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Convergence Rate of Empirical Autocovariance Operators in H-Valued Periodically Correlated Processes

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Abstract. This paper focuses on the empirical autocovariance operator of H-valued periodically correlated processes. It will be demonstrated that the empirical estimator converges to a limit with the same periodicity as the main process. Moreover, the rate of convergence of the empirical autocovariance operator in Hilbert-Schmidt norm is derived.

Keywords. Convergence Rate, Autocovariance Operator, H-Valued Periodically Correlated Processes, Strongly Second Order Processes.

MSC: 62G05, 62M10.

1 Introduction

Periodically correlated (PC) processes are random processes that exhibit a periodic rhythm in their structure which is more complicated than periodicity in the mean function. It seems that the notion of PC processes is first introduced by Bennett (1958) who observed their presence in a communication theoretic context and called

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them cyclostationary. Gudzenko (1959) took the first steps in studying the subject of nonparametric spectral analysis for PC processes. Although these processes are generally nonstationary, Gladyshev (1961, 1963) demonstrated that, when the period is known, PC processes are equivalent to vector-valued stationary processes. Besides, he established the spectral density properties. Tian (1988) studied the limiting property of sample autocovariance of PC processes. Afterwards, PC processes have found their applications in various areas, such as climatology, hydrology, medicine and biology, economics, mechanics and many fields in communication and signal processing. The bibliography by Serpedin et al. (2005) presents an exhaustive list of references on PC processes and their applications in various fields.

In recent years, extensive advances have occurred in data collection methods and storage techniques and make it possible to observe and record real life processes in great details. These progresses have a considerable impact on the analysis of financial transaction data, fMRI images, satellite photos, earth pollution distribution, etc. Due to the high dimensionality of such data, classical statistical tools become inadequate and inefficient. Functional Data Analysis (FDA) is referred to the statistical methodology for studying data that are in the form of functions, from the theoretical and practical points of view. Since, theoretically, functional data are of infinite dimensional nature, studying infinite dimensional processes attracts the attention of researchers, as well.

The concept of PC processes is extended to processes with values in Hilbert spaces by Makagon (1999), who presented the spectral analysis and the prediction method for forecasting the Hilbertian periodically correlated sequences, HPC processes in abbreviation. In particular, he obtained a moving average representation of a predictor and described its coefficients in the language of outer factors of spectral line densities of the sequence. The harmonizability, the structure of the autocovariance operator and the existence of a time dependent spectral density for HPC processes are considered by Soltani and Shishebor (2007). Besides, in 2011, Makagon introduced two transformations that map these sequences into T-dimensional stationary sequences and studied their properties. The spectral properties of such processes are studied by Makagon and Miamee (2013) and Soltani et al. (2010) and Shishebor et al. (2011) presented and examined the properties of their periodograms. Moreover, the class of PC autoregressive Hilbertian processes (PCARH) is considered by Soltani and Hashemi (2011) and the behavior of autocovariance and autocorrelation operators and their estimations are discussed in Haghbin et al. (2017) and Hashemi et al. (2019).

The goal of this paper is to obtain the convergence rate of empirical autocovariance operator of H-valued PC processes. Our approach is based on decomposing the empirical autocovariance operator into the sum of the autocovariance operator of the periodic mean function and the average of the periodic autocovariance operators. The autocovariance operator is commonly used for detecting possible periodicities and, consequently, it is possible to study the empirical autocovariance operator to check the periodic properties of a process.

The paper is organized as follows. After introducing the required concepts and notations in Section 2, we present the decomposition of the autocovariance operator in Section 3 and Section 4 is devoted to the convergence rate of the empirical estimator of the autocovariance operator. In the last section, some concluding notes and possible future works are mentioned in brief.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and H be a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. An H-valued discrete time stochastic process is defined as a random sequence $\{\xi_t, t \in \mathbb{Z}\}$ in H, where $\xi_t : \Omega \to H$ is $\mathcal{F}/\mathcal{B}$ measurable, $\mathcal{B}$ is the Borel field of H and $\mathbb{Z}$ is the set of integers. Let $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ stands for the Hilbert space of all complex random variables X defined on $(\Omega, \mathcal{F}, P)$ with finite second moment, which is equipped with the inner product $E(X\bar{Y})$, where E denotes the expectation. The process $\{\xi_t, t \in \mathbb{Z}\}$ is called strongly second order (SSO in abbreviation) if $\|\xi_t\| \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$.

