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E-Bayesian and Robust Bayesian Estimation and Prediction for the Exponential Distribution based on Record Values

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Abstract. This article deals with the problem of E-Bayesian and robust Bayesian estimation and prediction in the exponential distribution on the basis of record observations under the squared log error loss function. The E-Bayesian and robust Bayesian estimators of the scale parameter are computed and their performances are investigated using a simulation study. We extend the idea of E-Bayesian estimation to the E-Bayesian prediction of future record observation and perform a simulation study using a prequential analysis for comparison of proposed E-Bayesian and robust Bayesian predictors. Two real data sets are analyzed for illustrating the estimation and prediction results.

Keywords. E-Bayesian Estimation and Prediction, Exponential Distribution, Record Values, Robust Bayesian Estimation and Prediction, Squared Log Error Loss Function.

MSC: 62F15.

1 Introduction

Let $X_1, ..., X_n$ be a sequence of independent and identical distributed (i.i.d.) random variables from the exponential distribution with probability density function (p.d.f.)

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0, \;\; \theta > 0, \tag{1.1}$$

where θ is the unknown scale parameter. An observation x_j is said to be an upper record value if its value exceeds that of all previous observations. Thus, x_j is an upper (lower)

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record if $x_j > (<) x_i$ for every $i < j$. By convention x_1 is a record value. An analogous definition deals with lower record values. Data of this type arise in a wide variety of practical situations. Examples of application areas include industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys; see Arnold et al. (1998) for these types of applications.

Let R_m denote the mth upper record value. The joint density of the first m-records $\mathbf{R} = (R_1, ..., R_m)$ can be presented as

$$f_{R_1,\cdots,R_m}(r_1, ..., r_m) = f(r_m) \prod_{i=1}^{m-1} \frac{f(r_i)}{1 - F(r_i)}, \quad r_1 < r_2 < \cdots < r_m. \tag{1.2}$$

Also, the marginal p.d.f. of the mth record, R_m, is given by

$$f_{R_m}(x) = \frac{[-\log(1 - F(x))]^{m-1}}{(m-1)!} f(x). \tag{1.3}$$

Therefore, from (1.1) and (1.2), the likelihood function of θ based on $\mathbf{r} = (r_1, ..., r_m)$ is given by

$$L(\theta|\mathbf{r}) = \theta^m e^{-\theta r_m}, \;\; \theta > 0. \tag{1.4}$$

There are some methods in order to obtain an appropriate estimate of θ. In a classical method, we estimate θ based on the sample imformation, for example, the maximum likelihood estimator (MLE) denoted by T_m, and can be derived from the equation $\frac{\partial L(\theta|\mathbf{r})}{\partial \theta} = 0$, which is given by $T_m = m/R_m$. A Bayesian approach to a statistical problem requires defining a prior distribution over the parameter space and loss function. Many Bayesians believe that just one prior can be elicited. In practice, the prior knowledge is vague and any elicited prior distribution is only an approximation to the true one. Various solutions to this problem have been proposed. In robust Bayesian estimation or prediction, it is a common practice to construct optimal estimators and predictors by changing a prior whithin a class Γ of priors which seems to best match our personal beleifs, see Berger (1984), Rios Insua et al. (1995) and Arias-Nicolas et al. (2009) among others.

One of the recent proposed solutions is E-Bayesian estimation or prediction. The E-Bayesian estimator is the expectation of the Bayesian estimator of unknown parameter over the hyperparameter(s), which was first introduced by Han (1997). E-Bayesian estimation is investigated by Jaheen and Okasha (2011), Han (2017), Gonzalez-Lopez et al. (2017), Kiapour (2018), Han (2019), Okasha (2019), Piriaei et al. (2020) and Han (2021). We consider the E-Bayesian prediction of the future record observation as the expectation of the Bayesian predictor of future record observation over the hyperparameter(s).

In Bayesian inference, the most commonly used loss function is convex and symmetric squared error loss (SEL) function which is widely used in decision theory due to its simple mathematical properties. But in some cases, it does not represent the true loss structure. For example, it is not useful for estimation of the scale parameter and it assigns the same penalties to overestimation and underestimation. For estimation of the scale parameter θ, Brown (1968) proposed the Squared Log Error Loss (SLEL) function, which is given by

$$L(\theta, \delta) = (\ln \delta - \ln \theta)^2 = \left[\ln \frac{\delta}{\theta}\right]^2 = \ln^2 \Delta, \tag{1.5}$$

where both θ and δ are positive, see also Kiapour and Nematollahi (2011) for some estimation and prediction problems. This loss function is not symmetric and convex; it is convex for $\Delta = \frac{\delta}{\theta} \le e$ and concave otherwise, but has a unique minimum at $\Delta = 1$. Also when $\Delta > 1$, this loss increases sublinearly, while when $0 < \Delta < 1$, it rises rapidly to infinity at zero. Therefore it is useful in situations where underestimation is more serious than overestimation. It should be noticed here that an estimator δ of θ is risk-unbiased if $E(\ln \delta) = \ln \theta$. Therefore, the bias is defined as $E(\ln \delta) - \ln \theta$, see Lemma 4.1 in the Appendix A.

