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Preservation of Stochastic Orderings of Interdependent Series and Parallel Systems by Componentwise Switching to Exponentiated Models

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Abstract. This paper discusses the preservation of some stochastic orders between two interdependent series and parallel systems when the survival and distribution functions of all components switch to the exponentiated model. For the series systems, the likelihood ratio, hazard rate, usual, aging faster, aging intensity, convex transform, star, superadditive and dispersive orderings, and for the parallel systems the reversed hazard, usual, convex transform, star, superadditive and dispersive orderings are studied. Also, we present a necessary and sufficient condition for being finiteness of the moments of the switched series and switched parallel systems.

Keywords. Exponentiated Models, Max-stable Copulas, Parallel System, Series System, Stochastic Ordering

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1 Introduction

The topic of order statistics is one of the most attractive topics in statistics. They play an important role in applied probability, reliability theory, operations research, actuarial science, auction theory, hydrology and many other areas. For more details in applications of the order statistics, one may refer to Barlow and Proschan (1996), David and Nagaraja (2003), Li (2005), and Arnold et al. (2008).

Let $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the order statistics arising from random variables $X_1, \ldots, X_n$. The random variables $X_{1:n}$ and $X_{n:n}$ are the lifetimes of series and parallel systems, respectively. Many researchers have worked on the stochastic comparisons for the lifetimes of series and parallel systems in the literature. For a comprehensive review on this topic, one may refer to Dykstra et al. (1997), Khaledi and Kochar (2000), Kochar and Xu (2007, 2009), Zhao and Balakrishnan (2011, 2012, 2013, 2015), Balakrishnan et al. (2018), Patra et al. (2018), Nadeb and Torabi (2018), Das and Kayal (2020), and Nadeb et al. (2021).

Now, we recall some notions of stochastic orderings. Throughout the paper, we use the notations $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = (0, +\infty)$. Also, the term increasing means nondecreasing and decreasing means nonincreasing.

Let X and Y be two continuous and non-negative random variables with distribution functions F and G, density functions f and g, the survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, the right continuous inverses[1] F^{-1} and G^{-1} of F and G, the hazard rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, the reversed hazard rate functions $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$, and the aging intensity functions $L_X(x) = x r_X(x)/\int_0^x r_X(t)dt$ and $L_Y(x) = x r_Y(x)/\int_0^x r_Y(t)dt$, respectively. The following definition deals with some various orderings of random variables.

Definition 1.1. The random variable X is said to be smaller than Y in the

(i) usual stochastic ordering, denoted by $X \leq_{\text{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}$;

(ii) up proportional hazard rate order, denoted by $X \leq_{\text{phr}\uparrow} Y$, if for all $t \geq 0$ and $0 < \alpha \leq 1$, $\bar{G}(\alpha x)/\bar{F}(x + t)$ is increasing in $x \in \mathbb{R}_+$,

(iii) down proportional hazard rate order, denoted by $X \leq_{\text{phr}\downarrow} Y$, if for all $t \geq 0$ and $0 < \alpha \leq 1$, $\bar{G}(\alpha x + t)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}_+$,

[1]The right continuous inverse of an increasing function h is defined as $h^{-1}(u) = \sup\{x \in \mathbb{R} : h(x) \leq u\}$

(iv) up proportional reversed hazard rate order, denoted by $X \leq_{\text{prh}\uparrow} Y$, if for all $t \geq 0$ and $0 < \alpha \leq 1$, $G(\alpha x)/F(x+t)$ is increasing in $x \in \mathbb{R}_+$,

(v) likelihood ratio ordering, denoted by $X \leq_{\text{lr}} Y$, if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}_+$;

(vi) aging faster ordering, denoted by $X \leq_{\text{AF}} Y$, if $r_X(x)/r_Y(x)$ is increasing in $x \in \mathbb{R}_+$;

(vii) aging intensity ordering, denoted by $X \leq_{\text{AI}} Y$, if $L_Y(x) \leq L_X(x)$ for all $x \in \mathbb{R}_+$;

(viii) convex transform ordering, denoted by $X \leq_{\text{c}} Y$, if $G^{-1}F(x)$ is convex in $x \in \mathbb{R}_+$;

