

On Conditional Inactivity Time of Failed Components in an $(n - k + 1)$ -out-of- n System with Nonidentical Independent Components

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Abstract. In this paper, we study an $(n - k + 1)$ -out-of- n system by adopting their components to be statistically independent though nonidentically distributed. By assuming that at least m components at a fixed time have failed while the system is still working, we obtain the mixture representation of survival function for a quantity called the conditional inactivity time of failed components in the system. Moreover, this quantity for $(n - k + 1)$ -out-of- n system, in one sample with respect to k and m and in two samples, are stochastically compared.

Keywords. Conditional Inactivity Time, Nonidentical Components, Residual Lifetime.

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1 Introduction

The reliability properties of the conditional residual lifetime and the conditional inactivity time of $(n - k + 1)$ -out-of- n systems, as a special case of the coherent structures, have been studying for decades by several researchers. Most studies have concentrated on the cases where the components of such systems are assumed to be independent and identically distributed (IID). For more details, we refer interested readers to Navarro et. al (2013a), Li and Zhao (2006, 2008), Li and Zhang (2008), Navarro et. al (2005, 2013), Khaledi and Shaked (2007), Kochar et al. (1999), Zhang (2010), Eryılmaz (2013), Asadi (2006), Tavangar and Asadi (2010), and the references therein. Although, the aim at most related published papers is to study the $(n - k + 1)$ -out-of- n system with IID components, however, in recent years some authors have considered the $(n - k + 1)$ -out-of- n system with independent and nonidentically distributed (INID) components. For example, Zhao et. al (2008) studied the stochastic monotone properties of the residual life and the inactivity time of an k -out-of- n system with IIND components. For a parallel system with these properties, Sadegh (2008) studied the mean past lifetime and the mean residual life function. Gurler and Bairamov (2009) investigate the mean residual life function of a k -out-of- n : G system with INID components.

Let non-negative, independent random variables $\{T_i\}_{1 \leq i \leq n}$, be the lifetimes of components of an n -component system and let $T_{1:n} \leq T_{2:n} \leq \dots \leq T_{n:n}$ be the ordered lifetimes of these components. For an $(n - k + 1)$ -out-of- n system, $T_{k:n}$ corresponds to the lifetime of the system. Kochar and Xu (2010) studied the conditional residual lifetime of two cases: (a) given the condition that at time t , among n components, at least $n - m + 1$ of them are working and (b) given the condition that at least m components have failed while the system is still alive and is continuing to work at time t . In other words, by assuming the random variables

$$(T_{k:n} - t | T_{m:n} > t) \text{ for } 1 \leq m \leq k \leq n,$$

and

$$(T_{k:n} - t | T_{m:n} \leq t < T_{k:n}) \text{ for } 1 \leq m \leq k \leq n,$$

they obtain a mixture representation of survival functions for the residual lifetimes of k -out-of- n systems when the components are independent but not necessarily identically distributed. Salehi et. al (2012) studied the inactivity lifetime of an $(n - k + 1)$ -out-of- n system with INID components. Specifically, they studied the inactivity lifetime of such systems, given the condition that the system has broken down at time t , that is,

$$(t - T_{m:n} | T_{k:n} \leq t) \text{ for } m = 1, 2, \dots, k.$$

In this paper, we study the conditional inactivity time of the failed components in the $(n - k + 1)$ -out-of- n system with INID components, given that on that time a certain number of components have failed while the system is still alive. Particularly, we assume that at time $t > 0$ at least m components have stopped working but k^{th} component of the system still works. Furthermore, we consider the following random variables

$$IT_{m,k,n}(t) = (t - T_{m:n} | T_{m:n} < t < T_{k:n}), \quad m = 1, 2, \dots, k, \quad (1.1)$$

to refer to the conditional inactivity time of failed components of the system. This quantity for components of a coherent system, which consist of identical components with statistically independent lifetimes has been studied by Tavangar (2016). As already mentioned, adopting $(n - k + 1)$ -out-of- n systems with statistically INID components, Zhao et. al (2008) studied the random variables $(t - T_{i:n} | T_{k:n} < t < T_{k+1:n}), 1 \leq i < k \leq n$. Salehi and Tavangar (2019) studied this quantity for exchangeable component lifetimes of such a system.