The paper is organized as follows. In section 2, we obtain the Bayesian, robust Bayesian and E-Bayesian estimators of θ under the loss function (1.5) and compare their performances using a Monte Carlo simulation. A real data example is applied for illustrating the estimation results. In section 3, we obtain the Bayesian and robust Bayesian predictions and extend the idea of E-Bayesian estimation to E-Bayesian prediction. We use a prequential analysis for comparison of obtained E-Bayesian predictors and ilustrate the proposed predictors by an application example. We end the paper by a concluding remark.

2 Bayesian Methods of Estimation

Let $\mathbf{r} = (r_1, ..., r_m)$ be the first m-records coming from the exponential distribution with parameter θ given in (1.1). In this section, we obtain the Bayesian and E-Bayesian estimators of θ and compare them by a simulation study.

2.1 Bayesian Estimation

Considering a *Gamma*(α, β) distribution of θ with p.d.f.

$$\pi(\theta|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \alpha > 0,\ \beta > 0,\ \theta > 0. \tag{2.1}$$

Using the likelihood function (1.4), it can be shown that the posterior distribution is *Gamma*$(m + \alpha, r_m + \beta)$. In estimation of θ, let $L(\theta, \delta)$ be the loss function (1.5). So, the posterior risk of $\delta = \delta(\mathbf{r})$ can be expressed as

$$\rho(\pi, \delta) \quad = \quad E[L(\theta, \delta)|\mathbf{r}]$$

$$= \quad \ln^2\delta + E[\ln^2\theta|\mathbf{r}] - 2\ln\delta E[\ln\theta|\mathbf{r}]. \tag{2.2}$$

The Bayesian estimate of θ based on observation $\mathbf{r}$ is any estimate $\delta^{\pi}(\mathbf{r})$ that minimizes the posterior risk (2.2), which is given by $\delta^B(\mathbf{r}) = e^{E[\ln\theta|\mathbf{r}]}$. Using the fact $\theta|\mathbf{r} \sim Gamma(m+\alpha, r_m+\beta)$ and applying Lemma 4.2 (see Appendix A), we have $E[\ln\theta|\mathbf{r}] = \Psi(m+\alpha) - \ln(r_m+\beta)$, where $\Psi(\nu) = \frac{d}{d\nu}\ln\Gamma(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$ is the digamma function and $\Gamma(\nu)$ denotes the complete gamma function given by $\Gamma(\nu) = \int_0^{\infty} t^{\nu-1}e^{-t}dt$. Therefore, the Bayesian estimate of θ is given by

$$\delta^B(\mathbf{r}) = e^{E[\ln\theta|\mathbf{r}]} = \frac{e^{\Psi(m+\alpha)}}{r_m+\beta}. \tag{2.3}$$

2.2 Robust Bayesian Estimation

Suppose that the prior distribution is not exactly specified and consider the following class of priors for θ

$$\Gamma \quad = \quad \{\pi(\theta|\alpha,\beta), \alpha \in [\alpha_1,\alpha_2] \subset R^+, \beta \in [\beta_1,\beta_2] \subset R^+\}, \tag{2.4}$$

where α_1, α_2, β_1 and β_2 are known. There are several robust Bayesian estimators but we focus on the posterior Regret Γ-Minimax (PRGM) estimator. The estimator δ^{RB} is called the PRGM estimator if, for all $\mathbf{r}$,

$$\sup_{\pi\in\Gamma} R(\delta^{RB},\delta^{\pi}) = \inf_{\delta\in D}\sup_{\pi\in\Gamma} R(\delta,\delta^{\pi}), \tag{2.5}$$

where $R(\delta,\delta^{\pi}) = \rho(\pi,\delta) - \rho(\pi,\delta^{\pi})$ is the posterior regret, which measures the loss of optimality due to choosing the estimator $\delta(\mathbf{r})$ instead of the optimal Bayes estimator $\delta^{\pi}(\mathbf{r})$.

In the following theorem, we obtain the PRGM estimator of θ under the loss function (1.5) with respect to the class of priors given in (2.4). For the proof and more information on PRGM derivation, readers may refer to Kiapour and Nematollahi (2011).

Theorem 2.1. *Let $\boldsymbol{r} = (r_1,...,r_m)$ be a first m-record observations from the exponential distribution and the prior π belongs to a class Γ of priors. Suppose $\underline{\delta}(\boldsymbol{r}) = \inf_{\pi\in\Gamma}\delta^B(\boldsymbol{r})$ and $\overline{\delta}(\boldsymbol{r}) = \sup_{\pi\in\Gamma}\delta^B(\boldsymbol{r})$ are finite and $\underline{\delta} < \overline{\delta}$. Then, the PRGM estimates of θ corresponding to the prior given in (2.4) under the loss function (1.5) is given by*

$$\begin{aligned}\delta^{RB}(\boldsymbol{r}) &= \sqrt{\underline{\delta}(\boldsymbol{r})\overline{\delta}(\boldsymbol{r})},\\ &= \sqrt{\frac{e^{\psi(m+\alpha_1)+\psi(m+\alpha_2)}}{(r_m+\beta_1)(r_m+\beta_2)}}.\end{aligned} \tag{2.6}$$

2.3 E-Bayesian Estimation

Consider a prior $\pi(\theta|\alpha,\beta)$ for θ with hyperparameters α and β. Following Han (1997), the E-Bayesian estimator of θ is the expectation of the Bayesian estimator for the all hyperparameters which is defined as

$$\delta^{EB}(\mathbf{r}) = \int\int_D \delta^B(\mathbf{r})\pi(\alpha,\beta)d\alpha d\beta = E(\delta^B(\mathbf{R})), \tag{2.7}$$

where D is the domain of α and β and $\pi(\alpha,\beta)$ is the prior density function of α and β.

