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Quantile Approach of Generalized Cumulative Residual Information Measure of Order (α, β)

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Abstract. In this paper, we introduce the concept of quantile-based generalized cumulative residual entropy of order (α, β) for residual and past lifetimes and study their properties. Further, we study the proposed information measure for series and parallel systems when random variables are untruncated or truncated in nature and some characterization results are presented. At the end, we study generalized weighted dynamic cumulative residual entropy in terms of quantile functions.

Keywords. Generalized Entropy, Cumulative Residual Entropy, Order Statistics, Quantile Function, Weighted Entropy.

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1 Introduction

A probability distribution can be specified either in terms of distribution function or by the quantile function. Although both convey the same information about the

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distribution, with different interpretations, the concepts and methodologies based on distribution functions are traditionally employed in statistical theory and practice. Let X be a random variable with distribution function $F(x)$ and quantile function $Q(u)$. Then, the quantile function of X is defined by

$$Q(u) = F^{-1}(u) = \inf\{x \mid F(x) \geq u\}, \quad 0 \leq u \leq 1. \tag{1.1}$$

Here and throughout the article, X is a absolutely continuous nonnegative random variable with probability density function (pdf) $f(x)$ and survival function $\bar{F}(x)$. If $f(.)$ is the pdf of X, then $f(Q(u))$ and $q(u) = \frac{dQ(u)}{du}$ are, respectively, known as the density quantile function and the quantile density function. Using (1.1), we have $F(Q(u)) = u$ and, by differentiating with respect to u, we obtain

$$q(u)f(Q(u)) = 1. \tag{1.2}$$

The mean of X is assumed to be finite and is calculated as

$$E(X) = \int_0^1 Q(p)dp = \int_0^1 (1-p)q(p)dp. \tag{1.3}$$

An important quantile measure useful in reliability analysis is the hazard quantile function, defined as

$$K(u) = h(Q(u)) = \frac{f(Q(u))}{(1-u)} = \frac{1}{(1-u)q(u)}\ , \tag{1.4}$$

where $h(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of X. Another useful measure closely related to hazard quantile function is the mean residual quantile function, which is given by

$$M(u) = m(Q(u)) = (1-u)^{-1} \int_u^1 (1-p)q(p)dp\ , \tag{1.5}$$

where $m(t) = E(X - t|X > t)$ is the mean residual life function ($MRLF$) of X. Further the relationship between the quantile density function and mean residual quantile function is presented by

$$q(u) = \frac{M(u) - (1-u)M'(u)}{(1-u)}. \tag{1.6}$$

For a detailed and recent study on quantile function and its properties in modeling and analysis, we refer to Parzen (2004), Nair and Vinesh Kumar (2011), Nair et al. (2013), Sreelakshmi et al. (2018) and the references therein. There are some models that do not have any closed form expressions for distribution and density function, but have simple QFs or quantile density functions refer to, van Standen and Loots (2009) and Hankin and Lee (2006).

The average amount of uncertainty associated with the nonnegative continuous random variable X can be measured using the differential entropy function

$$H(f) = -\int_0^\infty f(x) \log f(x) dx \,, \tag{1.7}$$

a continuous counterpart of the Shannon (1948) entropy in the discrete case.

Rao et al. (2004) pointed out some basic shortcomings of the Shannon differential entropy measure. Rao et al. (2004) introduced an alternative measure of uncertainty called the *cumulative residual entropy (CRE)* of a random variable X with survival function $\bar{F}$, given by

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx \,.$$

Asadi and Zohrevand (2007) have considered the dynamic cumulative residual entropy (DCRE) as the cumulative residual entropy of the residual lifetime $X_t = [X - t|X > t]$ which is presented by

$$\xi(X_t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx. \tag{1.8}$$

Di Crescenzo and Longobardi (2009) introduced a dual measure based on the cumulative distribution function $F(x)$, called the cumulative entropy (CE) and its dynamic version which is analogous to CRE, as follow

$$\bar{\xi}(X) = -\int_0^\infty F(x) \log F(x) dx, \tag{1.9}$$

$$\bar{\xi}(X_t) = -\int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx \,. \tag{1.10}$$

There have been attempts by several authors for the parametric generalization of *CRE*. Kumar and Taneja (2011) introduced a generalized cumulative residual entropy of order (α, β) as

$$\xi_X^{(\alpha,\beta)} = \frac{1}{(\beta - \alpha)} \log\left(\int_0^\infty \bar{F}^{\alpha+\beta-1}(x) dx\right) \; ; \beta \neq \alpha, \beta - 1 < \alpha < \beta, \beta \geq 1 \,. \tag{1.11}$$

For a component/system which has survived for t units of time, the dynamic cumulative residual entropy of order (α, β) of random variable X (see, Kumar and Taneja, 2011)is given as

$$\xi_{X_t}^{(\alpha,\beta)} = \frac{1}{(\beta - \alpha)} \log\left(\int_t^\infty \frac{\bar{F}^{\alpha+\beta-1}(x)}{\bar{F}^{\alpha+\beta-1}(t)} dx \right). \tag{1.12}$$

This measure is much more flexible due to the parameters α and β, and enables several measurements of uncertainty within a given distribution and increases the scope of application. Also, it forms a parametric family of entropy measures that give completely different weights to extremely rare and regular events. Some properties and applications of these theoretical information measures in reliability engineering, computer vision, coding theory and finance have been also studied by several researcher, refer to Rao (2005), Wang and Vemuri (2007), Navarro et al. (2010), Sheraz et al. (2015), Kumar and Singh (2018), Khammar and Jahanshahi (2018), and Baratpour and Khammar (2018).

The study of entropy functions using quantile functions is of recent interest. Sankaran and Sunoj (2017) have introduced the quantile version of the dynamic cumulative residual entropy (DCRE), which is defined by

$$\xi(u) = \xi(X; Q(u)) = \frac{\log(1-u)}{(1-u)} \int_u^1 (1-p)q(p)dp - (1-u)^{-1} \int_u^1 \log(1-p)(1-p)q(p)dp. \tag{1.13}$$

When $u \longrightarrow 0$, (1.13) reduces to $\xi = -\int_0^1 (\log(1-p))(1-p)q(p)dp$, a quantile version of CRE. For more details and applications of quantile-based generalized CRE measures refer to Kang and Yan (2016), Sunoj et al. (2017) and Sunoj et al. (2018). When traditional approachs are either diffcult or failed in obtaining desired results, then quantile-based studies are carried out. Qunatile functions (QFs) have several properties that are not shared by distribution functions. these functions can be properly employed to formulate properties of entropy function and other information measures for nonnegative absolutely continuous random variables. We refer readers to Sunoj and Sankaran (2012), Sunoj et al. (2013), Sankaran et al. (2016), Qiu (2018), Kumar and Rani (2018), Kumar and Singh (2019) and many others.

Motivated by these, in the present study, we consider the survival and distribution function based generalized dynamic entropy measures based on Varma entropy in terms of quantile function. In the present manuscript, we introduce the quantile version of GCRE of order (α, β) for residual and reversed residual (past) lifetimes and prove some of their characterization results for extreme order statistics.

The text is organized as follows. In Section 2, we introduce the generalized cumulative residual quantile entropy (GCRQE) in residual and past lifetimes and various properties of these measures are discussed. Section 3 proves some characterization results based on the measures considered in Section 2. In Section 4, we extend the quantile-based generalized cumulative residual entropy in the context of extreme order statistics and study its properties. In Section 5, we derive the generalized weighted quantile entropy of order (α, β) and study some characterization results. Finally in Section 6, we give an example where generalized entropy has application in codding theory.

2 Generalized Cumulative Residual Quantile Entropy

The quantile version of the *generalized cumulative residual entropy (GCRE)* of the nonnegative random variable X is defined as

$$\text{Ḫ}_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\int_0^1 (1-p)^{\alpha+\beta-1} q(p) dp\right), \tag{2.1}$$

and it is called the *generalized cumulative residual quantile entropy (*GCRQE*)*. When $\alpha \to 1$ and $\beta = 1$, $\text{Ḫ}_X^{(\alpha,\beta)}$ reduces to $-\int_0^1 (\log(1-p))(1-p)q(p)dp$, a quantile version of *CRE*, sugggested by Sankaran and Sunoj (2017). There are some models that do not have any closed form expressions for cdf or pdf, but have simple quantile function or quantile density functions (see Nair et al. (2011)). Accordingly, in the following example, we obtain $\text{Ḫ}_X^{(\alpha,\beta)}$ for which $q(.)$ exists.

Examples 2.1. Suppose X is distributed with quantile density function $q(u) = (1-u)^{-A}(-\log(1-u))^{-M}$, $0 < u < 1$, where M and A are real constants. This quantile density functions contains several distributions as special cases, such as Weibull when A = 1, M = $\frac{\lambda-1}{\lambda}$ with shape parameter $\sigma = k\lambda$, uniform when A = 0, M = 0, Pareto when $A > 1$, M = 0 and rescaled beta when $A < 1$, M = 0. Then the quantile-based generalized cumulative residual entropy is obtained as

$$\text{Ḫ}_X^{(\alpha,\beta)} = \frac{1}{\beta-\alpha}\left\{\log(\gamma(1-M)) - (1-M)\log(1+(\alpha+\beta-1)-A)\right\},$$

where $\gamma(.)$ represents the gamma function.

Examples 2.2. A lambda family of distribution, which is of interest in reliability, is the Davis distribution proposed by Hankin and Lee (2006) with quantile function

$$Q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}; 0 < u < 1, C, \lambda_1, \lambda_2 \geq 0.$$

This is a flexible family for right skewed nonnegative data that provides good approximations to the exponential, gamma, lognormal and Weibull distributions. The GCRQE of the Davis distribution is given as:

$$Ḩ_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log \left\{ C\lambda_1 \beta(\lambda_1, (\alpha+\beta-1) - \lambda_2 + 1) + C\lambda_2 \beta(\lambda_1 + 1, (\alpha+\beta-1) - \lambda_2) \right\} . \tag{2.2}$$

As $\lambda_2 \longrightarrow 0$, (2.2) reduces to $Ḩ_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(C\lambda_1 \beta(\lambda_1, (\alpha+\beta)\right)$, corresponding to the power distribution. Also as $\lambda_1 \longrightarrow 0$, (2.2) reduces to $Ḩ_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\frac{C\lambda_2}{(\alpha+\beta-1)-\lambda_2}\right)$, corresponding to the Pareto I distribution.

Examples 2.3. If X be a random variable following the Govindarajulu's distribution (1977) with the quantile function $Q(u) = a\{(b+1)u^b - bu^{b+1}\}$, $0 \leq u \leq 1; a, b > 0$, then $Ḩ_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left\{ab(b+1)\beta(b, (\alpha+\beta-1)+2)\right\}$.