Additionally, let $\mathcal{L}(H)$ denote the space of all continuous linear operators from H to H, with operatorial norm $\|\cdot\|_{\mathcal{L}}$. An important subspace of $\mathcal{L}(H)$ is the space of Hilbert-Schmidt operators, $\mathcal{S}(H)$, which forms a Hilbert space equipped with the inner product $\langle A, B \rangle_{\mathcal{S}} = \sum_{k=1}^{\infty} \left\langle A\phi_k, B\phi_k \right\rangle$ and the norm $\|A\|_{\mathcal{S}} = \left\{ \sum_{k=1}^{\infty} \left\| A\phi_k \right\|^2 \right\}^{1/2}$, where $\left\{\phi_k\right\}$ is any orthonormal basis on H. The space of nuclear or trace class operators, $\mathcal{N}(H)$, is a notable subclass of $\mathcal{S}(H)$ and the associated norm is defined as

$$\|A\|_{\mathcal{N}} = \sum_{k=1}^{\infty} \left\langle |A| \, \phi_k, \phi_k \right\rangle = \sum_{k=1}^{\infty} \left\langle (A^* A)^{1/2} \, \phi_k, \phi_k \right\rangle, \tag{2.1}$$

where A^* is the adjoint of A, Conway (2000). If A is a self-adjoint nuclear operator with associated eigenvalues λ_k, then $\|A\|_{\mathcal{N}} = \sum_{k=1}^{\infty} |\lambda_k|$. If, in addition, A is non-negative, then $\|A\|_{\mathcal{N}} = \sum_{k=1}^{\infty} \left\langle A\phi_k, \phi_k \right\rangle = \sum_{k=1}^{\infty} \lambda_k$. Note that $\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{\mathcal{S}} \leq \|\cdot\|_{\mathcal{N}}$, Hsing and

Eubank (2015). For x and y in H, the tensorial product of x and y, $x \otimes y$, is a nuclear operator and is defined as $(x \otimes y)z := \langle y, z\rangle x$, $z \in H$, Schatten (2013).

An H-valued SSO stochastic process is called periodically correlated (HPC in abbreviation) if there exists an integer $T > 0$, such that for every $t \in \mathbb{Z}$:

$$\mu_t := \mathbb{E}(\xi_t) = \mathbb{E}(\xi_{t+T}) = \mu_{t+T}, \tag{2.2}$$

and

$$C_{t,\tau} := \mathbb{E}\{(\xi_t - \mu_t) \otimes (\xi_{t+\tau} - \mu_{t+\tau})\} = C_{t+T,\tau}. \tag{2.3}$$

In (2.2) and (2.3), $\mathbb{E}(\cdot)$ stands for the Bochner integral, and the smallest T, for which these equalities hold, is called the period of the process. If $T = 1$, then the process is called stationary. For more on Bochner integral and its properties we refer readers to Hsing and Eubank (2015).

3 Empirical Autocovariance Operator of H PC Processes

Consider the H-valued periodically correlated time series $\boldsymbol{\xi} = \{\xi_1, \xi_2, \cdots \xi_n\}$. A natural estimator for C_τ is the empirical autocovariance operator, $\hat{C}_\tau$, which is defined by

$$\hat{C}_\tau := \frac{1}{n-\tau} \sum_{t=1}^{n-\tau} (\xi_t - \bar{\xi}_n) \otimes (\xi_{t+\tau} - \bar{\xi}_n), \tag{3.1}$$

where $\bar{\xi}_n = \frac{1}{n}\sum_{t=1}^{n} \xi_t$. The following lemma is devoted to the asymptotic behavior of $\mathbb{E}(\bar{\xi}_n)$ and $\mathbb{E}\left(\hat{C}_\tau\right)$, as the number of observations tends to infinity.

Lemma 3.1. *Let $\boldsymbol{\xi}$ be a sequence of H-valued periodically correlated process with period T, for which $E\|\xi_t\|^4 < \infty$ and $\lim_{n\to\infty} \frac{1}{n^2}\sum_{t=1}^{n}\sum_{t'=1}^{n} \left\|C_{t,t-t'}\right\|_{\mathcal{S}} = 0$. Then,*

$$\lim_{n\to\infty} \left\|\mathbb{E}(\bar{\xi}_n) - \bar{\mu}\right\| = 0, \tag{3.2}$$

$$\lim_{n\to\infty} \left\|\mathbb{E}\left(\hat{C}_\tau\right) - C_{\mu,\tau} - \bar{C}_\tau\right\|_{\mathcal{S}} = 0, \tag{3.3}$$

where $\bar{\mu} = \frac{1}{T}\sum_{t=1}^{T} \mu_t$, $C_{\mu,\tau} = \frac{1}{T}\sum_{t=1}^{T} (\mu_t - \bar{\mu}) \otimes (\mu_{t+\tau} - \bar{\mu})$ and $\bar{C}_\tau = \frac{1}{T}\sum_{t=1}^{T} C_{t,\tau}$.