Based on Han (1997), the prior parameters α and β should be selected to guarantee that $\pi(\theta|\alpha,\beta)$ is a decreasing function of θ. If we take the conjugate prior (2.1), hyperparameters α and β should be in the ranges $0 < \alpha < 1$ and $\beta > 0$, respectively, due to $\frac{d\pi(\theta|\alpha,\beta)}{d\theta} < 0$. Prior distribution with thinner tail would worsen the robustness of the Bayesian distribution. Accordingly, β should not be too big while $0 < \alpha < 1$. For $\beta > 0$, there is a constant, say c, such that $0 < \beta < c$.

Assume that α and β are independent with bivariate density function $\pi(\alpha,\beta) = \pi(\alpha)\pi(\beta)$. We consider three prior distributions of the hyperparameters α and β for E-Bayesian estimation of θ:

$$\begin{aligned}
\pi_1(\alpha,\beta) &= \frac{2(c-\beta)}{c^2 B(u,v)}\alpha^{u-1}(1-\alpha)^{v-1}, \quad 0<\alpha<1,\ 0<\beta<c,\\
\pi_2(\alpha,\beta) &= \frac{1}{cB(u,v)}\alpha^{u-1}(1-\alpha)^{v-1}, \quad 0<\alpha<1,\ 0<\beta<c,\\
\pi_3(\alpha,\beta) &= \frac{2\beta}{c^2 B(u,v)}\alpha^{u-1}(1-\alpha)^{v-1}, \quad 0<\alpha<1,\ 0<\beta<c.
\end{aligned} \tag{2.8}$$

These distributions are used to investigate the influence of the different prior distributions on the E-Bayesian estimation, see Jaheen and Okasha (2011) and Naghizadeh Qomi and Kiapour (2020).

Theorem 2.2. *Let $\boldsymbol{r} = (r_1, ..., r_m)$ be a first m-record observations from the exponential distribution. Then, the E-Bayesian estimates of θ corresponding to the priors given in* (2.8) *under the loss function* (1.5) *are as follows:*

$$\begin{aligned}
\delta^{EB1}(\boldsymbol{r}) &= \frac{2I}{c^2}[(c+r_m)\ln(1+\frac{c}{r_m})-c], \qquad (2.9)\\
\delta^{EB2}(\boldsymbol{r}) &= \frac{I}{c}\ln(1+\frac{c}{r_m}),\\
\delta^{EB3}(\boldsymbol{r}) &= \frac{2I}{c^2}[c-r_m\ln(1+\frac{c}{r_m})],
\end{aligned}$$

where $I = \int_0^1 e^{\Psi(m+\alpha)}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{B(u,v)}d\alpha$.

Proof. For prior distribution $\pi_1(\alpha,\beta)$, the E-Bayesian estimate of θ under the loss function (1.5) is obtained as

$$\begin{aligned}
\delta^{EB_1}(\mathbf{r}) &= \int_0^c\int_0^1 \delta^B(\mathbf{r})\pi_1(\alpha,\beta)d\alpha d\beta \\
&= \int_0^1\int_0^c \frac{e^{\Psi(m+\alpha)}}{r_m+\beta}\frac{2(c-\beta)\alpha^{u-1}(1-\alpha)^{v-1}}{c^2B(u,v)}d\beta d\alpha \\
&= \int_0^1 e^{\Psi(m+\alpha)}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{B(u,v)}d\alpha\int_0^c\frac{2(c-\beta)}{c^2(r_m+\beta)}d\beta \\
&= \frac{2I}{c^2}\int_0^c\frac{c-\beta}{r_m+\beta}d\beta \\
&= \frac{2I}{c^2}[(c+r_m)\ln(1+\frac{c}{r_m})-c]. \qquad (2.10)
\end{aligned}$$

Similarly, if the prior distributions of (α,β) are given by $\pi_2(\alpha,\beta)$ and $\pi_3(\alpha,\beta)$, then, the corresponding E-Bayesian estimates under the loss function (1.5) will be obtained, respectively as

$$\begin{aligned}
\delta^{EB_2}(\mathbf{r}) &= \int_0^1\int_0^c \delta^B(\mathbf{r})\pi_2(\alpha,\beta)d\alpha d\beta \\
&= \int_0^1\int_0^c \frac{e^{\Psi(m+\alpha)}}{r_m+\beta}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{cB(u,v)}d\beta d\alpha \\
&= \int_0^1 e^{\Psi(m+\alpha)}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{B(u,v)}d\alpha\int_0^c\frac{1}{c(r_m+\beta)}d\beta \\
&= \frac{I}{c}\ln(1+\frac{c}{r_m}), \qquad (2.11)
\end{aligned}$$