In the context of reliability and survival analysis, when the current age of a component need to be taken into account. In such cases, measuring uncertainty using $Ḩ_X^{(\alpha,\beta)}$ is not appropriate and a modified version of $Ḩ_X^{(\alpha,\beta)}(u)$ is essential for such a residual random variable, $X_t = (X - t | X > t)$. An equivalent definition for the *generalized dynamic cumulative residual entropy (GDCRE)* in terms of quantile function is given by

$$Ḩ_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)} \log\left(\frac{1}{(1-u)^{\alpha+\beta-1}} \int_u^1 (1-p)^{\alpha+\beta-1} q(p)dp\right). \tag{2.3}$$

This measure can be considered as the *generalized dynamic cumulative residual quantile entropy (GDCRQE)* measure. When $\alpha \longrightarrow 1$, $\beta = 1$, the measure (2.3) reduces to (1.13). Rewriting (2.3) and using (1.6), we come to

$$e^{(\beta-\alpha)Ḩ_X^{(\alpha,\beta)}(u)} = \left(\frac{1}{(1-u)^{\alpha+\beta-1}} \int_u^1 (1-p)^{\alpha+\beta-2} M(p)dp - \frac{1}{(1-u)^{\alpha+\beta-1}} \int_u^1 (1-p)^{\alpha+\beta-1} M'(p)dp\right).$$

By applying integrating by parts on the last term and simplifying, we obtain

$$(1-u)^{\alpha+\beta-1} e^{(\beta-\alpha)Ḩ_X^{(\alpha,\beta)}(u)} - M(u)(1-u)^{\alpha+\beta-1} = (2-\beta-\alpha) \int_u^1 (1-p)^{\alpha+\beta-2} M(p)dp. \tag{2.4}$$

Differentiating (2.4) with respect to u on both sides, and using (1.4) reduces (2.4) to

$$q(u) = \left(\frac{(\alpha + \beta - 1)}{1 - u} - (\beta - \alpha) \text{Ḩ}'^{(\alpha,\beta)}_X(u) \right) e^{(\beta-\alpha)\text{Ḩ}^{(\alpha,\beta)}_X(u)}, \tag{2.5}$$

where prime denotes the derivative with respect to u. Equation (2.5) provides a direct relationship between quantile density function $q(u)$ and $\text{Ḩ}^{(\alpha,\beta)}_X(u)$. Therefore $\text{Ḩ}^{(\alpha,\beta)}_X(u)$ uniquely determines the underlying distribution. Table 1 provides the quantile functions of some important models and the corresponding proposed measures.

Table 1: Mean residual quantile function $M(u)$ and GDCRQE $\text{Ḩ}^{(\alpha,\beta)}_X(u)$ for some lifetime distributions

Distribution	Quantile function $Q(u)$	M(u)	$\text{Ḩ}^{(\alpha,\beta)}_X(u)$
Uniform	$a + (b - a)u$	$\frac{(b-a)(1-u)}{2}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{(b-a)(1-u)}{(\alpha+\beta-1)+1}\right)$
Exponential	$-\lambda^{-1}\log(1-u)$	$\frac{1}{\lambda}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{1}{\lambda(\alpha+\beta-1)}\right)$
Gompertz	$\frac{1}{\log c}\left(1 - \frac{\log c \log(1-u)}{B}\right)$	$\frac{1}{B}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{1}{B(\alpha+\beta-1)}\right)$
Pareto II	$a[(1-u)^{-\frac{1}{b}} - 1]$	$\frac{b(1-u)^{-\frac{1}{a}}}{a-1}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{b(1-u)^{-\frac{1}{a}}}{a(\alpha+\beta-1)-1}\right)$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right]$	$b(1-u)^{-\frac{a}{a+1}}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{b(1-u)^{-\frac{a}{a+1}}}{(a+1)(\alpha+\beta-1)-a}\right)$
Finite Range	$b\left(1 - (1-u)^{\frac{1}{a}}\right)$	$\frac{b(1-u)^{\frac{1}{a}}}{a+1}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{b(1-u)^{\frac{1}{a}}}{a(\alpha+\beta-1)+1}\right)$
Log logestic	$\frac{1}{a}\left(\frac{u}{(1-u)}\right)^{\frac{1}{b}}$	$\frac{\bar{\beta}_u(\frac{1}{b},1-\frac{1}{b})}{ab(1-u)}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{\bar{\beta}_u(b,(\alpha+\beta-1)-\frac{1}{b})}{ab(1-u)^{\alpha+\beta-1}}\right)$
Govindarajulu's	$a\{(b+1)u^b - bu^{b+1}\}$	$\frac{ab(b+1)}{(1-u)}\bar{\beta}_u(b,3)$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{ab(b+1)\bar{\beta}_u(b,(\alpha+\beta-1)+2)}{(1-u)^{\alpha+\beta-1}}\right)$
Tukey lambda	$\frac{u^\lambda-(1-u)^\lambda}{\lambda}$	$\frac{\bar{\beta}_u(\lambda,2)}{(1-u)} + \frac{(1-u)^\lambda}{\lambda+1}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{\bar{\beta}_u(\lambda,\alpha+\beta)}{(1-u)^{\alpha+\beta-1}} + \frac{(1-u)^\lambda}{(\alpha+\beta-1)+\lambda}\right)$
Power	$au^{\frac{1}{b}}$	$\frac{a\bar{\beta}_u(\frac{1}{b},2)}{b(1-u)}$	$\frac{1}{(\beta-\alpha)} \log\left(\frac{a\bar{\beta}_u(\frac{1}{b},\alpha+\beta)}{b(1-u)^{\alpha+\beta-1}}\right)$

Here, $\bar{\beta}_x(a, b)$ stands for the incomplete beta function defined as $\bar{\beta}_x(a, b) = \int_x^1 y^{a-1}(1-y)^{b-1}dy$, $a, b > 0, x > 0$.

In what follows, we see how the monotonicity of $\text{Ḩ}^{(\alpha,\beta)}_X(u)$ is affected by an increasing transformation. The following lemma helps us to prove the results on monotonicity of $\text{Ḩ}^{(\alpha,\beta)}_X(u)$.

Lemma 2.1. *Let* $f(u, x) : \Re^2_+ \longrightarrow \Re_+$ *and* $g : \Re_+ \longrightarrow \Re_+$ *be any two functions. If* $\int_u^\infty f(u, x)dx$ *is increasing and* $g(u)$ *is increasing (decreasing) in* u*, then* $\int_u^\infty f(u, x)g(x)dx$ *is increasing (decreasing) in* u*, provided the integrals exist.*

For more details, refer to Nanda et al. (2014).

Theorem 2.1. *Let X be a nonnegative and continuous random variable with quantile function $Q_X(.)$ and quantile density function $q_X(.)$. Define $Y = \phi(X)$, where $\phi(.)$ is a nonnegative, increasing and convex (concave) function.*
(i) For $0 < \alpha+\beta < 2$, $\text{Ḩ}_Y^{(\alpha,\beta)}(u)$ is increasing (decreasing) in u whenever $\text{Ḩ}_X^{(\alpha,\beta)}(u)$ is increasing (decreasing) in u.
(ii) For $\alpha + \beta > 2$, $\text{Ḩ}_Y^{(\alpha,\beta)}(u)$ is decreasing (increasing) in u whenever $\text{Ḩ}_X^{(\alpha,\beta)}(u)$ is increasing (decreasing) in u.

Proof. (i) The probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi\,'(\phi^{-1}(y))}$; hence the density quantile function is $g(Q_Y(u)) = \frac{1}{q_Y(u)} = \frac{f(Q(u))}{\phi' Q(u)} = \frac{1}{q_X(u)\phi'(Q_X(u))}$. Thus we have

$$\begin{aligned}\text{Ḩ}_Y^{(\alpha,\beta)}(u) &= \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_Y(p)dp\right)\\ &= \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_X(p)\phi'(Q_X(p))dp\right). \qquad (2.6)\end{aligned}$$

From the given condition we have $\frac{1}{(\beta-\alpha)}\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_X(p)dp\right)$ is increasing in u, which gives that $\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_Y(p)dp\right)$ is increasing in u. We can rewriten (2.6) as

$$(\beta-\alpha)\text{Ḩ}_Y^{(\alpha,\beta)}(u) = \log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_X(p)\phi'(Q_X(p))dp\right). \qquad (2.7)$$

Since $0 < \alpha+\beta < 2$ and ϕ is nonnegative, increasing and convex (concave), we have $[\phi'(Q(p))]^{2-\alpha-\beta}$ is increasing (decreasing) and nonnegative. Hence, by Lemma 2.1, (2.7) is increasing (decreasing). This completes the proof of (i). When $\alpha+\beta > 2$, $[\phi'(Q(p))]^{2-\alpha-\beta} = \frac{1}{[\phi'(Q(p))]^{\alpha+\beta-2}}$ is decreasing in p, since ϕ is increasing and convex. Hence we have

$$\text{Ḩ}_Y^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1(1-p)^{\alpha+\beta-1}q_Y(p)dp\right),$$

is decreasing (increasing) in u. Therefore, the proof is completed. □

Remark 1. For any absolutely continuous random variable X , let $Y = aX + b, a \geq 0, b \geq 0$ then

$$\text{Ḩ}_Y^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log a + \text{Ḩ}_X^{(\alpha,\beta)}(u). \qquad (2.8)$$

In many realistic situations, the random variable is not necessarily related to future only, but they can also refer to the past. Suppose at time t, one has undergone a medical test to check for a certain disease. Let us assume that the test is positive. If we denote by X the age when the patient was infected, then it is known that $X < t$. Now the question is, how much time has elapsed since the patient had been infected by this disease. In this situation, the random variable ${}_tX = [t - X|X \leq t]$, which is known as inactivity time, is suitable to describe the time elapsed between the failure of a system and the time when it is found to be 'down'.

