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Tests about a Set of Multivariate Simple Linear Models

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Abstract. In this article, the tests on parallelism, equal intercept and sets of lines intersected at a fixed value for a set of r simple linear models or a set of r linearizable regression models are generalized to the multivariate case, $r = 2, 3, \dots, R$. Likewise, the normality hypothesis is replaced assuming an elliptical matrix variate distribution, concluding that the tests obtained under normality are valid and are invariant under the whole family of elliptical matrix variate distributions. Finally, an application in an agricultural acarology context is provided.

Keywords. Elliptical Distributions, Likelihood Rate, Multivariate Linear Models, Parallelism, Union-Intersection Principles.

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1 Introduction

There are a number of research studies involving the behavior of a dependent variable Y as a function of one independent variable X. Sometimes the experiment accepts a simple linear model (or a linearizable regression model) and usually, this model is proposed for different experimental or observational situations. Then the following situations can emerge: the simple linear models have the same intercept, or these lines are parallel; or given a particular value of the independent variable X, say x_0, the lines are intersected in such value.

We illustrate these situations through the following examples:

Examples 1.1. Several diets are used to feed goats in order to determine the effect for losing or gaining weight. Three goat breeds are used, and for each breed the relationship between the gain or loss of weight in pounds per goat Y and the amount of diet in pounds ingested for each goat X is given by

$$y_{1j} = \alpha_1 + \beta_1 x_{1j} + \epsilon_{1j} \qquad y_{2k} = \alpha_2 + \beta_2 x_{2k} + \epsilon_{2k} \quad y_{3r} = \alpha_3 + \beta_3 x_{3r} + \epsilon_{3r},$$

$j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$, $r = 1, 2, \dots, n_3$, $n_s \geq 2$, $s = 1, 2, 3$. The investigator claims for a parallelism of the lines, that is, if $\beta_1 = \beta_2 = \beta_3$ (if the increase in the average weight of each goat per unit of diet is the same for all breeds). Or the researcher can ask for equality in the intercepts, that is, if $\alpha_1 = \alpha_2 = \alpha_3$ (if the average weight of each goat breed is the same when all breeds are fed with the same diet).

Examples 1.2. An assay is carried out to study the relationship between the elapsed time, t in minutes, and the number of bacteria Y in a Petri dish, for two hybrids of certain bacteria. In this situation, an exponential regression model is assumed for each hybrid bacteria. The models are

$$y_{1j} = \alpha_1 \exp(\beta_1 t_{1j}) \epsilon_{1j}, \quad \text{hybrid bacteria A,}$$

and

$$y_{2k} = \alpha_2 \exp(\beta_2 t_{2k}) \epsilon_{2k}, \quad \text{hybrid bacteria B;}$$

$j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$, $n_s \geq 2$, $s = 1, 2$. The researcher wants to know if $\alpha_1 \exp(\beta_1 t_0) = \alpha_2 \exp(\beta_2 t_0)$ (if at time t_0, the two Petri dishes contain the same number of hybrid bacteria).

Testing the equality of linearizable models, with the same functional structure, is not a common topic of research. Few cases are known, for example, Draper and Smith (1981) provide some insight about the interest of such problem; meanwhile, Graybill (1976) studied the problem in the univariate case by proposing the hypothesis test, the associated statistic and the decision rule. Now, some experiments require a test of parallel models. For instance, the design of new drugs usually demands the proof of parallelism between the dose-response curves of the standard and the new medicines. Then, the power in the preparation of the new brand in contrast with the standard drug can be determined. Jonkman and Sidik (2009) consider the union-intersection tests for proving the parallelism in a logistic response curve of four parameters. On the another hand, Novick *et al.* (2012) used a Bayesian approach in order to assert dissimilarity of biological responses under the effect of two different substances; they fitted two simple linear regression models and concluded the required difference by a parallelism test. Moreover, using non-linear regression models of the logistic type, they proposed a particular parallelism test for determining if two different biological environments reach similar dose-response curves under the same substance. Finally, Fleetwood *et al.* (2015) studied the same preceding problem but in the context of the classical statistic. We must highlight that those parallelism tests have been studied only for the univariate case and $r = 2$.

However, more realistic situations ask for the behavior of more than one dependent variable $\mathbf{y}' = (y_1, \dots, y_q)$ as a function of an independent variable X. In the statistical modeling of such situations, the *multivariate simple linear model (or associated linearizable regression models)* appears as an interesting alternative. In a wider context, the research can ask the same preceding hypothesis about the parallelism of a set of lines, or the same intercept, or a common given intersection point, but now from a multivariate point of view.

Thus, following the idea proposed by Graybill (1976) for the univariate case, the statistics for described multivariate hypothesis testing (parallelism of a set of lines, or the intercept, or same to common intersection point given), are obtained by rethinking the s simple linearizables models, as a general multivariate linear model, and then the three hypothesis tests can be obtained as particular cases of the multivariate extension of the general linear hypothesis.

Some preliminary results about matrix algebra, matrix multivariate distributions and general multivariate linear model are shown, see Section 2. By using likelihood rate and union-intersection principles, Section 3 derive the multivariate statistics versions for the above-mentioned hypotheses: same intercept, parallelism and intersection in a known point. Also, these results are extended to the elliptical case when the x's are fixed or random. Section 4 applies the developed theory in the context of agricultural acarology.

2 Preliminary Results

A detailed discussion of the univariate linear model and related topics may be found in Graybill (1976) and Draper and Smith (1981) and for the multivariate linear model see Rencher (2002) and Seber (1984), among many others. For completeness, we shall introduce some notations, although in general, we adhere to the standard notation forms.

