JIRSS (2019)
Vol. 18, No. 02, pp 1-20
DOI:10.29252/jirss.18.2.1

Goodness–of–Fit Tests for Birnbaum–Saunders Distributions

Saeed Darijani[1] **, Hojatollah Zakerzadeh** [1]**, and Hamzeh Torabi**[1]

[1] Department of Statistics, Yazd University, Yazd, Iran.

Received: 09/03/2018, Revision received: 26/12/2018, Published online: 29/08/2019

Abstract. Goodness-of-fit tests are constructed for the two-parameter Birnbaum-Saunders distribution in the case where the parameters are unknown and therefore are estimated from the data. In each test, the procedure starts by computing efficient estimators of the parameters. Then the data are transformed by a normal transformation and normality tests are applied on the transformed data, thereby avoiding reliance on parametric asymptotic critical values or the need for bootstrap computations. