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Analysis of Dependent Competing Risk Model in the Presence of Joint Type-II Censoring Using Bivariate Marshll-Olkin Family

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Abstract. Lifetime data has several applications in different fields such as Biology and Engineering. Failures for this type of data may occur due to several causes. In real world, causes of failures are interacting together which violates the independency assumption. Once dependency between failures is satisfied, bivariate families should be used to analyze the data. In literature, the majority of studies handle the case when data come from one source. However, in real life, data could come from different sources. One way to analyze data from different sources together and reduce the time and cost of the experiment is joint type-II censoring. To the best of our knowledge, joint type-II censoring was not yet derived using bivariate lifetime distributions. In this paper, we derive the likelihood function of joint type-II censoring using bivariate family in the presence of dependent competing risks. A simulation study is performed and two real datasets are analyzed.

Keywords. Bayesian, Bivariate Inverted Kumaraswamy Distribution, Bivariate Marshall-

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1 Introduction

Rapid enhancement in technology results in producing multi component machines. Failures of those complex structure machines can occur from different causes or factors which can be analyzed using competing risk models. Selection of the appropriate model depends on the relation between causes of failures.

Previous studies have mostly consider causes of failure to be independent. Once independence between causes is assumed, univariate distributions are considered to model lifetime data. For example, Kundu and Sarhan (2006) used Weibull distribution, Sarhan (2007) considered Generalized exponential distribution, Sarhan *et al* (2010) used exponential, Weibull and Chen distributions and Mazucheli and Achcar (2011) considered Lindley distribution.

The independence assumption is not always realistic. For example, the failure of one of a plane engine causes more load on the second engine which increases the risk of failure for this engine and the plane. In medical field, a patient with a disease that causes the blindness of one eye, would increase the pressure on the other eye and hence increase its risk of blindness. Accordingly, it is more reliable to consider dependence between causes of failures. For more details see Shen and Xu (2018).

Once dependent competing risk model is considered, bivariate distributions must be used as lifetime models. Since simultaneous occurrence of failure has positive probability, bivariate Marshall-Olkin family is widely used. Few works were performed under dependent competing risk model. Feizjavadian and Hashemi (2015) considered dependent competing risk model under progressive hybrid censoring using Marshall-Olkin bivariate Weibull distribution (MOBW). Shen and Xu (2018) studied dependent competing risk models under complete sampling assuming MOBW distribution. Bai et al. (2020) discussed statistical inference of constant-stress accelerated dependent competing risk model under Type II hybrid censoring assuming MOBW lifetime distribution. Wang et al. (2020) illustrated the statistical inference of Marshall-Olkin bivariate Kumaraswamy distribution under generalized pregressive hybrid censoring.

Previous studies handled the case when all data come from one source. However, in real world, data may come from two different sources. For example, suppose we have patients with a disease that causes the blindness of eyes from two different hospitals. To save time, we can analyze all data together after observing a prespecified number of failures (r). As an example in engineering, consider the units which are being manufactured by two different production lines within the same facility. To save time, the producer may like to analyze all data together. Also, he would like to reduce the cost by analyzing only part of the sample. To do so, one may use joint type-II censoring. This type of censoring analyze all data together and save cost by terminating the experiment after the occurrence of a pre-specified number of failures (r).

Joint type-II censoring can be explained as follows:

Let $\{x_1, ..., x_m\}$ be the lifetime of m specimens of production line A (or lifetimes of m patients from hospital A) with *pdf* $f(x)$ and *cdf* $F(x)$ and $\{y_1, ..., y_n\}$ be the lifetime of n specimens of production line B (or lifetimes of n patients from hospital B) with *pdf* $g(y)$ and *cdf* $G(y)$. Now, define $\{w_1 < w_2 < ... < w_N\}$ to be the order statistics of $\{x_1, ..., x_m, y_1, ..., y_n\}$, $N = n + m$. Then, under joint type-II censoring, the experiment is terminated after the occurrence of a predetermined number of failures (r). The likelihood function is as follows (for more details, see Balakrishnan and Rasouli (2008))

$$L = \frac{m!n!}{(m - m_r)(n - n_r)} \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{1-z_i} [1 - F(w_r)]^{m-m_r} [1 - G(w_r)]^{n-n_r}, \tag{1.1}$$

where m_r is the number of x failures in w, n_r is the number of y failures in w and

$$z_i = \begin{cases} 1, & \text{if } w_i \text{ from } x, \\ 0, & \text{if } w_i \text{ from } y. \end{cases}.$$

Dealing with univariate lifetime distributions, several work considered joint type-II censoring, using different lifetime distributions and different estimation techniques. See for example, Balakrishnan and Rasouli (2008), Ashour and Abo-Kasem (2014), Balakrishnan and Su (2015) and Abu-Zinadah (2017).

To the best of our knowledge, this type of censoring was not yet derived in the bivariate lifetime distributions. Here, we are interested in joint Type-II censoring under dependent competing risk model represented by a bivariate lifetime distribution.

The paper is organized as follows. In Section 2, the model is described. In Section

3, derivation of the likelihood function under joint type-II censoring in presence of dependent competing risk model is illustrated in details. Maximum likelihood and Bayesian approaches are used to estimate the unknown parameters in Section 4. In Section 5, a simulation study is performed and two real datasets are analyzed. Finally, the paper is concluded in Section 6.

2 The Model

In this Section, we illustrate the bivariate Marshall-Olkin family, the bivariate inverted Kumaraswamy (BIK) distribution and the form of the likelihood function under Type-II censoring in the presence of dependent competing risk model.

2.1 Bivariate Marshall-Olkin Family

Marshall and Olkin (1967) presented a bivariate exponential distribution with exponential marginals and loss of memory property. This type of bivariate distribution has the following form:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & if \quad x_1 < x_2 \\ f_2(x_1,x_2) & if \quad x_2 < x_1 \\ f_3(x) & if \quad x_1 = x_2 = x \end{cases}, \tag{2.1}$$

which has both an absolutely continuous part (i.e $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$) and singular part ($f_3(x)$).

Marshall-Olkin family has several applications in reliability engineering field such as shock model (see, Marshall and Olkin (1967)), maintenance model and stress model (Kundu and Gupta (2009)). Also, Marshall and Olkin (1967) illustrated some examples in which the case of $x_1 = x_2$ has positive probability. For example, if x_1 and x_2 are lifetimes, then the case of $x_1 = x_2$ may occur due to a simultaneous shock. Also, the case of $x_1 = x_2$ may occur if a jet engine explodes and the adjacent engine is destroyed by the explosion.

In this paper, we are applying the likelihood function on the bivariate inverted Kumaraswamy distribution (BIK) as a member of the bivariate Marshall-Olkin family. However, any other distribution, such as BMOW, can be used.