2.1 Notation, Matrix Algebra and Matrix Multivariate Distribution

For our purposes: if $\mathbf{A} \in \Re^{n\times m}$ denotes a matrix, that is, $\mathbf{A}$ has n rows and m columns, then $\mathbf{A}' \in \Re^{m\times n}$ denotes its transpose matrix, and if $\mathbf{A} \in \Re^{n\times n}$ has an inverse, it shall be denoted by $\mathbf{A}^{-1} \in \Re^{n\times n}$. An identity matrix shall be denoted by $\mathbf{I} \in \Re^{n\times n}$, to specify the size of the identity, we will use $\mathbf{I}_n$. A null matrix shall be denoted as $\mathbf{0} \in \Re^{n\times m}$. A vector of ones shall be denoted by $\mathbf{1} \in \Re^{n}$. For any matrix $\mathbf{A} \in \Re^{n\times m}$ there exists $\mathbf{A}^{-} \in \Re^{m\times n}$ which is termed Moore-Penrose inverse. Similarly given $\mathbf{A} \in \Re^{n\times m}$ there exists $\mathbf{A}^{c} \in \Re^{m\times n}$ such that $\mathbf{A}\mathbf{A}^{c}\mathbf{A} = \mathbf{A}$, $\mathbf{A}^{c}$ is termed conditional inverse. It is said that $\mathbf{A} \in \Re^{n\times n}$ is a symmetric matrix if $\mathbf{A} = \mathbf{A}'$ and if all the eigenvalues are positive, the matrix $\mathbf{A}$ is said to be positive definite, which shall be denoted as $\mathbf{A} > \mathbf{0}$. If $\mathbf{A} \in \Re^{n\times m}$ write it in terms of its m columns, $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m)$, $\mathbf{A}_j \in \Re^{n}$, $j = 1, 2\dots, m$, $\operatorname{vec}(\mathbf{A}) \in \Re^{nm}$ denotes the vectorization of $\mathbf{A}$, moreover, $\operatorname{vec}'(\mathbf{A}) = (\operatorname{vec}(\mathbf{A}))' = (\mathbf{A}'_1, \mathbf{A}'_2, \dots, \mathbf{A}'_m)$. Let $\mathbf{A} \in \Re^{r\times s}$ and $\mathbf{B} \in \Re^{n\times m}$, then $\mathbf{A}\otimes\mathbf{B} \in \Re^{sn\times rm}$ denotes its Kronecker product. Given a null matrix $\mathbf{A} \in \Re^{n\times n}$ with diagonal elements $a_{ii} \neq 0$ for at least one i, this shall be denoted by $\mathbf{A} = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$. Given $\mathbf{a} \in \Re^{n}$, a vector, its Euclidean norm shall be defined as $||\mathbf{a}|| = \sqrt{\mathbf{a}'\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

If a random matrix $\mathbf{Y} \in \Re^{n\times m}$ has a matrix multivariate normal distribution with matrix mean $\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu} \in \Re^{n\times m}$ and covariance matrix $\operatorname{Cov}(\operatorname{vec}\mathbf{Y}') = \boldsymbol{\Theta}\otimes\boldsymbol{\Sigma}$, $\boldsymbol{\Theta} = \boldsymbol{\Theta}' \in \Re^{n\times n}$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}' \in \Re^{m\times m}$ this fact shall be denoted as $\mathbf{Y} \sim \mathcal{N}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}\otimes\boldsymbol{\Sigma})$. Observe that, if $\mathbf{A} \in \Re^{n\times r}$, $\mathbf{B} \in \Re^{m\times s}$ and $\mathbf{C} \in \Re^{r\times s}$ matrices of constants,

$$\mathbf{A}'\mathbf{Y}\mathbf{B} + \mathbf{C} \sim \mathcal{N}_{r\times s}(A'\boldsymbol{\mu}B + \mathbf{C}, \mathbf{A}'\boldsymbol{\Theta}\mathbf{A}\otimes\mathbf{B}'\boldsymbol{\Sigma}\mathbf{B}). \tag{2.1}$$

Finally, consider that $\mathbf{Y} \sim \mathcal{N}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta} \otimes \boldsymbol{\Sigma})$ then the random matrix $\mathbf{V} = \mathbf{Y}'\boldsymbol{\Theta}^{-1}\mathbf{Y}$ has a noncentral Wishart distribution with n degrees of freedom and non-centrality parameter matrix $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}'\boldsymbol{\Theta}^{-1}\boldsymbol{\mu}/2$. This fact shall be denoted as $\mathbf{V} \sim \mathcal{W}_m(n, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$. Observe that if $\boldsymbol{\mu} = \mathbf{0}$, then $\boldsymbol{\Omega} = \mathbf{0}$, and $\mathbf{V}$ is said to have a central Wishart distribution and $\mathcal{W}_m(n, \boldsymbol{\Sigma}, \mathbf{0}) \equiv \mathcal{W}_m(n, \boldsymbol{\Sigma})$, see Srivastava and Khatri (1979) and Muirhead (2005).

2.2 General Multivariate Linear Model

Consider the general multivariate linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{2.2}$$

where, $\mathbf{Y} \in \Re^{n\times q}$ is the matrix of the observed values, $\boldsymbol{\beta} \in \Re^{p\times q}$ is the parameter matrix, $\mathbf{X} \in \Re^{n\times p}$ is the design matrix or the regression matrix of rank $r \le p$ and $n > p + q$, $\boldsymbol{\epsilon} \in \Re^{n\times q}$ is the error matrix which has a matrix multivariate normal distribution, specifically $\boldsymbol{\epsilon} \sim \mathcal{N}_{n\times q}(\mathbf{0}, \mathbf{I}_n \otimes \boldsymbol{\Sigma})$, (Muirhead , 2005, see p. 430) and $\boldsymbol{\Sigma} \in \Re^{q\times q}$, $\boldsymbol{\Sigma} > \mathbf{0}$. For this model, we want to test the hypothesis

$$H_0 : \mathbf{C}\boldsymbol{\beta}\mathbf{M} = \mathbf{0} \text{ versus } H_a : \mathbf{C}\boldsymbol{\beta}\mathbf{M} \neq \mathbf{0}, \tag{2.3}$$

where $\mathbf{C} \in \Re^{\nu_{\mathbf{H}}\times p}$ is of rank $\nu_{\mathbf{H}} \le r$ and $\mathbf{M} \in \Re^{q\times g}$ is of rank $g \le q$. As in the univariate case, the matrix $\mathbf{C}$ concerns with the hypothesis among the elements of the parameter matrix columns, while the matrix $\mathbf{M}$ allows hypothesis among the different response parameters. The matrix $\mathbf{M}$ plays a role in profile analysis, for example; in ordinary hypothesis testing it assumes the identity matrix, namely, $\mathbf{M} = \mathbf{I}_p$.

Let $\mathbf{S}_H$ be the matrix of sums of squares and sums of products due to the hypothesis and let $\mathbf{S}_E$ be the matrix of sums of squares and sums of products due to the error. These are defined as

$$\begin{array}{rcl} \mathbf{S}_H & = & (\mathbf{C}\widetilde{\boldsymbol{\beta}}\mathbf{M})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^c\mathbf{C}')^{-1}(\mathbf{C}\widetilde{\boldsymbol{\beta}}\mathbf{M}), \\ \mathbf{S}_E & = & \mathbf{M}'\mathbf{Y}'(\mathbf{I}_n - \mathbf{X}\mathbf{X}^c)\mathbf{Y}\mathbf{M}, \end{array} \tag{2.4}$$

respectively, where $\widetilde{\boldsymbol{\beta}} = \mathbf{X}^c\mathbf{Y}$. Note that, under the null hypothesis, $\mathbf{S}_H$ has a g-dimensional noncentral Wishart distribution with $\nu_{\mathbf{H}}$ degrees of freedom and parameter matrix $\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}$, i.e. $\mathbf{S}_H \sim \mathcal{W}_g(\nu_{\mathbf{H}}, \mathbf{M}'\boldsymbol{\Sigma}\mathbf{M})$. Similarly, $\mathbf{S}_E$ has a g-dimensional Wishart distribution with $\nu_{\mathbf{E}}$ degrees of freedom and parameter matrix $\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}$, i.e.