In a coherent system, if the times at which the failed components have not been monitoring continuously, then the lifetimes $T_{1:n}, \dots, T_{m:n}$ appears unknown. Hence, the knowledge of this kind of inactivity times, i.e., $IT_{m,k,n}(t)$, which has some information about the time that has elapsed from the m^{th} failure in the system, may help the reliability engineer to consider preventive maintenance or a replacement of the whole system at some reasonable epoch (Tavangar (2016); Zhao et. al (2008)).

The structure of the paper is as follows. In Section 2, we derive the survival function of $IT_{m,k,n}$. In Section 3, we stochastically compare the inactivity time of the failed components for an $(n - k + 1)$ -out-of- n system for both one and two samples.

2 The Inactivity Time of the Failed Components

Now, we assume that there is an $(n - k + 1)$ -out-of- n system when all of n components are independent. Let the independent random variables $\{T_i\}_{1 \leq i \leq n}$ represent the lifetimes of the components of this system with continuous distribution functions $G_1(t), G_2(t), \dots, G_n(t)$, respectively. In order to have the ratios well defined in the statements below, we assume that $G_i(t) > 0$ for all integers $1 \leq i \leq n$ and $t > 0$. It means that t is in the support of G_i . We denote the column vector of distributions of T 's by $\mathbf{G}(t) = (G_1(t), G_2(t), \dots, G_n(t))'$. Also we denote the inactivity time of T_i at time t by $(T_i) = (t - T_i | T_i < t)$ and its distribution and survival function by $G_{i,t}(y)$ and $\bar{G}_{i,t}(y)$ respectively, where $\bar{G}_{i,t}(y) = (G_i(t - y) / G_i(t)), 1 \leq i \leq n$ and $0 < y < t$. Similarly, we define $\mathbf{G}_t(y) =$

$(G_{1,t}(y), G_{2,t}(y), \dots, G_{n,t}(y))', \bar{\mathbf{G}}_t(y) = (\bar{G}_{1,t}(y), \bar{G}_{2,t}(y), \dots, \bar{G}_{n,t}(y))'$. Since we assumed that the underlying random variables are not identically distributed, we use the *permanents* representation for the joint distribution of the order statistics. Following Kochar and Xu (2010), for any $p \times p$ matrix $\mathbf{M} = (b_{i,j})$, the permanent of \mathbf{M} is defined as

$$\text{Per}(\mathbf{M}) = \sum_{\pi \in S_p} \prod_{i=1}^p b_{i,\pi(i)},$$

where S_p is the set of all permutations of $(1, \dots, p)$. For column vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ in \mathbb{R}^p , the permanent of the $p \times p$ matrix $(\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p)$ is then denoted by $[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p]$. Denoting

$$\left[\underbrace{\mathbf{b}_1}_{q_1}, \underbrace{\mathbf{b}_2}_{q_2}, \dots \right]$$

the permanent of the matrix is obtained by taking q_1 copies of \mathbf{b}_1 , q_2 copies of \mathbf{b}_2 and so on. When the permanent has those rows only in D we use the following notation:

$$\left[\underbrace{\mathbf{b}_1}_{q_1}, \underbrace{\mathbf{b}_2}_{q_2}, \dots \right]_D.$$

Finally we denote the survival function of $IT_{m,k,n}$ by $\bar{G}_{m,k,n}$, i.e.,

$$\begin{aligned} \bar{G}_{m,k,n}(y) &= \mathbb{P}(IT_{m,k,n}(t) > y) \\ &= \mathbb{P}(t - T_{m:n} > y | T_{m:n} < t < T_{k:n}). \end{aligned}$$

According to the following theorem, $\bar{G}_{m,k,n}$ has a mixture form.

Theorem 2.1. For $m = 1, 2, \dots, k, k = 1, 2, \dots, n$ and for all $y \in [0, t]$,

$$\bar{G}_{m,k,n}(y) = \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)},$$

where by \sum_{D_i} we mean that the summation is taken over all the sets $D_i \subset \{1, \dots, n\}$ with cardinality i , $D_i^c = \{1, \dots, n\} - D_i$,

$$\phi_i(t) = \prod_{s \in D_i} \frac{\bar{G}_s(t)}{G_s(t)}, \quad n - k - 1 \leq i \leq n - m, \tag{2.1}$$

and

$$\bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) = \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!} \left[\underbrace{\mathbf{1} - \bar{\mathbf{G}}_t(y)}_j, \underbrace{\bar{\mathbf{G}}_t(y)}_{n-i-j} \right]_{D_i^c},$$

where $\bar{G}_{n-i-m+1:n-i,t}^{D_i^c}$ is the survival function of the $(n - i - m + 1)^{th}$ order statistics from $(T_j)_t, j \in D_i^c$ (denoted by $T_{n-i-m+1:n-i,t}$).