(ix) star ordering, denoted by $X \leq_{*} Y$, if $G^{-1}F(x)/x$ is increasing in $x \in \mathbb{R}_+$;

(x) superadditive ordering, denoted by $X \leq_{\text{su}} Y$, if $G^{-1}F(x+y) \geq G^{-1}F(x) + G^{-1}F(y)$ for all $x, y \in \mathbb{R}_+$;

(xi) dispersive ordering, denoted by $X \leq_{\text{disp}} Y$, if $G^{-1}F(x) - x$ is increasing in $x \in \mathbb{R}_+$.

It is necessary to recall that, the cases (ii)-(vi) include some important orderings. When $\alpha = 1$, they imply the shifted stochastic orderings; i.e. (ii)-(vi) imply the up hazard rate ordering, denoted by $X \leq_{\text{hr}\uparrow} Y$, down hazard rate ordering, denoted by $X \leq_{\text{hr}\downarrow} Y$, and up reversed hazard rate ordering, denoted by $X \leq_{\text{rh}\uparrow} Y$, respectively. When $t = 0$, they imply the proportional stochastic orders; i.e. (ii) and (iii) imply the proportional hazard rate ordering, denoted by $X \leq_{\text{phr}} Y$, and (iv) implies the proportional reversed hazard rate ordering, denoted by $X \leq_{\text{prh}} Y$. When $\alpha = 1$ and $t = 0$, they imply the common stochastic orderings; i.e. (ii) and (iii) imply the hazard rate ordering, denoted by $X \leq_{\text{hr}} Y$ and (iv) implies the reversed hazard rate ordering, denoted by $X \leq_{\text{rh}} Y$.

For more details on the shifted stochastic orders, one may refer to Nakai (1995), Brown and Shanthikumar (1998), Shanthikumar and Yao (1986), Keilson and Sumita (1982) and Lillo et al. (2001). More discussions on the proportional stochastic orders can be found in Ramos Romero and Díaz (2001) and Belzunce et al. (2002). For comprehensive discussions on the aging faster ordering we refer to Sengupta and Deshpande (1994), and for the aging intensity functions and their orderings, one may refer to Jiang et al. (2003) and Nanda et al. (2007). The others can be found in Shaked and Shanthikumar (2007).

There are some researches in the study of the preservation of some stochastic orderings under the transformation of the distributions. For instance, one may refer to Abbasi et al. (2016) and Nadeb and Torabi (2020).

A way of developing a model is exponentiating its distribution or survival function. We say that a random variable follows the exponentiated model if its distribution function can be expressed as $F^{\lambda}(x)$ or if its survival function can be expressed as $\bar{F}^{\lambda}(x)$; where $\lambda > 0$, and $F(x)$ and $\bar{F}(x)$ are the baseline distribution and survival functions, respectively. This model includes some important distributions in statistics.

Generally, let $X_1, \ldots, X_n$ be continuous random variables with the joint distribution function [survival function] $H(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$ $[\bar{H}(x_1, \ldots, x_n) = P(X_1 > x_1, \ldots, X_n > x_n)]$, and marginal distribution functions [survival functions] $F_1, \ldots, F_n$ $[\bar{F}_1, \ldots, \bar{F}_n]$. In this setting, the joint distribution function [survival function] and the marginal distribution functions [survival functions] are linked through the relation $H(x_1, \ldots, x_n) = C\Big(F_1(x_1), \ldots, F_n(x_n)\Big)$ $[\bar{H}(x_1, \ldots, x_n) = \hat{C}\left(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)\right)]$ in view of the Sklar's Theorem; see Nelsen (2006). In this notation, the function C is called a copula and $\hat{C}$ is called a survival copula. In this paper, we consider the class of max-stable copulas with the following definition.