Aly and Abuelamayem (2020) derived bivariate and multivariate inverted Kumaraswamy distribution and illustrated its applications in different fields like engineering, medicine and sports. For example, as a lifetime distribution it can be used in reliability and life testing problems. Also it has applications in Maintenance models, shock models and failure rate models in medical research and biological studies such as frailty model. Moreover, it can be used in engineering studies like degradation of mechanical components. (For more details see, Aly and Abuelamayem (2022)). Moreover, In literature, BIK gave better results when applied in real datasets compared to bivariate generalized exponential, bivariate inverted Weibull and bivariate exponential distributions (For more details see, Aly and Abuelamayem (2020) and Aly and Abuelamayem (2022)). Here, we use the same methodology but using minimum instead of maximum in deriving the distribution. The joint survival function of (X_1, X_2) has the following form:

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} S_1(x_1,x_2) & if \quad x_1 < x_2 \\ S_2(x_1,x_2) & if \quad x_2 < x_1 \\ S_3(x) & if \quad x_1 = x_2 = x \end{cases}, \tag{2.2}$$

where

$$\begin{aligned} S_1(x_1,x_2) &= S_{IK}(x_1,\beta_1,\alpha)S_{IK}(x_2,\beta_2+\beta_3,\alpha), \\ S_2(x_1,x_2) &= S_{IK}(x_1,\beta_1+\beta_3,\alpha)S_{IK}(x_2,\beta_2,\alpha), \\ S_3(x) &= S_{IK}(x,\beta_1+\beta_2+\beta_3,\alpha), \\ S_{IK}(x,\alpha,\beta) &= 1-(1-(1+x)^{-\alpha})^{\beta}, x>0, \alpha>0, \beta>0. \end{aligned} \tag{2.3}$$

The joint *pdf* can be written as illustrated in (2.1) as follows:

$$\begin{aligned} f_1(x_1,x_2) &= f_{IK}(x_1,\beta_1,\alpha)f_{IK}(x_2,\beta_2+\beta_3,\alpha), \\ f_2(x_1,x_2) &= f_{IK}(x_1,\beta_1+\beta_3,\alpha)f_{IK}(x_2,\beta_2,\alpha), \\ f_3(x) &= \frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_{IK}(x,\beta_1+\beta_2+\beta_3,\alpha), \\ f_{IK}(x,\alpha,\beta) &= \alpha\beta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta-1}, \quad x>0, \ \ \alpha>0, \ \ \beta>0. \end{aligned} \tag{2.4}$$

2.2 Likelihood Function under Type-II Censoring

Let $\{(x_{11}, x_{21}), ..., (x_{1m}, x_{2m})\}$ be the lifetime of m specimens from production line A with p.d.f $f_{X_1,X_2}(x_1, x_2)$ and survival function $S_{X_1,X_2}(x_1, x_2)$. Let $u_i = min(x_{1i}, x_{2i})$, $i = 1, ..., m$. The likelihood function under type II censoring in presence of dependent competing risk model is derived using Feizjavadian and Hashemi (2015) as follows:

$$\begin{aligned} L \quad &\propto \quad \prod_{i=1}^{r} [f_{X_1,X_2}(u_i, u_i)]^{\delta_{i0}} [-\frac{\partial}{\partial x_1} S_{X_1,X_2}(x_1, x_2)|_{(u_i,u_i)}]^{\delta_{i1}} [-\frac{\partial}{\partial x_2} S_{X_1,X_2}(x_1, x_2)|_{(u_i,u_i)}]^{\delta_{i2}} \\ &\text{x} \quad [S_{X_1,X_2}(u_i, u_i)]^{m-r}, \end{aligned} \tag{2.5}$$

where

$$\delta_{i0} = \begin{cases} 1, & \text{if } x_1 = x_2, \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{i1} = \begin{cases} 1, & \text{if } x_1 < x_2, \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{i2} = \begin{cases} 1, & \text{if } x_1 > x_2, \\ 0, & \text{otherwise.} \end{cases},$$

and r is a predetermined number of failures.

3 Derivation of the Likelihood Function under Joint Type-II Censoring

In this Section, we illustrate the derivation of the likelihood function under joint type-II censoring in the presence of dependent competing risk model.

Let $\{(x_{11}, x_{21}), ..., (x_{1m}, x_{2m})\}$ be the lifetimes of m specimens from production line A (or lifetimes of m patients from hospital A) with *pdf* $f_{X_1,X_2}(x_1, x_2, \beta_1, \beta_2, \beta_3, \alpha)$ and survival function $S_{X_1,X_2}(x_1, x_2)$. Suppose $\{(y_{11}, y_{21}), ..., (y_{1n}, y_{2n})\}$ are the lifetimes of n specimens from production line B (or lifetimes of n patients from hospital B) with the same facilities as A, *pdf* $g_{Y_1,Y_2}(y_1, y_2, \beta_1^*, \beta_2^*, \beta_3^*, \alpha^*)$ and survival function $G_{Y_1,Y_2}(y_1, y_2)$.

For competing risk model, define $x_i^* = min(x_{1i}, x_{2i})$, $i = 1, ..., m$, and $y_j^* = min(y_{1j}, y_{2j})$, $j = 1, ..., n$ and let $w_1 < w_2 < ... < w_N$ are the order statistics of $\{x_1^*, ..., x_m^*, y_1^*, ..., y_n^*\}$ and $N = n + m$.

Now, to save cost and time, we apply joint type-II censoring and end the experiment after r failures and observe $\{w_1, ..., w_r\}$, $\{z_1, ..., z_r\}$, where

$$z_k = \begin{cases} 1, & \text{if failures from} x^*, \\ 0, & \text{if failures from} y^*. \end{cases},$$

$k = 1, ..., r$.

The dataset under the presented type of censoring consists of $\{(w_1, z_1, \delta_{10}, \delta_{11}, \delta_{12}), ..., (w_r, z_r, \delta_{r0}, \delta_{r1}, \delta_{r2})\}$, n, m, r, n_r and m_r, where

$$\delta_{k0} = \begin{cases} 1, & \text{if } x_1 = x_2, \\ & \text{or } y_1 = y_2 \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{k1} = \begin{cases} 1, & \text{if } x_1 < x_2, \\ & \text{or } y_1 < y_2 \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{k2} = \begin{cases} 1, & \text{if } x_1 > x_2, \\ & \text{or } y_1 > y_2 \\ 0, & \text{otherwise.} \end{cases}.$$

r is the predetermined number of failures, m_r is the total number of failures from x^* in w and n_r is the total number of failures from y^* in w.

The derived likelihood function is as follows:

$$\begin{aligned} L \quad \propto \quad & \prod_{k=1}^{r} \{[f_{X_1,X_2}(w_k, w_k)]^{z_k} [g_{Y_1,Y_2}(w_k, w_k)]^{1-z_k}\}^{\delta_{k0}} \\ & \text{x} \quad \{[-\frac{\partial}{\partial x_1} S_{X_1,X_2}(x_1, x_2)|_{(w_k,w_k)}]^{z_k} [-\frac{\partial}{\partial y_1} G_{Y_1,Y_2}(y_1, y_2)|_{(w_k,w_k)}]^{1-z_k}\}^{\delta_{k1}} \\ & \text{x} \quad \{[-\frac{\partial}{\partial x_2} S_{X_1,X_2}(x_1, x_2)|_{(w_k,w_k)}]^{z_k} [-\frac{\partial}{\partial y_2} G_{Y_1,Y_2}(y_1, y_2)|_{(w_k,w_k)}]^{1-z_k}\}^{\delta_{k2}} \\ & \text{x} \quad [S_{X_1,X_2}(w_r, w_r)]^{m-m_r} [G_{Y_1,Y_2}(w_r, w_r)]^{n-n_r}. \end{aligned} \tag{3.1}$$

