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Estimation for the Three-Parameter Exponentiated Weibull Distribution under Progressive Censored Data

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Abstract. In this paper, we consider the problem of estimating the unknown parameters of an exponentiated Weibull distribution when the data are observed in the presence of progressively Type II censoring. We observed that the maximum likelihood estimators do not have a closed form, and so require a numerical technique to compute, further the implementation of the EM algorithm still requires the numerical techniques. So we employ the stochastic expectation-maximization (SEM) algorithm to estimate the model parameters and further to construct the associated asymptotic confidence intervals of the unknown parameters. Moreover, under Bayesian approach, we consider symmetric and asymmetric loss functions and compute the Bayesian estimates using the Lindley's approximation and Gibbs sampler together with Metropolis Hastings algorithm. The highest posterior density (HPD) credible intervals are also constructed. The behavior of suggested estimators is assessed using a simulation study. Finally, a real life example is considered to illustrate the application and development of the inference methods.

Keywords. Gibbs sampling, Three-Parameter Exponentiated Weibull, Progressive Type II Censoring, Observed Fisher Information, SEM Algorithm.

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1 Introduction

One of the important features in reliability and life-testing experiments is to identify the underlying distribution that can adequately be utilized to explain the phenomenon under consideration. The Weibull distribution is the useful model to analyze the real data. In addition, exponentiation technique is an effective way of introducing an additional shape parameter to the standard distribution to attain robustness and flexibility distributions. The basic idea here is from the exponentiated class of models with cumulative and density functions as:

$$F(y;\omega,\alpha) = 1 - (1 - G(x;\omega))^{\alpha};\ x \in R,\ \alpha > 0,\ \omega \in \Omega,$$

and

$$f(y;\omega;\alpha) = \alpha g(y;\omega)(1 - G(y;\omega))^{\alpha-1}; y \in R;\ \alpha > 0;\ \omega \in \Omega.$$

The exponentiated distributions play a key role in reliability problems. Since then, several articles have been published on different exponentiated distributions. Gupta and Kundu (2001) studied the properties of the EE family. They considered the maximum likelihood estimators and their asymptotic results for the unknown parameters. Chung and Jung (2013) considered the Bayesian inference of three-parameter exponentiated Weibull distribution based on progressive Type-II censoring. Sobhi et al. (2016) derived the maximum likelihood and Bayes estimators for the two shape parameters of the exponentiated Weibull lifetime model under Adaptive Type-II progressive censoring. They estimated the two unknown parameters using Newton's method and importance sampling technique. They also compute different confidence intervals for the parameters. Kumar et al. (2017) established several explicit expressions and recurrence relations for single and product moments of lower record values from exponentiated Burr XII distribution. They used the method of maximum likelihood for estimating the model parameters based on lower record values. Sawadogo et al. (2017) studied the maximum likelihood estimation of the parameters of exponentiated generalized Weibull based on progressive Type-II censored data. Panahi (2018) derived different point and interval estimates of unknown parameters of an exponentiated Pareto distribution from classical and Bayesian viewpoints and examined the suitability of the EP distribution in modeling the main component of cumin essential oil data under progressive first-failure censoring scheme.

The exponentiated Weibull (EW) distribution is a significant continuous life time distribution. The probability density function and the corresponding cumulative distribution function of the EW distribution can be written as:

$$f(x;\alpha,\beta,\lambda) = \alpha\beta\lambda\, x^{\lambda-1}\, e^{-\beta x^{\lambda}}\left(1 - e^{-\beta x^{\lambda}}\right)^{\alpha-1}; x > 0, \alpha > 0, \beta > 0, \lambda > 0, \qquad (1.1)$$

$$F(x;\alpha,\beta,\lambda) = (1 - e^{-\beta x^{\lambda}})^{\alpha};\ x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{1.2}$$

The reliability and hazard rate functions can be written as:

$$R(t;\alpha,\beta,\lambda) = 1 - (1 - e^{-\beta t^{\lambda}})^{\alpha};\ t > 0,$$

and

$$h(t;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda\, t^{\lambda-1}\, e^{-\beta t^{\lambda}}(1 - e^{-\beta t^{\lambda}})^{\alpha-1}}{1 - (1 - e^{-\beta t^{\lambda}})^{\alpha}};\ t > 0.$$

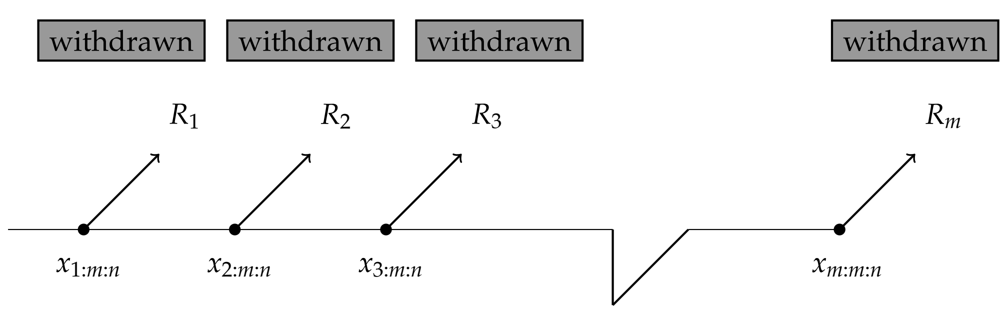
Here $\alpha, \beta, \lambda > 0$ are the unknown parameters. For $\lambda = 1$, it represents the exponentiated exponential distribution and for $\alpha = 1$, it represents the Weibull distribution. The two most common schemes are Type I and Type II censored schemes. Under Type I censored scheme, the experiment stops when it arrives at the predetermined time, while the experiment stops when it collects a specified amount of data under Type II censored scheme. The drawback of type I censoring is that one may not collect enough failure observations before the end of experiment, and the drawback of type II censoring is that the experimental time may be very long. So, Epstein (1954) proposed the Type I hybrid censoring scheme that the experiment terminates at a random time $T^{*} = min\{X_r, T\}$, where X is X_r the r^{th} failure time and T is a pre-specified time. Also, Childs et al. (2003) proposed the Type II hybrid censoring scheme that the experiment stops at a random time $T^{*} = max\{X, T\}$. The progressive censoring scheme (PCS) is a useful method for obtaining inferential conclusions for data which enables us to remove the data at intermediate stages. Some of the recent works on the progressive censoring scheme can be found in Habibirad et al. (2011), Huang and Wu (2012), Balakrishnan and Cohen (2014), Helu et al. (2015), Baratpour and Habibirad (2016), Singh and Tripathi (2018), Wang (2018) and Panahi (2020). In the present work, we consider estimation of three unknown parameters of an EW distribution under progressive type II censored sample from both classical and Bayesian perspective. The parameter estimation methods for EW distribution under progressive censoring is attractive and has research and practical application values. Kim et al. (2011) considered the two-parameter exponentiated Weibull distribution. They applied the Newton–Raphson method for estimating the unknown parameters. Moreover, they obtained the Bayesian estimates using the Lindley's approximation. Chung and Jung (2013) considered the Bayesian inference of three-parameter exponentiated Weibull distribution based on progressive Type-II censoring. They only studied the Bayesian estimations using Monte Carlo method. Sawadogo et al. (2017) studied the maximum likelihood estimation of the parameters of exponentiated generalized Weibull based on progressive Type-II censored data. They only considered the EM algorithm for estimating the unknown parameters.

However, on the one hand, notice that the maximum likelihood estimators of some unknown parameters from EW distribution based on progressively Type II censored do not exist in closed form. Therefore, it becomes difficult to evaluate accurate estimates, and the process of obtaining MLEs involves heavy computations. It looks like an alternative is to use other methods, such as EM and SEM algorithms. Further, the errors in point estimates can sometimes be large, so evaluating point estimates alone is not sufficient and interval estimation is also important for analyzing model parameters. Therefore, in this paper, we obtain maximum likelihood estimators (MLEs) of parameters using the SEM algorithm. In the sequel, the asymptotic confidence intervals are also constructed using Fisher information matrix. However, when much information is missing from the data, Bayesian analysis incorporated with prior information gives more precise inference. So, the Bayesian estimates of parameters against symmetric and asymmetric loss functions using different approximations are also computed. we derived the Bayesian estimates of the unknown model parameters, using the Lindley's approximation and the Markov Chain Monte Carlo ((MCMC) technique. Also, associated HPD credible intervals of the model parameters are obtained using the MCMC method.

The rest of the paper is arranged as follows. In Section 2, the process of generating progressively Type-II censored samples is described. In Section 3, we build the model and obtain the point estimation through the SEM algorithm. Through observed Fisher information, we gain the asymptotic confidence intervals. The Bayesian estimates by the two approximations are proposed in Section 4. In section 5, a simulation study is conducted to compare the proposed procedures. In section 6, a real life data example is presented to illustrate the application of the proposed inference procedures. Finally, in Section 7, some concluding remarks are added.

2 Data Descriptions

Suppose that n independent and identically distributed items are subjected to a life test simultaneously and a progressively censored sample of size $m(\leq n)$ is to be observed. At the first failure time, $(X_{1:m:n})$, R_1 of the remaining $(n-1)$ units are randomly removed from the experiment. Similarly, when the i^{th} failure occurs the R_i of the remaining $n-\sum_{j=1}^{i-1} R_j - i; i = 2, ..., m-1$ units are randomly removed from the experiment. Finally, at the occurrence of the m^{th} failure $(1 \leq m \leq n)$ all the remaining units $R_m = n - m - R_1 - R_2 - ... - R_{m-1}$ are removed. The observed failure times, $X_{1:m:n} < X_{2:m:n} < ... < X_{m:m:n}$, are called progressively censored sample with the progressive censoring scheme $(R_1, ..., R_m)$. The following figure shows the generation process of progressively Type-II censored scheme.