and

$$\begin{aligned}
\delta^{EB_3}(\mathbf{r}) &= \int_0^1\int_0^c \delta^B(\mathbf{r})\pi_3(\alpha,\beta)d\alpha d\beta \\
&= \int_0^1\int_0^c \frac{e^{\Psi(m+\alpha)}}{r_m+\beta}\frac{2\beta\alpha^{u-1}(1-\alpha)^{v-1}}{c^2B(u,v)}d\beta d\alpha \\
&= \int_0^1 e^{\Psi(m+\alpha)}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{B(u,v)}d\alpha\int_0^c\frac{2\beta}{c^2(r_m+\beta)}d\beta \\
&= \frac{2I}{c^2}\int_0^c\frac{\beta}{r_m+\beta}d\beta \\
&= \frac{2I}{c^2}[c-r_m\ln(1+\frac{c}{r_m})]. \qquad (2.12)
\end{aligned}$$

□

2.4 Monte Carlo Simulation and Comparisons

In this section, we perform a simulation study for comparison of proposed estimators of θ. For this purpose, we generate i.i.d. random observations $x_1, ..., x_n$ from the exponential distribution with parameter $\theta = 5$ and obtain record observations $r_1, ..., r_m$ for values $m = 3, 5, 7, 9$ by using $R_k = \sum_{i=1}^{k} X_i$, where $R_k \sim Gamma(k, \theta)$, see Awad and Raqab (2000). The performance of Bayesian, E-Bayesian and robust Bayesian estimates have been compared for repeated $M = 10000$ times simulation runs in terms of estimated bias as

$$EB(\delta^{(k)}) = \frac{1}{M} \sum_{i=1}^{M} (\ln \delta_i^{(k)} - \ln \theta), \tag{2.13}$$

and estimated risk as

$$ER(\delta^{(k)}) = \frac{1}{M} \sum_{i=1}^{M} (\ln \delta_i^{(k)} - \ln \theta)^2, \tag{2.14}$$

where $\delta^{(1)}$ is the Bayesian estimate δ^B in (2.3) with $\alpha = 0.6$ and $\beta = 3$, $\delta^{(2)}$ is the robust Bayesian estimate δ^{RB} in (2.6) with $\alpha_1 = 0.1$, $\alpha_2 = 0.7$, $\beta_1 = 1$ and $\beta_2 = 3$ and $\delta^{(3)}$, $\delta^{(4)}$ and $\delta^{(5)}$ are the E-Bayesian estimates δ^{EBi}, $i = 1, 2, 3$ given in (2.9) for selected values of $c = 3, 3.5$, $u = 3$ and $v = 2$.

The estimated values of bias and risk of the estimators are summarized in Table 1. It is observed from Table 1 that all E-Bayesian estimates are clearly biased. Moreover, δ^{EB1} has less bias when compared to other estimators. As can be observed, the robust Bayesian and E-Bayesian estimates performs well compared to the Bayesian estimates in terms of risk. Also, the estimated risk of the estimates decreases (increases) as the values of $m(c)$ increases.

Table 1: Estimated bias (in first row) and risk (in second row) for Bayesian, robust Bayesian and E-Bayesian estimates.

			δ^{EB1}		δ^{EB2}		δ^{EB3}	
m	δ^B	δ^{RB}	$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$
3	1.7437	1.3972	0.7231	0.9054	1.0042	1.2169	1.4049	1.6805
	3.0480	1.9720	0.6138	0.8951	1.0706	1.5315	1.9946	2.8377
5	1.3558	1.0407	0.5525	0.7062	0.7798	0.9658	1.0789	1.3225
	1.8489	1.1079	0.3806	0.5614	0.6613	0.9760	1.1878	1.7651
7	1.1109	0.8238	0.4266	0.5629	0.6231	0.7921	0.8709	1.0933
	1.2775	0.7062	0.2486	0.3728	0.4369	0.6669	0.7843	1.2130
9	0.9534	0.6965	0.3680	0.4889	0.5384	0.6915	0.7456	0.9478
	0.9224	0.5108	0.1881	0.2837	0.3294	0.5104	0.5794	0.9149

2.5 Real Data Analysis

Consider a data set of Dunsmore (1983), in which a rock crushing machine is kept working as long as the size of the crushed rock is larger than the rocks crushed before.

Otherwise it is reset. The following data show the sizes of the crushed rocks up to the third reset of the machine:

$$9.3 \quad 0.6 \quad 24.4 \quad 18.1 \quad 6.6 \quad 9.0 \quad 14.3 \quad 6.6 \quad 13 \quad 2.4 \quad 5.6 \quad 33.8.$$

The Kolmogorov-Smirnov (K-S) test with test statistics 0.2069 and a corresponding p-value = 0.6835 implies that the exponential distribution with mean 0.084 have a good fit to this data set. The observed upper record values $\mathbf{r} = (r_1, r_2, r_3)$ are obtained to be $\mathbf{r} = (9.3, 24.4, 33.8)$. Then, the maximum likelihood estimate is $t_3 = \frac{3}{r_3} = 0.089$. The Bayesian estimate δ^B with $\alpha = 0.6$ and $\beta = 3$, robust Bayesian estimate δ^{RB} with $\alpha_1 = 0.1$, $\alpha_2 = 0.7$, $\beta_1 = 1$ and $\beta_2 = 3$ and the E-Bayesian estimates δ^{EBi} $(i = 1, 2, 3)$ with values $u = 3$, $v = 2$ and $c = 3, 3.5$ and corresponding absolute error, $\Delta\delta = |\delta - 0.089|$, are summarized in Table 2. It is observed that the Bayesian, robust Bayesian and E-Bayesian estimates are close together and are all robust. Moreover, the E-Bayesian estimate δ^{EB1} outperforms other estimators.