Proof. Using the definition of conditional probability we have,

$$\begin{aligned} & \mathbb{P}(t - T_{m:n} > y | T_{m:n} < t < T_{k:n}) \\ &= \frac{\mathbb{P}(t - T_{m:n} > y, T_{m:n} < t < T_{k:n})}{\mathbb{P}(T_{m:n} < t < T_{k:n})} \\ &= \frac{\mathbb{P}(T_{m:n} < t - y, T_{k:n} > t)}{\mathbb{P}(T_{m:n} < t < T_{k:n})}. \end{aligned}$$

Following the argument in Kochar and Xu (2010) we have

$$\begin{aligned} & \mathbb{P}(T_{m:n} < t - y, T_{k:n} > t) \\ &= \sum_{i=n-k+1}^{n-m} \sum_{j=0}^{n-i-m} \mathbb{P}(t - y < \text{exactly } j \text{ of } T\text{'s are } < t, \text{ exactly } i \text{ of } T\text{'s are } > t) \\ &= \sum_{i=n-k+1}^{n-m} \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!i!} \left[\underbrace{\mathbf{G}(t-y)}_{n-i-j}, \underbrace{\mathbf{G}(t) - \mathbf{G}(t-y)}_j, \underbrace{\bar{\mathbf{G}}(t)}_i \right] \\ &= \sum_{i=n-k+1}^{n-m} \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!i!} \sum_{D_i} \left[\underbrace{\mathbf{G}(t-y)}_{n-i-j}, \underbrace{\mathbf{G}(t) - \mathbf{G}(t-y)}_j \right]_{D_i^c} \left[\underbrace{\bar{\mathbf{G}}(t)}_i \right]_{D_i} \end{aligned}$$

$$\begin{aligned}
&= \left(\prod_{i=1}^n G_i(t) \right) \sum_{i=n-k+1}^{n-m} \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!} \sum_{D_i} \prod_{s \in D_i} \frac{\bar{G}_s(t)}{G_s(t)} \left[\underbrace{\bar{\mathbf{G}}_t(\mathbf{y})}_{n-i-j}, \underbrace{\mathbf{1} - \bar{\mathbf{G}}_t(\mathbf{y})}_j \right]_{D_i^c} \\
&= \left(\prod_{i=1}^n G_i(t) \right) \sum_{i=n-k+1}^{n-m} \sum_{D_i} \prod_{s \in D_i} \frac{\bar{G}_s(t)}{G_s(t)} \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!} \left[\underbrace{\bar{\mathbf{G}}_t(\mathbf{y})}_{n-i-j}, \underbrace{\mathbf{1} - \bar{\mathbf{G}}_t(\mathbf{y})}_j \right]_{D_i^c} \\
&= \left(\prod_{i=1}^n G_i(t) \right) \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \sum_{j=0}^{n-i-m} \frac{1}{(n-i-j)!j!} \left[\underbrace{\mathbf{1} - \bar{\mathbf{G}}_t(\mathbf{y})}_j, \underbrace{\bar{\mathbf{G}}_t(\mathbf{y})}_{n-i-j} \right]_{D_i^c} \\
&= \left(\prod_{i=1}^n G_i(t) \right) \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(\mathbf{y}). \tag{2.2}
\end{aligned}$$

Setting $\mathbf{y} = 0$ in (2.2), we have

$$\mathbb{P}(T_{m:n} < t < T_{k:n}) = \left(\prod_{i=1}^n G_i(t) \right) \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t), \tag{2.3}$$

and the proof is completed. \square

Remark 1. Mean inactivity time of m^{th} strongest failed component when system is still working, has the following representation

$$\mathbb{E}(IT_{m,k,n}(t)) = \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \mu_{n,i,m}^{D_i^c}}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)},$$

where

$$\mu_{n,i,m}^{D_i^c} = \int_0^\infty \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(\mathbf{y}) d\mathbf{y}.$$