3 Stochastic EM (SEM) Algorithm

The EM algorithm which was originally discussed by Dempster et al. (1977) is a powerful method for calculating the maximum likelihood estimates of the unknown parameters in situations where data are censored in nature. Some merits of EM algorithm are (i) it can be applied to complex problems, (ii) log-likelihood increases at each iteration, (iii) computations are tedious but straightforward and (iv) the second and higher order derivatives are not required for calculation (For more details, one can refer to Habibi rad and Izanlo (2012), Tian et al. (2015), Gamchi et al. (2019), Panahi and Moradi (2020)). This algorithm contains two steps and starts with writing the likelihood function of complete sample put on the test, say $W = (W_1, W_2, ..., W_n)$. The complete sample W is the combination of observed data X and censored data Z. In our case, we have symbolized the observed progressive censored sample and censored observation by $X = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ and $Z = (Z_1, Z_2, ..., Z_m)$ respectively. Here Z_i denotes a $1 \times R_i$ random vector with $Z_i = (Z_{i1}, Z_{i2}, ..., Z_{iR_i})$; $i = 1, 2, ..., m$, items being removed at the time of x_i^{th} failure. Therefore, given the complete sample $W = (X, Z)$, the log-likelihood function ($l_c(W; \alpha, \beta, \lambda)$) can be written as:

$$
\begin{aligned}
l_c\,(W;\alpha,\beta,\lambda) &= n\ln\alpha + n\ln\beta + n\ln\lambda + (\lambda-1)\sum_{i=1}^{m}\ln x_{i:m:n} - \beta\sum_{i=1}^{m} x_{i:m:n}^{\lambda} \\
&+(\alpha-1)\sum_{i=1}^{m}\ln(1-e^{-\beta x_{i:m:n}^{\lambda}}) + (\lambda-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i}\ln z_{ik} - \beta\sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda} \\
&+(\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i}\ln(1-e^{-\beta z_{ik}^{\lambda}}).
\end{aligned}
$$

Now, we apply expectation step (E-step) of the algorithm by replacing the censored observations with expected ones as:

$$
\begin{aligned}
l_s\,(W;\alpha,\beta,\lambda) &= n\ln\alpha + n\ln\beta + n\ln\lambda + (\lambda-1)\sum_{i=1}^{m}\ln x_{i:m:n} - \beta\sum_{i=1}^{m} x_{i:m:n}^{\lambda} \\
&+(\alpha-1)\sum_{i=1}^{m}\ln(1-e^{-\beta x_{i:m:n}^{\lambda}}) + (\lambda-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i} E(\ln Z_{ik}\,|Z_{ik} > x_{i:m:n}) \\
&-\beta\sum_{i=1}^{m}\sum_{k=1}^{R_i} E(Z_{ik}^{\lambda}\,|Z_{ik} > x_{i:m:n}) + (\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i} E(\ln(1-e^{-\beta Z_{ik}^{\lambda}})\,|Z_{ik} > x_{i:m:n}),
\end{aligned}
$$

In EM algorithm, we should calculate the conditional expectations $E(\ln Z_{ik}|Z_{ik} > x_{i:m:n})$, $E(Z_{ik}^{\lambda}|Z_{ik} > x_{i:m:n})$ and $E(\ln(1 - e^{-\beta Z_{ik}^{\lambda}})|Z_{ik} > x_{i:m:n})$. These expectations are intractable and complex. Thus, we apply SEM algorithm by treating the data obtained from progressive censoring as a missing data problem (see for more details, Belaghi et al. (2019), Singh et al. (2019)). The details of the SEM steps are:

I. Generate the missing samples $Z_{ik};\ i = 1, 2, ..., m, k = 1, ..., R_i$, whose conditional distribution function is given by

$$\zeta(z_{ik}|z_{ik} > x_i) = \frac{F(z_{ik}) - F(x_i)}{1 - F(x_i)};\ z_{ik} > x_i.$$

Here, the conditional expectations can be approximated as:

$$E(\ln Z_{ik}|Z_{ik} > x_{i:m:n}) \sim \ln z_{ik},$$

$$E(Z_{ik}^{\lambda}|Z_{ik} > x_{i:m:n}) \sim z_{ik}^{\lambda},$$

$$E(\ln(1 - e^{-\beta Z_{ik}^{\lambda}})|Z_{ik} > x_{i:m:n}) \sim \ln(1 - e^{-\beta z_{ik}^{\lambda}}).$$

II. Assume that at the t^{th} stage, estimates of generate Z_{ik} through the condition density function $\zeta(z_{ik}|(z_{ik} > x_i)$.

III. In t^{th} iteration, obtain the estimation of α, β and λ as $\hat{\alpha}^t$, $\hat{\beta}^t$ and $\hat{\lambda}^t$.

IV. The maximum likelihood estimators of α, β and λ at the $(t+1)^{th}$ stage can be written as:

$$\alpha^{(t+1)} = n\left[\sum_{i=1}^{m} \ln(1 - e^{-\beta^{(g)} x_{i:m:n}^{\lambda^{(t)}}}) + \sum_{i=1}^{m}\sum_{k=1}^{R_i} \ln(1 - e^{-\beta z_{ik}^{\lambda^{(t)}}})\right]^{-1},$$

$$\beta^{(t+1)} = n\left[\sum_{i=1}^{m} x_{i:m:n}^{\lambda^{(t)}} - (\alpha^{(t+1)} - 1)\sum_{i=1}^{m} \frac{x_{i:m:n}^{\lambda^{(t)}} e^{-\beta^{(t)} x_{i:m:n}^{\lambda^{(t)}}}}{1 - e^{-\beta^{(t)} x_{i:m:n}^{\lambda^{(t)}}}} + \sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda^{(t)}}\right]^{-1},$$

and

$$\lambda^{(t+1)} = n\Bigg[-\sum_{i=1}^{m} \ln x_{i:m:n} + \beta^{(t+1)}\sum_{i=1}^{m} x_{i:m:n}^{\lambda^{(t)}} \ln x_{i:m:n} - \beta^{(t+1)}(\alpha^{(t+1)} - 1)$$
$$\times\sum_{i=1}^{m} \frac{x_{i:m:n}^{\lambda^{(t)}} \ln x_{i:m:n} e^{-\beta x_{i:m:n}^{\lambda^{(t)}}}}{1 - e^{-\beta^{(t+1)} x_{i:m:n}^{\lambda^{(gt)}}}} - \sum_{i=1}^{m}\sum_{k=1}^{R_i} \ln z_{ik} + \beta^{(t+1)}\sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda^{(t)}} \ln z_{ik}$$
$$-\beta^{(t+1)}(\alpha^{(t+1)} - 1)\sum_{i=1}^{m}\sum_{k=1}^{R_i} \frac{z_{ik}^{\lambda^{(t)}} \ln z_{ik} e^{-\beta z_{ik}^{\lambda^{(t)}}}}{1 - e^{-\beta^{(t+1)} z_{ik}^{\lambda^{(t)}}}}\Bigg].$$

The iterative process for maximum likelihood estimators of α, β and λ at $(t+1)^{th}$ stage can be stopped once $|\alpha^{(t+1)} - \alpha^t| + |\beta^{(t+1)} - \beta^t| + |\lambda^{(t+1)} - \lambda^t| \leq \epsilon$ for some given $\epsilon > 0$.

3.1 Asymptotic Confidence Intervals

In this subsection, we propose the asymptotic confidence intervals for the unknown parameters. The Fisher information matrix is

$$I(\alpha,\beta,\lambda) = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\lambda} \\ \frac{\partial^2 l}{\partial\lambda\partial\alpha} & \frac{\partial^2 l}{\partial\lambda\partial\beta} & \frac{\partial^2 l}{\partial\lambda^2} \end{bmatrix}_{(\alpha,\beta,\lambda)=(\hat\alpha,\hat\beta,\hat\lambda).}$$