The past lifetime random variable ${}_tX$ is related with two relevant ageing functions, *the reversed hazard rate* defined by $\mu_F(x) = \frac{f(x)}{F(x)}$, and *mean inactivity time (MIT)* defined by $m(t) = E(t - X|X < t) = \frac{1}{F(t)} \int_0^t F(x)dx$. The quantile versions of reversed hazard rate function and *mean inactivity time (MIT)* are given as

$$\bar{K}(u) = \bar{K}(Q(u)) = u^{-1} f(Q(u)) = [uq(u)]^{-1}, \tag{2.9}$$

and

$$\bar{M}(u) = m(Q(u)) = u^{-1} \int_0^u [Q(u) - Q(p)]dp = \frac{1}{u} \int_0^u pq(p)dp, \tag{2.10}$$

respectively. The relationship (1.6) for inactivity time becomes

$$q(u) = \frac{\bar{M}(u) + u\bar{M}(u)}{u}, \tag{2.11}$$

refer to Nair and Sankaran (2009). Analogous to generalized cumulative residual entropy of order (α, β), Minimol (2017) proposed a cumulative entropy *(CE)* measure of order (α, β) and its dynamic version, which are given as

$$\bar{\xi}_X^{(\alpha,\beta)} = \frac{1}{(\beta - \alpha)} \log\left(\int_0^\infty F^{(\alpha+\beta-1)}(x)dx\right), \tag{2.12}$$

and

$$\bar{\xi}_{X_t}^{(\alpha,\beta)} = \frac{1}{(\beta - \alpha)} \log\left(\int_0^t \frac{F^{(\alpha+\beta-1)}(x)dx}{F^{(\alpha+\beta-1)}(t)}\right), \; t \geq 0, \tag{2.13}$$

respectively. Sankaran and Sunoj (2017) have considered the quantile version of cumulative past entropy as

$$\bar{\xi}(u) = \bar{\xi}(X; Q(u)) = \frac{\log u}{u} \int_0^u pq(p)dp - u^{-1} \int_0^u p(\log p)q(p)dp. \tag{2.14}$$

In analogy to (2.3), we propose a generalized cumulative past quantile entropy (GCPQE) that computes the uncertainty related to past. It is defined as

$$\bar{\mathrm{H}}_X^{(\alpha,\beta)}(u) = \bar{\mathrm{H}}_X^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta - \alpha)} \log\left(\frac{1}{u^{\alpha+\beta-1}} \int_0^u p^{\alpha+\beta-1} q(p)dp\right). \tag{2.15}$$

Using (2.11), equation (2.15) can be written as

$$\bar{\mathrm{H}}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta - \alpha)} \log\left(\frac{1}{u^{\alpha+\beta-1}} \int_0^u p^{\alpha+\beta-2} \bar{M}(p)dp + \frac{1}{u^{\alpha+\beta-1}} \int_0^u p^{\alpha+\beta-1} \bar{M}'(p)dp\right). \tag{2.16}$$

After integration by parts on the last term and simplifying, we obtain

$$u^{\alpha+\beta-1}e^{(\beta-\alpha)\bar{H}_X^{(\alpha,\beta)}(u)} - \bar{M}(u)u^{\alpha+\beta-1} = (\beta-\alpha)\int_0^u p^{\alpha+\beta-2}\bar{M}(p)dp. \tag{2.17}$$

If we differentiating (2.17) with respect to u and simplify the resulting equation, we get

$$q(u) = e^{(\beta-\alpha)\bar{H}_X^{(\alpha,\beta)}(u)}\left((\beta-\alpha)\bar{H}'^{(\alpha,\beta)}_X(u) + \frac{(\alpha+\beta-1)}{u}\right). \tag{2.18}$$

Equation (2.18) provides a direct relationship between quantile density function $q(u)$ and $\bar{H}_X^{(\alpha,\beta)}(u)$. Therefore $\bar{H}_X^{(\alpha,\beta)}(u)$ uniquely determines the underlying distribution.

There are some models that do not have any closed form expressions for cdf or pdf, but have simple quantile function or quantile density functions (see Nair et al. (2013)). Accordingly in the following example, we obtain $\bar{H}_X^{(\alpha,\beta)}(u)$ for which $q(.)$ exists.

Examples 2.4. Let X be a random variable having the Tukey lambda distribution with the quantile function $Q(u) = \frac{u^\lambda-(1-u)^\lambda}{\lambda}$, $0 \le u \le 1$; defined for all nonzero lambda values. Then, the generalized cumulative past quantile entropy for Tukey lambda distribution is given as

$$\bar{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left[\frac{1}{u^{\alpha+\beta-1}}\int_0^u p^{\alpha+\beta-1}\{p^{\lambda-1} + (1-p)^{\lambda-1}\}dp\right],$$

which can be written as

$$\bar{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{u^\lambda}{(\alpha+\beta-1+\lambda)} + \frac{\beta_u(\alpha+\beta,\lambda)}{u^{\alpha+\beta-1}}\right),$$

where $\beta_u(a,b)$ represents the incomplete beta function.

Examples 2.5. Let X be distributed with quantile density function $q(u) = Ku^\delta(1-u)^{-(A+\delta)}$, $0 < u < 1$, where K, δ, and A are real constants. This form contains several distribution which include the exponential ($\delta = 0; A = 1$), Pareto ($\delta = 0; A < 1$), the rescalded beta ($\delta = 0; A > 1$), the log logestic distribution ($\delta = \lambda - 1; A = 2$) and Govindarajulu's distribution ($\delta = \beta - 1; A = -\beta$) with quantile function ($\theta + \sigma\{(\beta+1)u^\beta - \beta u^{\beta+1}\}$). Then GCPQE (2.15) becomes

$$\bar{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{K\beta_u(\alpha+\beta+\delta, 1-A-\delta)}{u^{\alpha+\beta-1}}\right).$$

3 Characterizing of Lifetime Distribution Functions

By considering a relationship between the *generalized dynamic cumulative residual quantile entropy* $H_X^{(\alpha,\beta)}(u)$ and the *hazard quantile function* $K(u)$, we characterize some lifetime distributions based on the quanlile entropy measure (2.3). We give the following theorem.

Theorem 3.1. *Let X be a random varible with hazard quantile function* $K(u)$*. The relationship*

$$\underset{\tilde{}}{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log c - \frac{1}{(\beta-\alpha)}\log K(u), \tag{3.1}$$

where $c = \left(\frac{a+1}{(\alpha+\beta-1)(a+1)-a}\right)$ *is a parameter, holds for all* $u \in (0,1)$ *if and only if X follows generalized Pareto distribution with quantile function* $Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right]$; $b > 0,\ a > -1$.

Proof. The hazard quantile function of generalized Pareto distribution is $K(u) = \frac{1}{(1-u)q(u)} = \frac{(a+1)(1-u)^{\frac{a}{a+1}}}{b}$. From Table 1, the quantile-based generalized residual entropy of order (α, β) of GPD is

$$\underset{\tilde{}}{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{b(1-u)^{-\frac{a}{a+1}}}{(a+1)(\alpha+\beta-1)-a}\right),$$

which can be written as

$$\underset{\tilde{}}{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{(a+1)}{(a+1)(\alpha+\beta-1)-a}\right) - \frac{1}{(\beta-\alpha)}\log\left(\frac{b}{a+1}\right)(1-u)^{-\frac{a}{a+1}}.$$

This proves the if part of the theorem. To prove the only if part, consider (3.1) to be valid. Substituting in (2.3) and simplifying, we get

$$\int_u^1 (1-p)^{(\alpha+\beta-1)} q(p)dp = \frac{c(1-u)^{(\alpha+\beta-1)}}{K(u)}.$$

Using (1.4), we have

$$\int_u^1 (1-p)^{(\alpha+\beta-1)} q(p)dp = c(1-u)^{\alpha+\beta} q(u).$$

After differentiation both sides with respect to u and some algebraic simplification, we have

$$\frac{q'(u)}{q(u)} = \left(\frac{c(\alpha+\beta)-1}{c}\right)\frac{1}{(1-u)}.$$

This gives

$$q(u) = A(1-u)^{\frac{1}{c}-(\alpha+\beta)},$$

where A is the constant of integration. Substituting the value of c, this gives

$$q(u) = A(1-u)^{-\frac{2a}{a+1}},$$

which characterizes the generalized Pareto distribution. Hence, the proof is completed. □

Corollary 3.1. *The relationship*

$$\underset{\varsigma}{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log c - \frac{1}{(\beta-\alpha)}\log K(u),$$

where c is some constant, holds if and only if for (i) $c = \frac{1}{(\alpha+\beta-1)}$, *X follows exponential distribution (ii)* $c < \frac{1}{(\alpha+\beta-1)}$, *X follows Pareto I distribution (iii)* $c > \frac{1}{(\alpha+\beta-1)}$, *X follow finite range distribution.*

Next we extend the result to a more general case where c is a function of u.

Theorem 3.2. *Let X be a nonnegative absolutely continuous random variable with hazard quantile function* $K(u)$ *and the* GDCRQE $\underset{\varsigma}{H}_X^{(\alpha,\beta)}(u)$ *be of the form*

$$\underset{\varsigma}{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log c(u) - \frac{1}{(\beta-\alpha)}\log K(u), u > 0, \tag{3.2}$$

then

$$q(u) = \exp\left(\frac{\int_0^u \frac{-du}{(1-u)c(u)}}{(1-u)^{\alpha+\beta}c(u)}\right). \tag{3.3}$$

Proof. Let (3.2) hold. Then

$$\frac{1}{\beta-\alpha}\log\left\{\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1 (1-p)^{\alpha+\beta-1}q(p)dp\right\} = \frac{1}{\beta-\alpha}\log\left(\frac{c(u)}{K(u)}\right).$$

Substituting the value of K(u) from (1.4), we have

$$\int_u^1 (1-p)^{\alpha+\beta-1}q(p)dp = c(u)q(u)(1-u)^{\alpha+\beta}.$$

If we differentiate both sides with respect to u and do some algebraic simplification, we get,

$$\frac{q'(u)}{q(u)} = \frac{(\alpha+\beta)}{(1-u)} - \frac{1}{(1-u)c(u)} - \frac{c'(u)}{c(u)},$$

Integrating with respect to u in the above expression and simplifying results in the following equation:

$$\log\left(c(u)(1-u)^{\alpha+\beta}q(u)\right) = -\int_0^u \frac{1}{(1-u)c(u)}.$$

□

In particular, if $c(u) = au + b$ and a, b ≥ 0 then

$$q(u) = \frac{1}{(1-u)^{\alpha+\beta}(au+b)}\left(\frac{au+b}{b(1-u)}\right)^{\frac{-1}{a+b}}. \tag{3.4}$$

Furthermore, we note that expression (3.4), for a = 0, gives the characterization result given by Theorem (3.1).

The following theorem gives another characterization of the generalized Pareto distribution using the relationship between $H_X^{(\alpha,\beta)}(u)$ and mean residual quantile function $M(u)$, the proof of which follows the same line as Theorem (3.1), hence omitted.

Theorem 3.3. *Let X be a random varible with mean residual quantile function $M(u)$ for all $u \in (0, 1)$. The relationship*

$$H_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)} \log c + \frac{1}{(\beta-\alpha)} \log M(u), \tag{3.5}$$

where c is SOME constant, holds if and only if X follows generalized Pareto distribution.

In the following theorem, we characterize the power distribution, when GCPQE $\bar{H}_X^{(\alpha,\beta)}(u)$ is expressed in terms of $\bar{K}(u)$.