Table 2: Results for estimates and corresponding absolute error of estimates.

δ^B	δ^{RB}	δ^{EB1}		δ^{EB2}		δ^{EB3}	
		$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$
0.0846	0.0866	0.08908	0.08788	0.08765	0.08590	0.08621	0.08393
$\Delta\delta^B$	$\Delta\delta^{RB}$	$\Delta\delta^{EB1}$		$\Delta\delta^{EB2}$		$\Delta\delta^{EB3}$	
		$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$	$c = 3$	$c = 3.5$
0.0044	0.0024	0.00008	0.0011	0.0014	0.0031	0.0028	0.0051

3 Bayesian Approches of Prediction

Making prediction is one of the fundamental objectives of statistical modeling, and a Bayesian approach can make this task reasonably straight forward. Let $\mathbf{r} = (r_1, ..., r_m)$ be the first m-records and r_{m+1} be a future record random variable coming from the exponential distribution with parameter θ given in (1.1). Our goal is to predict the future record r_{m+1} on the basis of currently observed data $\mathbf{r} = (r_1, ..., r_m)$. The predictive distribution is given by

$$h(r_{m+1}|\mathbf{r}) = \int_{\Theta} f(r_{m+1}|\theta)\pi(\theta|\mathbf{r})d\mu(\theta), \tag{3.1}$$

where $\theta \in \Theta$ has the prior density π w.r.t. σ-finite measure μ. In this section, we obtain the Bayesian and E-Bayesian predictors of r_{m+1} by δ_{m+1} from past observation $\mathbf{r}$ under the Squared Log Error Prediction Loss (SLEPL) function given by

$$L(r_{m+1}, \delta_{m+1}) = (\ln \delta_{m+1} - \ln r_{m+1})^2 = [\ln \frac{\delta_{m+1}}{r_{m+1}}]^2. \tag{3.2}$$

3.1 Bayesian Prediction

The posterior risk of the predictor δ_{m+1} unedr the loss function (3.2) is given by

$$\begin{aligned}\rho(\pi, \delta_{m+1}) &= E[L(r_{m+1}, \delta_{m+1}) \mid \mathbf{r}] \\ &= \ln^2 \delta_{m+1} + E[\ln^2 R_{m+1}|\mathbf{r}] - 2 \ln \delta_{m+1} E[\ln r_{m+1}|\mathbf{r}]. \end{aligned} \tag{3.3}$$

The Bayesian predictor is obtained by minimizing (3.3) over δ_{m+1} as

$$\hat{\delta}^B_{m+1}(\mathbf{r}) = e^{E[\ln r_{m+1}|\mathbf{r}]}. \tag{3.4}$$

From (1.3), we get $R_{m+1} \sim Gamma(m+1, \theta)$ and, using Lemma 4.2, we obtain $E[\ln R_{m+1}|\theta] = \Psi(m+1) - \ln\theta$. Therefore, we have

$$\begin{aligned}E[\ln R_{m+1}|\mathbf{r}] &= E\{E[\ln R_{m+1}|\theta]|\mathbf{r}\} \\ &= E[\Psi(m+1) - \ln\theta|\mathbf{r}] \\ &= \Psi(m+1) - \Psi(m+\alpha) + \ln(r_m + \beta). \end{aligned} \tag{3.5}$$

So, the Bayesian predictor of the random variable R_{m+1} under the loss function (3.2) is given by

$$\hat{\delta}^B_{m+1}(\mathbf{r}) = e^{E[\ln R_{m+1}|\mathbf{r}]} = (r_m + \beta) e^{\Psi(m+1) - \Psi(m+\alpha)}. \tag{3.6}$$

3.2 Robust Bayesian Prediction

We consider a class Γ of priors (2.4) and recall that the predictor $\hat{\delta}^{RB}_{m+1}$ is called the posterior Regret Γ-Minimax (PRGM) predictor if, for all $\mathbf{r}$,

$$\sup_{\pi\in\Gamma} R(\hat{\delta}^{RB}_{m+1}, \hat{\delta}^{\pi}_{m+1}) = \inf_{\delta_{m+1}\in D} \sup_{\pi\in\Gamma} R(\delta_{m+1}, \hat{\delta}^{\pi}_{m+1}), \tag{3.7}$$

where $R(\delta_{m+1}, \hat{\delta}^{\pi}_{m+1}) = \rho(\pi, \delta_{m+1}) - \rho(\pi, \hat{\delta}^{\pi}_{m+1})$ is the posterior regret, which measures the loss of optimality due to choosing the predictor $\delta_{m+1}(\mathbf{r})$ instead of the optimal Bayes predictor $\hat{\delta}^{\pi}_{m+1}(\mathbf{r})$.