Here,

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{n}{\alpha^2},$$

$$\frac{\partial^2 l}{\partial\beta^2} = -\frac{n}{\beta^2} - (\alpha-1)\sum_{i=1}^{m} \frac{x_{i:m:n}^{2\lambda} e^{-\beta x_{i:m:n}^{\lambda}}}{\left(1-e^{-\beta x_{i:m:n}^{\lambda}}\right)^2} - (\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i} \frac{z_{ik}^{2\lambda} e^{-\beta z_{ik}^{\lambda}}}{\left(1-e^{-\beta z_{ik}^{\lambda}}\right)^2},$$

$$\begin{aligned} \frac{\partial^2 l}{\partial\lambda^2} = &-\frac{n}{\lambda^2} - \beta\sum_{i=1}^{m} x_{i:m:n}^{\lambda}\ln(2x_{i:m:n}) + (\alpha-1)\beta\sum_{i=1}^{m} x_{i:m:n}^{\lambda} e^{-\beta x_{i:m:n}^{\lambda}}\ln(2x_{i:m:n}) \\ &\times\left[\frac{1-e^{-\beta x_{i:m:n}^{\lambda}}-\beta x_{i:m:n}^{\lambda}}{\left(1-e^{-\beta x_{i:m:n}^{\lambda}}\right)^2}\right] - \beta\sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda}\ln(2z_{ik}) \\ &+(\alpha-1)\beta\sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda} e^{-\beta z_{ik}^{\lambda}}\ln(2z_{ik})\times\left[\frac{1-e^{-\beta z_{ik}^{\lambda}}-\beta z_{ik}^{\lambda}}{\left(1-e^{-\beta z_{ik}^{\lambda}}\right)^2}\right], \end{aligned}$$

$$\frac{\partial^2}{\partial\alpha\partial\beta} = \sum_{i=1}^{m} \frac{x_{i:m:n}^{\lambda} e^{-\beta x_{i:m:n}^{\lambda}}}{1-e^{-\beta x_{i:m:n}^{\lambda}}} + \sum_{i=1}^{m}\sum_{k=1}^{R_i} \frac{z_{ik}^{\lambda} e^{-\beta z_{ik}^{\lambda}}}{1-e^{-\beta z_{ik}^{\lambda}}},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda} = \beta\sum_{i=1}^{m} \frac{x_{i:m:n}^{\lambda}\ln(x_{i:m:n}) e^{-\beta x_{i:m:n}^{\lambda}}}{1-e^{-\beta x_{i:m:n}^{\lambda}}} + \beta\sum_{i=1}^{m}\sum_{k=1}^{R_i} \frac{z_{ik}^{\lambda}\ln(z_{ik}) e^{-\beta z_{ik}^{\lambda}}}{1-e^{-\beta z_{ik}^{\lambda}}},$$

$$\begin{aligned} \frac{\partial^2 l}{\partial\beta\partial\lambda} = &-\sum_{i=1}^{m} x_{i:m:n}^{\lambda}\ln(x_{i:m:n}) + (\alpha-1)\sum_{i=1}^{m} x_{i:m:n}^{\lambda} e^{-\beta x_{i:m:n}^{\lambda}}\ln(x_{i:m:n}) \\ &\times\left[\frac{1-e^{-\beta x_{i:m:n}^{\lambda}}-\beta x_{i:m:n}^{\lambda}}{\left(1-e^{-\beta x_{i:m:n}^{\lambda}}\right)^2}\right] - \sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda}\ln(z_{ik}) \\ &+(\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{R_i} z_{ik}^{\lambda} e^{-\beta z_{ik}^{\lambda}}\ln(z_{ik})\left[\frac{1-e^{-\beta z_{ik}^{\lambda}}-\beta z_{ik}^{\lambda}}{\left(1-e^{-\beta z_{ik}^{\lambda}}\right)^2}\right]. \end{aligned}$$

By asymptotically normal property of the MLE, the asymptotic confidence intervals for the parameters α, β and λ are given by:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{Var(\hat{\alpha})},$$

$$\hat{\beta} \pm Z_{\gamma/2}\sqrt{Var(\hat{\beta})},$$

and

$$\hat{\lambda} \pm Z_{\gamma/2}\sqrt{Var(\hat{\lambda})},$$

where $Var(\hat{\alpha})$, $Var(\hat{\beta})$ and $Var(\hat{\lambda})$ are the entries on the main diagonal of the asymptotic variance covariance matrix $I_X^{-1}(\alpha, \beta, \lambda)$. Also $Z_{\frac{\gamma}{2}}$ denotes the upper $(\frac{\gamma}{2})^{th}$ percentile of the standard normal distribution.

4 Bayesian Estimation

In this Section, we obtain Bayes estimators of unknown parameters α, β and λ of the $EW(\alpha, \beta, \lambda)$ distribution against squared error and linex loss functions. The squared error (SE) and linex (LI) loss functions are defined as

$$L_{SE}(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2,$$

and

$$L_{LI}(\theta, \hat{\theta}) = d\left(e^{h(\hat{\theta}-\theta)}\right) - h(\hat{\theta} - \theta) - 1;\ h \neq 0,\ \ d > 0.$$

where $\hat{\theta}$ and h are the estimates of the unknown parameter θ and loss parameter respectively. The corresponding Bayes estimators of SE and LI loss functions are $\hat{\theta}_{SE} = E_\theta(\theta|\underline{X})$ and $\hat{\theta}_{LI} = -\frac{1}{h}\ln E_\theta(e^{-h\theta}|\underline{X})$ respectively. So, the Bayes estimate of any function of α, β and λ say $g(\theta)|_{\theta=(\alpha,\beta,\lambda)}$, under SE and LI loss functions, can be written as:

$$\hat{g}_{SE}(\theta) = E\left(g(\theta)\,|data\right) = \int_0^\infty \int_0^\infty \int_0^\infty g(\theta)\pi(\alpha, \beta, \lambda\,\big|data)d\alpha d\beta d\lambda\,, \tag{4.1}$$

and

$$\begin{aligned} \hat{g}_{LI}(\theta) &= \frac{-1}{h}\ln E\left(e^{-hg(\theta)}\,|data\right) \\ &= \frac{-1}{h}\ln \int_0^\infty \int_0^\infty \int_0^\infty e^{-hg(\theta)}\pi(\alpha, \beta, \lambda\,\big|data)d\alpha d\beta d\lambda\,, \end{aligned} \tag{4.2}$$

respectively, where $L(\alpha, \beta, \lambda)$ the likelihood function,

$$\pi(\alpha, \beta, \lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}\exp^{-b_1\alpha-b_2\beta-b_3\lambda},$$

and

$$\pi(\alpha,\beta,\lambda\,|data) = \frac{\pi(\alpha,\beta,\lambda)L(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\beta,\lambda)L(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}$$
$$\propto \alpha^{m+a_1-1}\beta^{m+a_2-1}\lambda^{m+a_3-1}e^{-b_1\alpha-\beta(b_2+\sum_{i=1}^{m}x_i^\lambda)-\lambda(b_3-\sum_{i=1}^{m}\ln x_i)}$$
$$\times\prod_{i=1}^{m}(1-e^{-\beta x_i^\lambda})^{\alpha-1}\left[1-(1-e^{-\beta x_i^\lambda})^{\alpha}\right]^{R_i}. \tag{4.3}$$

It is observed that the Bayesian estimators given by (4.1) and (4.2) do not admit closed form expressions, and so require some approximation methods. Various approaches have been discussed in the literature. The Lindley's approximation and Monte Carlo Markov Chain technique are illustrated in the next section.

4.1 Lindley's Approximation

This subsection, deals with the use of the Lindley's approximation technique (Lindley; 1980) to approximate the Bayes estimates. Based on the Lindley's approximation, the approximate Bayes estimates of α, β and λ under the SE loss function can be written as:

$$\hat{\alpha}_{SE} = \hat{\alpha} + \frac{1}{2}\left[2\hat{\rho}_1\hat{\sigma}_{11} + 2\hat{\rho}_2\hat{\sigma}_{12} + 2\hat{\rho}_3\hat{\sigma}_{13} + \hat{\sigma}_{11}A + \hat{\sigma}_{21}B + \hat{\sigma}_{31}C\right],$$
$$\hat{\beta}_{SE} = \hat{\beta} + \frac{1}{2}\left[2\hat{\rho}_1\hat{\sigma}_{21} + 2\hat{\rho}_2\hat{\sigma}_{22} + 2\hat{\rho}_3\hat{\sigma}_{23} + \hat{\sigma}_{12}A + \hat{\sigma}_{22}B + \hat{\sigma}_{32}C\right],$$

and

$$\hat{\lambda}_{SE} = \hat{\lambda} + \frac{1}{2}\left[2\hat{\rho}_1\hat{\sigma}_{31} + 2\hat{\rho}_2\hat{\sigma}_{32} + 2\hat{\rho}_3\hat{\sigma}_{33} + \hat{\sigma}_{13}A + \hat{\sigma}_{23}B + \hat{\sigma}_{33}C\right].$$

Similarly, under LI loss function, the approximate Bayes estimates of the unknown parameters are:

$$\hat{\alpha}_{LI} = -\frac{1}{h}\ln\left[e^{-h\hat{\alpha}}(1-{}^{h}/_{2}\left\{2\hat{\rho}_1\hat{\sigma}_{11} + 2\hat{\rho}_2\hat{\sigma}_{12} + 2\hat{\rho}_3\hat{\sigma}_{13} - (h-A)\hat{\sigma}_{11} + \hat{\sigma}_{21}B + \hat{\sigma}_{31}C\right\})\right],$$
$$\hat{\beta}_{LI} = -\frac{1}{h}\ln\left[e^{-h\hat{\beta}}(1-{}^{h}/_{2}\left\{2\hat{\rho}_1\hat{\sigma}_{21} + 2\hat{\rho}_2\hat{\sigma}_{22} + 2\hat{\rho}_3\hat{\sigma}_{23} - (h-B)\hat{\sigma}_{22} + \hat{\sigma}_{12}A + \hat{\sigma}_{32}C\right\})\right],$$

and

$$\hat{\lambda}_{LI} = -\frac{1}{h}\ln\left[e^{-h\hat{\lambda}}(1-{}^{h}/_{2}\left\{2\hat{\rho}_1\hat{\sigma}_{31} + 2\hat{\rho}_2\hat{\sigma}_{32} + 2\hat{\rho}_3\hat{\sigma}_{33} - (h-C)\hat{\sigma}_{33} + \hat{\sigma}_{13}A + \hat{\sigma}_{23}C\right\})\right],$$

respectively, where

$$A = \hat{\sigma}_{11}\hat{L}_{111} + 2\hat{\sigma}_{12}\hat{L}_{121} + 2\hat{\sigma}_{13}\hat{L}_{131} + 2\hat{\sigma}_{23}\hat{L}_{231} + \hat{\sigma}_{22}\hat{L}_{221} + \hat{\sigma}_{33}\hat{L}_{331},$$
$$B = \hat{\sigma}_{11}\hat{L}_{112} + 2\hat{\sigma}_{12}\hat{L}_{122} + 2\hat{\sigma}_{13}\hat{L}_{132} + 2\hat{\sigma}_{23}\hat{L}_{232} + \hat{\sigma}_{22}\hat{L}_{222} + \hat{\sigma}_{33}\hat{L}_{332},$$

$$C = \hat{\sigma}_{11}\hat{L}_{113} + 2\hat{\sigma}_{12}\hat{L}_{123} + 2\hat{\sigma}_{13}\hat{L}_{133} + 2\hat{\sigma}_{23}\hat{L}_{233} + \hat{\sigma}_{22}\hat{L}_{223} + \hat{\sigma}_{33}\hat{L}_{333},$$

$$\rho_1 = \frac{a_1 - 1}{\hat{\alpha}} - b_1; \rho_2 = \frac{a_2 - 1}{\hat{\beta}} - b_2; \rho_3 = \frac{a_3 - 1}{\hat{\lambda}} - b_3.$$