Applying the previous likelihood function on bivariate inverted Kumaraswamy distribution, we have:

$$\begin{aligned} f_{X_1,X_2}(w_k, w_k) &= \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} f_{IK}(w_k, \alpha, \beta_1 + \beta_2 + \beta_3), \\ g_{Y_1,Y_2}(w_k, w_k) &= \frac{\beta_3^*}{\beta_1^* + \beta_2^* + \beta_3^*} f_{IK}(w_k, \alpha^*, \beta_1^* + \beta_2^* + \beta_3^*), \\ -\frac{\partial}{\partial x_1} S_{X_1,X_2}(x_1, x_2)|_{(w_k,w_k)} &= f_{IK}(w_k, \alpha, \beta_1) S_{IK}(w_k, \alpha, \beta_2 + \beta_3), \\ -\frac{\partial}{\partial y_1} G_{Y_1,Y_2}(y_1, y_2)|_{(w_k,w_k)} &= f_{IK}(w_k, \alpha^*, \beta_1^*) S_{IK}(w_k, \alpha^*, \beta_2^* + \beta_3^*), \\ -\frac{\partial}{\partial x_2} S_{X_1,X_2}(x_1, x_2)|_{(w_k,w_k)} &= f_{IK}(w_k, \alpha, \beta_2) S_{IK}(w_k, \alpha, \beta_1 + \beta_3), \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial}{\partial y_2} G_{Y_1,Y_2}(y_1,y_2)|_{(w_k,w_k)} &= f_{IK}(w_k,\alpha^*,\beta_2^*) S_{IK}(w_k,\alpha^*,\beta_1^*+\beta_3^*),\\
S_{X_1,X_2}(w_r,w_r) &= S_{IK}(w_r,\alpha,\beta_1+\beta_2+\beta_3),\\
G_{Y_1,Y_2}(w_r,w_r) &= S_{IK}(w_r,\alpha^*,\beta_1^*+\beta_2^*+\beta_3^*), \qquad (3.2)
\end{aligned}
$$

where f_{IK} and S_{IK} are as defined in equations (2.1) and (2.2).

The likelihood function is as follows:

$$
\begin{aligned}
L \propto & \prod_{k=1}^{r} \{[\alpha\beta_3(1+w_k)^{-(\alpha+1)}(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_2+\beta_3-1}]^{z_k}\\
& \text{x } [\alpha^*\beta_3^*(1+w_k)^{-(\alpha^*+1)}(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_2^*+\beta_3^*-1}]^{1-z_k}\}^{\delta_{k0}}\\
& \text{x } \{[\alpha\beta_1(1+w_k)^{-(\alpha+1)}(1-(1+w_k)^{-\alpha})^{\beta_1-1}(1-(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3})]^{z_k}\\
& \text{x } [\alpha^*\beta_1^*(1+w_k)^{-(\alpha^*+1)}(1-(1+w_k)^{-\alpha^*})^{\beta_1^*-1}(1-(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*})]^{1-z_k}\}^{\delta_{k1}}\\
& \text{x } \{[\alpha\beta_2(1+w_k)^{-(\alpha+1)}(1-(1+w_k)^{-\alpha})^{\beta_2-1}(1-(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3})]^{z_k}\\
& \text{x } [\alpha^*\beta_2^*(1+w_k)^{-(\alpha^*+1)}(1-(1+w_k)^{-\alpha^*})^{\beta_2^*-1}(1-(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*})]^{1-z_k}\}^{\delta_{k2}}\\
& \text{x } [1-(1-(1+w_r)^{-\alpha})^{\beta_1+\beta_2+\beta_3}]^{m-m_r}[1-(1-(1+w_r)^{-\alpha^*})^{\beta_1^*+\beta_2^*+\beta_3^*}]^{n-n_r}. \qquad (3.3)
\end{aligned}
$$

4 Estimation

In this section, we estimate the unknown parameters using maximum likelihood and Bayesian approaches.

4.1 Maximum Likelihood Approach

In this sub-section, the maximum likelihood estimators (MLEs) of the unknown parameters are obtained by maximizing the logarithm of the likelihood function derived in Section 3. The derivatives of the log-likelihood with respect to the unknown parameters are illustrated in the Appendix. Maximum likelihood estimators can not be obtained in closed form, hence numerical analysis using monte carlo simulation is used. R program is used to do the numerical analysis using optim function and L-BFGS-B method which is a limited-memory modification of the BFGS quasi-Newton method. For more details see Byrd et al. (1995). Using the asymptotic distribution of the maximum likelihood estimators, the confidence intervals can be obtained as follows;

$$
\hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{\hat{var}(\hat{\lambda})}, \qquad (4.1)
$$

where $\hat{\lambda} = \hat{\beta_1}, \hat{\beta_2}, \hat{\beta_3}, \hat{\alpha}, \hat{\beta_1^*}, \hat{\beta_2^*}, \hat{\beta_3^*}, \hat{\alpha^*}$, $\hat{var}(\hat{\lambda})$ is the estimated variance, and $z_{\frac{\gamma}{2}}$ is the upper $\gamma/2$-th percentile of the standard normal distribution.

4.2 Bayesian Approach

Let (X_1, X_2) be two random variables from BIK($\underline{\theta}$), where $\underline{\theta} = (\alpha, \beta_1, \beta_2, \beta_3)$ is the vector of unknown parameters. Also assume that (Y_1, Y_2) be two random variables from BIK($\underline{\theta}^*$), where $\underline{\theta}^* = (\alpha, \beta_1^*, \beta_2^*, \beta_3^*)$ is a second vector of unknown parameters. The posterior pdf can be obtained as follows

$$P(\underline{\theta}, \underline{\theta}^*|(X_1, X_2, Y_1, Y_2)) \propto L(X_1, X_2, Y_1, Y_2; \underline{\theta}, \underline{\theta}^*)P(\underline{\theta})P(\underline{\theta*}), \tag{4.2}$$

where L is defined in equation (3.1), $P(\underline{\theta})$ and $P(\underline{\theta}^*)$ are the prior distributions.

Here, we considered a gamma prior distribution with the following pdf

$$P(\beta_i, a_i, b_i) \propto \beta_i^{a_i - 1} exp(-b_i\beta_i). \tag{4.3}$$

It can be seen that the Bayes estimators can not be obtained in explicit forms under square error loss function. Therefore, we obtain the posterior mean using MCMC method which is illustrated in the next section.

5 Numerical Results

In this Section, a simulation study is carried out to investigate the performance of the derived likelihood function under different schemes using different arbitrary chosen values of n, m, r and using different initial values. Moreover, an example and two real data sets are analyzed.

5.1 Simulation Results

In this subsection, we illustrate the simulation steps and the estimates obtained using the ML and Bayesian methods.

The following steps are applied in the simulation studies:

Step 1: Generate a sample $\{(x_{11}, x_{21}), ..., (x_{1m}, x_{2m})\}$ from BIK distribution (similar to Aly and Abuelamayem (2020)) as follows:

a) Generate U_1, U_2 and U_3 from uniform(0,1).

b) Compute $T_1 = (1 - U_1^{\beta_1})^{\frac{1}{\alpha}}, T_2 = (1 - U_2^{\beta_2})^{\frac{1}{\alpha}}$ and $T_3 = (1 - U_3^{\beta_3})^{\frac{1}{\alpha}}$.

c) Define $Z_1 = \frac{1}{T_1} - 1, Z_2 = \frac{1}{T_2} - 1$ and $Z_3 = \frac{1}{T_3} - 1$.

d) Obtain $X_1 = \min(Z_1, Z_3)$ and $X_2 = \min(Z_2, Z_3)$.

Use the same steps to generate a second sample of size n $\{(y_{11}, y_{21}),..., (y_{1n}, y_{2n})\}$.

Step 2: Define $x_i^* = min(x_{1i}, x_{2i}), i = 1, ..., m$ and $y_j^* = min(y_{1j}, y_{2j}), j = 1, ..., n$.