Proof. Let $n = k_nT + l_n$, then:

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\bar{\xi}_n\right)-\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(\xi_t\right)\right\| = \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left(\xi_t\right)-\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(\xi_t\right)\right\|$$
$$= \lim_{n\to\infty}\left\|\frac{T-l_n}{T\left(k_nT+l_n\right)}\sum_{t=1}^{l}\mathbb{E}\left(\xi_t\right)-\frac{l_n}{T\left(k_nT+l_n\right)}\sum_{t=l+1}^{T}\mathbb{E}\left(\xi_t\right)\right\|.$$

Since $1 \le l_n < T$ and $0 < T - l_n \le T - 1$, it can easily be seen that $\frac{T-l_n}{T(k_nT+l_n)}$ and $\frac{l_n}{T(k_nT+l_n)}$ tend to zero as n goes to infinity. Consequently,

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\bar{\xi}_n\right)-\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left(\xi_t\right)\right\| = 0.$$

For the proof of equation (3.3), we have:

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\hat{C}_\tau\right)-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}$$
$$= \lim_{n\to\infty}\left\|\mathbb{E}\left\{\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left(\xi_t-\bar{\xi}_n\right)\otimes\left(\xi_{t+\tau}-\bar{\xi}_n\right)\right\}-C_{\mu,\tau}-\bar{C}_\tau\left(\tau\right)\right\|_{\mathcal{S}}$$
$$= \lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\mathbb{E}\left\{\left(\xi_t-\mu_t+\mu_t-\bar{\xi}_n\right)\otimes\left(\xi_{t+\tau}-\mu_{t+\tau}+\mu_{t+\tau}-\bar{\xi}_n\right)\right\}-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}$$
$$= \lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left\{C_{t,\tau}+\mathbb{E}\left\{\left(\mu_t-\bar{\xi}_n\right)\otimes\left(\xi_{t+\tau}-\mu_{t+\tau}\right)\right.\right.\right.$$
$$\left.\left.\left.+\ \left(\xi_t-\mu_t\right)\otimes\left(\mu_{t+\tau}-\bar{\xi}_n\right)+\left(\mu_t-\bar{\xi}_n\right)\otimes\left(\mu_{t+\tau}-\bar{\xi}_n\right)\right\}\right\}-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}$$
$$= \lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left\{C_{t,\tau}+\mathbb{E}\left\{\left(\mu_t-\bar{\mu}_n+\bar{\mu}_n-\bar{\xi}_n\right)\otimes\left(\xi_{t+\tau}-\mu_{t+\tau}\right)\right.\right.\right.$$
$$+\ \left(\xi_t-\mu_t\right)\otimes\left(\mu_{t+\tau}-\bar{\mu}_n+\bar{\mu}_n-\bar{\xi}_n\right)$$
$$\left.\left.\left.+\ \left(\mu_t-\bar{\mu}_n+\bar{\mu}_n-\bar{\xi}_n\right)\otimes\left(\mu_{t+\tau}-\bar{\mu}_n+\bar{\mu}_n-\bar{\xi}_n\right)\right\}\right\}-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}.$$

Using the properties of tensorial product and the definition of autocovariance operator, given in equation (2.3), it can be shown that