In addition, we have

$$\hat{L}_{11} = -\frac{m}{\alpha^2} + \alpha \sum_{i=1}^{m} \frac{R_i x_i^{\lambda} \ln T_i T_i^{\alpha-1} e^{-\beta x_i^{\lambda}} \left[1 - 2T_i^{\alpha}\right]}{\left(1 - T_i^{\alpha}\right)^2},$$

$$\hat{L}_{12} = \hat{L}_{21} = \sum_{i=1}^{m} \frac{x_i^{\lambda} e^{-\beta x_i^{\lambda}}}{T_i} - \sum_{i=1}^{m} \frac{R_i x_i^{\lambda} T_i^{\alpha-1} e^{-\beta x_i^{\lambda}}}{1 - T_i^{\alpha}} - (\alpha - 1) \sum_{i=1}^{m} \frac{R_i x_i^{\lambda} \ln T_i T_i^{\alpha-1} e^{-\beta x_i^{\lambda}}}{\left(1 - T_i^{\alpha}\right)^2},$$

$$\hat{L}_{13} = \hat{L}_{31} = \beta \left(\sum_{i=1}^{m} \frac{x_i^{\lambda} \ln T_i e^{-\beta x_i^{\lambda}}}{T_i} - \sum_{i=1}^{m} \frac{R_i x_i^{\lambda} \ln T_i T_i^{\alpha-1} e^{-\beta x_i^{\lambda}} \left[1 + \alpha \ln T_i (1 + T_i^{\alpha})\right]}{\left(1 - T_i^{\alpha}\right)^2} \right),$$

$$\hat{L}_{22} = -\frac{m}{\beta^2} - (\alpha - 1) \sum_{i=1}^{m} \frac{x_i^{2\lambda} e^{-\beta x_i^{\lambda}}}{T_i^2}$$

$$+\alpha \sum_{i=1}^{m} \frac{R_i x_i^{2\lambda} T_i^{\alpha-1} e^{-\beta x_i^{\lambda}} \left((1 - T_i)[\alpha(\alpha - 1) T_i^{\alpha-2} e^{-\beta x_i^{\lambda}} + 1] + \alpha T_i^{\alpha-1} e^{-\beta x_i^{\lambda}}\right)}{\left(1 - T_i^{\alpha}\right)^2},$$

$$\hat{L}_{33} = -\frac{m}{\lambda^2} - \beta \sum_{i=1}^{m} x_i^{\lambda} \ln 2x_i + (\alpha - 1)\beta \sum_{i=1}^{m} \frac{x_i^{\lambda} \ln 2x_i e^{-\beta x_i^{\lambda}} (1 - \beta x_i^{\lambda} - e^{-\beta x_i^{\lambda}})}{T_i^2}$$

$$-\alpha\beta \sum_{i=1}^{m} \frac{R_i x_i^{\lambda} \ln 2x_i T_i^{\alpha-1} e^{-\beta x_i^{\lambda}} \left(1 - \beta x_i^{\lambda} + (\alpha - 1)\beta x_i^{\lambda} e^{-\beta x_i^{\lambda}} T_i^{\alpha-1} [1 - T_i^{\alpha}]\right)}{\left(1 - T_i^{\alpha}\right)^2},$$

where $T_i = 1 - e^{-\beta x_i^{\lambda}}$.

4.2 Monte Carlo Markov Chain Method

We consider Monte Carlo Markov Chain technique for the computation of Bayes estimates of the unknown parameters of the EW distribution. To apply this method, the conditional posterior densities of α, β and λ are

$$\pi_1^*(\alpha \,|\beta, \lambda, data) \propto \alpha^{m+a_1-1} e^{-b_1\alpha} \prod_{i=1}^{m} (1 - e^{-\beta x_i^{\lambda}})^{\alpha-1} \left[1 - (1 - e^{-\beta x_i^{\lambda}})^{\alpha}\right]^{R_i}, \tag{4.4}$$

$$\pi_2^*(\beta \,|\alpha, \lambda, data) \propto \beta^{m+a_2-1} e^{-\beta(b_2 + \sum_{i=1}^{m} x_i^{\lambda})} \prod_{i=1}^{m} (1 - e^{-\beta x_i^{\lambda}})^{\alpha-1} \left[1 - (1 - e^{-\beta x_i^{\lambda}})^{\alpha}\right]^{R_i}, \tag{4.5}$$

and

$$\pi_3^*(\lambda\left|\alpha,\beta,data\right.) \propto \lambda^{m+a_3-1}e^{-\lambda(b_3-\sum_{i=1}^{m}\ln x_i)}\prod_{i=1}^{m}(1-e^{-\beta x_i^\lambda})^{\alpha-1}\Big[1-(1-e^{-\beta x_i^\lambda})^{\alpha}\Big]^{R_i}. \quad (4.6)$$

Now, we use the Metropolis–Hastings algorithm within Gibbs sampling steps for obtaining the Bayesian estimates:

Step 1: Initialize the values of $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$.

Step 2: Set $j = 1$.

Step 3: Using the following Metropolis–Hastings algorithm (Metropolis et al. (1953) and Hastings (1970)), generate $\alpha^{(j)}$, $\beta^{(j)}$ and $\lambda^{(j)}$ from $\pi_1^*(\alpha^{(j-1)}|\beta^{(j-1)},\lambda^{(j-1)},data)$, $\pi_2^*(\beta^{(j-1)}|\alpha^{(j)},\lambda^{(j-1)},data)$ an $\pi_3^*(\lambda^{(j-1)}|\alpha^{(j)},\beta^{(j)},data)$ with the normal proposal distributions $Normal(\alpha^{j-1},Var(\alpha))$, $Normal(\beta^{j-1},Var(\beta))$ and $Normal(\lambda^{j-1},Var(\lambda))$ respectively.

Step 4: Generate $\ddot{\alpha}$ from $Normal(\alpha^{j-1},Var(\alpha))$, $\ddot{\beta}$ from $Normal(\beta^{j-1},Var(\beta))$ and $\ddot{\lambda}$ from $Normal(\lambda^{j-1},Var(\lambda))$.

Step 5: Generate a w_1, w_2 and w_3 from a Uniform(0,1) distribution.

Step 6: Put

$\alpha^{(j)} = \begin{cases} \ddot{\alpha} \text{ if } w_1 \le \vartheta_\alpha \\ \alpha^{(j-1)} \text{ if } w_1 > \vartheta_\alpha, \end{cases}$ $\beta^{(j)} = \begin{cases} \ddot{\beta} \text{ if } w_2 \le \vartheta_\beta \\ \beta^{(j-1)} \text{ if } w_2 > \vartheta_\beta, \end{cases}$ and $\lambda^{(j)} = \begin{cases} \ddot{\lambda} \text{ if } w_3 \le \vartheta_\lambda \\ \lambda^{(j-1)} \text{ if } w_3 > \vartheta_\lambda, \end{cases}$ where ϑ_α, ϑ_β and ϑ_λ are the acceptance probabilities as:

$$\vartheta_\alpha = \min\left(1, \frac{\pi_1^*(\ddot{\alpha}\left|\beta^{(j-1)},\lambda^{(j-1)},data\right.)}{\pi_1^*(\alpha^{(j-1)}\left|\beta^{(j-1)},\lambda^{(j-1)},data\right.)}\right),$$

$$\vartheta_\beta = \min\left(1, \frac{\pi_1^*(\ddot{\beta}\left|\alpha^{(j)},\lambda^{(j-1)},data\right.)}{\pi_1^*(\beta^{(j-1)}\left|\alpha^{(j)},\lambda^{(j-1)},data\right.)}\right),$$

$$\vartheta_\lambda = \min\left(1, \frac{\pi_1^*(\ddot{\lambda}\left|\alpha^{(j)},\beta^{(j)},data\right.)}{\pi_1^*(\lambda^{(j-1)}\left|\alpha^{(j)},\beta^{(j)},data\right.)}\right).$$

Step 7: Consider $j = j + 1$.

Step 8: Repeat steps 3 -7, N times.