Step 3: Let $w_1, ..., w_N$ be the order statistics of $x_1^*, ..., x_m^*, y_1^*, ..., y_n^*, N = n + m$.

Step 4: Define the number of failures r.

Step 5: Maximize the likelihood function in section 3.

Step 6: Repeat the five previous steps 5000 times (for ML).

Here, the following schemes are considered:

- $\beta_2 = 0.9$, β_3=2.6, $\beta_2^* = 0.8$, β_3^*=1.9.
- $\beta_2 = 0.4$, β_3=1.4, $\beta_2^* = 0.4$, β_3^*=1.2.
- $\beta_2 = 1.3$, β_3=3.7, $\beta_2^* = 1.2$, β_3^*=3.
- $\beta_2 = 0.6$, β_3=1.5, $\beta_2^* = 0.9$, β_3^*=2.3.

Maximum Likelihood Approach

Simulation results are obtained using R package with 5000 replications. The results are explained in Tables 1 and 2. Absolute Bias (ABias), mean square error (MSE), confidence width (CW) and coverage probability (CP) are obtained in each table. For simplicity, β_1, β_1^*, α and α^* are considered to be known.

Bayesian Approach

Using Bayesian approach, we need to obtain the posterior mean. However, it is hard to obtain it theoretically as we have four parameters to estimate. One can use MCMC simulation technique to obtain it numerically. MCMC method uses simulation techniques to obtain a Markov sequence such that they have a limiting distribution. Here, MCMC method can be used to set up a Markov chain of parameters θ with distribution $P(\theta|(X))$. Therefore, the mean of the sequence can be considered as the posterior mean.

To perform MCMC, we used WinBugs package. Gamma prior is used with the same sample sizes and usage rate used in ML approach. WinBugs is used with 1000

replications to generate the sequence of Markov chain. Absolute bias (ABias), mean square error (MSE) and credible interval width (CW) are obtained and presented in Tables 3 and 4. For analyzing the results,

First we consider the case when $n = m = 50$. Let $k = \frac{r}{n+m}$ be the rate of used data from the whole sample, we have the following values of k for Table 1.

- Set 1: $k = 0.95$, $r = 95$.
- Set 2: $k = 0.9$, $r = 90$.
- Set 3: $k = 0.85$, $r = 85$.
- Set 4: $k = 0.8$, $r = 80$.
- Set 5: $k = 0.75$, $r = 75$.

For ML approach, it can be seen that for all four schemes and for all four different parameters, as k (r) increases, the ABias and MSE decrease and CP increases. Also for the majority of cases, as k (r) increases CW deceases. For example, in Table 1 , using the first scheme ($\beta_2 = 0.9$, β_3=2.6, $\beta_2^* = 0.8$, β_3^*=1.9.), it can be seen that ABias for β_2 decreases from 0.209 at k =0.75 ($r = 75$) to 0.006 at k =0.95 ($r = 95$). Also, MSE decreases from 0.168 at k =0.75 ($r = 75$) to 0.095 at k =0.95 (r=95). Moreover, the CP increases from 95.7% at k =0.75 ($r = 75$) to 99% at k =0.95 ($r = 95$). Finally, CW decreases from 1.380 at $k = 0.75$ ($r = 75$) to 1.208 at $k = 0.95$ ($r = 95$). Also, using the same example, it can be seen that ABias for β_3^* decreases from 0.201 at $k = 0.75$ ($r = 75$) to 0.036 at $k = 0.95$ ($r = 95$). MSE defreases from 0.348 at k =0.75 ($r = 75$) to 0.318 at $k = 0.95$ ($r = 95$). The CP increases from 93.2 % at k =0.75 ($r = 75$) to 97.7 % at k =0.95 ($r = 95$). Similarly, the same conclusion can be obtained for β_3 and β_2^*.

For Bayesin approach, it can be noticed that for all four different used schemes and for all four different parameters, in more than half of the cases as k (r) increases, the ABias, MSE, CW decrease and CP increases. For example, in Table 3, using the third scheme ($\beta_2 = 1.3$, β_3=3.7, $\beta_2^* = 1.2$, β_3^*=3.), it can be seen that ABias for β_3 decreases from 0.051 at k =0.75 ($r = 75$) to 0.001 at k =0.95 ($r = 95$). Also, MSE decreases from 0.010 at k =0.75 ($r = 75$) to 0.007 at k =0.95 (r=95). Finally, CW decreases from 0.344 at $k = 0.75$ ($r = 75$) to 0.340 at $k = 0.95$ ($r = 95$). Also, using the same example, it can be seen that ABias for β_2^* decreases from 0.072 at $k = 0.75$ ($r = 75$) to 0.047 at $k = 0.95$ ($r = 95$) and MSE defreases from 0.009 at k =0.75 ($r = 75$) to 0.006 at $k = 0.95$ ($r = 95$). Finally, CW decreases from 0.241 at $k = 0.75$ ($r = 75$) to 0.232 at $k = 0.95$ ($r = 95$)

Second, we consider the case when n and m are different (i.e. $n = 40$ and $m = 30$), Table 2, the following values for k are considered

- Set 1: $k = 0.93$, $r = 65$.
- Set 2: $k = 0.86$, $r = 60$.
- Set 3: $k = 0.77$, $r = 55$.
- Set 4: $k = 0.71$, $r = 50$.
- Set 5: $k = 0.64$, $r = 45$.

For ML approach, it can be noticed that for all used different populations, as k increases, the ABias and MSE decrease, CP increases and for majority of the cases CW decreases. For example, in Table 2, when $\beta_3^* = 3$, the ABias decreases from 0.706 at $k = 0.64$ to 0.198 at k =0.93. Also, MSE decreases from 1.447 at $k = 0.64$ to 0.918 at $k = 0.93$. Moreover, the CP increases from 82.9 % at $k = 0.64$ to 94.3 % at $k = 0.93$. Finally, CW decreases from 3.818 to 3.676 at $k = 0.64$ and k =0.93, respectively. Also, using the same example, it can be seen that ABias for β_2 decreases from 0.423 at $k = 0.64$ (r =45) to 0.050 at k = 0.93 ($r = 65$). MSE defreases from 1 at $k = 0.64$ (r =45) to 0.183 at $k = 0.93$ ($r = 65$). The CP increases from 94.7 % at $k = 0.64$ (r =45) to 98.3 % at k =0.93 (r = 93). Similarly, the same conclusion can be reached for β_3 and β_2^*.

For Bayesin approach, it can be observed that for all four different used schemes and for all four different parameters, in more than half of the cases as k (r) increases, the ABias, MSE, CW decrease and CP increases. For example, in Table 4, using the fourth scheme ($\beta_2 = 0.6$, β_3=1.5, $\beta_2^* = 0.9$, β_3^*=2.2.), it can be seen that ABias for β_3 decreases from 0.116 at k =0.64 ($r = 45$) to 0.069 at k =0.93 ($r = 65$). Also, MSE decreases from 0.017 at k =0.64 ($r = 45$) to 0.008 at k =0.93 (r=65). Finally, CW decreases from 0.222 at k = 0.64 ($r = 45$) to 0.212 at $k = 0.93$ ($r = 65$). Besides, using the same example, it can be seen that ABias for β_3^* decreases from 0.158 at $k = 0.64$ ($r = 45$) to 0.119 at $k = 0.93$ (r = 65). MSE defreases from 0.030 at k =0.64 ($r = 45$) to 0.019 at $k = 0.93$ ($r = 65$).