In the following theorem, we obtain the PRGM predictor of R_{m+1} under the loss function (3.2) with respect to the class of priors given in (2.4). For the proof and more information on PRGM derivation, readers may refer to Kiapour and Nematollahi (2011).

Theorem 3.1. *Let $\boldsymbol{r} = (r_1, ..., r_m)$ be a first m-record observations from the exponential distribution and the prior π belongs to a class Γ of prior. Suppose $\underline{\hat{\delta}}_{m+1}(\boldsymbol{r}) = \inf_{\pi\in\Gamma} \hat{\delta}^B_{m+1}(\boldsymbol{r})$ and $\overline{\hat{\delta}}_{m+1}(\boldsymbol{r}) = \sup_{\pi\in\Gamma} \hat{\delta}^B_{m+1}(\boldsymbol{r})$ are finite and $\underline{\hat{\delta}}_{m+1} < \overline{\hat{\delta}}_{m+1}$. Then, the PRGM predictor of r_{m+1} corresponding to the prior given in* (2.4) *under the loss function* (3.2) *is given by*

$$\begin{aligned}\hat{\delta}^{RB}_{m+1}(\boldsymbol{r}) &= \sqrt{\underline{\hat{\delta}}_{m+1}(\boldsymbol{r}) \overline{\hat{\delta}}_{m+1}(\boldsymbol{r})}, \\ &= e^{\psi(m+1)} \sqrt{(r_m + \beta_1)(r_m + \beta_2) e^{-(\psi(m+\alpha_1) + \psi(m+\alpha_2))}}. \end{aligned} \tag{3.8}$$

3.3 E-Bayesian Prediction

The E-Bayesian predictor of the future record r_{m+1} is defined as the expectation of the Bayesian predictor w.r.t. the prior $\pi(\alpha,\beta)$ and is given as follows:

$$\hat{\delta}^{EB}_{m+1}(\mathbf{r}) = \int\int_D \hat{\delta}^{B}_{m+1}(\mathbf{r})\pi(\alpha,\beta)d\alpha d\beta = E(\hat{\delta}^{B}_{m+1}(\mathbf{r})). \tag{3.9}$$

In the next theorem, we obtain the E-Bayesian predictors under the loss function (3.2) w.r.t. the prior distributions given in (2.8).

Theorem 3.2. *Let $\boldsymbol{r} = (r_1, ..., r_m)$ be a first m-record random observations from the exponential distribution. Then, the E-Bayesian predictors of r_{m+1} corresponding to the priors given in (2.8) under the loss function* (3.2) *are given by*

$$\begin{aligned}
\hat{\delta}^{EB_1}_{m+1}(\boldsymbol{r}) &= J(r_m + \frac{c}{3}), \qquad (3.10)\\
\hat{\delta}^{EB_2}_{m+1}(\boldsymbol{r}) &= J(r_m + \frac{c}{2}),\\
\hat{\delta}^{EB_3}_{n+1}(\boldsymbol{r}) &= J(r_m + \frac{2c}{3}),
\end{aligned}$$

where $J = \frac{e^{\Psi(m+1)}}{B(u,v)}\int_0^1 e^{-\Psi(m+\alpha)}\alpha^{u-1}(1-\alpha)^{v-1}d\alpha.$

Proof. For $\pi_1(\alpha,\beta)$, the E-Bayesian predictor under the loss function (3.2) is obtained as

$$\begin{aligned}
\hat{\delta}^{EB_1}_{m+1}(\mathbf{r}) &= \int_0^1\int_0^c \hat{\delta}^{B}_{m+1}(\mathbf{r})\pi_1(\alpha,\beta)d\beta d\alpha\\
&= \int_0^1\int_0^c (r_m+\beta)e^{\Psi(m+1)-\Psi(m+\alpha)}\frac{2(c-\beta)\alpha^{u-1}(1-\alpha)^{v-1}}{c^2B(u,v)}d\beta d\alpha\\
&= \frac{e^{\Psi(m+1)}}{B(u,v)}\int_0^1 e^{-\Psi(m+\alpha)}\alpha^{u-1}(1-\alpha)^{v-1}d\alpha\int_0^c\frac{2(r_m+\beta)(c-\beta)}{c^2}d\beta\\
&= J\int_0^c\frac{2(r_m+\beta)(c-\beta)}{c^2}d\beta\\
&= J[R_m + \frac{c}{3}]. \qquad (3.11)
\end{aligned}$$