Step 9: The Bayesian estimates of $\zeta = (\alpha,\beta,\lambda)$ under SE and LL are:

$$\vartheta_\lambda = \min\left(1, \frac{\pi_1^*(\ddot{\lambda}\left|\alpha^{(j)},\beta^{(j)},data\right.)}{\pi_1^*(\lambda^{(j-1)}\left|\alpha^{(j)},\beta^{(j)},data\right.)}\right),$$

and

$$\hat{\zeta}_{MCMC-LL} = -\frac{1}{h}\log\left[\frac{1}{N-M}\sum_{j=M+1}^{N}\exp(-s\zeta^{(j)})\right],$$

respectively, where M is the burn-in-period of Markov Chain which is considered for removing the affection of the selection of initial values and guarantying the convergence of the algorithm.

Step 10: For constructing the credible intervals of α, β and λ, order $\alpha^{(j)}$, $\beta^{(j)}$ and $\lambda^{(j)}$ as $\left\{\alpha^{(1)} < ... < \alpha^{(N)}\right\}$, $\left\{\beta^{(1)} < ... < \beta^{(N)}\right\}$ and $\left\{\lambda^{(1)} < ... < \lambda^{(N)}\right\}$. Then, the $100(1-\gamma)\%$ credible intervals of $\zeta = (\alpha, \beta, \lambda)$ become:

$$\left[\zeta_{(N\gamma/2)} \ , \ \zeta_{(N(1-\gamma/2))}\right].$$

5 Simulation Study

In this section, we carry out a detailed simulation study to observe the performance of the different point and interval estimators under progressive Type II censoring scheme. We use the following algorithm to generate 10000 PCS samples (Balakrishnan and Sandhu (1995)).

Step 1: Generate m independent U(0,1) observations $Q_1, Q_2, ..., Q_m$.

Step 2: For given values of n, m, and $(R_1, ..., R_m)$, we set $T_i = Q_i^{1/(i+\sum_{j=m-i+1}^{m})}$; $i = 1, 2, ..., m$.

Step 3: Set $S_i = 1 - T_m T_{m-1} ... T_{m-i+1}$ for $i = 1, 2, ..., m$. Then $S_1, S_2, ..., S_m$ is a progressive Type II censored sample of size m from U(0,1) distribution.

Step 4: For given values of the parameters α, β and λ, $X_i = F^{-1}(S_i)$ is the required progressive Type II censored sample of size m from the EW distribution.

We compute the MLEs and the Bayes estimates of different parameters. To evaluate the MLEs, we apply the SEM algorithm. To compute the Bayes estimates of parameters, two approximation techniques, Lindley's approximation and Metropolis–Hastings algorithm, are used. For Metropolis–Hastings algorithm, we generate $N = 10000$ samples and $M = 20\%$ of N is considered as burn-in period. The initial values of α, β and λ for running the MCMC sampler algorithm were taken to be their maximum likelihood estimates. We consider three censoring schemes (CS) as:

$CS1 : R_1 = n - m \ and \ R_2 = ... = R_m = 0,$

$CS2 : R_1 = ... = R_{m-1} = 0 \ and \ R_m = n - m \ ,$

$CS3 : R_1 = ... = R_{m-1} = 1 \ and \ R_m = n - 2m + 1.$

Notice that the CS2 is the conventional Type II censoring scheme and $n - m$ survival units are totally removed when the m^{th} failure happens. While CS1 is opposite of CS2, it removes $n - m$ survival units at the first failure time from the experiment. Note that the simulation is performed in R-3:5:1 software for 10000 sets of random observations. We consider $(n, m) = (20, 5), (30, 5), (50, 5), (20, 10), (30, 10), (50, 10)$ and the true values of the parameters to be estimated are $(\alpha, \beta, \lambda) = (1.7, 1.5, 0.9)$. We get the values of hyper-parameters as $a_1 = a_2 = a_3 = 1$, $b_1 = 1.7, b_2 = 1.5, b_3 = 0.9$. Further, for computing the Bayes estimates with respect to the LI loss function, we assume $h = 0.25$. Based on these set up assumptions, we show the numerical results in the Tables 1-5. Tables 1, 2 and 3 show the average estimate and MSE of the estimations of the unknown parameters to compare the MLE and Bayes methods under different n, m and censoring schemes. The 95% approximate confidence and credible intervals (with average length and coverage probability) are displayed in Tables 4 and 5. Note that the notation used in censoring schemes like (0^{*7}) denotes $(0, 0, 0, 0, 0, 0, 0)$ and (4^{*3}) stands for $(4, 4, 4)$. The Tables reveal that the bias and MSE decrease when m increases for fixed n. Hence, increase in effective sample size increases the accuracy of the estimates obtained. Also, as m increases, the average interval length shortens. The Bayes estimates have notably better performance than the estimates of MLE method in most cases. In Bayes estimations, we consider and study two different loss functions, and the simulation results show that the estimations with LI loss have lower Bias and MSE than the estimates with squared error loss. It is observed that Metropolis-Hastings algorithm gives better result than that of the Lindley's approximation technique. Among the two intervals, it is observed that the HPD method performs better than the intervals obtained via normal approximation. Further, the average lengths of the confidence intervals decrease when effective sample size increases.

Table 1: Average estimates and MSEs of α for different choices of n, m, R.

(n,m)	Scheme	ML Estimates (MSE)	Bayes Estimates (MSE) MH SE	MH LINEX	Lindley SE	Lindley LINEX
(20,5)	$(15, 0^{(*4)})$	1.6994229	1.708199	1.705312	1.646909	1.699333
		(0.149167)	(0.061701)	(0.046492)	(0.105258)	(0.088039)
	$(0^{(*4)}, 15)$	1.6909439	1.757962	1.695813	1.660721	1.684290
		(0.154638)	(0.078316)	(0.056349)	(0.120623)	(0.095328)
	$(1^{(*4)}, 11)$	1.6879145	1.735490	1.705116	1.767699	1.685243
		(0.150018)	(0.278580)	(0.063200)	(0.114899)	(0.100055)
(30,5)	$(25, 0^{(*4)})$	1.6858344	1.708170	1.695073	1.667610	1.699976
		(0.140508)	(0.050800)	(0.037183)	(0.071501)	(0.062292)
	$(0^{(*4)}, 25)$	1.6855011	1.710084	1.695721	1.759375	1.749736
		(0.150243)	(0.059119)	(0.043898)	(0.068791)	(0.067030)
	$(1^{(*4)}, 21)$	1.6815538	1.737144	1.715558	1.774017	1.738911
		(0.138814)	(0.056404)	(0.039336)	(0.074895)	(0.057197)
(50,5)	$(45, 0^{(*4)})$	1.6805102	1.696946	1.699988	1.728674	1.742178
		(0.064168)	(0.035262)	(0.028900)	(0.039696)	(0.037024)
	$(0^{(*4)}, 45)$	1.6759953	1.716733	1.702127	1.750326	1.734980
		(0.089662)	(0.040435)	(0.027708)	(0.052446)	(0.050620)
	$(1^{(*4)}, 41)$	1.6186766	1.732743	1.728669	1.750961	1.745763
		(0.075851)	(0.043369)	(0.035588)	(0.055748)	(0.044168)
(20,10)	$(10, 0^{(*9)})$	1.7283333	1.724817	1.693523	1.747513	1.7331083
		(0.054638)	(0.025654)	(0.015002)	(0.028980)	(0.026817)
	$(0^{(*9)}, 10)$	1.6707711	1.715614	1.698269	1.736751	1.727407
		(0.062414)	(0.027684)	(0.022115)	(0.033911)	(0.031265)
	$(1^{(*10)})$	1.6197016	1.724787	1.713932	1.747854	1.742031
		(0.050018)	(0.029094)	(0.024787)	(0.042430)	(0.031500)
(30,10)	$(20, 0^{(*9)})$	1.6608084	1.713913	1.704055	1.7428504	1.730786
		(0.049167)	(0.012926)	(0.009197)	(0.021154)	(0.019975)
	$(0^{(*9)}, 20)$	1.6584634	1.699459	1.706226	1.717528	1.692497
		(0.058620)	(0.017375)	(0.010062)	(0.023463)	(0.021055)
	$(1^{(*9)}, 11)$	1.7452595	1.719358	1.70161	1.7383	1.72622
		(0.051118)	(0.021016)	(0.009627)	(0.025968)	(0.024764)
(50,10)	$(40, 0^{(*9)})$	1.6474022	1.694019	1.698008	1.742824	1.733056
		(0.015985)	(0.008601)	(0.007990)	(0.011161)	(0.008784)
	$(0^{(*9)}, 40)$	1.7375154	1.717927	1.695757	1.739566	1.735032
		(0.026656)	(0.009905)	(0.008479)	(0.012956)	(0.011182)
	$(1^{(*9)}, 31)$	1.7257515	1.710329	1.695981	1.733563	1.716198
		(0.020527)	(0.009610)	(0.008705)	(0.017406)	(0.010128)

Table 2: Average estimates and MSEs of β for different choices of n, m, R.