Third, for analyzing the effect of increasing the total sample size $N = n + m$ with approximately the same value of k, the following sets are considered:

- Set 1: $k = 0.95$, $r = 95$.
- Set 2: $k = 0.93$, $r = 65$.
- Set 3: $k = 0.85$, $r = 85$.
- Set 4: $k = 0.86$, $r = 60$.
- Set 5: $k = 0.8$, $r = 80$.
- Set 6: $k = 0.79$, $r = 55$.

We compared set 1 with set 2, set 3 with set 4 and set 5 with set 6 for all different initial values assumed. It can be seen that as N increases MSE and CW decrease. For example, from Table 1 (set 1) and Table 2 (set 2), MSE for β_3 decreases from 1.324 to 0.751. Also, CW decreases from 4.488 to 3.394. Also, from Table 3 (set 1) and Table 4 (set 2), MSE for β_3 decreases from 0.040 to 0.012 and CW decreases from 0.360 to 0.315.

Fourth, for analyzing the effect of changing the parameters, the following sets are considered:

- Set 1: β_2 and $\beta_2^* < 0.5$. (i.e. $\beta_2 = 0.4$, β_3=1.4, $\beta_2^* = 0.4$, β_3^*=1.2).
- Set 2: $0.5 < \beta_2$ and $\beta_2^* < 1$. (i.e. $\beta_2 = 0.9$, β_3=2.6, $\beta_2^* = 0.8$, β_3^*=1.9. and $\beta_2 = 0.6$, β_3=1.5, $\beta_2^* = 0.9$, β_3^*=2.3).
- Set 3: β_2 and $\beta_2^* > 1$. (i.e. $\beta_2 = 1.3$, β_3=3.7, $\beta_2^* = 1.2$, β_3^*=3).

Comparing set 1, set 2 and set 3 for all different values of r, n and m. It can be seen that as the values of β_2 and β_2^* decrease MSE and CW decrease. For example, comparing set 1 and 3 in Table 1 (n=m =50) at r = 95, MSE for β_2^* decreases from 0.134 to 0.019 and CW decreases from 1.264 to 0.540. Also, comparing set 1 and 3 in Table 2 (n=40, m =30) at r = 65 MSE for β_2^* decreases from 0.170 to 0.025 and CW decreases from 1.424 to 0.620. Similarly, same conclusion can be reached for β_2, β_3 and β_3^* for different values of r and different sets (i.e. comparing set 1 and 3 or set 2 and set 3).

To summarize, it is observed that for majority of cases coverage probability is above 90 %.

5.2 An example

In this subsection, an example is explained to illustrate the applicability of joint Type-II censoring with dependent competing risk model.

Here, data presents the lifetime from two different production lines with two dependent causes of failures. Data are generated from bivariate inverted Kumaraswamy distribution using R package with n=100, m=90 and r=170, 150. Maximum likelihood estimates are illustrated in Table 5 and it can be seen that Joint type-II censoring provides results close to that in complete case but with lower cost and shorter time.

Table 1: The results of MLE (n=m=50).

	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
r=95									
ABias	0.006	0.023	0.109	0.036		0.039	0.394	0.007	0.089
MSE	0.095	0.751	0.056	0.318		0.017	0.239	0.019	0.137
CW	1.208	3.394	0.822	2.208		0.480	1.136	0.540	1.408
CP	99.0	97.4	94.8	97.7		95.6	84.1	98.9	93.9
r=90									
ABias	0.066	0.194	0.125	0.024		0.055	0.433	0.030	0.153
MSE	0.101	0.800	0.059	0.310		0.018	0.270	0.020	0.151
CW	1.220	3.467	0.812	2.180		0.480	1.130	0.630	1.402
CP	98.2	95.0	93.7	96.9		94.2	80.0	98.1	91.0
r=85									
ABias	0.116	0.342	0.142	0.085		0.069	0.471	0.049	0.200
MSE	0.116	0.877	0.063	0.312		0.020	0.304	0.022	0.171
CW	1.258	3.418	0.812	2.164		0.480	1.122	0.554	1.418
CP	97.4	93.2	92.6	95.9		92.3	75.9	97.3	88.1
r=80									
ABias	0.163	0.467	0.161	0.144		0.083	0.504	0.067	0.244
MSE	0.136	0.977	0.069	0.325		0.023	0.337	0.025	0.193
CW	1.294	3.416	0.812	2.162		0.496	1.130	0.568	1.430
CP	96.8	90.8	91.0	94.5		90.6	71.9	95.9	85.3
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
r=75									
ABias	0.209	0.575	0.181	0.201		0.097	0.537	0.084	0.283
MSE	0.168	1.098	0.076	0.348		0.025	0.372	0.031	0.216
CW	1.380	3.434	0.812	2.176		0.496	1.136	0.608	1.446
CP	95.7	88.2	89.2	93.2		88.9	66.9	95.4	82.2
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	1.3	3.7	1.2	3		0.6	1.5	0.9	2.2
r = 95									
ABias	0.042	0.155	0.173	0.173		0.071	0.015	0.090	0.026
MSE	0.183	1.289	0.134	0.707		0.033	0.214	0.074	0.474
CW	1.668	4.408	1.264	3.226		0.656	1.814	1.008	2.696
CP	98.8	96.3	93.9	95.5		96.2	97.3	97.0	96.8
r = 90									
ABias	0.112	0.357	0.209	0.283		0.094	0.089	0.124	0.138
MSE	0.196	1.397	0.148	0.747		0.037	0.221	0.081	0.487
CW	1.676	4.418	1.264	3.202		0.656	1.810	1.008	2.682
CP	97.1	93.4	91.9	92.8		94.8	95.7	95.9	95.4
r = 85									
ABias	0.174	0.505	0.244	0.387		0.114	0.149	0.159	0.137
MSE	0.218	1.576	0.164	0.807		0.041	0.232	0.091	0.489
CW	1.700	4.506	1.264	3.178		0.656	1.796	1.008	2.688
CP	97.0	91.7	91.0	90.8		93.4	93.9	94.1	93.0
r = 80									
ABias	0.228	0.642	0.279	0.475		0.132	0.198	0.192	0.324
MSE	0.252	1.789	0.182	0.890		0.046	0.253	0.105	0.583
CW	1.754	4.600	1.270	3.194		0.668	1.814	1.022	2.710
CP	96.0	88.8	89.1	88.3		91.9	92.2	92.5	90.9
r = 75									
ABias	0.284	0.767	0.312	0.555		0.148	0.242	0.225	0.407
MSE	0.304	2.005	0.205	0.997		0.052	0.277	0.122	0.658
CW	1.852	4.666	1.288	3.254		0.678	1.830	1.044	2.750
CP	94.9	86.2	86.6	85.3		90.4	90.3	90.0	88.1

Table 2: The results of MLE (n=40, m=30).