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\mathbb{E}\left(\hat{C}_\tau\right)-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}\\
&\quad=\lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}C_{t,\tau}-\frac{1}{n\,(n-\tau)}\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}\mathbb{E}\left(\xi_{t'}-\mu_{t'}\right)\otimes\left(\xi_{t+\tau}-\mu_{t+\tau}\right)\right.\\
&\qquad\quad-\frac{1}{n\,(n-\tau)}\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}\mathbb{E}\left(\xi_t-\mu_t\right)\otimes\left(\xi_{t'}-\mu_{t'}\right)+\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left(\mu_t-\bar{\mu}_n\right)\otimes\left(\mu_{t+\tau}-\bar{\mu}_n\right)\\
&\qquad\quad\left.+\frac{1}{n^2}\sum_{t=1}^{n}\sum_{t'=1}^{n}\mathbb{E}\left(\xi_t-\mu_t\right)\otimes\left(\xi_{t'}-\mu_{t'}\right)-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}\\
&\quad=\lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}C_{t,\tau}-\frac{1}{n\,(n-\tau)}\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}C_{t',t+\tau-t'}-\frac{1}{n\,(n-\tau)}\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}C_{t,t'-t}\right.\\
&\qquad\quad\left.+\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left(\mu_t-\bar{\mu}_n\right)\otimes\left(\mu_{t+\tau}-\bar{\mu}_n\right)+\frac{1}{n^2}\sum_{t=1}^{n}\sum_{t'=1}^{n}C_{t,t'-t}-C_{\mu,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}\\
&\quad\leq\lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}C_{t,\tau}-\bar{C}_\tau\right\|_{\mathcal{S}}\\
&\qquad\quad+\lim_{n\to\infty}\left\|\frac{1}{n^2\,(n-\tau)}\left\{(n-\tau)\sum_{t=1}^{n}\sum_{t'=1}^{n}C_{t,t'-t}-n\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}C_{t',t+\tau-t'}-n\sum_{t=1}^{n-\tau}\sum_{t'=1}^{n}C_{t,t'-t}\right\}\right\|_{\mathcal{S}}\\
&\qquad\quad+\lim_{n\to\infty}\left\|\frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left(\mu_t-\bar{\mu}_n\right)\otimes\left(\mu_{t+\tau}-\bar{\mu}_n\right)-C_{\mu,\tau}\right\|_{\mathcal{S}}\\
&\quad=0,
\end{aligned}
$$

which completes the proof. □

Examples 3.1. Let $\boldsymbol{\xi}=\{\xi_n,\ n\in\mathbb{Z}\}$ be a periodically correlated autoregressive Hilbertian process of order 1 with period T, PCARH(1) in abbreviation, associated with $\left(\varepsilon,\phi\right)$. This process is periodic and it satisfies the following formulation:

$$
\xi_t-\mu_t=\phi_n\left(\xi_{t-1}-\mu_{t-1}\right)+\varepsilon_t, \tag{3.4}
$$

where $\boldsymbol{\varepsilon}=\{\varepsilon_t,\ t\in\mathbb{Z}\}$ is a sequence of PC Hilbertian white noise $\left[0,\sigma^2,T\right]$, called PCHWN$\left[0,\sigma^2,T\right]$ in short, $\{\mu_t,\ t\in\mathbb{Z}\}$ is a T-periodic sequence in H and $\left\{\phi_t,\ t\in\mathbb{Z}\right\}$

is a T-periodic sequence in $\mathcal{L}(H)$. For more on PCARH(1) processes, we refer readers to Soltani and Hashemi (2011). Let $n = k_nT + l_n$. The limiting equation (3.2) obviously holds true. In order to demonstrate (3.3), note that

$$\begin{aligned}\xi_{t+\tau} - \mu_{t+\tau} &= \phi_{t+\tau}\left(\xi_{t+\tau-1} - \mu_{t+\tau-1}\right) + \varepsilon_{t+\tau} \\ &= \phi_{t+\tau}\phi_{t+\tau-1}\left(\xi_{t+\tau-2} - \mu_{t+\tau-2}\right) + \phi_{t+\tau}\varepsilon_{t+\tau-1} + \varepsilon_{t+\tau} \\ &\vdots \\ &= \phi_{t+\tau}\phi_{t+\tau-1}\ldots\phi_{t+1}\left(\xi_t - \mu_t\right) + \phi_{t+\tau}\phi_{t+\tau-1}\ldots\phi_{t+2}\varepsilon_{t+1} + \ldots + \phi_{t+\tau}\varepsilon_{t+\tau-1} + \varepsilon_{t+\tau},\end{aligned}$$

and, consequently,

$$C_{t,\tau} = \phi_{t+\tau}\phi_{t+\tau-1}\ldots\phi_{t+1}C_{t,0}.$$

Therefore, $\mathbb{E}\left(\hat{C}_\tau\right)$ can be approximated by $\frac{1}{T}\sum_{t=1}^{T}\phi_{t+\tau}\phi_{t+\tau-1}\ldots\phi_{t+1}C_{t,0} + C_{\mu,\tau}$ in the Hilbert-Schmidt norm.