(n,m)	Scheme	ML Estimates (MSE)	Bayes Estimates (MSE) MH SE	MH LINEX	Lindley SE	Lindley LINEX
(20,5)	$(15, 0^{(*4)})$	1.5168502	1.6114999	1.604202	1.617905	1.624029
		(0. 24692)	(0.108244)	(0.060366)	(0.169190)	(0.148201)
	$(0^{(*4)}, 15)$	1.5991581	1.595916	1.5654167	1.639273	1.6066569
		(0.269790)	(0.106845)	(0.065025)	(0.158960)	(0.120200)
	$(1^{(*4)}, 11)$	1.6160154	1.592563	1.5175801	1.6422225	1.619169
		(0. 25698)	(0.087954)	(0.080276)	(0.172550)	(0.151407)
(30,5)	$(25, 0^{(*4)})$	1.4166679	1.591972	1.5349031	1.649218	1.620705
		(0.16863)	(0.055621)	(0.054823)	(0.110119)	(0.078037)
	$(0^{(*4)}, 25)$	1.4382118	1.618657	1.606168	1.630311	1.6150385
		(0. 21750)	(0.059676)	(0.049193)	(0.131842)	(0.098959)
	$(1^{(*4)}, 21)$	1.4559754	1.566141	1.527924	1.553207	1.537548
		(0. 17284)	(0.079632)	(0.058960)	(0.152159)	(0.080278)
(50,5)	$(45, 0^{(*4)})$	1.5371053	1.540547	1.5063534	1.571252	1.556528
		(0. 13331)	0.032184	(0.023519)	(0.081626)	(0.040900)
	$(0^{(*4)}, 45)$	1.6392846	1.5778320	1.5459021	1.626674	1.5835723
		(0. 16822)	(0.039129)	(0.032958)	(0.080505)	(0.050915)
	$(1^{(*4)}, 41)$	1.5729498	1.5314748	1.5143534	1.606177	1.5402078
		(0. 15494)	(0.048811)	(0.040595)	(0.071563)	(0.059719)
(20,10)	$(10, 0^{(*9)})$	1.6125896	1.541740	1.5030469	1.5739358	1.5603318
		(0.063429)	(0.028914)	(0.015808)	(0.055029)	(0.032077)
	$(0^{(*9)}, 10)$	1.6055554	1.5239807	1.501903	1.5318721	1.4624903
		(0.086512)	(0.025938)	(0.012870)	(0.057153)	(0.033917)
	$(1^{(*10)})$	1.5928135	1.6073044	1.5343337	1.628108	1.6150000
		(0.070163)	(0.023775)	(0.019508)	(0.069690)	(0.037697)
(30,10)	$(20, 0^{(*9)})$	1.5726783	1.679164	1.5583333	1.5118612	1.577635
		(0.049497)	(0.016256)	(0.006599)	(0.037893)	(0.030375)
	$(0^{(*9)}, 20)$	1.5606455	1.532979	1.4937966	1.5709883	1.551809
		(0.052299)	(0.018199)	(0.008920)	(0.048562)	(0.028330)
	$(1^{(*9)}, 11)$	1.5488888	1.528492	1.5104764	1.53875	1.541518
		(0.052173)	(0.021251)	(0.010957)	(0.041459)	(0.025673)
(50,10)	$(40, 0^{(*9)})$	1.5386683	1.4588715	1.5030030	1.5215169	1.5753069
		(0.035499)	(0.010732)	(0.001200)	(0.018775)	(0.017517)
	$(0^{(*9)}, 40)$	1.5258612	1.4993016	1.5014766	1.5292965	1.5303916
		(0.041866)	(0.012844)	(0.001806)	(0.023460)	(0.021161)
	$(1^{(*9)}, 31)$	1.4886420	1.5090797	1.4998359	1.5215494	1.5142079
		(0.036898)	(0.015479)	(0.004064)	(0.032626)	(0.022605)

Table 3: Average estimates and MSEs of λ for different choices of n, m, R.

(n,m)	Scheme	ML Estimates (MSE)	Bayes Estimates (MSE)			
			MH		Lindley	
			SE	LINEX	SE	LINEX
(20,5)	$(15, 0^{(*4)})$	0. 905058	0.934505	0.928671	0.956111	0.947554
		(0. 35192)	(0.297249)	(0.225306)	(0.337363)	(0.302164)
	$(0^{(*4)}, 15)$	0. 853306	0.925321	0.921998	0.944647	0.937500
		(0. 50277)	(0.286554)	(0.260112)	(0.352095)	(0.304087)
	$(1^{(*4)}, 11)$	0. 980234	0.920338	0.899202	0.954674	0.946888
		(0. 41815)	(0.245195)	(0.242329)	(0.394773)	(0.353580)
(30,5)	$(25, 0^{(*4)})$	0. 838128	0.925698	0.909869	0.958764	0.930903
		(0.221253)	(0.138571)	(0.084311)	(0.205419)	(0.156504)
	$(0^{(*4)}, 25)$	0. 925789	0.917829	0.902327	0.877821	0.919888
		(0. 44171)	(0.146017)	(0.093094)	(0.213053)	(0.190984)
	$(1^{(*4)}, 21)$	0. 931758	0.932537	0.916314	0.941982	0.938950
		(0. 32812)	(0.165230)	(0.108186)	(0.226668)	(0.191586)
(50,5)	$(45, 0^{(*4)})$	0.9424042	0.920989	0.917361	0.937266	0.919738
		(0. 24406)	(0.082352)	(0.078820)	(0.172845)	(0.084325)
	$(0^{(*4)}, 45)$	0.961113	0.938697	0.905894	0.877932	0.908586
		(0. 31607)	(0.115600)	(0.064075)	(0.125589)	(0.121338)
	$(1^{(*4)}, 41)$	0. 951653	0.927440	0.916924	0.894583	0.914329
		(0.307508)	(0.089176)	(0.045698)	(0.142023)	(0.107323)
(20,10)	$(10, 0^{(*9)})$	0.9711246	0.911596	0.908144	0.935303	0.920928
		(0.093229)	(0.039327)	(0.029708)	(0.067514)	(0.059061)
	$(0^{(*9)}, 10)$	0.9755523	0.927382	0.898417	0.941941	0.931410
		(0. 16262)	(0.044925)	(0.029182)	(0.079327)	(0.060760)
	$(1^{(*10)})$	0. 970426	0.915577	0.890628	0.951591	0.940845
		(0.117252)	(0.051305)	(0.038391)	(0.065227)	(0.057113)
(30,10)	$(20, 0^{(*9)})$	0. 954797	0.916033	0.908008	0.931885	0.927812
		(0.080069)	(0.019406)	(0.007282)	(0.030177)	(0.029618)
	$(0^{(*9)}, 20)$	0.8392846	0.919814	0.908004	0.959424	0.938397
		(0. 10138)	(0.009467)	(0.009399)	(0.033917)	(0.030714)
	$(1^{(*9)}, 11)$	0. 992795	0.919067	0.892437	0.940182	0.931076
		(0.099479)	(0.010368)	(0.007216)	(0.049406)	(0.047657)
(50,10)	$(40, 0^{(*9)})$	0. 97788	0.881292	0.907898	0.879053	0.928585
		(0.023870)	(0.002937)	(0.002014)	(0.021181)	(0.017774)
	$(0^{(*9)}, 40)$	0. 973283	0.889757	0.906250	0.935694	0.928457
		(0.052107)	(0.003062)	(0.001943)	(0.023195)	(0.021684)
	$(1^{(*9)}, 31)$	0. 891976	0.916897	0.902522	0.937081	0.923580
		(0.028989)	(0.005017)	(0.002995)	(0.027508)	(0.025798)

Table 4: The average approximate intervals when $\alpha = 1.7$, $\beta = 1.5$ and $\lambda = 0.9$.

(n,m)	Scheme	%95 Approximate interval					
		α	Length (CP)	β	Length (CP)	λ	Length (CP)
(20,5)	$(15, 0^{(*4)})$	[1.3484,2.0460]	0.6980 (0.940)	[0.6435,2.4840]	1.8405 (0.930)	[0.8536,1.1731]	0.3195 (0.941)
	$(0^{(*4)}, 15)$	[1.2756,2.3609]	1.0853 (0.928)	[0.8816 2.4740]	1.5924 (0.926)	[0.7720,1.2269]	0.4549 (0.929)
	$(1^{(*4)}, 11)$	[1.4257,2.1614]	0.7357 (0.935)	[0.6717,2.2071]	1.5354 (0.937)	[0.7550,1.1088]	0.3538 (0.939)
(30,5)	$(25, 0^{(*4)})$	[1.4069,1.9652]	0.5583 (0.943)	[1.1920 1.8669]	0.6749 (0.943)	[0.8305,1.0338]	0.2033 (0.931)
	$(0^{(*4)}, 25)$	[1.3511,1.9411]	0.5900 (0.942)	[1.2895,2.0897]	0.8002 (0.941)	[0.7875,1.0273]	0.2398 (0.932)
	$(1^{(*4)}, 21)$	[1.0438,1.9013]	0.8575 (0.959)	[0.9394,2.1482]	1.2088 (0.938)	[0.8979,1.1451]	0.2472 (0.940)
(50,5)	$(45, 0^{(*4)})$	[1.5114,1.9224]	0.4110 (0.942)	[1.3531,1.9934]	0.6403 (0.944)	[0.8806,1.0509]	0.1703 (0.960)
	$(0^{(*4)}, 45)$	[1.4241,1.9079]	0.4838 (0.937)	[1.2771,2.0096]	0.7325 (0.958)	[0.8241,0.9994]	0.1753 (0.931)
	$(1^{(*4)}, 41)$	[1.6109,2.1238]	0.5129 (0.956)	[1.3776,2.0273]	0.6497 (0.940)	[0.8710,1.0493]	0.1783 (0.942)
(20,10)	$(10, 0^{(*9)})$	[1.4614,1.9386]	0.4772 (0.943)	[1.3676,1.7898]	0.4222 (0.934)	[0.8025,0.9550]	0.1525 (0.936)
	$(0^{(*9)}, 10)$	[1.3966,1.8064]	0.4098 (0.938)	[1.3684,1.8890]	0.5206 (0.946)	[0.7986,0.9618]	0.1632 (0.941)
	$(1^{(*10)})$	[1.4830,1.8812]	0.3982 (0.941)	[1.3377,1.8733]	0.5356 (0.934)	[0.8524,0.9797]	0.1273 (0.940)
(30,10)	$(20, 0^{(*9)})$	[1.5523,1.8992]	0.3469 (0.946)	[1.3299,1.6701]	0.3402 (0.933)	[0.8606,1.0244]	0.1638 (0.930)
	$(0^{(*9)}, 20)$	[1.5615,1.9198]	0.3583) (0.940)	[1.4679,1.8178]	0.3499 (0.959)	[0.8648,1.0032]	0.1384 (0.943)
	$(1^{(*9)}, 11)$	[1.4930,1.8951]	0.4021 (0.940)	[1.2965,1.7038]	0.4073 (0.960)	[0.8782,1.0193]	0.1411 (0.958)
(50,10)	$(40, 0^{(*9)})$	[1.5546,1.8457]	0.2911 (0.946)	[1.3419,1.6579]	0.3160 (0.957)	[0.8378,0.9489]	0.1111 (0.946)
	$(0^{(*9)}, 40)$	[1.4791,1.7647]	0.2856 (0.942)	[1.4810,1.8142]	0.3332 (0.949)	[0.8926,1.0172]	0.1246 (0.950)
	$(1^{(*9)}, 31)$	[1.4315,1.7832]	0.3517 (0.955)	[1.4190,1.7468]	0.3278 (0.956)	[0.8373,0.9689]	0.1316 (0.953)