Definition 1.2. A copula C is max-stable if for every $r > 0$ and $\boldsymbol{u} \in [0, 1]^n$,

$$C(\boldsymbol{u}) = C^r\left(u_1^{1/r}, \ldots, u_n^{1/r}\right).$$

There are some ways to construct a max-stable copula. Here, we provide the proposed way by Pickands (1981). Suppose that a function $A : [0, \infty)^n \to [0, \infty)$ satisfies the following properties:

- $A(0, \ldots, 0, x_i, 0, \ldots, x_n) = x_i$, $i = 1, \ldots, n$;
- $A(sx_1, \ldots, sx_n) = sA(x_1, \ldots, x_n)$, for every $s \geq 0$;
- A is convex;
- $\max(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} x_i$,

then

$$C(\boldsymbol{u}) = \exp\left\{-A\left(-\ln u_1, \ldots, -\ln u_n\right)\right\}, \qquad (u_1, \ldots, u_n) \in [0, 1]^n, \tag{1.1}$$

is a max-stable copula.

Considering $A(\boldsymbol{x}) = \left(\sum_{i=1}^{n} x_i^{\theta}\right)^{1/\theta}$, $\theta \geq 1$, we get the Gumbel-Hougaard copula, which was introduced by Gumbel (1960). For $A(\boldsymbol{x}) = \sum_{i=1}^{n} x_i$ the independence copula and for $A(\boldsymbol{x}) = \max(x_1, \ldots, x_n)$ the copula $M(\boldsymbol{u}) = \min(u_1, \ldots, u_n)$ are obtained. Each random variable in a set is almost surely a strictly increasing function of any of the others if and only if the corresponding copula is M. Based on Theorems 3.3.5 and 4.5.2 of Nelsen (2006), the Gumbel-Hougaard copula is the only Archimedean max-stable copula. There are several other max-stable copulas based on structure (1.1) such as the Galambos copula, Hüsler-Reiss copula, and t-copula, and some other structures. For more details on this topic, one may refer to Joe (2014).

Suppose that we have two series [parallel] systems, say I and II. System I consists of n components whose lifetimes are $X_1, \ldots, X_n$ with corresponding absolutely continuous survival [distribution] functions $\bar{F}_1, \ldots, \bar{F}_n$ $[F_1, \ldots, F_n]$, which are linked through an arbitrary max-stable survival copula [copula]. Similarly, system II consists of n components whose lifetimes are $Y_1, \ldots, Y_n$ with corresponding absolutely continuous survival [distribution] functions $\bar{G}_1, \ldots, \bar{G}_n$ $[G_1, \ldots, G_n]$, which are linked through an equal survival copula [copula] to system I. Also, consider the survival [distribution] functions $\bar{F}_1^\lambda, \ldots, \bar{F}_n^\lambda$ $[F_1^\lambda, \ldots, F_n^\lambda]$ are linked through an equal survival copula [copula] to system I. Now, consider two new series [parallel] systems, one consisting of the component lifetimes $X_1^*, \ldots, X_n^*$ with the survival [distribution] functions $\bar{F}_1^\lambda, \ldots, \bar{F}_n^\lambda$ $[F_1^\lambda, \ldots, F_n^\lambda]$, and another with the component lifetimes $Y_1^*, \ldots, Y_n^*$ with the survival [distribution] functions $\bar{G}_1^\lambda, \ldots, \bar{G}_n^\lambda$ $[G_1^\lambda, \ldots, G_n^\lambda]$, and the unchanged survival copula [copula].

Recently, Balakrishnan et al. (2020) discussed the preservation of the usual, hazard rate, reversed hazard rate, star and dispersive orderings of these series and parallel systems under exponentiation procedure with independence structure. Here, we generalize the obtained results by Balakrishnan et al. (2020) under the exponentiation procedure and unchanging the dependence structure for the case that the dependence structure is a max-stable copula. Further, we discuss more stochastic orderings than Balakrishnan et al. (2020) such as the up hazard rate, down hazard rate, proportional hazard rate, up proportional hazard rate, down proportional hazard rate, up reversed hazard rate, up proportional reversed hazard rate, aging faster, aging intensity, convex transform, and superadditive orderings. Hereafter the series [parallel] systems I and II are called the original series (OS) [original parallel (OP)] systems, and the systems after componentwise exponentiation are called the switched series (SS) [switched parallel (SP)] systems.