$\mathbf{S}_E \sim \mathcal{W}_g(\nu_{\mathbf{E}}, \mathbf{M}'\mathbf{\Sigma}\mathbf{M})$. Specifically, $\nu_{\mathbf{H}}$ and $\nu_{\mathbf{E}}$ denote the degrees of freedom of the hypothesis and the error, respectively. All the results given below are true for $\mathbf{M} \neq \mathbf{I}_q$, just compute $\mathbf{S}_H$ and $\mathbf{S}_E$ from (2.4) and replace q by g. Now, let $\lambda_1, \cdots, \lambda_s$ be the $s = \min(\nu_{\mathbf{H}}, g)$ non-null eigenvalues of the matrix $\mathbf{S}_H\mathbf{S}_E^{-1}$ such that $0 < \lambda_s < \cdots < \lambda_1 < \infty$ and let $\theta_1, \cdots, \theta_s$ be the s non-null eigenvalues of the matrix $\mathbf{S}_H(\mathbf{S}_H + \mathbf{S}_E)^{-1}$ with $0 < \theta_s < \cdots < \theta_1 < 1$; here we note $\lambda_i = \theta_i/(1-\theta_i)$ and $\theta_i = \lambda_i/(1+\lambda_i)$, $i = 1, \cdots, s$. Various authors have proposed a number of different criteria for testing the hypothesis (2.3). But it is known, that all the tests can be expressed in terms of the eigenvalues $\lambda's$ or $\theta's$, see for example Kres (1983). The likelihood ratio test statistics termed Wilks's Λ, given next, is one of such statistics.

The likelihood ratio test of size α of $H_0 : \mathbf{C}\boldsymbol{\beta}\mathbf{M} = \mathbf{0}$ against $H_a : \mathbf{C}\boldsymbol{\beta}\mathbf{M} \neq \mathbf{0}$ reject if $\Lambda \leq \Lambda_{\alpha,q,\nu_{\mathbf{H}},\nu_{\mathbf{E}}}$, where

$$\Lambda = \frac{|\mathbf{S}_{\mathbf{E}}|}{|\mathbf{S}_{\mathbf{E}} + \mathbf{S}_{\mathbf{H}}|} = \prod_{i=1}^{s} \frac{1}{(1+\lambda_i)} = \prod_{i=1}^{s} \frac{1}{(1-\theta_i)}. \tag{2.5}$$

Exact critical values of $\Lambda_{\alpha,q,\nu_{\mathbf{H}},\nu_{\mathbf{E}}}$ can be found in Table A.9 of Rencher (2002) or Table 1 of Kres (1983).

3 Test About a Set of Multivariate Simple Linear Models

Consider the following R multivariate simple linear models

$$\mathbf{Y}_r = \mathbf{X}_r\boldsymbol{B}_r + \boldsymbol{\epsilon}_r, \tag{3.1}$$

$\mathbf{Y}_r \in \Re^{n_r \times q}$, $\mathbf{X}_r \in \Re^{n_r \times 2}$ such that its rank is 2; $\boldsymbol{B}_r \in \Re^{2\times q}$, $n_r > 2$ and $n_r > q+2$ for all $r = 1, 2, \ldots, R$; $\sum_{r=1}^{R} n_r = N$ and $\boldsymbol{\epsilon}_r \sim \mathcal{N}_{n_r \times q}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \mathbf{\Sigma})$, with $\mathbf{\Sigma} > \mathbf{0}$; where

$$\boldsymbol{B}_r = \begin{pmatrix} \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rq} \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rq1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_r' \\ \boldsymbol{\beta}_r' \end{pmatrix}, \quad \mathbf{X_r} = \begin{pmatrix} 1 & x_{r1} \\ 1 & x_{r2} \\ \vdots & \vdots \\ 1 & x_{rn_r} \end{pmatrix} = (\mathbf{1}_{n_r} \mathbf{x}_r).$$

We are interested in the following hypothesis

i) $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_R$, that is, the set of lines are parallel.

ii) $H_0 : \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \cdots = \boldsymbol{\alpha}_R$, that is, the set of lines have a common vector intercept.

iii) $H_0 : \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1 x_0 = \boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2 x_0 = \cdots = \boldsymbol{\alpha}_R + \boldsymbol{\beta}_R x_0$, ($x_0$ known), that is, the set of lines intersect at the x value x_0 which is specified in advance.

First, observe that the R multivariate simple linear models can be written as a general multivariate linear model, $\mathbb{Y} = \mathbb{X}\mathbb{B} + \mathbb{E}$, such that

$$\underset{N\times q}{\mathbb{Y}} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_R \end{pmatrix}, \underset{N\times 2R}{\mathbb{X}} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_R \end{pmatrix}, \underset{2R\times q}{\mathbb{B}} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_R \end{pmatrix}, \underset{N\times q}{\mathbb{E}} = \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_R \end{pmatrix}.$$

Namely, $\mathbb{E} \sim \mathcal{N}_{N\times q}(\mathbf{0}, \mathbf{I}_{NR} \otimes \boldsymbol{\Sigma})$. Thus

$$\mathbb{X}'\mathbb{X} = \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2'\mathbf{X}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_R'\mathbf{X}_R \end{pmatrix}, \quad \mathbb{X}'\mathbb{Y} = \begin{pmatrix} \mathbf{X}_1'\mathbf{Y}_1 \\ \mathbf{X}_2'\mathbf{Y}_2 \\ \vdots \\ \mathbf{X}_R'\mathbf{Y}_R \end{pmatrix},$$

and by (Graybill , 1976, Theorem 1.3.1, p. 19)

$$(\mathbb{X}'\mathbb{X})^{-1} = \begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_2'\mathbf{X}_2)^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & (\mathbf{X}_R'\mathbf{X}_R)^{-1} \end{pmatrix}.$$

Therefore by Muirhead (2005) (Theorem 10.1.1, p. 430) and see also Rencher (2002)(Equation 10.46, p. 339),

$$\widehat{\mathbb{B}} = (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Y} = \begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_2'\mathbf{X}_2)^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & (\mathbf{X}_R'\mathbf{X}_R)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1'\mathbf{Y}_1 \\ \mathbf{X}_2'\mathbf{Y}_2 \\ \vdots \\ \mathbf{X}_R'\mathbf{Y}_R \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{Y}_1 \\ (\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{Y}_2 \\ \vdots \\ (\mathbf{X}_R'\mathbf{X}_R)^{-1}\mathbf{X}_R'\mathbf{Y}_R \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1^-\mathbf{Y}_1 \\ \mathbf{X}_2^-\mathbf{Y}_2 \\ \vdots \\ \mathbf{X}_R^-\mathbf{Y}_R \end{pmatrix},$$

that is, $\mathbf{B}_r = (\mathbf{X}_r'\mathbf{X}_r)^{-1}\mathbf{X}_r'\mathbf{Y}_r = \mathbf{X}_r^-\mathbf{Y}_r$ and

$$\begin{aligned} \mathbf{S_E} &= (\mathbb{Y} - \widehat{\mathbb{B}}'\mathbb{X}'\mathbb{Y})'(\mathbb{Y} - \widehat{\mathbb{B}}'\mathbb{X}'\mathbb{Y}) = \sum_{r=1}^{R} \mathbf{Y}_r\mathbf{Y}_r - \sum_{r=1}^{R} \mathbf{B}_r'\mathbf{X}_r'\mathbf{Y}_r \\ &= \sum_{r=1}^{R} \mathbf{Y}_r' \left(\mathbf{I}_{n_r} - \mathbf{X}_r\mathbf{X}_r^-\right) \mathbf{Y}_r \in \Re^{q\times q}. \end{aligned} \tag{3.2}$$