Remark 2. In the case of which T_1, T_2, \dots, T_n are IID random variables, we have

$$\mathbb{P}(IT_{m,k,n}(t) > y) = \mathbb{P}(t - T_{m:n} > y | T_{m:n} < t < T_{k:n})$$

$$\begin{aligned}
 &= \frac{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i \bar{G}_{n-i-m+1:n-i,t}(y)}{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i} \\
 &= \frac{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i \sum_{j=0}^{n-i-m} \frac{(n-i)!}{(n-i-j)!j!} [1 - \bar{G}_t(y)]^j [\bar{G}_t(y)]^{n-i-j}}{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i} \\
 &= \frac{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i \sum_{j=0}^{l-m} \binom{l}{j} [G(t) - G(t-y)]^j [G(t-y)]^{l-j} \frac{1}{[G(t)]^l}}{\sum_{i=n-k+1}^{n-m} \binom{n}{i} \left(\frac{\bar{G}(t)}{G(t)}\right)^i} \\
 &= \frac{\sum_{l=m}^{k-1} \sum_{j=0}^{l-m} \binom{n}{l} \binom{l}{j} \frac{[\bar{G}(t)]^{n-l}}{[G(t)]^n} [G(t) - G(t-y)]^j [G(t-y)]^{l-j}}{\sum_{l=m}^{k-1} \binom{n}{l} \left(\frac{\bar{G}(t)}{G(t)}\right)^{n-l}} \\
 &= \frac{\sum_{l=m}^{k-1} \sum_{j=0}^{l-m} \binom{n}{l} \binom{l}{j} [\bar{G}(t)]^{-l} [G(t) - G(t-y)]^j [G(t-y)]^{l-j}}{\sum_{l=m}^{k-1} \binom{n}{l} \left(\frac{G(t)}{\bar{G}(t)}\right)^l},
 \end{aligned}$$

which is the same as Equation (3) in Tavangar (2016).

Examples 2.1. Let T_1, T_2 and T_3 be independent random variables which represent the lifetime of components of a $(3-3+1)$ -out-of-3 system (a parallel system consist of three components). If at time t system is still working with at least 2 failed components, then

the survival function of the inactivity time $IT_{2,3,3}(t)$ can be represented as

$$\bar{G}_{2,3,3}(y) = \frac{\sum_{D_1} \phi_1(t) \bar{G}_{1:2,t}^{D_1^c}(y)}{\sum_{D_1} \phi_1(t)}.$$

Therefore if the underlying distribution of each T_i , is exponentially distributed with mean $(1/\lambda_i)$, then we have

$$\sum_{D_1} \phi_1(t) = \sum_{j=1}^3 \frac{1}{e^{\lambda_j t} - 1},$$

and

$$\sum_{D_1} \phi_1(t) \bar{G}_{1:2,t}^{D_1^c}(y) = \sum_{D_1, j \in D_1} \frac{1}{e^{\lambda_j t} - 1} \prod_{k \in D_1^c} \frac{1 - e^{\lambda_k(t-y)}}{1 - e^{\lambda_k t}}.$$

3 Stochastic Comparisons

In the following section first the monotone properties of $IT_{m,k,n}$ with respect to m as well as k will be studied. Then there will be a comparison between two systems with independent but nonidentical components from two independent samples.

Theorem 3.1. For all $t > 0$ and $m = 2, \dots, n, k = 2, \dots, n, m < k$ we have

$$IT_{m,k,n}(t) \geq_{st} IT_{m-1,k,n}(t). \quad (3.1)$$

Proof. To show (3.1) we observe the simple fact that for $y, t \geq 0$,

$$\mathbb{P}(IT_{m-1,k,n}(t) > y) \leq \mathbb{P}(IT_{m,k,n}(t) > y),$$

has the same sign as

$$\frac{\sum_{i=n-k+1}^{n-m+1} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+2:n-i,t}^{D_i^c}(y)}{\sum_{i=n-k+1}^{n-m+1} \sum_{D_i} \phi_i(t)} - \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)}.$$