Theorem 3.4. *Let X be a nonnegative continuous random variable with reverse hazard quantile function $\bar{K}(u)$ for all $u \in (0, 1)$ and GCPQE $\bar{H}_X^{(\alpha,\beta)}(u)$ is given by*

$$\bar{H}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)} \log c - \frac{1}{(\beta-\alpha)} \log \bar{K}(u), \tag{3.6}$$

if and only if X follows the power distribution function.

Proof. The reverse hazard quantile function of the power distribution is $\bar{K}(u) = \frac{bu^{\frac{-1}{b}}}{a}$. Taking $c = \frac{b}{b(\alpha+\beta-1)+1}$ proves the if part of the theorem. To prove the only if part, consider (3.6) to be valid. Using (2.12), it gives

$$\frac{\int_0^u p^{\alpha+\beta-1} q(p)dp}{u^{\alpha+\beta-1}} = \frac{c}{\bar{K}(u)}.$$

Substituting $\bar{K}(u) = \frac{1}{uq(u)}$, results in

$$\int_0^u p^{\alpha+\beta-1} q(p)dp = cu^{\alpha+\beta} q(u).$$

By differentiating both sides with respect to u and simplifying, it can be shown that

$$\frac{q'(u)}{q(u)} = \left(\frac{1-c(\alpha+\beta)}{c}\right)\frac{1}{u},$$

which leads to

$$q(u) = Au^{\frac{1}{c}-(\alpha+\beta)},$$

where A is a constant. This expression characterizes the power distribution function for $c = \frac{b}{b(\alpha+\beta-1)+1}$. □

Next we characterize the lifetime models when GCPQE (2.15) is expressed in terms of quantile version of mean inactivity time $\bar{M}(u)$. The proof follows on the same line as Theorem (3.4), hence omitted.

Theorem 3.5. *Let X be a nonnegative continuous random variable with mean residual quantile function $\bar{M}(u)$. The relationship*

$$\bar{\underset{\sim}{H}}_X^{(\alpha,\beta)}(u) = \frac{1}{(\beta-\alpha)}\log c + \frac{1}{(\beta-\alpha)}\log \bar{M}(u),$$

where c is constant, hold for all $u \in (0,1)$, if and only if, X follow the power distribution function.

4 GDCRQE for Order Statistics $X_{i:n}$

Suppose $X_1, X_2, ..., X_n$ be a random sample from a population with probability density function f and cumulative distribution function $F(.)$ and let $X_{1:n} \leq X_{2:n} \leq ...X_{n:n}$ be the order statistics obtained by arranging the preceding random sample in increasing order of magnitude. Then the pdf of the i^{th} order statistics $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)}(F(x))^{i-1}(\bar{F}(x))^{n-i}f(x),$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx;\ a, b > 0$, is the beta function. The corresponding quantile-based density function of $f_{i:n}(x)$ is

$$f_{i:n}(u) = f_{i:n}(Q(u)) = \frac{u^{i-1}(1-u)^{n-i}}{B(i, n-i+1)q(u)}.$$

Sunoj et al. (2017) introduced a quantile-based entropy of order statistics and studied its properties. Order statistics play an important role in system reliability. These statistics have been used in a wide range of problems, including robust statistical estimation, detection of outliers, characterization of the probability distribution and goodness-of-fit tests, analysis of censored samples, reliability analysis, quality control and strength of materials. For more details, we refer readers to Arnold et al. (1992), David and Nagaraja (2003), and references therein. Similar results on generalized residual entropy for order statistics have been derived by Abbasnejad and Arghami (2011), Zarezadeh and Asadi (2010), Thapliyal et al. (2015), Kayal (2016), and Kumar (2018).

Thapliyal and Taneja (2012) proposed the two parameter generalized entropy for the i^{th} order statistics $X_{i:n}$ as

$$H_{X_{i:n}}^{(\alpha,\beta)}(f) = \frac{1}{(\beta-\alpha)}\log\left\{\int_0^\infty (f_{i:n}(x))^{\alpha+\beta-1}dx\right\}, \beta \neq \alpha, \beta-1 < \alpha < \beta, \beta \geq 1. \tag{4.1}$$

and studied some of its properties. Kumar and Nirdesh (2019) proposed quantile-based generalized entropy for the i^{th} order statistics, which is given as

$$\mathfrak{H}_{X_{i:n}}^{(\alpha,\beta)}(Q(p)) = \frac{1}{(\beta-\alpha)} \log \int_0^1 \left(\frac{1}{B(i, n-i+1)}\right)^{\alpha+\beta-1} p^{(\alpha+\beta-1)(i-1)}(1-p)^{(\alpha+\beta-1)(n-i)}(q(p))^{2-\alpha-\beta}dp,$$

and studied its properties. Unlike (4.1), $\mathfrak{H}_{X_{i:n}}^{(\alpha,\beta)}(f)$ will be more useful in cases we do not have a tractable distribution function but have a closed quantile function. In analogy with (1.11), the generalized cumulative residual entropy for the i^{th} order statistic $X_{i:n}$ is defined as

$$\begin{aligned} H_{X_{i:n}}^{(\alpha,\beta)} &= \frac{1}{(\beta-\alpha)} \log \int_0^\infty \bar{F}_{i:n}^{\alpha+\beta-1}(x)dx \\ &= \frac{1}{(\beta-\alpha)} \log \int_0^\infty \left(\frac{\bar{\beta}_{F(x)}(i, n-i+1)}{\beta(i, n-i+1)}\right)^{\alpha+\beta-1} dx, \end{aligned} \tag{4.2}$$

where $\bar{F}_{i:n}(x) = \frac{\bar{\beta}_{F(x)}(i,n-i+1)}{\beta(i,n-i+1)}$ is the survival function of the i^{th} order statistics. The generalized cumulative residual quantile entropy of order (α, β), (4.2) becomes

$$\mathfrak{H}_{X_{i:n}}^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log \int_0^1 \left(\frac{\bar{\beta}_p(i, n-i+1)}{\beta(i, n-i+1)}\right)^{\alpha+\beta-1} q(p)dp, \tag{4.3}$$

where $\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}$ is the quantile form of survival function $\bar{F}_{i:n}(x)$. In system reliability, the first order statistic represents the lifetime of a series system while the n^{th} order statistic measure the lifetime of a parallel system. For a series system $(i = 1)$, we have

$$\mathfrak{H}_{X_{1:n}}^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\int_0^1 (1-p)^{n(\alpha+\beta-1)}q(p)dp\right), \tag{4.4}$$

and, for or the parallel system $(i = n)$, we have

$$\mathfrak{H}_{X_{n:n}}^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\int_0^1 (1-p^n)^{(\alpha+\beta-1)}q(p)dp\right). \tag{4.5}$$

The residual lifetime of a system, when it is still operating at time t, is $(X_t = X - t|X > t)$ which has the probability density function $f(x,t) = \frac{f(x)}{\bar{F}(t)}, x \geq t > 0$. The generalized dynamic cumulative residual entropy (GDCRE) measure for $X_{i:n}$ is given by

$$H_{X_{i:n}}^{(\alpha,\beta)}(t) = \frac{1}{(\beta-\alpha)} \log\left(\int_t^\infty \frac{\bar{F}_{i:n}^{\alpha+\beta-1}(x)}{\bar{F}_{i:n}^{\alpha+\beta-1}(t)}dx\right).$$

For the i^{th} order statistics $X_{i:n}$, the quantile version of GDCRQE is

$$\begin{aligned}
\text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}(u) = \text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}(Q(u)) &= \frac{1}{(\beta-\alpha)}\log\left\{\int_u^1 \left(\frac{\bar{\beta}_p(i,n-i+1)}{\beta(i,n-i+1)}\right)^{\alpha+\beta-1}\left(\frac{\beta(i,n-i+1)}{\bar{\beta}_u(i,n-i+1)}\right)^{\alpha+\beta-1} q(p)dp\right\} \\
&= \frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(\bar{\beta}_u(i,n-i+1))^{\alpha+\beta-1}}\int_u^1 (\bar{\beta}_p(i,n-i+1))^{\alpha+\beta-1} q(p)dp\right\}, \quad (4.6)
\end{aligned}$$

where $\frac{\beta_u(i,n-i+1)}{\beta(i,n-i+1)}$ is the quantile form of survival function $\bar{F}_{i:n}(x)$, see David and Nagaraja (2003). An equivalent representation of (4.6) is of the form

$$\exp^{(\beta-\alpha)\text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}}(\bar{\beta}_u(i,n-i+1))^{\alpha+\beta-1} = \int_u^1 (\bar{\beta}_p(i,n-i+1))^{\alpha+\beta-1} q(p)dp.$$

Differentiating both sides of (4.6) with respect to u and doing some algebraic simplification, we have

$$q(u) = \left\{\frac{(\alpha+\beta-1)u^{i-1}(1-u)^{n-i}}{\bar{\beta}_u(i,n-i+1)} - (\beta-\alpha)\text{Ḩ}_{X_{i:n}}^{\prime(\alpha,\beta)}\right\} e^{(\beta-\alpha)\text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}}. \quad (4.7)$$

Equation (4.7) provides a direct relationship between quantile density function $q(u)$ and $\text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}(u)$, which shows that $\text{Ḩ}_{X_{i:n}}^{(\alpha,\beta)}(u)$ uniquely determines the underlying distribution.

In system reliability, the minimum and maximum values are examples of extreme order statistics and are defined by $X_{1:n} = min\{X_1, X_2, ..., X_n\}$ and $X_{n:n} = max\{X_1, X_2, ..., X_n\}$. The extreme $X_{1:n}$ and $X_{n:n}$ are of special interest in many practical problems of distribution analysis. The extremes aries in the statistical study of floods and droughts, as well as in problems of breaking strength and fatigue failure. Substituting ($i = 1$) in (4.6), the *GDCRQE* of the first order statistic $X_{1:n}$ is given as

$$\begin{aligned}
\text{Ḩ}_{X_{1:n}}^{(\alpha,\beta)}(u) &= \frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(\bar{\beta}_u(1,n))^{\alpha+\beta-1}}\int_u^1 (\bar{\beta}_p(1,n))^{\alpha+\beta-1} q(p)dp\right\} \\
&= \frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(1-u)^{n(\alpha+\beta-1)}}\int_u^1 (1-p)^{n(\alpha+\beta-1)} q(p)dp\right\}. \quad (4.8)
\end{aligned}$$

The *GDCRQE* for the sample maxima $X_{n:n}$ can be obtained from (4.6) by taking ($i = n$), as

$$\begin{aligned}
\text{Ḩ}_{X_{n:n}}^{(\alpha,\beta)}(u) &= \frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(\bar{\beta}_u(n,1))^{\alpha+\beta-1}}\int_u^1 (\bar{\beta}_p(n,1))^{\alpha+\beta-1} q(p)dp\right\} \\
&= \frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(1-u^n)^{\alpha+\beta-1}}\int_u^1 (1-p^n)^{\alpha+\beta-1} q(p)dp\right\}. \quad (4.9)
\end{aligned}$$

For various specific univariate continuous distributions, the expression (4.8) is evaluated as in Table 2.