	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
					$r = 65$				
ABias	0.050	0.115	0.113	0.034		0.044	0.365	0.014	0.092
MSE	0.183	1.324	0.069	0.430		0.030	0.314	0.025	0.184
CW	1.664	4.488	0.928	2.568		0.656	1.668	0.620	1.644
CP	98.3	96.6	94.2	98.0		95.1	86.4	97.9	94.6
					$r = 60$				
ABias	0.144	0.374	0.139	0.061		0.067	0.430	0.043	0.175
MSE	0.244	1.361	0.075	0.421		0.035	0.354	0.028	0.203
CW	1.852	4.332	0.928	2.532		0.690	1.612	0.632	1.626
CP	98.0	94.3	92.8	96.4		94.0	82.5	97.1	91.6
					$r = 55$				
ABias	0.234	0.610	0.168	0.148		0.090	0.491	0.071	0.242
MSE	0.360	1.492	0.084	0.435		0.044	0.401	0.037	0.231
CW	2.164	4.148	0.928	2.520		0.744	1.568	0.702	1.626
CP	96.6	91.3	90.8	95.0		91.7	78.9	96.0	87.8
					$r = 50$				
ABias	0.324	0.793	0.200	0.219		0.112	0.536	0.097	0.304
MSE	0.592	1.759	0.099	0.472		0.057	0.451	0.049	0.263
CW	2.736	4.168	0.952	2.552		0.822	1.588	0.606	1.622
CP	95.6	87.7	89.6	93.5		90.5	76.5	95.7	85.2
					$r = 45$				
ABias	0.423	0.940	0.235	0.307		0.139	0.590	0.125	0.362
MSE	1.000	2.129	0.117	0.515		0.082	0.516	0.077	0.310
CW	3.552	4.374	0.976	2.544		0.984	1.606	0.968	1.658
CP	94.7	85.0	86.5	90.8		89.5	73.0	94.2	82.1
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	1.3	3.7	1.2	3		0.6	1.5	0.9	2.2
					$r = 65$				
ABias	0.098	0.256	0.194	0.198		0.082	0.016	0.103	0.744
MSE	0.337	2.195	0.170	0.918		0.318	0.414	0.097	1.184
CW	2.242	5.720	1.424	3.676		0.876	2.522	1.150	3.112
CP	97.7	96.0	94.1	94.3		95.7	97.0	96.2	97.3
					$r = 60$				
ABias	0.197	0.505	0.247	0.348		0.116	0.125	0.153	0.192
MSE	0.393	2.438	0.194	0.994		0.066	0.398	0.110	0.661
CW	2.332	5.792	1.430	3.662		0.902	2.422	1.156	3.096
CP	96.6	93.5	91.4	92.2		94.5	95.3	94.2	95.0
					$r = 55$				
ABias	0.289	0.739	0.298	0.478		0.145	0.213	0.202	0.319
MSE	0.490	2.693	0.226	1.116		0.080	0.410	0.132	0.729
CW	2.498	5.744	1.450	3.694		0.952	2.368	1.182	3.104
CP	95.8	90.6	88.4	89.2		92.7	93.5	92.4	92.3

	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	1.3	3.7	1.2	3		0.6	1.5	0.9	2.2
					$r = 50$				
ABias	0.384	0.987	0.347	0.588		0.172	0.284	0.249	0.436
MSE	0.609	2.976	0.263	1.267		0.099	0.441	0.167	0.835
CW	2.664	5.546	1.482	1.267		1.030	2.352	1.270	3.148
CP	94.5	88.1	85.5	86.0		90.6	91.4	90.7	89.0
					$r = 45$				
ABias	0.480	1.222	0.392	0.706		0.203	0.356	0.302	0.568
MSE	0.822	3.372	0.317	1.447		0.133	0.480	0.224	0.965
CW	3.016	5.374	1.582	3.818		1.188	2.330	1.430	3.140
CP	93.7	85.6	84.1	82.9		89.7	89.5	87.5	85.2

Table 3: The results of Bayesian (n=m=50).

	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
r=95									
ABias	0.056	0.240	0.127	0.023		0.029	0.073	0.034	0.211
MSE	0.005	0.063	0.019	0.005		0.001	0.012	0.002	0.049
CW	0.143	0.298	0.195	0.274		0.063	0.315	0.098	0.246
r=90									
ABias	0.060	0.229	0.128	0.046		0.029	0.121	0.013	0.142
MSE	0.005	0.059	0.019	0.006		0.001	0.022	0.0007	0.024
CW	0.145	0.321	0.203	0.238		0.069	0.319	0.091	0.229
r=85									
ABias	0.053	0.247	0.126	0.038		0.029	0.068	0.027	0.043
MSE	0.004	0.067	0.018	0.006		0.001	0.011	0.001	0.006
CW	0.147	0.301	0.195	0.251		0.063	0.321	0.096	0.248
r=80									
ABias	0.057	0.217	0.090	0.080		0.030	0.067	0.025	0.060
MSE	0.005	0.053	0.010	0.011		0.001	0.011	0.001	0.008
CW	0.153	0.301	0.182	0.266		0.065	0.327	0.091	0.256
r=75									
ABias	0.056	0.226	0.122	0.030		0.031	0.063	0.017	0.108
MSE	0.005	0.057	0.017	0.005		0.001	0.011	0.0008	0.015
CW	0.146	0.296	0.196	0.251		0.064	0.346	0.086	0.249
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	1.3	3.7	1.2	3		0.6	1.5	0.9	2.2
r = 95									
ABias	0.041	0.001	0.047	0.038		0.046	0.065	0.134	0.124
MSE	0.004	0.007	0.006	0.087		0.003	0.004	0.021	0.020
CW	0.188	0.340	0.232	0.353		0.111	0.215	0.226	0.266
r = 90									
ABias	0.034	0.007	0.018	0.023		0.053	0.017	0.135	0.123
MSE	0.003	0.008	0.004	0.007		0.004	0.003	0.021	0.020
CW	0.095	0.353	0.245	0.325		0.109	0.217	0.219	0.285
r = 85									
ABias	0.034	0.011	0.058	0.066		0.060	0.008	0.139	0.126
MSE	0.003	0.009	0.007	0.011		0.004	0.003	0.023	0.021
CW	0.182	0.369	0.225	0.304		0.113	0.224	0.224	0.142
r = 80									
ABias	0.052	0.050	0.011	0.079		0.039	0.056	0.135	0.125
MSE	0.005	0.010	0.004	0.013		0.002	0.006	0.022	0.021
CW	0.185	0.364	0.238	0.315		0.108	0.232	0.235	0.282
r = 75									
ABias	0.031	0.051	0.072	0.078		0.016	0.141	0.156	0.106
MSE	0.003	0.010	0.009	0.014		0.001	0.023	0.028	0.016
CW	0.187	0.344	0.241	0.336		0.107	0.208	0.231	0.280

Table 4: The results of Bayesian (n=40,m=30).