Hence by (Muirhead , 2005, Theorem 10.1.2, p. 431) and (Srivastava and Khatri, 1979, equation 6.3.8, p. 171) we have that $\widehat{\mathbb{B}} \sim \mathcal{N}_{2R\times q}\left(\mathbb{B}, (\mathbb{X}'\mathbb{X})^{-1} \otimes \mathbf{\Sigma}\right)$.

Note that

$$\widehat{\mathbf{B}}_r = \left(\underset{1}{\mathbf{0}} \cdots \underset{r}{\mathbf{I}_2} \cdots \underset{R}{\mathbf{0}}\right)\widehat{\mathbb{B}} = \left(\underset{1}{\mathbf{0}} \cdots \underset{r}{\mathbf{I}_2} \cdots \underset{R}{\mathbf{0}}\right)\begin{pmatrix} \widehat{\mathbf{B}}_1 \\ \widehat{\mathbf{B}}_2 \\ \vdots \\ \widehat{\mathbf{B}}_R \end{pmatrix},$$

thus, $\widehat{\mathbf{B}}_r \sim \mathcal{N}_{2\times q}\left(\mathbf{B}_r, (\mathbf{X}_r'\mathbf{X}_r)^{-1} \otimes \mathbf{\Sigma}\right)$ and $\mathbf{S_E} \sim \mathcal{W}_q(N-2R, \mathbf{\Sigma})$. Observe that $\widehat{\mathbf{B}}_r$ is computed from the data for the rth model and $\mathbf{S_E}$ is computed by pooling the estimators of $\mathbf{S_E}$ from each model $\mathbf{S}_{\mathbf{E}_r}$.

Generalizing the results in (Graybill , 1976, Example 6.2.1, pp. 177-178) and using matrix notation in the multivariate case, we have

$$\widehat{\mathbf{B}}_r = \begin{pmatrix} \widehat{\boldsymbol{\alpha}}_r' \\ \widehat{\boldsymbol{\beta}}_r' \end{pmatrix} = \begin{pmatrix} \left(\bar{\mathbf{Y}}_r - \widehat{\boldsymbol{\beta}}_r\bar{x}_r\right)' \\ \left(\dfrac{\mathbf{Y}_r'\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r}{\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}\right)' \end{pmatrix},$$

where $\bar{\mathbf{Y}}_r = \mathbf{Y}_r'\mathbf{1}_{n_r}/n_r$ and $\bar{x}_r = \mathbf{x}_r'\mathbf{1}_{n_r}/n_r$, $r = 1, 2, \dots, R$. And

$$\mathbf{S_E} = \sum_{r=1}^{R} \mathbf{S}_{\mathbf{E}_r},$$

where

$$\mathbf{S}_{\mathbf{E}_r} = \mathbf{Y}_r'\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{Y}_r - \frac{\mathbf{Y}_r'\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\mathbf{x}_r'\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{Y}_r}{\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}.$$

Theorem 3.1. *Given the R multivariate simple linear models (3.1) and known constants a and b, the likelihood ratio test of size α of*

$$H_0 : a\boldsymbol{\alpha}_1 + b\boldsymbol{\beta}_1 = a\boldsymbol{\alpha}_2 + b\boldsymbol{\beta}_2 = \cdots = a\boldsymbol{\alpha}_R + b\boldsymbol{\beta}_R$$

versus

$$H_1 : \textit{ at least an equality is an inequality,}$$

is given by

$$\Lambda = \frac{|\mathbf{S}_E|}{|\mathbf{S}_E + \mathbf{S}_H|}, \tag{3.3}$$

where

$$\mathbf{S}_E = \sum_{r=1}^{R} \mathbf{Y}_r'(\mathbf{I}_{n_r} - \mathbf{X}_r\mathbf{X}_r^-)\mathbf{Y}_r, \tag{3.4}$$

$$\mathbf{S}_H = \left(\mathbf{D}^{1/2}\mathbf{Z}\right)'\left(\mathbf{I}_R - \mathbf{D}^{1/2}\mathbf{1}_R\mathbf{1}_R'\mathbf{D}^{1/2}/\mathbf{1}_R'\mathbf{D}\mathbf{1}_R\right)\left(\mathbf{D}^{1/2}\mathbf{Z}\right), \tag{3.5}$$

with $\mathbf{D} = \mathrm{diag}(d_{11}, d_{22}, \ldots, d_{RR})$,

$$d_{rr} = \frac{n_r\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}{\|(a\mathbf{x}_r - b\mathbf{1}_{n_r})\|^2}$$

and

$$\mathbf{Z} = \left(a\begin{pmatrix} \widehat{\boldsymbol{\alpha}}_1' \\ \widehat{\boldsymbol{\alpha}}_2' \\ \vdots \\ \widehat{\boldsymbol{\alpha}}_R' \end{pmatrix} + b\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1' \\ \widehat{\boldsymbol{\beta}}_2' \\ \vdots \\ \widehat{\boldsymbol{\beta}}_R' \end{pmatrix} \right) \in \mathfrak{R}^{R\times q}.$$

We reject H_0 if

$$\Lambda \leq \Lambda_{\alpha,1,\nu_{\mathbf{H}},\nu_{\mathbf{E}}},$$

where $\nu_{\mathbf{H}} = (R-1)$, $\nu_{\mathbf{E}} = N - 2R$.