Table 2: Quantile function and $\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u)$ for various lifetime distributions

Distribution	Quantile function $Q(u)$	$\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u)$
Uniform	$a + (b-a)u$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(b-a)(1-u)}{n(\alpha+\beta-1)+1}\right\}$
Exponential	$-\lambda^{-1}\log(1-u)$	$\frac{1}{\beta-\alpha}\log\left\{\frac{1}{n\lambda(\alpha+\beta-1)}\right\}$
Pareto I	$b(1-u)^{-\frac{1}{a}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{b(1-u)^{\frac{-1}{a}}}{na(\alpha+\beta-1)-1}\right\}$
Folded Cramer	$\frac{u}{\theta(1-u)}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{1}{\theta\{n(\alpha+\beta-1)-1\}(1-u)}\right\}$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$\frac{1}{\beta-\alpha}\log\left\{\frac{b(1-u)^{\frac{-a}{a+1}}}{n(a+1)(\alpha+\beta-1)-a)}\right\}$
Finite Range	$b\left\{1-(1-u)^{\frac{1}{a}}\right\}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{b(1-u)^{\frac{1}{a}}}{na(\alpha+\beta-1)+1}\right\}$
Log logestic	$\frac{1}{a}\left(\frac{u}{(1-u)}\right)^{\frac{1}{b}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{\bar{\beta}_u[b,n(\alpha+\beta-1)-\frac{1}{b}]}{ab(1-u)^{n(\alpha+\beta-1)}}\right\}$
Generalized lambda	$\lambda_1+\frac{1}{\lambda_2}\left(\frac{u^{\lambda_3-1}}{\lambda_3}-\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(\lambda_3-1)\bar{\beta}_u[\lambda_3-1,n(\alpha+\beta-1)+1]}{\lambda_2\lambda_3(1-u)^{n(\alpha+\beta-1)}}+\frac{(1-u)^{\lambda_4}}{\lambda_2\{n(\alpha+\beta-1)+\lambda_4\}}\right\}$
Skew lambda	$\alpha u^{\lambda}-(1-u)^{\lambda}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{\alpha\lambda\bar{\beta}_u[\lambda,n(\alpha+\beta-1)+1]}{(1-u)^{n(\alpha+\beta-1)}}+\frac{\lambda(1-u)^{\lambda}}{n(\alpha+\beta-1)+\lambda}\right\}$
Govindarajulu's	$a\left\{(b+1)u^b-bu^{(b+1)}\right\}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{ab(b+1)\bar{\beta}_u[b,n(\alpha+\beta-1)+2]}{(1-u)^{n(\alpha+\beta-1)}}\right\}$

Examples 4.1. A lambda family of distribution that is of interest in reliability is the Davis Distribution, proposed by Hankin and Lee (2006) with quantile function $Q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}$, $0 < u < 1, C, \lambda_1, \lambda_2 \geq 0$. This is a flexible family for right skewed nonnegative data and provides a good approximation to the exponential, gamma, lognormal and Weibull distributions. A special feature of these families is that they are expressed in terms of quantile functions and the distribution functions are not available in closed form to facilitate the conventional analysis. The *GDCRQE* entropy of sample minima for Davis distribution is given by

$$\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log\left(C\lambda_1\frac{\bar{\beta}_u(\lambda_1, n(\alpha+\beta-1)-\lambda_2+1)}{(1-u)^{n(\alpha+\beta-1)}} + C\lambda_2\frac{\bar{\beta}_u(\lambda_1+1, n(\alpha+\beta-1)-\lambda_2)}{(1-u)^{n(\alpha+\beta-1)}}\right). \tag{4.10}$$

As $\lambda_1 \longrightarrow 0$, (4.10) reduces to $\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{C\lambda_2}{(1-u)^{\lambda}(n(\alpha+\beta-1)-\lambda_2)}\right)$, corresponding to the Pareto I distribution.

Also, as $\lambda_2 \longrightarrow 0$, (4.10) reduces to $\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log\left(\frac{C\lambda_1\bar{\beta}_u(\lambda_1,n(\alpha+\beta-1)+1)}{(1-u)^{n(\alpha+\beta-1)}}\right)$, corresponding to the power distribution.

Next, we obtain the characterization result based on the first (minima) and the last (maxima) in a random sample $X_1, X_2, ..., X_n$ of size n from a positive and continuous random variable X.

Theorem 4.1. *Let* $X_{1:n}$ *denote the first order statistic with survival function* $\bar{F}_{1:n}(x)$ *and hazard quantile* *function* $K_{X_{1:n}}(u)$*. Then*

$$\underset{\sim}{H}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log c - \frac{1}{(\beta-\alpha)}\log K_{X_{1:n}}(u), \tag{4.11}$$

where c is some constant, holds for all $u \in (0,1)$ *if and only if X follows the generalized Pareto distribution (GPD).*

Proof. The hazard quantile function for the sample minima $X_{1:n}$ for GPD is given by

$$K_{X_{1:n}}(u) = \frac{f_{1:n}(Q(u))}{(1-F(Q(u)))^n} = \frac{n}{(1-u)q(u)} = \frac{n(a+1)(1-u)^{\frac{a}{a+1}}}{b}.$$

Based on Table 2, the if part of the theorem is proved. To prove the only if part, let (4.11) hold. Then

$$\int_u^1 (1-p)^{n(\alpha+\beta-1)}q(p)dp = \frac{c(1-u)^{n(\alpha+\beta-1)}}{K_{X_{1:n}}(u)}.$$

By substituting the value of $K_{X_{1:n}}(u)$ and simplifying, we can get

$$n\int_u^1 (1-p)^{n(\alpha+\beta-1)}q(p)dp = cq(u)(1-u)^{n(\alpha+\beta-1)+1}.$$

A similar calculation as described previously shows that

$$\frac{q'(u)}{q(u)} = \left\{\frac{n-c\,(n(\alpha+\beta-1)+1)}{c}\right\}\frac{1}{(1-u)}.$$

This gives

$$q(u) = A(1-u)^{\frac{n}{c}-n(\alpha+\beta-1)-1}.$$

Substituting the value of $c = \frac{n(a+1)}{n(a+1)(\alpha+\beta-1)-a}$, we obtain

$$q(u) = A(1-u)^{-\left(\frac{2a+1}{a+1}\right)},$$

which characterizes the generalized Pareto distribution. This completes the proof. □

Corollary 4.1. *Let* $X_{1:n}$ *denote the first order statistic with the survival function* $\bar{F}_{1:n}(x)$ *and the hazard quantile function* $K_{X_{1:n}}(u)$ *for all* $u \in (0,1)$*. Then,*

$$\underset{\sim}{H}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log c - \frac{1}{(\beta-\alpha)}\log K_{X_{1:n}}(u).$$

holds, if and only if for (i) $c = \frac{1}{\alpha+\beta-1}$*, X follows the exponential distribution (ii)* $c < \frac{1}{\alpha+\beta-1}$*, X follows the Pareto I distribution (iii)* $c > \frac{1}{\alpha+\beta-1}$*, X follows the finite range distribution.*

In the following theorem, we give the characterization result of some well known distributions in terms of GDCRQE for the sample minima $X_{1:n}$.

Theorem 4.2. *Let $X_{1:n}$ denote the first order statistic with survival function $\bar{F}_{1:n}(x)$. Then, for $u \in (0,1)$, the $\H^{(\alpha,\beta)}_{X_{1:n}}(u)$ is given by*

$$(\beta - \alpha)\H'^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{-C}{1-u}, \tag{4.12}$$

where C is some constant, if and only if X is distributed as
(i) the uniform distribution for $C = 1$.
(ii) the exponential distribution for $C = 0$.
(iii) the Pareto I distribution for $C = \frac{-1}{a}$.

Proof. The necessity part follows from Table 2. For sufficient part, let us assume that the relationship (4.12) holds. From (4.8) and (4.12), we get

$$-\frac{(1-u)^{n(\alpha+\beta-1)}q(u)}{\int_u^1 (1-p)^{n(\alpha+\beta-1)}q(u)dp} + \frac{n(\alpha+\beta-1)}{1-u} = -\frac{C}{1-u}.$$

After simplifying, we have

$$q(u)(1-u)^{n(\alpha+\beta-1)+1} = \{n(\alpha+\beta-1)+C\}\int_u^1 (1-p)^{n(\alpha+\beta-1)}q(p)dp.$$

After differentiation both sides with respect to u and doing some algebraic simplification, it can be shown that

$$-q(u)(1-u)^{n(\alpha+\beta-1)}\{n(\alpha+\beta-1)+1\} + (1-u)^{n(\alpha+\beta-1)+1}q'(u) = -\{n(\alpha+\beta-1)+C\}(1-u)^{n(\alpha+\beta-1)}q(u),$$

or equivalently

$$\frac{q'(u)}{q(u)} = \frac{1-C}{1-u},$$

which leads to

$$q(u) = (1-u)^{C-1}e^{A},$$

where A is the constant of integration. Now, if C = 1 and $A = \log(b-a)$; $b > a$, it is implies that $Q(u) = a + (b-a)u$. Thus, we have the uniform distribution. If $C = 0$ and $A = -\log\lambda$; $\lambda \geq 0$, It can be concluded that $Q(u) = -\lambda^{-1}\log(1-u)$. Thus, we have the exponential distribution with parameter λ. If, $C = \frac{-1}{a}$ and $A = \log(\frac{b}{a})$, such that a and b are positive constants we have $Q(u) = b(1-u)^{-\frac{1}{a}}$. This means, we have the Pareto I distribution. □

Theorem 4.3. *Let $X_{1:n}$ denote the first order statistic with survival function $\bar{F}_{1:n}(x)$ and hazard quantile function $K_{X_{1:n}}(u)$. Then, for all $u \in (0,1)$, the $\underset{\smile}{H}^{(\alpha,\beta)}_{X_{1:n}}(u)$ is expressed as*

$$\{n(\alpha+\beta-1)+1-C\}\, e^{(\beta-\alpha)\underset{\smile}{H}^{(\alpha,\beta)}_{X_{1:n}}(u)} = \frac{n}{K_{X_{1:n}}(u)}, \tag{4.13}$$

where C is some constant, if and only if X is distributed as
(i) the uniform distribution for $C = 0$.
(ii) the exponential distribution for $C = 1$.
(iii) the Pareto I distribution for $C = 1 + \frac{1}{a}$.