Table 5: MCMC intervals when $\alpha = 1.7$, $\beta = 1.5$ and $\lambda = 0.9$.

(n,m)	Scheme	%95 HPD credible intervals					
		α	Length (CP)	β	Length (CP)	λ	Length (CP)
(20,5)	$(15, 0^{(*4)})$	[1.3633,1.9093]	0.5460 (0.940)	[1.3385,2.1700]	0.8315 (0.938)	[0.8642,1.0917]	0.2275 (0.939)
	$(0^{(*4)}, 15)$	[1.6944,2.0456]	0.3512 (0.938)	[1.0142,2.1192]	1.1050 (0.930)	[0.7797,1.0215]	0.2418 (0.942)
	$(1^{(*4)}, 11)$	[1.5089,1.8509]	0.3420 (0.939)	[1.0827,2.4839]	1.4012 (0.940)	[0.7739,1.0473]	0.2734 (0.942)
(30,5)	$(25, 0^{(*4)})$	[1.5504,1.8515]	0.3011 (0.946)	[1.1908,1.8425]	0.6517 (0.960)	[0.7919,0.9960]	0.2041 (0.934)
	$(0^{(*4)}, 25)$	[1.3122,1.6427]	0.3305 (0.944)	[1.3191,1.8908]	0.5717 (0.958)	[0.8067,0.9935]	0.1868 (0.947)
	$(1^{(*4)}, 21)$	[1.4197,1.8468]	0.4271 (0.949)	[1.1520,1.9337]	0.7817 (0.942)	[0.8780,1.0495]	0.1715 (0.946)
(50,5)	$(45, 0^{(*4)})$	[1.5442,1.7985]	0.2543 (0.956)	[1.4291,1.8210]	0.3919 (0.937)	[0.8722,0.9840]	0.1118 (0.957)
	$(0^{(*4)}, 45)$	[1.6789,1.9616]	0.2827 (0.938)	[1.3185,1.7114]	0.3929 (0.948)	[0.8793,1.0105]	0.1312 (0.944)
	$(1^{(*4)}, 41)$	[1.3779,1.7416]	0.3637 (0.942)	[1.4105,1.8781]	0.4676 (0.955)	[0.8788,1.0488]	0.1700 (0.941)
(20,10)	$(10, 0^{(*9)})$	[1.3592,1.6408]	0.2816 (0.941)	[1.4414,1.7507]	0.3093 (0.934)	[0.8644,0.9660]	0.1016 (0.959)
	$(0^{(*9)}, 10)$	[1.5933,1.8734]	0.2801 (0.958)	[1.4669,1.8313]	0.3644 (0.942)	[0.8594,0.9687]	0.1093 (0.943)
	$(1^{(*10)})$	[1.4834,1.7750]	0.2916 (0.936)	[1.4038,1.7923]	0.3885 (0.956)	[0.8511,0.9571]	0.106 (0.940)
(30,10)	$(20, 0^{(*9)})$	[1.6858,1.9267]	0.2409 (0.945)	[1.3457,1.6543]	0.3086 (0.941)	[0.8828,0.9809]	0.0981 (0.943)
	$(0^{(*9)}, 20)$	[1.5882,1.8384]	0.2502 (0.947)	[1.3954,1.6896]	0.2942 (0.946)	[0.8769,0.9757]	0.0988 (0.941)
	$(1^{(*9)}, 11)$	[1.6447,1.9085]	0.2638 (0.941)	[1.4307,1.6876]	0.2569 (0.944)	[0.8501,0.9499]	0.0998 (0.960)
(50,10)	$(40, 0^{(*9)})$	[1.6895 1.7435]	0.0540 (0.951)	[1.4791,1.6106]	0.1315 (0.942)	[0.8817,0.9516]	0.0998 (0.947)
	$(0^{(*9)}, 40)$	[1.6558 1.7903]	0.1345 (0.944)	[1.4968,1.6374]	0.1406 (0.953)	[0.8667,0.9489]	0.0822 (0.945)
	$(1^{(*9)}, 31)$	[1.6807,1.8871]	0.2064 (0.942)	[1.4518,1.6332]	0.1814 (0.947)	[0.8989,0.9768]	0.0779 (0.943)

6 Real Life Data Example

This section illustrates application of EW distribution using a real data set. The real data contains 100 observations which are the fracture stresses of carbon fibers. Nichols and Padgett (2006) considered this data for Bootstrap control chart. The data are as follows:

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.8, 1.84, 1.84, 1.87, 1.89, 1.92, 2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.6, 3.65, 3.68, 3.68, 3.68, 3.7, 3.75, 4.2, 4.38, 4.42, 4.7, 4.9, 4.91, 5.08, 5.56.

We compare the fits of the EW distribution with some of the related models such as exponentiated Rayleigh (ER), exponentiated exponential (EE) and Gumbel (GU) distributions. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and log-likelihood (Log-L) for the fitted models are listed in Table 6. According to the criteria AIC, BIC and Log-L, we found that EW distribution is the best fitted model compared to the other proposed models. We have obtained the MLEs by using SEM algorithm by taking initial values with the help of contour and 3D profile plot given in Figure 1.

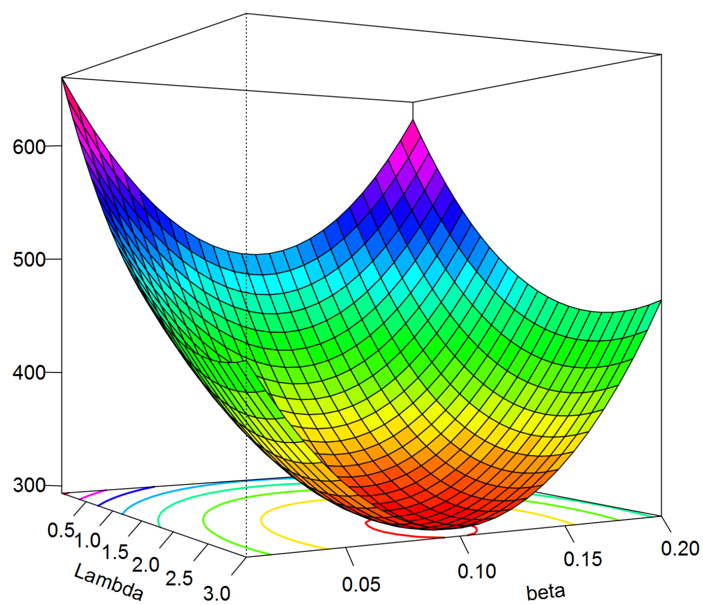


Figure 1: Contour plot and 3-D profile plot of minus log likelihood for real data set ($\alpha \simeq 1.32$).

Now, we generate three progressive Type II censored samples (Sc) from the above dataset as:

• Sc1: n=100, m=90, $(0^{*89}, 10)$:

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.8, 1.84, 1.84, 1.87, 1.89, 1.92, 2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.6, 3.65, 3.68, 3.68, 3.68.

• Sc2: n=100, m=90, $(10, 0^{*89})$:

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.8, 1.84, 1.84, 1.87, 1.89, 1.92, 2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.6, 3.65, 3.68, 3.68, 3.68, 3.7, 3.75, 4.2, 4.38, 4.42, 4.7, 4.9, 4.91, 5.08, 5.56.

• Sc3: n=100, m=50, (1^{*50}):

0.39, 0.85, 1.47, 1.57, 1.57, 1.59, 1.61, 1.69, 1.73, 1.84, 1.87, 1.92, 2.03, 2.05, 2.17, 2.35, 2.41, 2.48, 2.48, 2.5, 2.53, 2.55, 2.56, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.82, 2.85, 2.88, 2.95, 2.97, 3.11, 3.15, 3.22, 3.27, 3.31, 3.33, 3.39, 3.6, 3.68, 3.68, 3.75, 4.38, 4.7, 4.91, 5.56.