As can be seen, the expected value of the empirical autocovariance operator can be approximated by the sum of two terms, autocovariance operator of the periodic mean function μ_t (at lag τ), called $C_{\mu,\tau}$, and the average of the periodic autocovariance operators $C_{t,\tau}$, $t = 1, 2, \cdots, T$, called $\bar{C}_\tau$. By definition, it can easily be seen that $C_{\mu,\tau}$ is periodic in τ with the same period as μ_t and, if the process is periodic, $\bar{C}_\tau$ decays as τ increases. Therefore, equation (3.3) demonstrates the possibility of studying the periodic properties of an HPC process using its empirical autocovariance operator.

4 Convergence Rate of the Sample Autocovariances

Consider the sequence of HPC processes $\boldsymbol{\xi}$. It can be demonstrated that this process has a unique Wold decomposition, Zamani et al. (2020), and it can be presented as

$$\xi_t = \mu_t + \sum_{j=0}^{\infty}\rho_{j,t}\left(\varepsilon_{t-j}\right), \tag{4.1}$$

where $\mathcal{I} = \{\varepsilon_m, m \in \mathbb{Z}\}$ is a set of orthonormal innovation processes, $\rho_{j,t}$ are linear operators on H for which $\sum_{t=1}^{T}\sum_{j\geq 0}\left\|\rho_{j,t}\right\|_{\mathcal{L}}^2 < \infty$, and $\rho_{j,t+kT} = \rho_{j,t}$ for every j, k and t, with $t - j \geq 0$.

Let $\boldsymbol{\varepsilon} = \{\varepsilon_t,\ t \in \mathbb{Z}\}$ be a sequence of martingale differences with respect to an increasing sequence of σ-fields $\mathcal{F}_t$; i.e.,

$$\mathbb{E}^{\mathcal{F}_{t-1}}(\varepsilon_t) = 0, \quad \text{a.s.} \quad \forall t \in \mathbb{Z}, \tag{4.2}$$

where 0 is the zero of H. If, for some $a > 0$, $E\|\varepsilon_t\|^a < \infty$, we call $\boldsymbol{\varepsilon}$ an L_a-martingale. Assume that

$$\mathbb{E}(\varepsilon_t \otimes \varepsilon_t | \mathcal{F}_{t-1}) = C^{\varepsilon}, \quad \text{a.s. } \forall t \in \mathbb{Z}, \tag{4.3}$$

$$\sup_t E\|\varepsilon_t\|^{\lambda} < \infty, \quad \text{for some } \lambda > 4. \tag{4.4}$$

Theorem 4.1. *Let $\boldsymbol{\xi}$ be a sequence of H-valued periodically correlated process with period T, for which moving average representation* (4.1) *holds. Then, for some $\delta > 0$ and $1 \le G(n) = o(n)$,*

$$\max_{1 \le \tau \le G(n)} \left\| \hat{C}_{\tau} - \left(C_{\mu,\tau} + \bar{C}_{\tau}\right) \right\|_{\mathcal{S}} = o\left(n^{-1/2} (G(n) \log n)^{2/\lambda} (\log\log n)^{2(1+\delta)/\lambda}\right). \tag{4.5}$$

As the first step to prove this theorem, the following lemma is required, which is presented in Moricz (1976) and Chow and Teicher (2012).

Lemma 4.1. *Let $\{\eta_i,\ i = 1, 2, \cdots\}$ be a stochastic sequence such that there exist positive constants C, ν and a positive sequence of $f(n)$ for which $\limsup f(2n)/f(n) < \infty$ and for all n, $x > 0$, $P(\max_{1 \le i \le n} \eta_i > x) < Cf(n)x^{-\nu}$. Then, for any $\delta > 0$,*

$$\eta_n = o\left(\left(f(n)(\log n)(\log\log n)^{1+\delta}\right)^{1/\nu}\right). \tag{4.6}$$