	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
r=65									
ABias	0.056	0.245	0.121	0.044		0.027	0.178	0.013	0.153
MSE	0.005	0.066	0.017	0.006		0.001	0.040	0.0007	0.027
CW	0.149	0.301	0.213	0.250		0.068	0.360	0.094	0.218
r=60									
ABias	0.060	0.215	0.128	0.042		0.031	0.072	0.018	0.090
MSE	0.005	0.052	0.019	0.006		0.001	0.012	0.0008	0.012
CW	0.147	0.313	0.220	0.256		0.066	0.341	0.094	0.253
r=55									
ABias	0.064	0.199	0.114	0.058		0.029	0.157	0.011	0.165
MSE	0.005	0.046	0.016	0.007		0.001	0.032	0.0007	0.031
CW	0.142	0.317	0.208	0.255		0.067	0.331	0.096	0.228
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	0.9	2.6	0.8	1.9		0.4	1.4	0.4	1.2
r=50									
ABias	0.070	0.071	0.090	0.071		0.031	0.069	0.016	0.131
MSE	0.006	0.009	0.010	0.009		0.001	0.012	0.0008	0.021
CW	0.153	0.254	0.184	0.254		0.063	0.320	0.090	0.243
r=45									
ABias	0.035	0.186	0.106	0.073		0.032	0.078	0.014	0.137
MSE	0.003	0.040	0.014	0.010		0.001	0.013	0.0009	0.022
CW	0.147	0.309	0.204	0.259		0.064	0.329	0.101	0.231
	β_2	β_3	β_2^*	β_3^*		β_2	β_3	β_2^*	β_3^*
	1.3	3.7	1.2	3		0.6	1.5	0.9	2.2
r = 65									
ABias	0.030	0.005	0.008	0.034		0.036	0.069	0.137	0.119
MSE	0.003	0.008	0.004	0.008		0.002	0.008	0.022	0.019
CW	0.175	0.351	0.224	0.303		0.113	0.212	0.224	0.282
r = 60									
ABias	0.040	0.039	0.016	0.026		0.024	0.129	0.170	0.079
MSE	0.004	0.010	0.004	0.007		0.001	0.020	0.032	0.011
CW	0.180	0.359	0.236	0.308		0.114	0.222	0.231	0.287
r = 55									
ABias	0.031	0.012	0.031	0.010		0.022	0.130	0.159	0.097
MSE	0.003	0.007	0.005	0.007		0.001	0.020	0.029	0.014
CW	0.174	0.325	0.231	0.330		0.102	0.199	0.229	0.278
r = 50									
ABias	0.026	0.022	0.004	0.041		0.021	0.134	0.103	0.151
MSE	0.003	0.008	0.004	0.008		0.001	0.021	0.014	0.028
CW	0.178	0.342	0.238	0.314		0.103	0.214	0.230	0.261
r = 45									
ABias	0.033	0.003	0.007	0.074		0.028	0.116	0.117	0.158
MSE	0.003	0.008	0.004	0.012		0.001	0.017	0.017	0.030
CW	0.178	0.334	0.252	0.324		0.098	0.222	0.239	0.276

Table 5: The results of MLE for production lines data.

		β_2	β_3	β_2^*	β_3^*
Complete data	estimates	0.987	2.052	0.697	1.776
Joint type II cencoring (r=170)	estimates	0.944	1.796	0.613	1.587
Joint type II cencoring (r=150)	estimates	0.758	1.747	0.562	1.458

5.3 Real Life Data Set

In this subsection, two real dataset are analyzed to show the applicability of joint type-II censoring with dependent competing risk model.

1) Diabetic Retinopathy Dataset

Data come from a multicenter clinical trials sponsored by the national eye institute to test the effect of laser treatment in delaying blindness in patients with diabetic retinopathy. The data consists of failure time for 71 patients. For each patient, one eye was randomly selected for treatment, while the other eye received no treatment. (For more details see, Csorgo and Welsh (1985)).

It was mentioned that data come from different centers but no other variable or indicator was given to illustrate different groups. Here, we divide the data randomly to two groups (see the Appendix) such that

- Group 1: patients from center A with 35 patients.
- Group 2: patients from center B with 36 patients.

For each patient in groups one and two, minimum time to blindness (i.e. x^* and y^*) is recorded. Now, we describe our data in a similar way to that illustrated in section 3.

$$z_k = \begin{cases} 1, & \text{if failures from group 1} \\ 0, & \text{if failures from group 2.} \end{cases}, \quad \delta_{k1} = \begin{cases} 1, & \text{Treated eyes firstly failed} \\ 0, & \text{otherwise.} \end{cases},$$

$$\delta_{k2} = \begin{cases} 1, & \text{Untreated eyes firstly failed} \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{k0} = \begin{cases} 1, & \text{both eyes failed together} \\ 0, & \text{otherwise.} \end{cases}.$$

Besides, $k = 1, \ldots, r$, r =46, 54 and 65 and data , data are divided by 365 to be recorded in years.

For each group, Cramer-Von Mises test is applied and we found that inverted Kumaraswamy distribution fitted the data well. Maximum likelihood and Bayesian estimates are illustrated in Table 6 for different values of r. It can be seen that when 92% of the data is used (r=65), joint Type-II censoring provides results close to the complete case but with lower cost and shorter time. Also, when 76 % and 66% of the data is used in the analysis (r =54 and 47, respectively) the results are still quite welll.

Table 6: The results of MLE for Diabetic Retinopathy data.

			β_2	β_3	β_2^*	β_3^*	
Complete sampling		estimates	3.069	15.603	4.955	16.771	
Joint type II cencoring (MLE)	r=65	estimates	3.086	13.945	3.793	16.572	
	r=54	estimates	2.157	11.349	3.174	16.444	
	r=47	estimates	1.482	10.148	2.587	16.235	
Joint type II cencoring (Bayesian)	r=65	estimates	3.016	14.401	4.085	15.56	
	r=54	estimates	2.957	11.85	4.57	17.48	
	r=47	estimates	2.932	11.63	4.873	15.17	

2) Cancer Dataset

The data consists of 506 patients as illustrated in Andrews and Herzberg (1985). Here, we restrict our study to 338 patients with three causes of failures: prostatic cancer, cerebrovascular accident and other causes of death(heart or vascular disease, pulmonary embolus, another cancer, respiratory disease, specific non-cancer cause, unspecified non-cancer cause and unknown causes). We have two main groups;

- Group 1: patients with history of cardiovascular disease, number of patients is 163.
- Group 2: patients with no history of cardiovascular disease, number of patients is 175.

Since both groups enter the study with no cardiovascular disease, we can analyze both groups together.

For each group, the causes are combined as follows (for more details, see Feizjavadian and Hashemi (2015));

- Prostatic cancer or other causes.
- cerebrovascular or other causes.

Clearly, the causes are dependent because when the cause of death is other causes, the two risks occur simultaneously. Hence, bivariate inverted Kumaraswamy distribu-

tion can be used to analyze this data. Now, we will describe our data in a similar way to that illustrated in Section 3:

$$z_k = \begin{cases} 1, & \text{if failures from group 1} \\ 0, & \text{if failures from group 2.} \end{cases}, \quad \delta_{k1} = \begin{cases} 1, & \text{if failure from prostatic cancer} \\ 0, & \text{otherwise.} \end{cases},$$

$$\delta_{k2} = \begin{cases} 1, & \text{if failure from cerebrovascular} \\ 0, & \text{otherwise.} \end{cases}, \quad \delta_{k0} = \begin{cases} 1, & \text{if failure from other causes} \\ 0, & \text{otherwise.} \end{cases},$$

where $k = 1, ..., r$.

For each group, Cramer-Von Mises test is applied and we found that inverted Kumaraswamy distribution fit the data well. Maximum likelihood and Bayesian estimates are illustrated in Table 7. It can be seen that, for different values of r and different estimation methods, joint type-II censoring provides results close to that in complete case but with lower cost and shorter time.

Table 7: The results of MLE for cancer data.

			β_2	β_3	β_2^*	β_3^*
Complete sampling		estimates	0.449	0.784	0.369	0.762
Joint type II cencoring (MLE)	r=320	estimates	0.408	0.604	0.379	0.883
	r=300	estimates	0.411	0.574	0.361	0.849
	r=237	estimates	0.361	0.542	0.323	0.743
	r=169	estimates	0.306	0.531	0.289	0.606
Joint type II cencoring (Bayesian)	r=320	estimates	0.459	0.787	0.374	0.745
	r=300	estimates	0.474	0.862	0.369	0.734
	r=237	estimates	0.459	0.809	0.354	0.694
	r=169	estimates	0.300	0.312	0.441	0.991

6 Conclusion

In this paper, the likelihood function is derived for the bivariate Marshall-Olkin family under joint type-II censoring in the presence of dependent competing risk model. The derived likelihood function is applied on the Marshall-Olkin bivariate inverted Kumaraswamy lifetime distribution. Maximum likelihood and Bayesian approaches are used to obtain the estimates of the unknown parameters. A simulation study is performed, an example and two real data sets are analyzed. The results after applying joint type-II censoring are compared with the case of complete sampling. It is found that joint type-II censoring provided similar estimates but with lower cost and shorter

time. Hence, we recommend the use of joint Type-II censoring in the bivariate case to save time and cost.