Similarly, if the prior distributions of α and β are given by $\pi_2(\alpha,\beta)$ and $\pi_3(\alpha,\beta)$ respectively, then the corresponding E-Bayesian predictors under the loss function (3.2) will be, given as

$$\begin{aligned}
\hat{\delta}^{EB_2}_{m+1}(\mathbf{r}) &= \int_0^1\int_0^c \hat{\delta}^{B}_{m+1}(\mathbf{r})\pi_2(\alpha,\beta)d\beta d\alpha\\
&= \int_0^1\int_0^c (r_m+\beta)e^{\Psi(m+1)-\Psi(m+\alpha)}\frac{\alpha^{u-1}(1-\alpha)^{v-1}}{cB(u,v)}d\beta d\alpha\\
&= \frac{e^{\Psi(m+1)}}{B(u,v)}\int_0^1 e^{-\Psi(m+\alpha)}\alpha^{u-1}(1-\alpha)^{v-1}d\alpha\int_0^c\frac{(r_m+\beta)}{c}d\beta
\end{aligned}$$

$$= \quad J(r_m + \frac{c}{2}), \tag{3.12}$$

and

$$\begin{aligned}
\hat{\delta}^{EB_3}_{m+1}(\mathbf{r}) &= \int_0^1 \int_0^c \hat{\delta}^B_{m+1}(\mathbf{r})\pi_3(\alpha,\beta)d\alpha d\beta \\
&= \int_0^1 \int_0^c (r_m+\beta)e^{\Psi(m+1)-\Psi(m+\alpha)} \frac{2\beta\alpha^{u-1}(1-\alpha)^{v-1}}{c^2 B(u,v)} d\beta d\alpha \\
&= \frac{e^{\Psi(m+1)}}{B(u,v)} \int_0^1 e^{-\Psi(m+\alpha)}\alpha^{u-1}(1-\alpha)^{v-1}d\alpha \int_0^c \frac{2\beta(r_m+\beta)}{c^2}d\beta \\
&= J(r_m + \frac{2c}{3}). \qquad (3.13)
\end{aligned}$$

□

3.4 Prequential Analysis for Comparison of Predictors

In this section, the performance of the proposed predictors are investigated through a prequential analysis, see Kiapour and Nematollahi (2011). For this purpose, we conduct a simulation study as follows:

1. Generate i.i.d. random observations $x_1, ..., x_{n+1}$ from exponential distribution with parameter $\theta = 3$ and obtain $r_1, ..., r_{m+1}$ using the algorithm mentioned in section 2.4.

2. Compute $\hat{\delta}^{(1)}$ (the Bayesian predictor $\hat{\delta}^B_{m+1}$ in (3.6)) with $\alpha = 0.6$ and $\beta = 0.2$, $\hat{\delta}^{(2)}$ (the robust Bayesian predictor $\hat{\delta}^{RB}_{m+1}$ in (3.8)) with $\alpha_1 = 0$, $\alpha_2 = 0.7$, $\beta_1 = 0$ and $\beta_2 = 0.5$ and $\hat{\delta}^{(3)}$, $\hat{\delta}^{(4)}$, $\hat{\delta}^{(5)}$ (the E-Bayesian predictors $\hat{\delta}^{EBi}_{m+1}$, $i = 1,2,3$ given in (3.10)) with $c = 0.5, 0.75, 1$, $u = 3$ and $v = 2$.

3. Calculate the prediction loss for r_{m+1} as $(\ln \hat{\delta}^{(k)}_{m+1} - \ln r_{m+1})^2$.

4. Increase m by 1 and repeat Steps 2 and 3 until $m = n$, when $n = 3, 5, 7, 9$.

5. Average all of the one-step-ahead prediction errors computed in Step 3 and compute Average Prediction Error (APE) for each predictor as

$$APE(k) = \frac{1}{n}\sum_{m=1}^{n}(\ln \hat{\delta}^{(k)} - \ln r_{m+1})^2. \tag{3.14}$$

6. Repeat Steps 1-5 for $N = 10^5$ times and compute the Simulated APE (SAPE) as a measure of the prediction error by avaraging the APE given in (3.14).

The results of SAPE for Bayesian and E-Bayesian predictors are presented in Table 3. The conclusions, as can be observed from this table, are that:

1. The E-Bayesian predictors $\hat{\delta}^{EB2}$ and $\hat{\delta}^{EB3}$ performs well against the Bayesian predictor for all values of c, while the E-Bayesian predictor $\hat{\delta}^{EB1}$ has good performance against the Bayesian predictor for $c \in \{0.75, 1\}$.

2. We observe the following inequality for E-Bayesian predictors in term of simulated APE:

$$SAPE(\delta^{EB3}) < SAPE(\delta^{EB2}) < SAPE(\delta^{EB1}) \tag{3.15}$$

3. Also, the estimated risk of the estimates decreases as the values of n and c increases.

Table 3: Results of SAPE for Bayesian, robust Bayesian and E-Bayesian predictors.

n	c	$\hat{\delta}^{B}$	$\hat{\delta}^{RB}$	$\hat{\delta}^{EB1}$	$\hat{\delta}^{EB2}$	$\hat{\delta}^{EB3}$
3	0.5	1.614743	1.372065	1.736646	1.406910	1.152514
	0.75			1.406911	1.045817	0.785720
	1			1.152514	0.785720	0.538034
5	0.5	1.098426	0.9319076	1.176755	0.974982	0.811832
	0.75			0.974982	0.741583	0.566063
	1			0.811832	0.566063	0.393650
7	0.5	0.408006	0.344423	0.436278	0.365650	0.306523
	0.75			0.365650	0.280622	0.215199
	1			0.306523	0.215199	0.151172
9	0.5	0.182201	0.1604618	0.193683	0.165764	0.1428950
	0.75			0.165764	0.133134	0.1096857
	1			0.142895	0.109685	0.0899195

3.5 A Numerical Example

We use a real data set concerning the times (in minutes) between 24 consecutive telephone calls to a companys switchboard which is presented by Castillo et al. (2005). Data are as follows:

1.34	0.14	0.33	1.68	1.86	1.31	0.83	0.33	2.20	0.62	3.20	1.38
0.96	0.28	0.44	0.59	0.25	0.51	1.61	1.85	0.47	0.41	1.46	0.09.