Proof of Theorem 4.1. Let us first define $\hat{C}_{i,\tau}$ as:

$$\hat{C}_{i,\tau} = (h_n + 1)^{-1} \sum_{m=0}^{h_n} \left(\xi_{i+mT} - \bar{\xi}_{n-\tau}(i, T)\right) \otimes \left(\xi_{i+\tau+mT} - \bar{\xi}_n(i + \tau, T)\right), \tag{4.7}$$

where

$$\bar{\xi}_n(i, d) = (h(n - i, d) + 1)^{-1} \sum_{j=0}^{h(n-i,d)} \xi_{i+jd}, \qquad 1 \le i, d < n,$$

and

$$h_n = h(n - \tau - i, T) = \left[\frac{n - \tau - i}{T}\right].$$

The relation presented in (4.7) can be rewritten as:

$$
\begin{aligned}
\hat{C}_{i,\tau} &= (h_n+1)^{-1}\sum_{m=0}^{h_n}\left(\xi_{i+mT}-\bar{\xi}_{n-\tau}(i,T)\right)\otimes\left(\xi_{i+\tau+mT}-\bar{\xi}_n(i+\tau,T)\right)\\
&= (h_n+1)^{-1}\sum_{m=0}^{h_n}\left(\xi_{i+mT}\otimes\xi_{i+\tau+mT}\right)-\left[\bar{\xi}_{n-\tau}(i,T)\otimes\left((h_n+1)^{-1}\sum_{m=0}^{h_n}\xi_{i+\tau+mT}\right)\right]\\
&\quad-\left[\left((h_n+1)^{-1}\sum_{m=0}^{h_n}\xi_{i+mT}\right)\otimes\bar{\xi}_n(i+\tau,T)\right]+\bar{\xi}_{n-\tau}(i,T)\otimes\bar{\xi}_n(i+\tau,T)\\
&= (h_n+1)^{-1}\sum_{m=0}^{h_n}\left(\xi_{i+mT}\otimes\xi_{i+\tau+mT}\right)-\bar{\xi}_{n-\tau}(i,T)\otimes\bar{\xi}_n(i+\tau,T)\,.
\end{aligned}
$$

Besides,

$$
\begin{aligned}
\hat{C}_{\tau} &= \frac{1}{n-\tau}\sum_{t=1}^{n-\tau}\left(\xi_t-\bar{\xi}_n\right)\otimes\left(\xi_{t+\tau}-\bar{\xi}_n\right)\\
&= \frac{1}{n-\tau}\sum_{i=1}^{T}\sum_{m=0}^{h_n}\left(\xi_{i+mT}-\bar{\xi}_n\right)\otimes\left(\xi_{i+\tau+mT}-\bar{\xi}_n\right)\\
&= \frac{1}{n-\tau}\left[\sum_{i=1}^{T}\sum_{m=0}^{h_n}\xi_{i+mT}\otimes\xi_{i+\tau+mT}-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\xi_{i+\tau+mT}\right.\\
&\quad\left.-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\xi_{i+mT}\otimes\bar{\xi}_n+\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\bar{\xi}_n\right]\\
&= \frac{1}{n-\tau}\left[\sum_{i=1}^{T}(h_n+1)\left(\frac{1}{h_n+1}\sum_{m=0}^{h_n}\xi_{i+mT}\otimes\xi_{i+\tau+mT}\right)-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\xi_{i+\tau+mT}\right.\\
&\quad\left.-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\xi_{i+mT}\otimes\bar{\xi}_n+\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\bar{\xi}_n\right]\\
&= \frac{1}{n-\tau}\left[\sum_{i=1}^{T}(h_n+1)\left(\hat{C}_{i,\tau}+\bar{\xi}_{n-\tau}(i,T)\otimes\bar{\xi}_n(i+\tau,T)\right)\right.\\
&\quad\left.-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\xi_{i+\tau+mT}-\sum_{i=1}^{T}\sum_{m=0}^{h_n}\xi_{i+mT}\otimes\bar{\xi}_n+\sum_{i=1}^{T}\sum_{m=0}^{h_n}\bar{\xi}_n\otimes\bar{\xi}_n\right],
\end{aligned}
$$

which can be written as

$$\hat{C}_\tau = \sum_{i=1}^{T} \frac{(h_n+1)}{n-\tau} \hat{C}_{i,\tau} + \sum_{i=1}^{T} \frac{(h_n+1)}{n-\tau} \left(\bar{\xi}_{n-\tau}(i,T) \otimes \bar{\xi}_n(i+\tau,T)\right)$$
$$- \bar{\xi}_n \otimes \bar{\xi}_n(\tau+1,1) + \bar{\xi}_{n-\tau} \otimes \bar{\xi}_n + \bar{\xi}_n \otimes \bar{\xi}_n.$$