Proof. The necessity part follows from Table 2. For sufficiency part, let us assume that the relationship (4.13) holds. Substituting $K_{X_{1:n}}(u) = \frac{n}{(1-u)q(u)}$ and (4.8), we have

$$\{n(\alpha+\beta-1)+1-C\}\,\frac{\int_u^1 (1-p)^{n(\alpha+\beta-1)} q(u)dp}{(1-u)^{n(\alpha+\beta-1)}} = (1-u)q(u).$$

Differentiating both sides with respect to u and after some algebraic simplification, we get

$$\{n(\alpha+\beta-1)+1-C\}\,(1-u)^{n(\alpha+\beta-1)}q(u) = \{n(\alpha+\beta-1)+1\}\,(1-u)^{n(\alpha+\beta-1)}q(u) - (1-u)^{n(\alpha+\beta-1)+1}q'(u).$$

By the above equation, we have

$$\frac{q'(u)}{q(u)} = \frac{C}{1-u}.$$

This gives

$$q(u) = (1-u)^{-C}e^{A},$$

where A is the constant of integration. Now, for $C = 0$, $C = 1$ and $C = 1+\frac{1}{a}$ and with appropriate A's, similar Theorem (4.2), we get the stated results. □

The mean residual quantile function for the sample minima $X_{1:n}$ is given as

$$M_{X_{1:n}}(u) = M_{X_{1:n}}(Q(u)) = \frac{\int_{Q(u)}^{\infty} \bar{F}_{1:n}(Q(p))d(Q(u))}{\bar{F}_{1:n}(Q(u))}$$
$$= \frac{\int_{Q(u)}^{\infty} (\bar{F}(Q(p)))^n d(Q(u))}{(\bar{F}(Q(u)))^n} = (1-u)^{-n}\int_u^1 (1-u)^n q(p)dp.$$

For the sample minima $X_{1:n}$, the It can be concluded that (1.6) becomes $(1-u)q(u) = nM_{X_{1:n}}(u) - (1-u)M'_{X_{1:n}}(u)$. We state a characterization result using the relationship between $\underset{\smile}{H}^{(\alpha,\beta)}_{X_{1:n}}(u)$ and $M_{X_{1:n}}(u)$, the proof of which follows the same line as given in Theorem (3.4), hence omitted.

Theorem 4.4. *Let* $X_{1:n}$ *denote the first order statistic with survival function* $\bar{F}_{1:n}(x)$ *and mean residual quantile function* $M_{X_{1:n}}(u)$*, for all* $u \in (0,1)$*. Then the relationship*

$$\underset{\text{ș}}{H}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log c + \frac{1}{(\beta-\alpha)}\log M_{X_{1:n}}(u), \tag{4.14}$$

where c is a constant, holds if and only if X has the generalized Pareto distribution with quantile function

$$Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right];\ b>0,\ a>-1.$$

Corollary 4.2. *Let* $X_{1:n}$ *denote the first order statistic with survival function* $\bar{F}_{1:n}(x)$ *and mean quantile function* $M_{X_{1:n}}(u)$*. Then the relationship*

$$\underset{\text{ș}}{H}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{(\beta-\alpha)}\log c + \frac{1}{(\beta-\alpha)}\log M_{X_{1:n}}(u).$$

where c is some constant, holds for all $u \in (0,1)$ *if and only if for (i)* $c = \frac{1}{(\alpha+\beta-1)}$*, X follows the exponential distribution (ii)* $c < \frac{1}{(\alpha+\beta-1)}$*, X follows the Pareto I distribution (iii)* $c > \frac{1}{(\alpha+\beta-1)}$*, X follows the finite range distribution.*

Let $X_{n:n}$ be the largest order statistic in a random sample of size n from an absolutely continuous nonnegative random variable X. Then, the generalized cumulative past entropy for sample maxima is as follow

$$\bar{H}^{(\alpha,\beta)}_{X_{n:n}}(t) = \frac{1}{(\beta-\alpha)}\log\left(\int_0^t \frac{F^{(\alpha+\beta-1)}_{n:n}(x)}{F^{(\alpha+\beta-1)}_{n:n}(t)}dx\right).$$

The quantile version of the above equation can be expressed as

$$\bar{\underset{\text{ș}}{H}}^{(\alpha,\beta)}_{X_{n:n}}(u) = \bar{\underset{\text{ș}}{H}}^{(\alpha,\beta)}_{X_{n:n}}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{u^{n(\alpha+\beta-1)}}\int_0^u p^{n(\alpha+\beta-1)}q(p)dp\right). \tag{4.15}$$

Let X have the power distribution with quantile function $au^{\frac{1}{b}}$. In the following theorem, we show that the power distribution can be characterize in terms of $\bar{\underset{\text{ș}}{H}}^{(\alpha,\beta)}_{X_{n:n}}(u)$.

Theorem 4.5. *Let* $X_{n:n}$ *denotes the last order statistics with the survival function* $\bar{F}_{n:n}(x)$ *and the reverse hazard quantile function* $\bar{K}_{X_{n:n}}(u)$*, then the generalized cumulative past quantile entropy for sample maxima* $\bar{\underset{\text{ș}}{H}}^{(\alpha,\beta)}_{X_{n:n}}(u)$ *is expressed as*

$$\bar{\underset{\text{ș}}{H}}^{(\alpha,\beta)}_{X_{n:n}}(u) = \frac{1}{\beta-\alpha}\log c - \frac{1}{\beta-\alpha}\log \bar{K}_{X_{n:n}}(u), \tag{4.16}$$

if and only if X has the power distribution function.

Proof. The reverse hazard quantile function for the sample maxima $X_{n:n}$ of power distribution is $\bar{K}_{X_{n:n}}(u) = \frac{f_{n:n}(Q(u))}{F_{n:n}(Q(u))} = \frac{nf(Q(u))}{F(Q(u))} = n(uq(u))^{-1} = \frac{nbu^{\frac{-1}{b}}}{a}$. Taking $c = \frac{nb}{nb(\alpha+\beta-1)+1}$ gives the if part of the theorem. To prove the only if part, consider (4.16) be valid. Using (4.15), it gives

$$\frac{\int_0^u p^{n(\alpha+\beta-1)}q(p)dp}{u^{n(\alpha+\beta-1)}} = \frac{c}{\bar{K}_{X_{n:n}}(u)}.$$

Substituting $\bar{K}_{X_{n:n}}(u) = \frac{n}{uq(u)}$, we have

$$n\int_0^u p^{n(\alpha+\beta-1)}q(p)dp = cu^{n(\alpha+\beta-1)+1}q(u).$$

By differentiating both sides with respect to u and simplifying, we have

$$\frac{q'(u)}{q(u)} = \left(\frac{n - c\left\{n(\alpha+\beta-1)+1\right\}}{c}\right)\frac{1}{u},$$

which leads to

$$q(u) = Au^{\frac{n}{c}-\left\{n(\alpha+\beta-1)+1\right\}},$$

where A is a constant. This equation characterizes the power distribution for $c = \frac{nb}{nb(\alpha+\beta-1)+1}$. □

Also we have this characterization in terms of $\bar{M}_{X_{n:n}}(u)$. The proof follows on the same line as Theorem (4.6), hence omitted.

Theorem 4.6. *Let $X_{n:n}$ denotes the last order statistics with the survival function $\bar{F}_{n:n}(x)$ and the quantile version of mean inactivity time $\bar{M}_{X_{n:n}}(u)$, then the generalized cumulative past quantile entropy for sample maxima $\bar{\mathcal{H}}^{(\alpha,\beta)}_{X_{n:n}}(u)$ is expressed as*

$$\bar{\mathcal{H}}^{(\alpha,\beta)}_{X_{n:n}}(u) = \frac{1}{(\beta-\alpha)}\log c + \frac{1}{(\beta-\alpha)}\log \bar{M}_{X_{n:n}}(u), \tag{4.17}$$

where c is a constant. If and only if X has the power distribution function.

Remark 2. If $c = \frac{n+1}{n(\alpha+\beta-1)+1}$, then equation (4.16) presents a characterization of the uniform distribution.

5 Generalized Weighted Quantile Entropy of Order (α, β)

Sometimes, in statistical modeling, standard distributions are not suitable for our data and we need to study weighted distributions. This concept has been applied in many areas of statistics, such as analysis of family size, human heredity, world life population study, renewal theory, biomedical and statistical ecology. Associated to a random variable X with pdf $f(x)$ and to a nonnegative real function $w(x)$, we can define the weighted random variable X^w with density function $f^w(x) = \frac{w(x)f(x)}{E(w(X))}$, $0 < E(w(X)) < \infty$. When $w(x) = x$, X^w is called the length (size) biased random variable. Using $f^w(x)$, the corresponding density quantile function is given by

$$f^w(Q(u)) = \frac{w(Q(u))f(Q(u))}{\mu},$$

where $\mu = \int_0^1 w(Q(p))f(Q(p))d(Q(p)) = \int_0^1 w(Q(p))dp$. The weighted entropy has been used to balance the amount of information and the degree of homogeneity associated with a partition of data in classes. The quantile-based generalized weighted entropy is of the form

$$\text{H}_w^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{\mu^{\alpha+\beta-1}}\int_0^1 [w(Q(p))]^{\alpha+\beta-1}(q(p))^{2-\alpha-\beta}dp\right).$$

In case of the length (size) biased random variable, the above expression is known as *length biased weighted generalized quantile entropy* and is given by

$$\text{H}_L^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{\mu^{\alpha+\beta-1}}\int_0^1 (Q(p))^{(\alpha+\beta-1)}(q(p))^{2-\alpha-\beta}dp\right). \tag{5.1}$$

For some specific univariate continuous distributions, the expression (5.1) is evaluated as given below in Table 3.