Proof. This theorem is a special case of the results obtained for testing the hypotheses (2.3) and it can be proved by selecting the proper **C** and **M** matrices into Equation (2.4) [1]. Alternatively, we extend the proof in (Graybill , 1976, Theorem 8.6.1, p. 288) for a univariate case into the multivariate case. The result follows from (2.5), we just need to define explicit matrices of sums of squares and products $\mathbf{S}_E$ and $\mathbf{S}_H$. First define the random vectors $\mathbf{z}_r = a\widehat{\boldsymbol{\alpha}}_r + b\widehat{\boldsymbol{\beta}}_r$, $r = 1, 2, \dots, R$, where a and b are known constants to be defined later. Hence, given that $\widehat{\mathbf{B}}_r \sim \mathcal{N}_{2\times q}\left(\mathbf{B}_r, (\mathbf{X}_r'\mathbf{X}_r)^{-1} \otimes \boldsymbol{\Sigma}\right)$, we have

$$\mathrm{E}(\mathbf{z}_r) = \mathrm{E}\left(a\widehat{\boldsymbol{\alpha}}_r + b\widehat{\boldsymbol{\beta}}_r\right) = a\boldsymbol{\alpha}_r + b\boldsymbol{\beta}_r.$$

Also note that,

$$\mathbf{z}_r = \widehat{\mathbf{B}}_r'\begin{pmatrix} a \\ b \end{pmatrix} = a\widehat{\boldsymbol{\alpha}}_r + b\widehat{\boldsymbol{\beta}}_r,$$

thus

$$\begin{aligned}
\mathrm{Cov}(\mathbf{z}_r) &= \mathrm{Cov}(\mathbf{z}_r) = \mathrm{Cov}\left(\mathrm{vec}\,\widehat{\mathbf{B}}_r'\begin{pmatrix} a \\ b \end{pmatrix}\right) \\
&= \mathrm{Cov}\left(\left(\begin{pmatrix} a \\ b \end{pmatrix}' \otimes \mathbf{I}_q\right)\mathrm{vec}\,\widehat{\mathbf{B}}_r'\right) \\
&= \left((a, b) \otimes \mathbf{I}_q\right)\left((\mathbf{X}_r'\mathbf{X}_r)^{-1} \otimes \boldsymbol{\Sigma}\right)\left(\begin{pmatrix} a \\ b \end{pmatrix} \otimes \mathbf{I}_q\right) \\
&= (a, b)\,(\mathbf{X}_r'\mathbf{X}_r)^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \otimes \boldsymbol{\Sigma} \\
&= d_{rr}^{-1} \otimes \boldsymbol{\Sigma} = d_{rr}^{-1}\boldsymbol{\Sigma},
\end{aligned}$$

[1] In our case taking, $\mathbf{M} = \mathbf{I}_q$ and

$$\mathbf{C} = \begin{pmatrix} a & b & -a & -b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & -a & -b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & b & -a & -b \end{pmatrix} \in \Re^{R-1\times 2R},$$

into to Equation 2.4, the desired result is obtained.

with

$$\begin{aligned}
d_{rr}^{-1} &= (a,b)\,(\mathbf{X}_r'\mathbf{X}_r)^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \\
&= (a,b)\begin{pmatrix} \|\mathbf{1}_{nr}\|^2 & \mathbf{1}_{nr}'\mathbf{x}_r \\ \mathbf{x}_r'\mathbf{1}_{nr} & \|\mathbf{x}_r\|^2 \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{1}{n_r\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}(a,b)\begin{pmatrix} \|\mathbf{x}_r\|^2 & -\mathbf{1}_{nr}'\mathbf{x}_r \\ -\mathbf{x}_r'\mathbf{1}_{nr} & n_r \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{\|'(a\mathbf{x}_r - b\mathbf{1}_{n_r})\|^2}{n_r\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}.
\end{aligned}$$

Therefore,

$$\mathbf{z}_r = a\widehat{\boldsymbol{\alpha}}_r + b\widehat{\boldsymbol{\beta}}_r \sim \mathcal{N}_q\left(a\boldsymbol{\alpha}_r + b\boldsymbol{\beta}_r, d_{rr}^{-1}\boldsymbol{\Sigma}\right).$$

Now, consider the random matrix $\mathbf{Z}$ defined by

$$\mathbf{Z} = \begin{pmatrix} \mathbf{z}_1' \\ \mathbf{z}_2' \\ \vdots \\ \mathbf{z}_R' \end{pmatrix} = \left(a\begin{pmatrix} \widehat{\boldsymbol{\alpha}}_1' \\ \widehat{\boldsymbol{\alpha}}_2' \\ \vdots \\ \widehat{\boldsymbol{\alpha}}_R' \end{pmatrix} + b\begin{pmatrix} \widehat{\boldsymbol{\beta}}_1' \\ \widehat{\boldsymbol{\beta}}_2' \\ \vdots \\ \widehat{\boldsymbol{\beta}}_R' \end{pmatrix}\right) \in \Re^{R\times q}.$$

Thus

$$\mathrm{E}(\mathbf{Z}) = \left(a\begin{pmatrix} \boldsymbol{\alpha}_1' \\ \boldsymbol{\alpha}_2' \\ \vdots \\ \boldsymbol{\alpha}_R' \end{pmatrix} + b\begin{pmatrix} \boldsymbol{\beta}_1' \\ \boldsymbol{\beta}_2' \\ \vdots \\ \boldsymbol{\beta}_R' \end{pmatrix}\right).$$

and

$$\mathrm{Cov}(\mathrm{vec}\,\mathbf{Z}') = \mathrm{Cov}((\mathbf{z}_1', \mathbf{z}_2', \ldots \mathbf{z}_R')') = \mathbf{D}^{-1} \otimes \boldsymbol{\Sigma},$$

where $\mathbf{D} = \operatorname{diag}(d_{11}, d_{22}, \dots, d_{RR})$. Thus

$$\mathbf{Z} \sim \mathcal{N}_{R\times q}\left(a \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1' \\ \widetilde{\boldsymbol{\alpha}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_R' \end{pmatrix} + b \begin{pmatrix} \widetilde{\boldsymbol{\beta}}_1' \\ \widetilde{\boldsymbol{\beta}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\beta}}_R' \end{pmatrix}, \mathbf{D}^{-1} \otimes \boldsymbol{\Sigma} \right).$$

furthermore

$$\mathbf{D}^{1/2}\mathbf{Z} \sim \mathcal{N}_{R\times q}\left(\mathbf{D}^{1/2} \left(a \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1' \\ \widetilde{\boldsymbol{\alpha}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_R' \end{pmatrix} + b \begin{pmatrix} \widetilde{\boldsymbol{\beta}}_1' \\ \widetilde{\boldsymbol{\beta}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\beta}}_R' \end{pmatrix} \right), \mathbf{I}_R \otimes \boldsymbol{\Sigma} \right).$$

Consider the constant matrix $\left(\mathbf{I}_R - \mathbf{D}^{1/2}\mathbf{1}_R\mathbf{1}_R'\mathbf{D}^{1/2}/\mathbf{1}_R'\mathbf{D}\mathbf{1}_R\right)$, which is symmetric and idempotent. Then

$$\mathbf{S_H} = \left(\mathbf{D}^{1/2}\mathbf{Z}\right)' \left(\mathbf{I}_R - \mathbf{D}^{1/2}\mathbf{1}_R\mathbf{1}_R'\mathbf{D}^{1/2}/\mathbf{1}_R'\mathbf{D}\mathbf{1}_R\right)\left(\mathbf{D}^{1/2}\mathbf{Z}\right),$$

moreover, $\mathbf{S_H}$ has a Wishart distribution and is independently distributed of $\mathbf{S_E}$ (see Equation (3.2)), where $\mathbf{S_H} \sim \mathcal{W}_q(R-1, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$ and $\mathbf{S_E} \sim \mathcal{W}_q(N-2R, \boldsymbol{\Sigma})$; in addition,