Note that

$$\frac{h_n+1}{n-\tau} \to \frac{1}{T}, \qquad \text{as } n \to \infty,$$

and consequently, it suffices to prove that

$$\max_{i \le i \le G(n)} \left\| \bar{\xi}_n(i,T) - \mu_{r(i)} \right\| = o(\alpha(n)) \qquad \text{a.s.,} \tag{4.8}$$

$$\max_{i \le i \le G(n)} \left\| \hat{C}_{i,\tau} - C_{i,\tau} \right\|_{\mathcal{S}} = o(\alpha(n)) \qquad \text{a.s.,} \tag{4.9}$$

where $\alpha(n) = n^{-1/2}(G(n)\log n)^{2/\lambda}(\log\log n)^{2(1+\delta)/\lambda}$ and

$$r(i) = \begin{cases} T & \text{if } i \ mod \ T = 0 \\ i \ mod \ T & \text{otherwise .} \end{cases}$$

In order to prove (4.8), set $\zeta_t = \xi_t - \mu_t$. Define $S(n,i) = (h(n-i,T)+1)\bar{\zeta}_n(i,T)$, or equivalently,

$$S(n,i) = \sum_{k=0}^{\infty} \sum_{j=0}^{h(n-i,T)} \rho_{k,i}\left(\varepsilon_{i+jT-k}\right).$$

Let $\eta_m(i,j) = \sum_{l=0}^{m} \varepsilon_{i+lT-j}$. It can easily be demonstrated that $\eta_m(i,j)$ is an L_λ-martingale relative to the family of σ-fields $\mathcal{F}_t$ and the process $\|\eta_m(i,j)\|^\lambda$ is a real-valued submartingale, Gawarecki and Mandrekar (2010). Consequently, by Doob's and Burkholder's inequalities, we have:

$$E(\max_{1 \le m \le M} \|\eta_m(i,j)\|^\lambda)^{\frac{1}{\lambda}} \le \frac{\lambda}{\lambda-1}(E\|\eta_M(i,j)\|^\lambda)^{\frac{1}{\lambda}} \le CM^{0.5}, \tag{4.10}$$

where C is a positive constant independent of i and j. From equations (4.4) and (4.10), it follows that

$$E \max_{1 \le n \le N} \|S(n,i)\| \le CN^{\frac{\lambda}{2}},$$

and, by the Markov inequality, we get:

$$P(\max_{1\leq n\leq N} \|S(n,i)\| \geq x) \leq CN^{\frac{\lambda}{2}} x^{-\lambda}, \tag{4.11}$$

which results in that:

$$P(\max_{1\leq n\leq N} \max_{1\leq i\leq G(N)} \|S(n,i)\| \geq x) \leq C\ G(N)N^{\frac{\lambda}{2}} x^{-\lambda}. \tag{4.12}$$

By substituting $f(n) = n^{\frac{\lambda}{2}} G(n)$ and $\nu = \lambda$ in Lemma 4.1, equation (4.8) have been proved. Similarly, we can prove equation (4.9) and the proof is completed. □

For a periodic Gaussian process, equation (4.4) holds for any integer λ, and consequently the following corollary holds..

Corollary 4.1. *Let ξ be a sequence of Gaussian H-valued periodically correlated processes with period T. For an integer sequence $G(n) = o(n)$ and any $\delta > 0$,*

$$\max_{1\leq\tau\leq G(n)} \left\| \hat{C}_\tau - \left(C_{\mu,\tau} + \bar{C}_\tau\right) \right\|_{\mathcal{S}} = o\left(n^{-1/2+\delta}\right). \tag{4.13}$$

5 Discussion and Conclusion

In this paper, we consider the empirical autocovariance operator of H-valued periodically correlated processes. It is proved that the empirical estimator of the autocovariance operator converges to a limit with the same periodicity as the main process. Besides, the rate of convergence of the empirical autocovariance operator in H-valued and Gaussian H-valued periodically correlated processes are obtained.

In practice, an important issue in the study of PC processes is determining the value of the period, T. In this study and other similar researches, the value of T is considered to be known. However, in practical problems, the value of T should be estimated. Lemma 3.1, Theorem 4.1 and Corollary 4.1 indicate the possibility of studying the periodic properties of HPC processes, using the empirical autocovariance operator behavior. However, if the mean of the HPC process is zero, $\hat{C}_\tau$ will not provide any useful information concerning the value of T. These points will be focus of future researches.

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