \bar{G}Q(u_0)$ | $D(Y,u_0)$ | $\sigma^2_{Q(u_0)}$ |
|---|---|---|---|---|
| 1.5 | 0.75 | 2.5533 | 12.2149 | 18.9 |
| 2 | | 2.8409 | 16.2865 | 33.6 |
| 4 | | 3.5341 | 32.573 | 134.4 |
| 1.5 | 2.5 | 1.0498 | 7.2135 | 17.5 |
| 2 | | 1.3375 | 9.618 | 31.1111 |
| 4 | | 2.0306 | 19.236 | 124.444 |
| 1.5 | 3.5 | 0.8657 | 1.8189 | 9.66 |
| 2 | | 1.1535 | 2.4254 | 17.1733 |
| 4 | | 1.8465 | 4.8506 | 68.6933 |
| 1.5 | 4 | 0.8082 | 1.2574 | 8.5 |
| 2 | | 1.0959 | 1.6766 | 15.1111 |
| 4 | | 1.7890 | 3.3532 | 60.4444 |

From Table 3, we see that $\log \bar{G}Q(u_0)$ increases as $\alpha$ increases. At certain points in the interval $1 \le \beta \le 2$, the risk measures $D(Y;u_0)$ and variance do not exist but our proposed risk measure $\log \bar{G}Q(u_0)$ exists in the same interval and can be used for the same purpose.

## 5 Simulation Study and Application to Real Life Data

In this section, the quantile-based QPGVF is proposed for some distributions. However, based on the available real data and to keep the simulation study related to application part, we investigate the performance of the quantile-based QPGVF for the power distribution.

### 5.1 Simulation Study

We conducted simulation studies to investigate the efficiency of the QPGVF estimators of the largest order statistics for power distribution in terms of the average bias and mean squared error (MSE), based on sample sizes 10, 25, 100, 200 and 500 for different parameter combinations. The estimation of parameter $\beta$ was achieved using ML estimation, and the process was repeated 2000 times.

From the result of the simulation study (see Table 4 and Table 5), conclusions are drawn regarding the behaviour of the estimator in general, which are summarized below: (1) The ML estimates of $\log \bar{G}_{n:n}(Q(u))$ approaches to actual value when sample size $n$ increases. (2) When sample size $n$ increases, the MSE of $\log \bar{G}_{n:n}(Q(u))$ decreases.

### 5.2 Application to Real Life Data

The real data in this section represents 20 oral irrigators described in Jiang and Murthy (1998) for estimating the parameters of the model. The data values are: 1.175, 7.02, 7.58, 9.76, 15.02, 15.57, 17.39, 19.55, 22.47, 23.24,23.96, 25.05, 32.44, 36.87, 42.76, 43.14, 43.81, 46.95, 56.33, 56.68. We use this data for two primary purposes: (i) for investigating the performance of our QPGVF using the power distribution case and (ii) for comparing $\log \bar{G}_{n:n}(Q(u))$ with the quantile-based Tsallis entropy ($\bar{H}_{Y_{n:n}}$). The Tsallis entropy associated with $Y_{r:n}$ was proposed by Vikas Kumar and Rekha (2018) and is defined as

$$\bar{H}^{\alpha}_{Y_{r:n}} = \frac{1}{1-\alpha}\left(\frac{1}{B_u(r, n-r+1)}\int_0^u (g_r(p))^{\alpha}(q(p))^{1-\alpha}dp - 1\right).$$

Based on this data, we first used the maximum likelihood method to estimate the power distribution parameter, $\hat{\beta} = \frac{\sum_{i=1}^{n} log[u_i]}{n} = 3.03$. Then for different values of $u$, varied from 0.1 to 0.9, we calculate the estimated values of $\log \bar{G}_{n:n}(Q(u))$ and $\bar{H}_{Y_{n:n}}$ under power distribution. The results are displayed in Table 6. It should be noted that the estimated values of $\log \bar{G}_{n:n}(Q(u))$ increase with $u$. Also, the results in Table 6 clearly indicates that the estimated values based on $\log \bar{G}_{n:n}(Q(u))$ are less than those given by $\bar{H}_{Y_{n:n}}$.