The rest of this paper is organized as follows. In Section 2, we discuss on preservation the orderings of series system by componentwise switching to the exponentiated model with unchanged survival copula. Section 3 considers the preservation of some orderings of a parallel system by componentwise switching to the exponentiated model with unchanged copula. Finally, in Section 4, we investigate the being finiteness of the moments of the SS and SP systems based on some characterizations on the hazard rate functions of the OS and OP systems.

2 Preservation of Stochastic Orderings of Series Systems

In this section, we discuss the preservation of some stochastic orderings of series systems. First, we state the following lemma of Balakrishnan et al. (2020) which is useful in proving some of the main results.

Lemma 2.1. *For $\lambda \geq 1$, the function $h : (0,1) \longrightarrow (0,\infty)$ given by $h(x) = \frac{x^{\lambda-1}-x^{\lambda}}{1-x^{\lambda}}$ is increasing in $x \in (0,1)$.*

It is easily seen that the survival functions of two OS systems are given by

$$\bar{F}_{X_{1:n}}(x) = C\left(\bar{F}_1(x), \dots, \bar{F}_n(x)\right), \quad \bar{G}_{Y_{1:n}}(x) = C\left(\bar{G}_1(x), \dots, \bar{G}_n(x)\right),$$

and the survival functions of two SS systems are given by

$$\bar{F}_{X^*_{1:n}}(x) = C\left(\bar{F}_1^{\lambda}(x), \dots, \bar{F}_n^{\lambda}(x)\right), \quad \bar{G}_{Y^*_{1:n}}(x) = C\left(\bar{G}_1^{\lambda}(x), \dots, \bar{G}_n^{\lambda}(x)\right).$$

Consequently, after some simple calculations, we get the following relations:

$$\bar{F}_{X^*_{1:n}}(x) = \bar{F}^{\lambda}_{X_{1:n}}(x), \qquad \bar{G}_{Y^*_{1:n}}(x) = \bar{G}^{\lambda}_{Y_{1:n}}(x), \tag{2.1}$$

$$r_{X^*_{1:n}}(x) = \lambda\, r_{X_{1:n}}(x), \qquad r_{Y^*_{1:n}}(x) = \lambda\, r_{Y_{1:n}}(x), \tag{2.2}$$

$$\tilde{r}_{X^*_{1:n}}(x) = \lambda\, \tilde{r}_{X_{1:n}}(x) h\left(\bar{F}_{X_{1:n}}(x)\right), \quad \tilde{r}_{Y^*_{1:n}}(x) = \lambda\, \tilde{r}_{Y_{1:n}}(x) h\left(\bar{G}_{Y_{1:n}}(x)\right), \tag{2.3}$$

where the function h was introduced in Lemma 2.1.

Now, the two following theorems consider some stochastic orderings between two SS systems based on comparing the OS systems. These theorems indicate that many well-known orders are preserved between two SS systems after componentwise exponentiation of the OS systems.

Theorem 2.1. *Let $\leq_{\text{ordering}}$ stands for any of the orderings $\leq_{\text{st}}, \leq_{\text{hr}}, \leq_{\text{hr}\uparrow}, \leq_{\text{hr}\downarrow}, \leq_{\text{phr}}, \leq_{\text{phr}\uparrow}, \leq_{\text{phr}\downarrow}, \leq_{\text{AF}}, \leq_{\text{AI}}, \leq_{\text{c}}, \leq_{*}, \leq_{\text{su}}, \leq_{\text{disp}}$. Then, $X^*_{1:n} \leq_{\text{ordering}} Y^*_{1:n}$ if and only if $X_{1:n} \leq_{\text{ordering}} Y_{1:n}$.*