So, we consider point and interval estimates based on this data set for different censoring schemes. In Tables 7-9, we have presented maximum likelihood and Bayes estimates of and respectively. For the purpose of Bayes estimates, we use non-informative prior. The %95 approximate confidence intervals of the parameters based on MLEs and the corresponding HPD credible intervals are presented in Tables 10 and 11 respectively. We have plotted the fitted distribution functions based on maximum likelihood and Bayes estimators in Figure 2 for all schemes. It indicates the closeness of the different estimates. As seen in Tables 7-11, two types of point estimates of parameters are observed: The MLEs and Bayes estimates are quite similar. Comparing approximate and credible intervals derived from Bayesian method, the latter are noticeably smaller in interval lengths than the former. We also plotted the fitted distribution functions (based on MLEs, Lindley and MCMC) using the complete data and the data in the three censoring schemes in Figure 3. The plot indicates that different schemes are close to complete case. We further present the trace plots of the first 12000 MCMC outputs for posterior distribution of and in Figure 4. We discard the first 3000 values as the burn-in samples $M = 3000$. It is observed that the MCMC procedure converges very well.

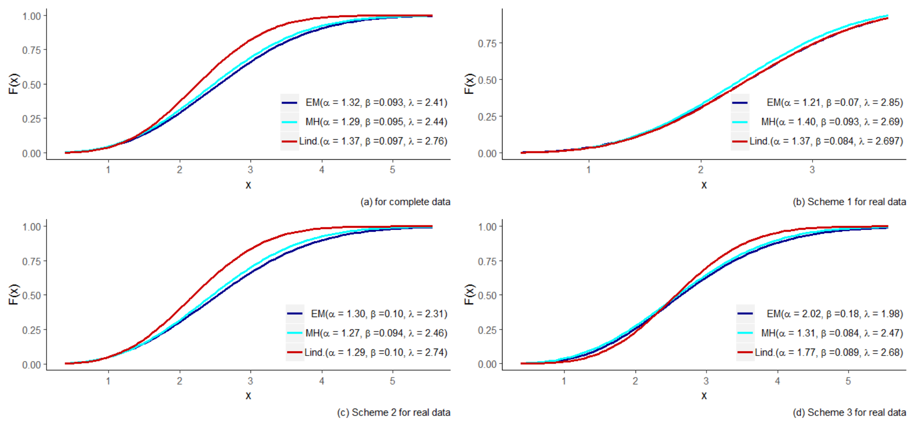


Figure 2: The fitted distribution functions based on MLEs and Bayes estimators (Lindley and MCMC) for various progressive censoring schemes.

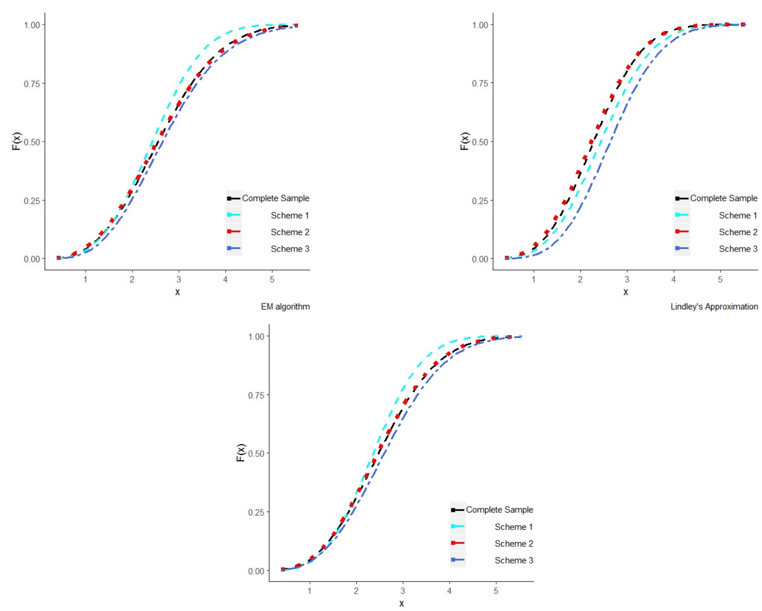


Figure 3: The fitted distribution functions for complete and different censoring schemes.

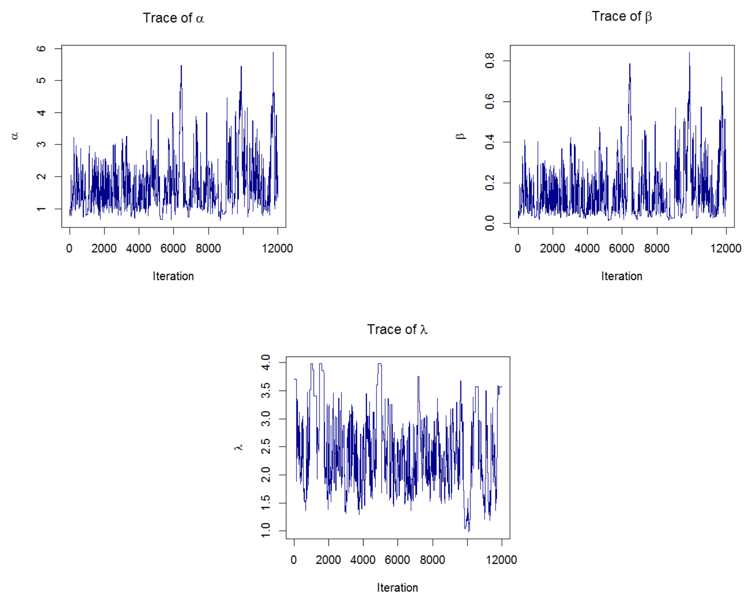


Figure 4: The Trace plots of 12000 iterations of complete sample for α, β and λ.

Table 6: The model selection criteria for different models..

models	AIC	BIC	Log-L
EW	288.6641	296.4796	-141.332
ER	353.6813	358.8917	-174.8407
EE	296.3646	301.5749	-146.1823
GU	350.2879	355.4982	-173.144

Table 7: The MLEs and Bayesian estimates of α for different censoring schemes.

m	Scheme	ML Estimates	Bayes Estimates			
			MH		Lindley	
			SE	LINEX	SE	LINEX
90	$(0^{(*89)}, 10)$	1.2074780	1.401999	1.399336	1.35625667	1.36515677
90	$(10, 0^{(*89)})$	1.3049673	1.267034	1.266763	1.2422398	1.2947901
50	$(1^{(*50)})$	2.0160582	1.305492	1.305333	1.70568279	1.76773479
100	$(0^{(*100)})$	1.31630695	1.29092	1.290441	1.32161122	1.36505301

Table 10: The%95 approximate confidence intervals for real data.

m	Scheme	α	Length	β	Length	λ	Length
90	$(0^{(*89)}, 10)$	[0.783829,1.631127]	0.847	[0.022790,0.108936]	0.086	[2.364218,3.331764]	0.967
90	$(10, 0^{(*89)})$	[0.826550,1.783384]	0.957	[0.071469,0.133788]	0.062	[2.075969,2.552973]	0.477
50	$(1^{(*50)})$	[[0.725373,3.306743]	2.581	[0.007437,0.351090]	0.344	[1.394148,2.566253]	1.172
100	$(0^{(*100)})$	[0.731970,1.900644]	1.169	[0.016690,0.168826]	0.152	[1.886561,2.932246]	1.046

Table 8: The MLEs and Bayesian estimates of β for different censoring schemes.

m	Scheme	ML Estimates	Bayes Estimates			
			MH		Lindley	
			SE	LINEX	SE	LINEX
90	$(0^{(*89)}, 10)$	0.0658613	0.09305555	0.09300073	0.08370209	0.08380249
90	$(10, 0^{(*89)})$	0.1026281	0.09439744	0.09430456	0.0996575	0.1011385
50	$(1^{(*50)})$	0.1792627	0.08356092	0.08355689	0.08707891	0.08906368
100	$(0^{(*100)})$	0.09275743	0.0953368	0.0953207	0.09631612	0.09734263

Table 9: The MLEs and Bayesian estimates of λ for different censoring schemes.

m	Scheme	ML Estimates	Bayes Estimates			
			MH		Lindley	
			SE	LINEX	SE	LINEX
90	$(0^{(*89)}, 10)$	2.8479910	2.691114	2.686073	2.68611896	2.696619029
90	$(10, 0^{(*89)})$	2.3144712	2.459094	2.458932	2.6635793	2.7362913
50	$(1^{(*50)})$	1.9801980	2.47313	2.469045	2.60971940	2.67843428
100	$(0^{(*100)})$	2.40940313	2.445127	2.443417	2.69920804	2.75821442

Table 11: The HPD credible intervals for real data.

m	Scheme	α	Length	β	Length	λ	Length
90	$(0^{(*89)}, 10)$	[1.169731,1.601055]	0.431	[0.063454,0.137771]	0.074	[2.316318,2.951086]	0.635
90	$(10, 0^{(*89)})$	[1.045238,1.574544]	0.529	[0.037438,0.101257]	0.064	[1.91418,2.531702]	0.617
50	$(1^{(*50)})$	[1.127915,1.677771]	0.549	[0.042536,0.122022]	0.079	[2.132352,2.796492]	0.664
100	$(0^{(*100)})$	[1.141814,1.396714]	0.255	[0.075921,0.122702]	0.047	[2.163763,2.644759]	0.481

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