Here, the K-S test was used for checking the validity of the exponential distribution based on the parameter $\theta = 1.0059$. The test statistic K-S= 0.1489 with a corresponding p-value= 0.6618 implies that the exponential distribution have a good fit to the above data. The observed record values $\mathbf{r} = (r_1, ..., r_5)$ are obtained to be

$$\mathbf{r} = (1.34, 1.68, 1.86, 2.20, 3.20). \tag{3.16}$$

Our interest is to predict $r_5 = 3.2$. The predicted values of r_5 and corresponding absolute errors $\Delta\hat{\delta} = |\hat{\delta} - r_5|$ are listed in Table 4 for selected values $\alpha = 0.6$, $\beta = 0.2$, $u = 3$, $v = 2$

$\alpha_1 = 0.5$, $\alpha_2 = 1$, $\beta_1 = 0$, $\beta_2 = 0.2$ and $c = 1$. It is observed that the E-Bayesian predictor $\hat{\delta}^{EB3}$ has good performance.

Table 4: The predicted values of 5th record and corresponding absolute errors.

$\hat{\delta}^{B}$	$\hat{\delta}^{RB}$	$\hat{\delta}^{EB1}$	$\hat{\delta}^{EB2}$	$\hat{\delta}^{EB3}$
2.6331	2.4366	2.786	2.9693	3.1526
$\Delta\hat{\delta}^{B}$	$\Delta\hat{\delta}^{RB}$	$\Delta\hat{\delta}^{EB1}$	$\Delta\hat{\delta}^{EB2}$	$\Delta\hat{\delta}^{EB3}$
0.5669	0.7634	0.4140	0.2307	0.0474

4 Concluding Remarks

The problem of Bayesian, robust Bayesian and E-Bayesian estimation and prediction from the exponential distribution based on record values is considered. The Estimators and predictors are obtained under the loss functions (1.5) and (3.2), respectively. We compared the performance of Bayesian, robust Bayesian and E-Bayesian estimators and predictors using a simulation study and prequential analysises, respectively. Our findings show that tha E-Bayesian and robust Bayesian estimators and predictors work better than the Bayesian counterparts. We also considered two real data sets for illustrating the results. The results of these analysises agree with the simulation results.

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Appendix

Lemma 4.1. *With respect to loss, the estimator δ of θ is risk-unbiased if*

$$E[\ln\delta] = \ln\theta, \quad \forall\theta.$$

Proof. Following Lehmann (1951), an estimator δ of θ is said to be risk-unbiased under the loss function $L(\theta, \delta)$ if it satisfies

$$E[L(\theta,\delta)] \leq E[L(\theta',\delta)], \quad \forall\theta' \neq \theta. \tag{4.1}$$

Under the loss, we have

$$E\left[\ln^2\frac{\delta}{\theta}\right] - E\left[\ln^2\frac{\delta}{\theta'}\right] = (\ln^2\theta - \ln^2\theta') - 2(\ln\theta - \ln\theta')E[\ln\delta]. \tag{4.2}$$

If we consider $E[\ln\delta] = \ln\theta$, we conclude that

$$E\left[\ln^2\frac{\delta}{\theta}\right] - E\left[\ln^2\frac{\delta}{\theta'}\right] = -(\ln\theta - \ln\theta')^2 < 0. \tag{4.3}$$

Therefore, an estimator δ of θ is risk unbiased under the loss if it satisfies in the condition $E[\ln\delta] = \ln\theta$. Thus the biase of δ is $E[\ln\delta] - \ln\theta$. □

Lemma 4.2. *Let U be a random variable with Gamma(ν, b), distribution and let $\Psi(\nu) = \frac{d}{d\nu}\ln\Gamma(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$ be the digamma function, $\Psi'(\nu) = \frac{d}{d\nu}\Psi(\nu)$ be the trigamma function and $\Gamma(\nu)$ denote the complete gamma function given by*

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1}e^{-t}dt. \tag{4.4}$$

Then we have $E[\ln U] = \Psi(\nu) - \ln b$.

Proof. By differentiating both sides of (4.4) with respect to ν and dividing by $\Gamma(\nu)$, we get

$$\Psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)} = \int_0^\infty (\ln t)\,\frac{t^{\nu-1}e^{-t}}{\Gamma(\nu)}dt. \tag{4.5}$$

Now, using the transformation $u = t/b$ in the integral, it reduces to

$$\Psi(\nu) = \int_0^\infty (\ln u + \ln b)\frac{b^\nu}{\Gamma(\nu)}u^{\nu-1}e^{-bu}du = \ln b + E[\ln U], \tag{4.6}$$

where $U \sim$ *Gamma*(ν, b), which completes the proof.

□