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Appendix

a) Here, derivatives of the log-likelihood with respect to the unknown parameters are illustrated.

$$
\begin{aligned}
\frac{\partial logL}{\partial \alpha} &= \frac{\sum_{k=1}^{r}(\delta_{k0}+\delta_{k1}+\delta_{k2})z_k}{\alpha} - \sum_{k=1}^{r}(\delta_{k0}+\delta_{k1}+\delta_{k2})z_k log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k0}z_k(\beta_1+\beta_2+\beta_3-1)}{1-(1+w_k)^{-\alpha}}(1+w_k)^{-\alpha}log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k1}z_k(\beta_1-1)}{1-(1+w_k)^{-\alpha}}(1+w_k)^{-\alpha}log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k2}z_k(\beta_2-1)}{1-(1+w_k)^{-\alpha}}(1+w_k)^{-\alpha}log(1+w_k) \\
&- \sum_{k=1}^{r}\frac{\delta_{k1}z_k(\beta_2+\beta_3)(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3-1}(1+w_k)^{-\alpha}log(1+w_k)}{1-(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3}} \\
&- \sum_{k=1}^{r}\frac{\delta_{k2}z_k(\beta_1+\beta_3)(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3-1}(1+w_k)^{-\alpha}log(1+w_k)}{1-(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3}}. \\
\frac{\partial logL}{\partial \beta_1} &= \frac{\sum_{k=1}^{r}z_k\delta_{k1}}{\beta_1} + \sum_{k=1}^{r}\delta_{k0}z_k log(1-(1+w_k)^{-\alpha}) \\
&+ \sum_{k=1}^{r}\delta_{k1}z_k log(1-(1+w_k)^{-\alpha}) \\
&- \sum_{k=1}^{r}\frac{\delta_{k2}z_k(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3}log(1-(1+w_k)^{-\alpha})}{1-(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3}}. \\
\frac{\partial logL}{\partial \beta_2} &= \frac{\sum_{k=1}^{r}z_k\delta_{k2}}{\beta_2} + \sum_{k=1}^{r}\delta_{k0}z_k log(1-(1+w_k)^{-\alpha}) \\
&+ \sum_{k=1}^{r}\delta_{k2}z_k log(1-(1+w_k)^{-\alpha}) \\
&- \sum_{k=1}^{r}\frac{\delta_{k1}z_k(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3}log(1-(1+w_k)^{-\alpha})}{1-(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3}}. \\
\frac{\partial logL}{\partial \beta_3} &= \frac{\sum_{k=1}^{r}z_k\delta_{k0}}{\beta_3} + \sum_{k=1}^{r}\delta_{k0}z_k log(1-(1+w_k)^{-\alpha}) \\
&- \sum_{k=1}^{r}\frac{\delta_{k1}z_k(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3}log(1-(1+w_k)^{-\alpha})}{1-(1-(1+w_k)^{-\alpha})^{\beta_2+\beta_3}} \\
&- \sum_{k=1}^{r}\frac{\delta_{k2}z_k(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3}log(1-(1+w_k)^{-\alpha})}{1-(1-(1+w_k)^{-\alpha})^{\beta_1+\beta_3}}.
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial logL}{\partial \alpha^*} &= \frac{\sum_{k=1}^{r}(\delta_{k0}+\delta_{k1}+\delta_{k2})z_k}{\alpha^*} - \sum_{k=1}^{r}(\delta_{k0}+\delta_{k1}+\delta_{k2})z_k log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k0}z_k(\beta_1^*+\beta_2^*+\beta_3^*-1)}{1-(1+w_k)^{-\alpha^*}}(1+w_k)^{-\alpha^*}log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k1}z_k(\beta_1^*-1)}{1-(1+w_k)^{-\alpha^*}}(1+w_k)^{-\alpha^*}log(1+w_k) \\
&+ \sum_{k=1}^{r}\frac{\delta_{k2}z_k(\beta_2^*-1)}{1-(1+w_k)^{-\alpha^*}}(1+w_k)^{-\alpha^*}log(1+w_k) \\
&- \sum_{k=1}^{r}\frac{\delta_{k1}z_k(\beta_2^*+\beta_3^*)(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*-1}(1+w_k)^{-\alpha^*}log(1+w_k)}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*}} \\
&- \sum_{k=1}^{r}\frac{\delta_{k2}z_k(\beta_1^*+\beta_3^*)(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*-1}(1+w_k)^{-\alpha^*}log(1+w_k)}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*}}. \\
\frac{\partial logL}{\partial \beta_1^*} &= \frac{\sum_{k=1}^{r}z_k\delta_{k1}}{\beta_1^*} + \sum_{k=1}^{r}\delta_{k0}z_k log(1-(1+w_k)^{-\alpha^*}) \\
&+ \sum_{k=1}^{r}\delta_{k1}z_k log(1-(1+w_k)^{-\alpha^*}) \\
&- \sum_{k=1}^{r}\frac{\delta_{k2}z_k(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*}log(1-(1+w_k)^{-\alpha^*})}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*}}. \\
\frac{\partial logL}{\partial \beta_2^*} &= \frac{\sum_{k=1}^{r}z_k\delta_{k2}}{\beta_2^*} + \sum_{k=1}^{r}\delta_{k0}z_k log(1-(1+w_k)^{-\alpha^*}) \\
&+ \sum_{k=1}^{r}\delta_{k2}z_k log(1-(1+w_k)^{-\alpha^*}) \\
&- \sum_{k=1}^{r}\frac{\delta_{k1}z_k(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*}log(1-(1+w_k)^{-\alpha^*})}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*}}.
\end{aligned}$$

$$\begin{aligned}\frac{\partial logL}{\partial \beta_3^*} &= \frac{\sum_{k=1}^{r} z_k \delta_{k0}}{\beta_3^*} + \sum_{k=1}^{r} \delta_{k0} z_k log(1-(1+w_k)^{-\alpha^*}) \\ &- \sum_{k=1}^{r} \frac{\delta_{k1} z_k (1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*} log(1-(1+w_k)^{-\alpha^*})}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_2^*+\beta_3^*}} \\ &- \sum_{k=1}^{r} \frac{\delta_{k2} z_k (1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*} log(1-(1+w_k)^{-\alpha^*})}{1-(1-(1+w_k)^{-\alpha^*})^{\beta_1^*+\beta_3^*}}.\end{aligned}$$

b) Diabetic Retinopathy Dataset

Group 1	Group 2	Group 1	Group 2
266	1653	272	503
91	427	1137	423
154	699	1484	285
285	36	315	315
583	667	287	727
547	588	1252	210
79	471	717	409
622	126	642	584
707	350	141	355
469	663	407	1302
93	567	356	277
1313	966		
805	203		
344	84		
790	392		
125	1140		
777	901		
306	1247		
415	448		
307	904		
637	276		
577	520		
178	485		
517	248		