Consider a random variable Y with density function $f_Y(x) = \frac{\bar{F}(x)}{\mu}$, where $\mu = E(X) < \infty$. Then, Y is called the equilibrium random variable of the original random variable X, and its distribution is known as equilibrium distribution of original random variable. The equilibrium distribution arises as the limiting distribution of the forward recurrence time in a renewal process. We have $f_Y(Q(u)) = \frac{\bar{F}(Q(u))}{\mu} = \frac{1-u}{\mu}$. Thus quantile density function for equilibrium distribution is $q_Y(u) = \frac{1}{f_Y(Q(u))} = \frac{\mu}{1-u}$. From (2.3), the *generalized dynamic cumulative residual quantile entropy (*GDCRQE*)* for equilibrium distribution is

$$\text{H}_Y^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{1}{(1-u)^{\alpha+\beta-1}}\int_u^1 (1-p)^{\alpha+\beta-1}q_Y(p)dp\right). \tag{5.2}$$

Theorem 5.1. *Let X be a nonnegative random variable. Then the relation* $\text{H}_Y^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{\mu}{(\alpha+\beta-1)}\right)$ *holds if and only if X follow the equilibrium distribution.*

Proof. The if part of the theorem is easy to prove. For the proof of the only if part, let us assume that $\text{H}_Y^{(\alpha,\beta)}(Q(u)) = \frac{1}{(\beta-\alpha)}\log\left(\frac{\mu}{(\alpha+\beta-1)}\right)$. From equation (5.2), we have

$$\int_u^1 (1-p)^{(\alpha+\beta-1)}q_Y(p)dp = \frac{\mu(1-u)^{\alpha+\beta-1}}{(\alpha+\beta-1)}.$$

Table 3: Length biased generalized weighted quantile entropy $H_L^{(\alpha,\beta)}(Q(u))$ for some lifetime distributions

Distribution	Quantile function $Q(u)$	$H_L^{(\alpha,\beta)}(Q(u))$
Uniform	$a+(b-a)u$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(b^{\alpha+\beta}-a^{\alpha+\beta})2^{\alpha+\beta-1}}{(a+b)^{\alpha+\beta-1}(b-a)^{\alpha+\beta-1}(\alpha+\beta)}\right\}$
Exponential	$-\lambda^{-1}\log(1-u)$	$\frac{1}{\beta-\alpha}\log\left\{\frac{\lambda^{\alpha+\beta-2}\gamma(\alpha+\beta)}{(\alpha+\beta-1)^{\alpha+\beta}}\right\}$
Power	$au^{\frac{1}{b}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{a^{2-\alpha-\beta}(b+1)^{\alpha+\beta-1}}{b(\alpha+\beta-1)+1}\right\}$
Pareto I	$b(1-u)^{-\frac{1}{a}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{b^{2-\alpha-\beta}(a-1)^{\alpha+\beta-1}}{a(\alpha+\beta-1)-1}\right\}$
Finite Range	$b\left\{1-(1-u)^{\frac{1}{a}}\right\}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(a+1)^{\alpha+\beta-1}\beta[\alpha+\beta,(\alpha+\beta-1)(a-1)+1]}{a^{1-\alpha-\beta}b^{\alpha+\beta-2}}\right\}$
Log logestic	$\frac{1}{a}\left(\frac{u}{1-u}\right)^{\frac{1}{b}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{a^{\alpha+\beta-2}b^{\alpha+\beta-2}\beta(\alpha+\beta-1+\frac{1}{b},\alpha+\beta-1-\frac{1}{b})}{(\beta(1+\frac{1}{b},1-\frac{1}{b}))^{\alpha+\beta-1}}\right\}$
Weibull	$\left\{-\frac{1}{a}\log(1-u)\right\}^{\frac{1}{b}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(ab)^{\alpha+\beta-2}(\frac{1}{a})^{(2-\alpha-\beta)(\frac{1}{b}-1)}\gamma(\alpha+\beta+\frac{1}{b}-1)}{(\alpha+\beta-1)^{\alpha+\beta+\frac{1}{b}}(\gamma(1+\frac{1}{b}))^{\alpha+\beta-1}}\right\}$
Generalized Pareto	$\frac{b}{a}\left\{(1-u)^{-\frac{a}{a+1}}-1\right\}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{(-1)^{\alpha+\beta}(\frac{b}{a+1})^{2-\alpha-\beta}\beta[\alpha+\beta,(2-\alpha-\beta)(1+\frac{1}{a})(\frac{2a+1}{a+1})-\frac{1}{a}-1]}{a^{\alpha+\beta-1}}\right\}$
Rayleigh	$\left\{-\frac{1}{a}\log(1-u)\right\}^{\frac{1}{2}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{2^{\alpha+\beta-1}(\frac{1}{a})^{\frac{1}{2}(\alpha+\beta-2)}\gamma(\alpha+\beta-\frac{1}{2})}{\pi^{\frac{\alpha+\beta-1}{2}}(\alpha+\beta-1)^{\alpha+\beta-\frac{1}{2}}(2a)^{1-\alpha}}\right\}$

Differentiating with respect to u on both sides and after some simplification, we get $q_Y = \frac{\mu}{(1-u)}$, which is the quantile density function for equilibrium distribution. Hence, the proof is completed. □

Remark 3. The mean residual quantile function satisfies the relation $M(Y;Q(u)) = \mu$ if and only if X follow equilibrium distribution.

5.1 Weighted Cumulative Residual Generalized Entropy

Misagh et al. (2011) proposed a weighted information which is based on the *CRE*, called weighted cumulative residual entropy (*WCRE*). This measure is defined as

$$\bar{\xi}_w(X) = -\int_0^\infty x\bar{F}(x)\log\bar{F}(x)dx\,. \tag{5.3}$$

Several authors studied properties of (5.3) and its dynamic version, refer to Kayal and Moharana (2017) and Mirali et al. (2017). As pointed out by Misagh et al. (2011), in some practical situations of reliability and neurobiology, a shift-dependent measure of uncertainty is desirable. Also, an important feature of the human visual system is that it can recognize objects in a scale and translation invariant manner. However, achieving this desirable behavior using biologically realistic network is a challenge. The notion of weighted entropy addresses this requirement.

In, analogy to (5.3), the generalized weighted cumulative residual entropy (*GWCRE*) and its residual form are defined as

$$H_w^{(\alpha,\beta)}(X) = \frac{1}{(\beta-\alpha)} \log\left(\int_0^\infty x\bar{F}^{\alpha+\beta-1}(x)dx\right), \beta \neq \alpha, \beta-1<\alpha<\beta, \beta \geq 1\ , \tag{5.4}$$

and

$$H_w^{(\alpha,\beta)}(X;t) = \frac{1}{(\beta-\alpha)} \log\left(\frac{\int_t^\infty x\bar{F}^{\alpha+\beta-1}(x)dx}{\bar{F}^{\alpha+\beta-1}(t)}\right), \tag{5.5}$$

respectively. The factor x in the integral on the right-hand side yields a "length-biased" shift dependent information measure assigning greater importance to larger values of the random variable X. For more details and applications of generalized weighted cumulative residual entropy measures, we refer to Toomaj and Di Crescenzo (2020). From (1.1) and (5.5), we propose the quantile version of *GWCRE* and its residual form for a nonnegative random variable X to be defined as

$$\text{Ḩ}_w^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\int_0^1 Q(u)(1-u)^{\alpha+\beta-1}q(u)du\right), \tag{5.6}$$

and

$$\text{Ḩ}_w^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha} \log\left\{\frac{\int_u^1 Q(p)(1-p)^{\alpha+\beta-1}q(p)dp}{(1-u)^{\alpha+\beta-1}}\right\}, \tag{5.7}$$

respectively. The measure (5.7) may be considered as the *generalized weighted dynamic cumulative residual quantile entropy (*GWDCRQE*)* measure. An alternative expression for the *GWDCRQE* in terms of mean residual quantile function $M(u)$ of random variable X is

$$\text{Ḩ}_w^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha} \log\left\{\frac{\int_u^1 (1-p)^{\alpha+\beta-2}Q(p)M(p)dp}{(1-u)^{\alpha+\beta-1}} - \frac{\int_u^1 (1-p)^{\alpha+\beta-1}Q(p)M'(p)dp}{(1-u)^{\alpha+\beta-1}}\right\}.$$

For some well-known univariate continuous families of distributions, the expression (5.7) is evaluated as given in Table 4.

Examples 5.1. Let X follows lambda family of distribution as given in example (2.2), then the generalized weighted dynamic cumulative residual quantile entropy (*GWDCRQE*), (5.6) is given as

$$\text{Ḩ}_w^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left\{C^2\lambda_1\beta(2\lambda_1, \alpha+\beta-2\lambda_2) + C^2\lambda_2\beta(2\lambda_1+1, (\alpha+\beta-1)-2\lambda_2)\right\}. \tag{5.8}$$

As $\lambda_1 \longrightarrow 0$, (5.8) reduces to $\text{Ḩ}_w^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\frac{C^2\lambda_2}{(\alpha+\beta-1)-2\lambda_2}\right)$, corresponding to the Pareto II distribution. Also, as $\lambda_2 \longrightarrow 0$, (5.8) reduces to $\text{Ḩ}_w^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(C^2\lambda_1\beta(2\lambda_1, \alpha+\beta)\right)$, correspondi-ng to the power distribution.

Table 4: *GWDCRQE* for several well-known families of distributions

Distribution	Quantile function $Q(u)$	$\underset{\smile}{H}_w^{(\alpha,\beta)}(u)$
Uniform	$a+(b-a)u$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{a(b-a)(1-u)}{\beta+\alpha}+\frac{(b-a)^2(1-u)}{\alpha+\beta}-\frac{(b-a)^2(1-u)^2}{1+\alpha+\beta}\right\}$
Exponential	$-\lambda^{-1}\log(1-u)$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{\bar{\gamma}_{-\log(1-u)}(2,\alpha+\beta-1)}{\lambda^2(1-u)^{\alpha+\beta}}\right\}$
Power	$au^{\frac{1}{b}}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{a^2\bar{\beta}_u(\frac{2}{b},\alpha+\beta)}{b(1-u)^{\alpha+\beta-1}}\right\}$
Pareto I	$b(1-u)^{-\frac{1}{a}}$	$\frac{1}{\beta-\alpha}\log\left\{\frac{b^2(1-u)^{\frac{-2}{a}}}{a(\alpha+\beta)-2}\right\}$
Folded Cramer	$\frac{u}{\theta(1-u)}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{\bar{\beta}_u(2,\alpha+\beta-3)}{\theta^2(1-u)^{\alpha+\beta-1}}\right\}$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{b^2(1-u)^{\frac{-2a}{a+1}}}{a(a\alpha+\alpha-2a)}-\frac{b^2(1-u)^{\frac{-a}{a+1}}}{a(a\alpha+\alpha-a)}\right\}$
Finite Range	$b\{1-(1-u)^{\frac{1}{a}}\}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{b^2(1-u)^{\frac{1}{a}}}{a(\alpha+\beta-1)+1}-\frac{b^2(1-u)^{\frac{2}{a}}}{a(\alpha+\beta-1)+2}\right\}$
Log logestic	$\frac{1}{a}\left(\frac{u}{(1-u)}\right)^{\frac{1}{b}}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{\bar{\beta}_u(\frac{2}{b},\alpha+\beta-1-\frac{2}{b})}{a^2b(1-u)^{\alpha+\beta-1}}\right\}$
Weibull	$\left\{-\frac{1}{a}\log(1-u)\right\}^{\frac{1}{b}}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{(\frac{1}{a})^{\frac{2}{b}-1}\bar{\gamma}_{-\log(1-u)}(\frac{2}{b},\alpha+\beta-1)}{ab(1-u)^{(\alpha+\beta-1)}}\right\}$
Rayleigh	$\left\{-\frac{1}{a}\log(1-u)\right\}^{\frac{1}{2}}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{2a(\alpha+\beta-1)}\right\}$
Gompertz	$\frac{1}{\log C}\left\{1-\frac{\log C\log(1-u)}{B}\right\}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{1}{(\alpha+\beta-1)B\log C}+\frac{\bar{\gamma}_{-\log(1-u)}(2,\alpha+\beta-1)}{B^2(1-u)^{\alpha+\beta-1}}\right\}$
Govindarajulu's	$a\left\{(b+1)u^b-bu^{b+1}\right\}$	$\frac{1}{(\beta-\alpha)}\log\left\{\frac{a^2b(b+1)^2\bar{\beta}_u(2b,\alpha+\beta+1)}{(1-u)^{\alpha+\beta-1}}-\frac{a^2b^2(b+1)\bar{\beta}_u(2b+1,\alpha+\beta+1)}{(1-u)^{\alpha+\beta-1}}\right\}$

In order to provide some characterization results for *GWDCRQE* of a non-negative random variable X, let us define the quantile version of the weighted mean residual lifetime (*WMRE*), as follows

$$M^w(u)=m^w(Q(u))=\frac{\int_u^1(1-p)Q(p)q(p)dp}{(1-u)}, \tag{5.9}$$

here $m^w(t)=\frac{\int_t^\infty x\bar{F}(x)dx}{\bar{F}(t)}$ is the *WMRE* of random variable X. In the following theorem, we characterize rayleigh distribution using a relationship between *GWDCRQE* and the quantile-based *WMRE*.