Proof. The result immediately follows from (2.1) for the orderings $\leq_{\text{st}}, \leq_{\text{hr}}, \leq_{\text{hr}\uparrow}, \leq_{\text{hr}\downarrow}, \leq_{\text{phr}}, \leq_{\text{phr}\uparrow}, \leq_{\text{phr}\downarrow}, \leq_{\text{AF}}, \leq_{\text{AI}}$. On the other hand, using (2.1) we can see that $G^{-1}_{Y^*_{1:n}}(x) = G^{-1}_{Y_{1:n}}\left(1-(1-x)^{1/\lambda}\right)$. Thus, we have $G^{-1}_{Y^*_{1:n}} F_{X^*_{1:n}}(x) = G^{-1}_{Y_{1:n}} F_{X_{1:n}}(x)$. Hence, Definition 1.1 (viii)-(xi) complete the proof for the orderings $\leq_{\text{c}}, \leq_{*}, \leq_{\text{su}}, \leq_{\text{disp}}$. □

Theorem 2.2. *For $\lambda \geq 1$, if $X_{1:n} \leq_{\text{rh}} Y_{1:n}$, then $X^*_{1:n} \leq_{\text{rh}} Y^*_{1:n}$.*

Proof. Based on the assumption, we have $\tilde{r}_{X_{1:n}}(x) \le \tilde{r}_{X_{1:n}}(x)$ for all $x \ge 0$, and consequently we conclude that $\bar{F}_{X_{1:n}}(x) \le \bar{G}_{Y_{1:n}}(x)$. On the other hand Lemma 2.1 implies that $h\left(\bar{F}_{X_{1:n}}(x)\right) \le h\left(\bar{G}_{Y_{1:n}}(x)\right)$. So, we have $\tilde{r}_{X_{1:n}}(x)h\left(\bar{F}_{X_{1:n}}(x)\right) \le \tilde{r}_{X_{1:n}}(x)h\left(\bar{G}_{Y_{1:n}}(x)\right)$, and the desired result is obtained by (2.3). □

The following theorem provides the likelihood ratio ordering between two SS systems based on the comparison of the OS systems.

Theorem 2.3. *If* $X_{1:n} \le_{\mathrm{hr}} Y_{1:n}$ *and* $Y_{1:n} \le_{\mathrm{AF}} X_{1:n}$*, then* $X^*_{1:n} \le_{\mathrm{lr}} Y^*_{1:n}$.

Proof. By (2.1) and (2.2) we can write

$$\frac{g_{Y^*_{1:n}}(x)}{f_{X^*_{1:n}}(x)} = \frac{r_{Y_{1:n}}(x)}{r_{X_{1:n}}(x)}\left(\frac{\bar{G}_{Y_{1:n}}(x)}{\bar{F}_{X_{1:n}}(x)}\right)^{\lambda}.$$

Hence, the assumption clearly imply the increasingness property of $\frac{g_{Y^*_{1:n}}(x)}{f_{X^*_{1:n}}(x)}$, which completes the proof. □

Note that the assumptions of Theorem 2.3 can be satisfied. For instance, suppose that $\bar{F}_{X_{1:n}}(x) = \exp\left\{-\sum_{i=1}^{n}\beta_i x^{\alpha}\right\}$, $x > 0$, and $\bar{F}_{Y_{1:n}}(x) = \exp\left\{-\sum_{i=1}^{n}\gamma_i x^{\alpha}\right\}$, $x > 0$. It is easily seen that, by assuming $\sum_{i=1}^{n}\gamma_i \le \sum_{i=1}^{n}\beta_i$, the two comparisons $X_{1:n} \le_{\mathrm{hr}} Y_{1:n}$ and $Y_{1:n} \le_{\mathrm{AF}} X_{1:n}$ are jointly satisfied.

3 Preservation of Stochastic Orderings of Parallel Systems

In this section, we discuss the preservation of some stochastic orderings of parallel systems.

It is easily seen that the distribution functions of two OP systems are given by

$$F_{X_{n:n}}(x) = C\left(F_1(x), \ldots, F_n(x)\right), \quad G_{Y_{n:n}}(x) = C\left(G_1(x), \ldots, G_n(x)\right),$$

and the survival functions of two SP systems are given by

$$F_{X^*_{n:n}}(x) = C\left(F_1^{\lambda}(x), \ldots, F_n^{\lambda}(x)\right), \quad G_{Y^*_{n:n}}(x) = C\left(G_1^{\lambda}(x), \ldots, G_n^{\lambda}(x)\right).$$