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\Sigma}^{-1}\left(\mathbf{D}^{1/2}\operatorname{E}(\mathbf{Z})\right)' \left(\mathbf{I}_R - \mathbf{D}^{1/2}\mathbf{1}_R\mathbf{1}_R'\mathbf{D}^{1/2}/\mathbf{1}_R'\mathbf{D}\mathbf{1}_R\right)\left(\mathbf{D}^{1/2}\operatorname{E}(\mathbf{Z})\right),$$

and observe that $\boldsymbol{\Omega} = \mathbf{0}$ if an only if $a\boldsymbol{\alpha}_1 + b\boldsymbol{\beta}_1 = a\boldsymbol{\alpha}_2 + b\boldsymbol{\beta}_2 = \cdots = a\boldsymbol{\alpha}_R + b\boldsymbol{\beta}_R$. Which is the desired result. □

As we mentioned before, different test statistics have been proposed for verifying the hypothesis (2.3). Next, we propose three of them in our particular case.

Theorem 3.2. *Given the R multivariate simple linear models (3.1) and known constants a and b, the union-intersection test, Pillai test and Lawley-Hotelling test of size α of*

$$H_0 : a\boldsymbol{\alpha}_1 + b\boldsymbol{\beta}_1 = a\boldsymbol{\alpha}_2 + b\boldsymbol{\beta}_2 = \cdots = a\boldsymbol{\alpha}_R + b\boldsymbol{\beta}_R$$

versus

$$H_1: \text{ at least ane equality is an inequality,}$$

are given respectively by

1.
$$\theta_1 = \frac{\lambda_1}{1+\lambda_1}, \tag{3.6}$$

which is termed Roy's largest root test. Where λ_1 *is the maximum eigenvalue of* $\left(\mathbf{S}_H\mathbf{S}_E^{-1}\right)$, *where* $\mathbf{S}_H$ *and* $\mathbf{S}_E$ *are given by (3.5) and (3.4), respectively. We reject* H_0 *if* $\theta \geq \theta_{\alpha,s,m,h}$. *Exact critical values of* $\theta_{\alpha,s,m,h}$ *are found in Table A.10 of Rencher (2002) or Tables 2, 4 and 5 of Kres (1983).*

2.
$$V^{(s)} = \text{tr}[\mathbf{S}_H(\mathbf{S}_E + \mathbf{S}_H)^{-1}] = \sum_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} = \sum_{i=1}^{s} \theta_i. \tag{3.7}$$

This way we reject H_0 *if*
$$V^{(s)} \geq V^{(s)}_{\alpha,s,m,h},$$
where the exact critical values of $V^{(s)}_{\alpha,s,m,h}$ *are found in Table A.11 of Rencher (2002) or Table 7 of Kres (1983).*

3.
$$U^{(s)} = \text{tr}[\mathbf{S}_H\mathbf{S}_E^{-1}] = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \frac{\theta_i}{1-\theta_i}. \tag{3.8}$$

We reject H_0 *if*
$$U^{(s)} \geq U^{(s)}_{\alpha,s,m,h}.$$
The upper percentage points, $U^{(s)}_{\alpha,s,m,h}$, *are given in Table 6 of Kres (1983).*

The parameters s, m and h are defined as

$$s = \min(1, \nu_{\mathbf{H}}), \quad m = (|1-\nu_{\mathbf{H}}| - 1)/2, \quad h = (\nu_{\mathbf{E}} - 2)/2.$$

where $\nu_{\mathbf{H}} = (R-1)$, $\nu_{\mathbf{E}} = N - 2R$ *and* $N = \sum_{r=1}^{R} n_r$.

As a special case of Theorem 3.1 (and Theorem 3.2), we obtain the test of the hypotheses **i**), **ii**) and **iii**) established above.

Theorem 3.3. *Consider the R multivariate simple linear models (3.1). The likelihood ratio test of size α of tests of hypotheses **i**), **ii**) and **iii**) are given as follows:*

The test of H_0 vs. H_1 is this: Reject H_0 if and only if

$$\Lambda = \frac{|\mathbf{S}_E|}{|\mathbf{S}_E + \mathbf{S}_H|} \leq \Lambda_{\alpha,1,\nu_{\mathbf{H}},\nu_{\mathbf{E}}},$$

where

$$\mathbf{S}_E = \sum_{r=1}^{R} \mathbf{Y}_r'(\mathbf{I}_{n_r} - \mathbf{X}_r\mathbf{X}_r^-)\mathbf{Y}_r, \tag{3.9}$$

$$\mathbf{S}_H = \left(\mathbf{D}^{1/2}\mathbf{Z}\right)'\left(\mathbf{I}_R - \mathbf{D}^{1/2}\mathbf{1}_R\mathbf{1}_R'\mathbf{D}^{1/2}/\mathbf{1}_R'\mathbf{D}\mathbf{1}_R\right)\left(\mathbf{D}^{1/2}\mathbf{Z}\right). \tag{3.10}$$

i) *With $H_0 : \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \cdots = \boldsymbol{\alpha}_R$ (R set of lines with the same vector intercept) vs. $H_1 : \boldsymbol{\alpha}_i = \boldsymbol{\alpha}_j$ for at least one $i \neq j$, $i, j = 1, 2, \ldots, R$. Where $\mathbf{D} = \operatorname{diag}(d_{11}, d_{22}, \ldots, d_{RR})$,*

$$d_{rr} = \frac{n_r\|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2}{\|\mathbf{x}_r\|^2},$$

and

$$\mathbf{Z} = \begin{pmatrix} \widetilde{\boldsymbol{\alpha}}_1' \\ \widetilde{\boldsymbol{\alpha}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\alpha}}_R' \end{pmatrix} \in \Re^{R\times q}.$$

ii) *$H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_R$ (R set of lines are parallel) vs. $H_1 : \boldsymbol{\beta}_i = \boldsymbol{\beta}_j$ for at least one $i \neq j$, $i, j = 1, 2, \ldots, R$. With $\mathbf{D} = \operatorname{diag}(d_{11}, d_{22}, \ldots, d_{RR})$,*

$$d_{rr} = \|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}_{n_r}'/n_r\right)\mathbf{x}_r\|^2,$$

and

$$\mathbf{Z} = \begin{pmatrix} \widetilde{\boldsymbol{\beta}}_1' \\ \widetilde{\boldsymbol{\beta}}_2' \\ \vdots \\ \widetilde{\boldsymbol{\beta}}_R' \end{pmatrix} \in \Re^{R\times q}.$$