Theorem 5.2. *Let X be an absolutely continuous random variable. Then the relation*

$$\underset{\smile}{H}_w^{(\alpha,\beta)}(u)=\frac{1}{(\beta-\alpha)}\log\left(\frac{M^w(u)}{(\alpha+\beta-1)}\right), \tag{5.10}$$

holds if and only if X follows the rayleigh distribution.

Proof. The quantile-based *WMRE* (5.9), for the rayleigh distribution is given as

$$M^{w}(u) = \frac{1}{2a(1-u)} \int_{u}^{1} \left(\frac{-1}{a}\log(1-p)\right)^{\frac{1}{2}} \left(\frac{-1}{a}\log(1-p)\right)^{\frac{-1}{2}} dp = \frac{1}{2a}.$$

Besides, the generalized weighted cumulative residual quantile entropy (5.7), for the rayleigh distribution is

$$\underset{\sim}{\mathrm{H}}_{w}^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha} \log\left(\frac{1}{2a(\alpha+\beta-1)}\right).$$

This prove the if part of the Theorem. To prove the only if part, let (5.10) holds. Then

$$(\alpha+\beta-1)\int_{u}^{1}(1-p)^{\alpha+\beta-1}Q(p)q(p)dp = (1-u)^{\alpha+\beta-2}\int_{u}^{1}(1-p)Q(p)q(p)dp.$$

Differentiating both sides with respect to u, we have

$$(1-u)^{\alpha+\beta-1}q(u)Q(u) = (1-u)^{\alpha+\beta-3}\int_{u}^{1}(1-p)Q(p)q(p)dp.$$

Using (5.9) and (5.10), we get

$$Q(u)q(u) = \frac{M^{w}(u)}{1-u}\,. \tag{5.11}$$

By differentiating both sides with respect to u, we have

$$\frac{dM^{w}(u)}{du} - \frac{M^{w}(u)}{1-u} = -Q(u)q(u).$$

Substituting (5.11), gives $\frac{dM^{w}(u)}{du} = 0$ or equivalently $M^{w}(u) = k$ (constant), which characterizes the rayleigh distribution. □

Theorem 5.3. *Let X be a nonnegative random variable for which the relationship*

$$\underset{\sim}{\mathrm{H}}_{w}^{(\alpha,\beta)}(u) = C, \tag{5.12}$$

holds, where C is a constant. Then, X follows the rayleigh distribution.

Proof. The necessary part follows from Table 4. For the sufficiency part, let us assume (5.12) holds. From (5.7), we have

$$e^{(\beta-\alpha)\underset{\sim}{\mathrm{H}}_{w}^{(\alpha,\beta)}(u)}(1-u)^{(\alpha+\beta-1)} = \int_{u}^{1} Q(p)(1-p)^{(\alpha+\beta-1)}q(p)dp.$$

Taking derivative on both sides with respect to u and after some algebraic simplification, it can be shown that

$$\left((\beta-\alpha)\underset{\sim}{\mathrm{H}}_{w}^{\prime(\alpha,\beta)}(u) - \frac{(\alpha+\beta-1)}{(1-u)}\right)e^{(\beta-\alpha)\underset{\sim}{\mathrm{H}}_{w}^{(\alpha,\beta)}(u)} = -q(u)Q(u).$$

Using (1.4), this gives

$$\left((\beta-\alpha)\underset{\sim}{\mathrm{H}}_w'^{(\alpha,\beta)}(u)-\frac{(\alpha+\beta-1)}{(1-u)}\right)e^{(\beta-\alpha)\underset{\sim}{\mathrm{H}}_w^{(\alpha,\beta)}(u)}=-\frac{Q(u)}{(1-u)K(u)}. \tag{5.13}$$

From (5.13), we get $\underset{\sim}{\mathrm{H}}_w'^{(\alpha,\beta)}(u)=0$. If this value is substituted in the above expression, we get

$$(\alpha+\beta-1)e^{(\beta-\alpha)\underset{\sim}{\mathrm{H}}_w^{(\alpha,\beta)}(u)}K(u)-Q(u)=0,$$

which leads to $\frac{K(u)}{Q(u)}=\frac{1}{(\alpha+\beta-1)e^{C(\beta-\alpha)}}=2a$ (*constant*). Thus, X follows the rayleigh distribution with the survival function $\bar{F}(x)=exp\left(-\frac{x}{2a^2}\right)$. Hence, the proof is completed. □

6 Application in Coding Theory

In many situations, for describing the lifetime of devices, discrete time is appropriate. For instance, actuaries and bio-statisticians are interested in the lifetimes of persons or organisms, measured in months, weeks, or days. Also, in case of equipment operating in cycles, the random variable of interest is the successful number of cycles before the failure. Let X denote a discrete lifetime random variable taking values on $N=\{1,2,3,\ldots,n\}$ where $1<n\leq\infty$ is integer with $P(X=i)=p_i$. Shannon (1948) conceived the statistical nature of the communication signal associated with X and introduced the measure of information (or, uncertainty) as

$$H(P)=-\sum_{i=1}^{n}p_i\log p_i\ ,\quad 0\leq p_i\leq 1,\quad \sum_{i=1}^{n}p_i=1, \tag{6.1}$$

associated with this experiment. Corresponding to (6.1), the discrete version of the generalized information measure of order (α,β) is defined as

$$H_\alpha^\beta(P)=\frac{1}{\beta-\alpha}\log\sum_{i=0}^{n}(p(k))^{\alpha+\beta-1}\ ;\ \beta-1<\alpha<\beta,\ \beta\geq 1. \tag{6.2}$$

Shannon measure of entropy finds beautiful applications in coding theory. One of the important applications of the information measure for noiseless channel is that it gives bounds for suitable encoding information under the condition of uniquely decipherable code. Let a finite set of n input symbols $X=(x_1,x_1,...,x_n)$ with probabilities $P=(p_1,p_2,...,p_n)$ and utilities $U=(u_1,u_2,...,u_n)$ be encoded using a code alphabet of size $D(\geq 2)$ and let $l_1,l_2,...,l_n$ be the lengths of the transmitted codewords. Here, u_i is a nonnegative real number, independent of p_i, accounting for the utility of occurrence of x_i. Guiasu and Picard (1971) considered the problem of encoding the letter output by means of a single letter prefix code and introduced the quantity

$$L(P,U)=\sum_{i=1}^{i=n}\frac{u_ip_il_i}{\sum_{i=1}^{i=n}u_ip_i}, \tag{6.3}$$

as the 'useful' mean codeword length. In the literature of coding theory, researchers have considered various generalized codeword lengths, shown their correspondence with information measures and obtained different coding theorems under the condition of uniquely decipherability. In parallel to that, we define the generalized 'useful' mean codeword length of order (α, β) as follows

$$L_{\alpha}^{\beta}(P, U) = \left(\frac{\alpha+\beta-1}{\beta-\alpha}\right)\log_D\left(\sum_{i=1}^{n} p_i D^{l_i\left(\frac{\beta-\alpha}{\alpha+\beta-1}\right)}\right); \;\; \beta-1 < \alpha < \beta,\; \beta \geq 1. \tag{6.4}$$

In particular if $l_1 = l_2 = ... = l_n = l$, then $L_{\alpha}^{\beta}(P, U) = l$. For $\beta = 1$, (6.4) reduces to the useful mean codeword length of Taneja et al. (1985). Also when $\alpha \to 1$ and $\beta = 1$, (6.4) reduces to (6.3) defined by Guiasu and Picard (1971). We take the following example to illustrate the application of generalized entropy measure of order (α, β) in coding theory.

Examples 6.1. Consider the following information transmission scenario where Alice attempts to communicate the outcome X of rolling a dice experiments to her friend Bob. Assume that Alice is using an irregular five sided dice for the experiment. Suppose the outcomes (source alphabets in a communication setup) follow the probability distribution as given in Table 5. The problem here is to encode the source alphabets with minimum bits possible. Alice chooses

Table 5: Probability Distribution

X	1	2	3	4	5
P_i	.5	.2	.1	.1	.1

an encoding method and encodes the outcomes 1, 2, 3, 4, 5 respectively with 1, 3, 3, 3, 3 bit-strings. From equations (6.2) and (6.4), we get the results provided in Table 6. Thus we observe that for a fixed value of parameter β and various values of parameter α, the value of discrete generalized entropy $H_{\alpha}^{\beta}(P)$ gives the lower bound of the mean codeword length of the message passes through the communication channel.

Table 6: Generalized entropy is lower bound of mean codeword length

β	α	$H_{\alpha}^{\beta}(P)$	$L_{\alpha}^{\beta}(P)$
1	.10	2.287	2.888
1	.15	2.270	2.823
1	.20	2.253	2.757
1	.25	2.235	2.674
1	.30	2.217	2.595

7 Conclusion

The quantile-based entropy measures possess some unique properties compared with the distribution function approach. The quantile-based generalized cumulative residual entropy of order (α, β) has several advantages. The computation of proposed measure is quite simple in cases where the distribution function are not tractable while the quantile functions have simpler forms. The results obtained in this article are general in the sense that they are reduced to some of the result for the quantile-based Shannon and *Rényi* entropies, when parameters approaches unity.

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