This quantity is non-positive if and only if

$$\begin{aligned} & \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k+1}^{n-m+1} \sum_{D_i} \phi_i(t) \overline{G}_{n-i-m+2:n-i,t}^{D_i^c}(y) \right] \\ & - \left[\sum_{i=n-k+1}^{n-m+1} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \end{aligned} \quad (3.2)$$

is non-positive. But, since Equation (3.2) can be rewritten as

$$\begin{aligned} & \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \left[\overline{G}_{n-i-m+2:n-i,t}^{D_i^c}(y) - \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \right] \\ & + \sum_{i=n-k+1}^{n-m} \sum_{D_i} \sum_{D_{n-m+1}} \phi_{n-m+1}(t) \phi_i(t) \left[\overline{G}_{1:m-1,t}^{D_i^c}(y) - \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right], \end{aligned}$$

it is enough to show that

$$\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \left[\overline{G}_{n-i-m+2:n-i,t}^{D_i^c}(y) - \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \leq 0,$$

and for all $n-k+1 \leq i \leq n-m$,

$$\left[\overline{G}_{1:m-1,t}^{D_i^c}(y) - \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \leq 0.$$

But by telescoping sum formula and the fact that $D_{n-k+1} \subset D_{n-m}$, we have

$$\begin{aligned} & \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \left[\overline{G}_{n-i-m+2:n-i,t}^{D_i^c}(y) - \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \\ & \leq \sum_{D_{n-m}} \phi_{n-m}(t) \left[\overline{G}_{k-m+1:k-1,t}^{D_{n-k+1}^c}(y) - \overline{G}_{1:m,t}^{D_{n-m}^c}(y) \right]. \end{aligned}$$

Since $D_{n-m}^c \subset D_{n-k+1}^c$ therefore,

$$\overline{G}_{1:m,t}^{D_{n-m}^c}(y) \leq \overline{G}_{k-m:k-1,t}^{D_{n-k+1}^c}(y),$$

and since $G_{k-m:k-1,t}^{D_{n-k+1}^c}(y) \leq G_{k-m+1:k-1,t}^{D_{n-k+1}^c}(y)$ we have

$$[\bar{G}_{k-m+1:k-1,t}^{D_c^{n-k+1}}(y) - \bar{G}_{1:m,t}^{D_c^{n-m}}(y)] \leq 0.$$

Now to show the difference $[G_{1:m-1,t}^{D_c^{n-m+1}}(y) - G_{n-i-m+1:n-i,t}^{D_i^c}(y)]$ is not positive, we note that since $D_{n-m+1}^c \subset D_i^c$, we have

$$\bar{G}_{1:m-1,t}^{D_c^{n-m+1}}(y) \leq \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y).$$

On the other hand, $G_{n-i-m+1:n-i,t}^{D_i^c}(y) \leq G_{n-i-m+1:n-i,t}^{D_c^{n-m+1}}(y)$ and hence,

$$\bar{G}_{1:m-1,t}^{D_c^{n-m+1}}(y) - \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \leq 0.$$

□

In the next theorem we compare the inactivity time of failed components of $(n-k+1)$ -out-of- n system stochastically with respect to parameter k .

Theorem 3.2. For all $t > 0$ and $m, k = 1, \dots, n-1, m < k$, we have

$$IT_{m,k,n}(t) \leq_{st} IT_{m,k+1,n}(t). \quad (3.3)$$

Proof. To show (3.3) observe that the sign of

$$\mathbb{P}(IT_{m,k,n}(t) > y) \leq \mathbb{P}(IT_{m,k+1,n}(t) > y)$$

is the same as that of

$$\frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)} - \frac{\sum_{i=n-k}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y)}{\sum_{i=n-k}^{n-m} \sum_{D_i} \phi_i(t)},$$

which is non-positive, if and only if, the following difference is non-positive:

$$\begin{aligned} & \left[\sum_{i=n-k}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \\ & - \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right]. \end{aligned}$$

But this quantity equals to

$$\begin{aligned}
& \left[\sum_{D_{n-k}} \phi_{n-k}(t) + \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \\
& - \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{D_{n-k}} \phi_{n-k}(t) \overline{G}_{k-m+1:k,t}^{D_{n-k}^c}(y) \right] \\
& + \sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \\
& = \left[\sum_{D_{n-k}} \phi_{n-k}(t) \right] \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \overline{F}_{n-i-m+1:n-i,t}^{D_i^c}(y) \right] \\
& - \left[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \right] \left[\sum_{D_{n-k}} \phi_{n-k}(t) \overline{G}_{k-m+1:k,t}^{D_{n-k}^c}(y) \right] \\
& = \sum_{i=n-k+1}^{n-m} \sum_{D_i} \sum_{D_{n-k}} \phi_i(t) \phi_{n-k}(t) \left[\overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) - \overline{G}_{k-m+1:k,t}^{D_{n-k}^c}(y) \right].
\end{aligned}$$

Since $i \in [n-k+1, n-m]$ thus, $D_i^c \subset D_{n-k}^c$ and hence by Lemma 4.1 in Kochar and Xu (2010),

$$\overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(y) \leq \overline{G}_{k-m+1:k,t}^{D_{n-k}^c}(y).$$

□

Next Theorem compares two $(n-k+1)$ -out-of- n systems. We assume that the components are independent but they do not follow the same distribution.