iii) *$H_0 : \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1 x_0 = \boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2 x_0 = \cdots = \boldsymbol{\alpha}_R + \boldsymbol{\beta}_R x_0$ (all R set of lines intersect at $x = x_0$, known) vs. H_1 at least one equality is an inequality (all R set of lines do not intersect at $x = x_0$). Where $\mathbf{D} = \operatorname{diag}(d_{11}, d_{22}, \ldots, d_{RR})$,*

$$d_{rr} = \frac{n_r \|\left(\mathbf{I}_{n_r} - \mathbf{1}_{n_r}\mathbf{1}'_{n_r}/n_r\right)\mathbf{x}_r\|^2}{\|(\mathbf{x}_r - x_0\mathbf{1}_{n_r})\|^2},$$

and

$$\mathbf{Z} = \begin{pmatrix} \widetilde{\boldsymbol{\alpha}'_1} + \widetilde{\boldsymbol{\beta}'_1} x_0 \\ \widetilde{\boldsymbol{\alpha}'_2} + \widetilde{\boldsymbol{\beta}'_2} x_0 \\ \vdots \\ \widetilde{\boldsymbol{\alpha}'_R} + \widetilde{\boldsymbol{\beta}'_R} x_0 \end{pmatrix} \in \Re^{R\times q}.$$

Where $\nu_{\mathbf{H}} = (R-1)$, $\nu_{\mathbf{E}} = N - 2R$.

Proof. This is a simple consequence of Theorem 3.1. To test that a set of R lines have the same vector intercept, take $a = 1$ and $b = 0$; to test whether a set of R lines are parallel, we set $a = 0$ and $b = 1$, and to test that a set of R lines intersect at $x = x_0$, we set $a = 1$ and $b = x_0$.

□

3.1 Test about a Set of Multivariate Simple Linear Models under Matrix Multivariate Elliptical Model

In order to consider phenomena and experiments under more flexible and robust conditions than the usual normality, various works have appeared in the statistical literature since the 80's. Those efforts have been collected in various books and papers which are consolidated in the so-termed generalised multivariate analysis or multivariate statistics analysis under elliptically contoured distributions, see Gupta and Varga (1993) and Fang and Zhang (1990), among other authors. These new techniques generalize the classical matrix multivariate normal distribution by a robust family of matrix multivariate distributions with elliptical contours.

Recall that $\mathbf{Y} \in \Re^{n\times m}$ has a matrix multivariate elliptically contoured distribution if its density with respect to the Lebesgue measure is given by

$$dF_{\mathbf{Y}}(\mathbf{Y}) = \frac{1}{|\boldsymbol{\Sigma}|^{n/2}|\boldsymbol{\Theta}|^{m/2}} h\left\{\operatorname{tr}\left[(\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}\right]\right\}(d\mathbf{Y}),$$

where $\boldsymbol{\mu} \in \Re^{n\times m}$, $\boldsymbol{\Sigma} \in \Re^{m\times m}$, $\boldsymbol{\Theta} \in \Re^{n\times n}$, $\boldsymbol{\Sigma} > \mathbf{0}$ and $\boldsymbol{\Theta} > \mathbf{0}$ and $(d\mathbf{Y})$ is the Lebesgue measure. The function $h : \Re \to [0,\infty)$ is termed the generator function and satisfies $\int_0^\infty u^{mn-1}h(u^2)du < \infty$. Such a distribution is denoted by $\mathbf{Y} \sim \mathcal{E}_{n\times m}(\boldsymbol{\mu}, \boldsymbol{\Theta}\otimes\boldsymbol{\Sigma}, h)$, see Gupta and Varga (1993). Observe that this class of matrix multivariate distributions includes normal, contaminated normal, Pearson type II and VI, Kotz, logistic, power exponential, and so on; these distributions have tails that are weighted more or less, and/or they have a greater or smaller degree of kurtosis than the normal distribution.

Among other properties of this family of distributions, the invariance of some test statistics under this family of distributions stands out, that is, some test statistics have the same distribution under normality as under the whole family of elliptically contoured distributions, see theorems 5.3.3 and 5.3.4 of pp. 185-186 in Gupta and Varga (1993). Therefore, the distributions of Wilks, Roy, Lawley-Hotelling and Pillai test statistics are invariant under the whole family of elliptically contoured distributions, see (Gupta and Varga , 1993, pp. 297-299). As an immediate consequence, all hypothesis tests proposed in Theorems 3.1 - 3.3 are invariant under the family of matrix variate elliptical distributions. Therefore, no particular form of the function $h()$ must be defined when an application fits the conclusions based on the hypothesis tests proposed in this article.

Note that finally, in a multivariate linear model, it was assumed that the x's were fixed. However, in many applications, the x's are random variables. Then, as in the normal case, see Section 10.8, p. 358 in Rencher (2002), if we assume that $\left(y_1, y_2, \ldots, y_q, x\right)$ has a multivariate elliptically contoured distribution, then all estimations and tests have the same formulation as in the fixed-x case. Thus there is no essential difference in our procedures between the fixed-x case and the random-x case.

4 Application

The rosebush (*Rosa sp. L.*) is the ornamental species of major importance in the *State of Mexico, Mexico*, being the red spider (*Tetranychus urticae Koch*) (*Acari: Tetranychidae*) its main acarological problem, the control has been based almost exclusively using acaricide, which has caused this plague to acquire resistance in a short time. In order to counteract this problem in part, an experiment was carried out using the variety of *red petals Vega* in two greenhouses located in the *Ejido*[2] *"Los Morales", in Tenancingo, State of Mexico, Mexico*, from October 2008 to August 2009. In a greenhouse, chemical control was applied exclusively, while in the other, combined control (chemical and biological) was used, where applications of acaricide were reduced and releases of two predatory mites were made: *Phytoseiulus persimilis Athias-Henriot* and *Neoseiulus californicus McGregor* (*Acari: Phytoseiidae*). The red spider infestations decrease the length of the stem (Y_1) and the size of the floral button (Y_2), preponderant characteristics so that the final product reaches the best commercial value, so that a total of 15 stems were measured randomly and weekly from each greenhouse, their respective floral button, to quantify their length and diameter in centimeters, respectively, for a total of 15 weeks (X), see Preciado–Ramírez (2014). The measurements of the variables were carried out from January to April 2009 and the application of the treatments was initiated in week 44 of 2008.