Consequently, after some simple calculations we get the following relations:

$$
\begin{aligned}
&F_{X^*_{n:n}}(x) = F^{\lambda}_{X_{n:n}}(x), && G_{Y^*_{n:n}}(x) = G^{\lambda}_{Y_{n:n}}(x),\\
&\tilde{r}_{X^*_{n:n}}(x) = \lambda\, \tilde{r}_{X_{n:n}}(x), && \tilde{r}_{Y^*_{n:n}}(x) = \lambda\, \tilde{r}_{Y_{n:n}}(x),\\
&r_{X^*_{n:n}}(x) = \lambda\, r_{X_{n:n}}(x) h\left(F_{X_{n:n}}(x)\right), && r_{Y^*_{n:n}}(x) = \lambda\, r_{Y_{n:n}}(x) h\left(G_{Y_{n:n}}(x)\right).
\end{aligned}
\tag{3.1}
$$

The following theorems discuss the orderings of the SP systems based on the orderings of the OP systems. These theorems indicate that many well-known orders are preserved between two SP systems after componentwise exponentiation of the OP systems. The proofs are almost the same as the proofs in the previous section and thus are omitted.

Theorem 3.1. *Let $\leq_{\text{ordering}}$ stands for any of the orderings $\leq_{\text{st}}, \leq_{\text{rh}}, \leq_{\text{rh}\uparrow}, \leq_{\text{prh}\uparrow}, \leq_{\text{c}}, \leq_{*}, \leq_{\text{su}}$, $\leq_{\text{disp}}$. Then, $X^*_{n:n} \leq_{\text{ordering}} Y^*_{n:n}$ if and only if $X_{n:n} \leq_{\text{ordering}} Y_{n:n}$.*

Theorem 3.2. *For $\lambda \geq 1$, if $X_{n:n} \leq_{\text{hr}} Y_{n:n}$, then $X^*_{n:n} \leq_{\text{hr}} Y^*_{n:n}$.*

4 Being Finiteness of the Moments

Main problem in this section is evaluating the being finiteness of the moments of the SS and SP systems by knowing some information of the OS and OP systems. With this goal, this section investigates being finiteness of the moments of the SS and SP systems based on some characterizations on the hazard rate functions of the OS and OP systems, respectively. The two following theorems deal with this problem.

Theorem 4.1. *Let $xr_{X_{1:n}}(x)$ be an increasing function in $x \geq 0$ and $\lim_{x\to\infty} xr_{X_{1:n}}(x) = l$, where l is possibly infinite. Then for each $\lambda > 0$, $E[X^{*k}_{1:n}], k > 0$, is finite if and only if, $k < \lambda l$.*

Proof. Under the assumptions and using (2.2) it is easily seen that $xr_{X^*_{1:n}}(x)$ is increasing in $x \geq 0$ and $\lim_{x\to\infty} xr_{X^*_{1:n}}(x) = \lambda l$. Thus, Theorem 2 in Lariviere (2006) implies that $E[X^{*k}_{1:n}]$ is finite if and only if $k < \lambda l$. □

Theorem 4.2. *Let $xr_{X_{n:n}}(x)$ be an increasing function in $x \geq 0$ and $\lim_{x\to\infty} xr_{X_{n:n}}(x) = l$, where l is possibly infinite. Then for each $\lambda \geq 1$, $E[X^{*k}_{n:n}], k > 0$, is finite if and only if, $k < l$.*

Proof. Under the assumptions and using (3.1) and Lemma 2.1, it is easily seen that for $\lambda \geq 1$, the function $xr_{X^*_{n:n}}(x)$ is increasing in $x \geq 0$ and $\lim_{x\to\infty} xr_{X^*_{n:n}}(x) = l$. Thus, Theorem 2 in Lariviere (2006) implies that $E[X^{*k}_{n:n}]$ is finite if and only if $k < l$. □

Discussion and Conclusions

In this paper the preservation of some stochastic orderings between two interdependent series and parallel systems when the survival and distribution functions of all components have a max-stable copula dependence structure and switch to the exponentiated model, were discussed. Also, a necessary and sufficient condition for being finiteness of the moments of the switched series and switched parallel systems was presented.

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