Theorem 3.3. Let $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be independent random variables such that for all integers $1 \leq i, j \leq n$, $X_i \leq_{hr} Y_j$. Then for all $t > 0$

$$(t - X_{m:n} | X_{m:n} < t < X_{k:n}) \geq_{st} (t - Y_{m:n} | Y_{m:n} < t < Y_{k:n}). \quad (3.4)$$

Proof. Let G and H be the cumulative distribution functions of X and Y , respectively. To show Equation (3.4), it is enough to show for $a > 0$,

$$\frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t) \overline{H}_{n-i-m+1:n-i,t}^{D_i^c}(a)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)} \leq \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \overline{G}_{n-i-m+1:n-i,t}^{D_i^c}(a)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)},$$

where $\Psi_i(t) = \prod_{s \in D_i} \frac{\bar{H}_s(t)}{\bar{H}_s(t)}$ and $\phi_i(t) = \prod_{s \in D_i} \frac{\bar{G}_s(t)}{\bar{G}_s(t)}$. Since $X_i \leq_{hr} Y_i$ then $X_i \leq_{st} Y_i$ and therefore $X_{i:n} \leq_{st} Y_{j:n}$ for $i \leq j$. Also for each $s \in D_i$, $\bar{H}_s(t) \leq \bar{G}_s(t)$ and hence,

$$\frac{\Psi_i(t)}{\phi_i(t)} = \prod_{s \in D_i} \frac{\bar{H}_s(t) \bar{G}_s(t)}{\bar{H}_s(t) \bar{G}_s(t)} \leq 1.$$

By definition of $\bar{H}_{n-i-m+1:n-i,t}^{D_i^c}(a)$ and $\bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(a)$ and the fact that $t - Y_i \leq_{st} t - T_i$ we have $\bar{H}_{n-i-m+1:n-i,t}^{D_i^c}(a) \leq \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(a)$. Also,

$$\begin{aligned} & \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t) \bar{H}_{n-i-m+1:n-i,t}^{D_i^c}(a)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)} - \frac{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(a)}{\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)} \\ &= \frac{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t) \bar{H}_{n-i-m+1:n-i,t}^{D_i^c}(a)]}{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)]} \\ & \quad - \frac{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t) \bar{G}_{n-i-m+1:n-i,t}^{D_i^c}(a)]}{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)]} \\ &= \frac{[\sum_{j=n-k+1}^{n-m} \sum_{i=n-k+1}^{n-m} \sum_{C_j} \sum_{D_i} \phi_j(t) \Psi_i(t) \bar{H}_{n-i-m+1:n-i,t}^{D_i^c}(a)]}{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)]} \end{aligned}$$

$$- \frac{[\sum_{j=n-k+1}^{n-m} \sum_{i=n-k+1}^{n-m} \sum_{C_j} \sum_{D_i} \Psi_i(t) \phi_j(t) \overline{G}_{n-j-m+1:n-j,t}^{D_i^c}(a)]}{[\sum_{i=n-k+1}^{n-m} \sum_{D_i} \Psi_i(t)] [\sum_{i=n-k+1}^{n-m} \sum_{D_i} \phi_i(t)]}.$$

Since for $i = j$, $\overline{H}_{n-i-m+1:n-i,t}^{D_i^c}(a) - \overline{G}_{n-j-m+1:n-j,t}^{D_j^c}(a) < 0$, and for $i > j$, $D_i^c \subset D_j^c$, thus we have

$$\begin{aligned} \overline{H}_{n-i-m+1:n-i,t}^{D_i^c}(a) &\leq \overline{H}_{n-j-m+1:n-j,t}^{D_j^c}(a) \\ &\leq \overline{G}_{n-j-m+1:n-j,t}^{D_j^c}(a). \end{aligned}$$