The investigator considers[3] that a multivariate simple linear model for the results of each greenhouse is the appropriate model to relate the two dependent variables Y_1 and Y_2 in terms of the independent variable X. The corresponding multivariate simple linear models are

$$\mathbf{Y}_r = \mathbf{X}_r\boldsymbol{\beta}_r + \boldsymbol{\epsilon}_r, \quad r = 1, 2,$$

and in terms of the Section 3.1 we can assume that: $\boldsymbol{\epsilon}_r \sim \mathcal{E}_{n_r\times 2}(\mathbf{0}, \mathbf{I}_{n_r} \otimes \boldsymbol{\Sigma}, h)$, $\boldsymbol{\Sigma} \in \Re^{2\times 2}$, $\boldsymbol{\Sigma} > \mathbf{0}$, with $n_1 = 15$, and $n_2 = 15$ and

$$\boldsymbol{\beta}_r = \begin{pmatrix} \alpha_{r1} & \alpha_{r2} \\ \beta_{r1} & \beta_{r2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}'_r \\ \boldsymbol{\beta}'_r \end{pmatrix}.$$

The researcher asks for the following hypotheses testing.

i) $H_{01} : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$, that is, the set of lines are parallel (if the average stem length and the average floral button diameter of each sample of roses per week are the same under the two methods of pest control).

ii) $H_{02} : \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$, that is, the set of lines have a common vector intercept (if the average

[2]A piece of land farmed communally, pasture land, other uncultivated lands, and the fundo legal, or townsite, under a system supported by the state.

[3]In the original work, the analysis was made based on univariate statistical techniques only.

stem length and the average floral button diameter of each sample roses in week zero are the same under the two methods of pest control).

The results of the experiment are presented in the next Table 1.

Table 1: Experimental results of length of the stem (cms) and the diameter of the floral button (cms) of Vega rose variety.

Biological control			Chemical control		
X	Y_1	Y_2	X	Y_1	Y_2
1	67.32	4.87	1	55.74	4.82
2	68.92	4.89	2	58.63	4.97
3	69.33	5.07	3	61.14	5.01
4	71.66	5.19	4	62.46	5.06
5	72.26	5.26	5	62.96	5.13
6	76.55	5.73	6	64.55	5.22
7	81.41	5.82	7	66.87	5.28
8	82.71	6.09	8	67.93	5.34
9	83.09	6.15	9	68.38	5.37
10	83.59	6.17	10	68.88	5.39
11	83.91	6.24	11	69.76	5.40
12	84.67	6.30	12	71.31	5.42
13	85.34	6.33	13	72.98	5.54
14	87.41	6.61	14	74.33	5.65
15	88.21	6.62	15	76.44	5.74

Thus the matrices $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and $\mathbf{S}_E$ are given by

$$\boldsymbol{\beta}_1 = \begin{pmatrix} 66.521429 & 4.75238095 \\ 1.571321 & 0.13378571 \end{pmatrix}, \quad \boldsymbol{\beta}_2 = \begin{pmatrix} 56.416286 & 4.83647619 \\ 1.300964 & 0.05660714 \end{pmatrix},$$

and

$$\mathbf{S}_E = \begin{pmatrix} 65.625451 & 3.9069754 \\ 3.906975 & 0.3025506 \end{pmatrix}.$$

Moreover,

i) from the Part (ii) of Theorem 3.3, we have

$$\mathbf{S}_H = \begin{pmatrix} 10.233018 & 2.9212089 \\ 2.921209 & 0.8339145 \end{pmatrix}, \quad \text{and}$$

Table 2: Four criteria to prove $H_{01} : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$.

Criteria	Statistic	α Critical value
Wilks[a]	0.1159631	0.860199
Roy	0.8840369	0.775
Pillai	0.8840369	0.775
Lawley-Hotelling	7.62343	4.225201[b]

[a]Remember that for this tests, the decision rule is: statistic ≤ critical value
[b]Using an F approximation, see equation (6.26) in (Rencher, 2002, p.166).

Thus, from Table 2, there is no doubt that the four criterions reject the null hypothesis $H_{01} : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ for $\alpha = 0.05$.

ii) Similarly, from the Part (i) of Theorem 3.3, the matrix $\mathbf{S_H}$ is given by

$$\mathbf{S}_H = \begin{pmatrix} 172.934851 & -1.43916793 \\ -1.439168 & 0.01197679 \end{pmatrix},$$

and

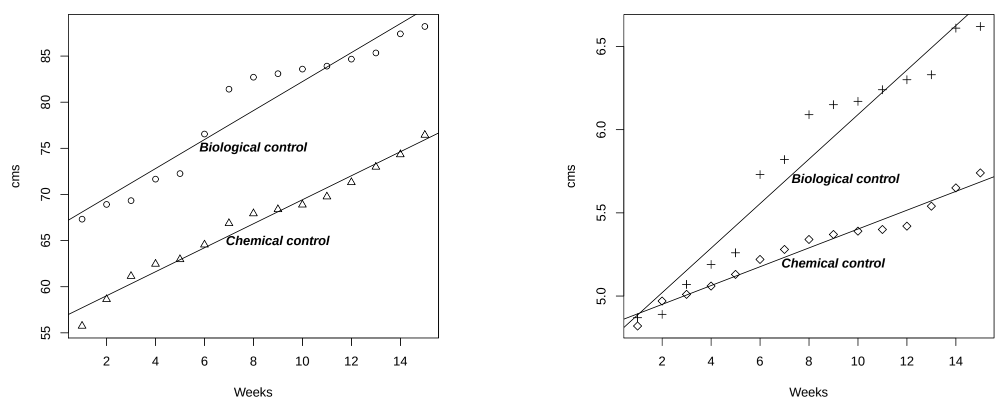
Table 3: Four criteria to prove $H_{02} : \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$.

Criteria	Statistic	α Critical value
Wilks[a]	0.06658425	0.860199
Roy	0.9334158	0.808619
Pillai	0.9334158	0.808619
Lawley-Hotelling	14.01857	4.225201[b]

[a]Remember that for these tests, the decision rule is: statistic ≤ critical value
[b]Using an F approximation, see equation (6.26), p. 166 in Rencher (2002).

From Table 3 we can conclude that under the four criterions of test the hypothesis $H_{02} : \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$ is rejected for a level of significance of $\alpha = 0.05$.

Given that $R = 2$, we can easily check graphically the conclusions reached in the hypothesis testing. Figure 1 (b) shows the intersection of lines for the floral button diameters, which explains the rejection of parallelism hypothesis. However, Figure 1(a) shows parallel lines, which certainly implies that the average length of the stem for each sample per week is the same for both pest controls. Similarly, Figure 1(a) depicts very different intercepts associated to the length with the stem, explaining the rejecting of the hypothesis for equal intercepts. Also, Figure 1(b) shows equal intercepts, which implies that the average floral button diameter for each sample in week zero is the same for both pest controls.



(a) Length of the stem (b) Diameter of the floral button

Figure 1: Observations and adjusted values.

The thesis Preciado–Ramírez (2014) concludes that the biological control method reduces the infestation of the pest and as a consequence both the stem length and the button size are increased. This aspect promotes a higher sale price, but this result was not incorporated in the addressed work. Our analysis confirms these conclusions, but in a robust way that include all the decision variables simultaneously.

5 Conclusions

As a consequence of Subsection 3.1 the three hypotheses testing of this paper are valid under the complete family of elliptically contoured distribution, i.e. in any practical circumstance we can assume that our information has a matrix multivariate elliptically contoured distribution instead of considering the usual nonrealistic normality.

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