Table 4: Average estimates, Bias and MSE for $\log \bar{G}_{n:n}(Q(u))$ under power distribution for different values of $\lambda$ and fixed values of $\beta$ =0.3

| n | Criterion | $\lambda = 0.2, \beta = 0.3$ | $\lambda = 0.8, \beta = 0.3$ | $\lambda = 1.6, \beta = 0.3$ |
|---|---|---|---|---|
| 10 | $E(\log \bar{G}_{n:n}(Q(u)))$ | -1.48053 | -0.10244 | 0.58852 |
| | Bias | -0.86356 | -0.85536 | -0.85317 |
| | MSE | 0.74641 | 0.73201 | 0.72819 |
| 25 | $E(\log \bar{G}_{n:n}(Q(u)))$ | -1.55748 | -0.17510 | 0.51743 |
| | Bias | -0.34582 | -0.34190 | -0.34129 |
| | MSE | 0.11973 | 0.11696 | 0.11653 |
| 100 | $E(\log \bar{G}_{n:n}(Q(u)))$ | -1.59643 | -0.21107 | 0.48184 |
| | Bias | -0.08647 | -0.08554 | -0.08529 |
| | MSE | 0.00749 | 0.00732 | 0.00728 |
| 200 | $E(\log \bar{G}_{n:n}(Q(u)))$ | -1.64617 | -0.21707 | 0.47590 |
| | Bias | -0.04324 | -0.04281 | -0.04263 |
| | MSE | 0.00187 | 0.00183 | 0.00181 |
| 500 | $E(\log \bar{G}_{n:n}(Q(u)))$ | -1.60686 | -0.22072 | 0.47236 |
| | Bias | -0.34190 | -0.17111 | -0.01705 |
| | MSE | 0.000298 | 0.000292 | 0.000290 |

Table 5: Average estimates, Bias and MSE for $\log \bar{G}_{n:n}Q(u)$ under power distribution for different values of $\beta$ and fixed valuse of $\lambda$ =1.2

| n | Criterion | $\lambda = 1.2, \beta = 0.3$ | $\lambda = 1.2, \beta = 0.9$ | $\lambda = 1.2, \beta = 1.8$ |
|---|---|---|---|---|
| 10 | $E(\log \bar{G}_{n:n}Q(u))$ | 0.301242 | 0.321468 | 0.337189 |
| | Bias | -0.853578 | -0.384033 | -0.277315 |
| | MSE | 0.728914 | 0.148702 | 0.079216 |
| 25 | $E(\log \bar{G}_{n:n}Q(u))$ | 0.230024 | 0.238206 | 0.243773 |
| | Bias | -0.341566 | -0.153838 | -0.110429 |
| | MSE | 0.116718 | 0.023907 | 0.012602 |
| 100 | $E(\log \bar{G}_{n:n}Q(u))$ | 0.194321 | 0.196289 | 0.190005 |
| | Bias | -0.085343 | -0.038456 | -0.027715 |
| | MSE | 0.007287 | 0.001494 | 0.000797 |
| 200 | $E(\log \bar{G}_{n:n}Q(u))$ | 0.188307 | 0.189351 | 0.190005 |
| | Bias | -0.040272 | -0.0192738 | -0.013806 |
| | MSE | 0.001826 | 0.000375 | 0.000198 |
| 500 | $E(\log \bar{G}_{n:n}Q(u))$ | 0.184715 | 0.185073 | 0.184699 |
| | Bias | 0.017086 | 0.007649 | 0.005527 |
| | MSE | 0.000292 | 0.000059 | 0.000031 |

Table 6: Estimates of $\log \hat{\hat{G}}_{n:n}(Q(u))$ and $\hat{H}_{Y_{n:n}}$ for power distribution for different values of u.

| u | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| $\log \hat{\hat{G}}_{n:n}Q(u)$ | 1.380 | 1.607 | 1.742 | 1.837 | 1.911 | 1.971 | 2.022 | 2.066 | 2.105 |
| $\hat{\hat{H}}_{Y_{n:n}}$ | 5.916 | 6.722 | 7.229 | 7.605 | 7.691 | 8.159 | 8.379 | 8.572 | 8.746 |

## 6 Conclusion

We have introduced past GVF of order statistics in terms of quantile function. Also, we have established that the quantile-based past GVF of order statistics determines the quantile function uniquely using a simple relationship between quantile function and quantile-based past GVF of order statistics. The performance of the quantile-based past GVF is investigated by simulation studies and using real data applications. We found that the ML estimate of the quantile-based GVF of order statistics approaches true value when sample size $n$ increases for the simulation part. Further in the application part, we have compared our quantile-based past GVF of order statistics with the existing Tsallis entropy measure and the result showed that the quantile-based past GVF is smaller compared to Tsallis entropy.

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