For $i < j$, $D_j^c \subset D_i^c$ and since $0 < \overline{H}(a) < 1$, then we obtain

$$\begin{aligned} \overline{H}_{n-i-m+1:n-i,t}^{D_i^c}(a) &\leq \overline{H}_{n-i-m+1:n-i,t}^{D_j^c}(a) \\ &\leq \overline{G}_{n-i-m+1:n-i,t}^{D_j^c}(a). \end{aligned}$$

But since $n - m + 1 - j < n - m + 1 - i$, by the definition of $\overline{G}_{n-j-m+1:n-j,t}^{D_j^c}(a)$ we have $\overline{G}_{n-i-m+1:n-i,t}^{D_j^c}(a) \leq \overline{G}_{n-j-m+1:n-j,t}^{D_j^c}(a)$. □

References

- Asadi , M. (2006), On the mean past lifetime of the components of a parallel system. *Journal of Statistical Planning and Inference*, **136**(4), 1197- 1206 .
- Eryilmaz, S. (2013), On residual lifetime of coherent systems after the r^{th} failure. *Statistical Papers*, **54**, 243-250.
- Gurler, S. and Bairamov, I. (2009), Parallel and k-out-of-n: G systems with nonidentical components and their mean residual life functions. *Applied Mathematical Modelling*, **33**, 1116-1125.
- Kochar, S., Mukerjee, H. and Samaniego, F. J. (1999), The “signature” of a coherent system and its application to comparison among systems. *Naval Research Logistics*, **46**, 507-523.

- Kochar, S. and Xu, M. (2010), On residual lifetimes of k -out-of- n systems with nonidentical components. *Probability in the Engineering and Informational Sciences*, **24**, 109-127.
- Khaledi, B. E. and Shaked, M. (2007), Ordering conditional lifetimes of coherent systems. *Journal of Statistical Planning and Inference*, **137**, 1173-1184.
- Li, X. and Zhao, P. (2006), Some aging properties of the residual life of k -out-of- n systems. *IEEE Transactions on Reliability*, **55**, 535-541.
- Li, X. and Zhao, P. (2008), Stochastic comparison on general inactivity time and general residual life of k -out-of- n system. *Communication in Statistics- Simulation and Computation*, **37**, 1005-1019.
- Li, X. and Zhang, Z. (2008), Some stochastic comparisons of conditional coherent systems. *Applied Stochastic Models in Business and Industry*, **24**, 541-549.
- Navarro, J., Ruiz, J. M. and Sandoval, C. J. (2005), A note on comparisons among coherent systems with dependent components using signatures. *Statistics and Probability Letters*, **72**, 179-185.
- Navarro, J., Aguila, Y., Sordo, M. A. and Suarez-Llorens, A. (2013), Stochastic ordering properties for systems with dependent identically distributed components. *Applied Stochastic Models in Business and Industry*, **29**, 264-278.
- Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2013a), Mixture representations for the joint distribution of lifetimes of two coherent systems with shared components. *Advances in Applied Probability*, **45**, 1011-1027.
- Sadegh, M. K. (2008), Mean past and mean residual Life of parallel system with nonidentical components. *Communications in Statistics Theory and Methods*, **37**, 1137-1145.
- Salehi, E. T. , Asadi, M. and Eryilmaz, S. (2012), On the mean residual lifetime of consecutive k -out-of- n systems. *Test*, **21**, 93-115.
- Salehi, E. and Tavangar M. (2019), Stochastic comparisons on conditional residual lifetime and inactivity time of coherent systems with exchangeable components, *Statistics and Probability Letters*, **145**, 327-337.
- Tavangar, M. (2016), Conditional inactivity time of components in a coherent operating system. *IEEE Transactions on Reliability*, **65**, 359-369.

Tavangar, M. and Asadi, M. (2010), A study on the mean past lifetime of the components of $(n - k + 1)$ -out-of- n system at the system level. *Metrika*, **72**(1), 59-73.

Zhang, Z. (2010), Ordering conditional general coherent systems with exchangeable components. *Journal of Statistical Planning and Inference*. **140**, 454-460.

Zhao, P. Li, X. and Balakrishnan, N. (2008), Conditional ordering of k -out-of- n systems with independent but nonidentical components. *Journal of Applied Probability*, **45**, 1113-1125.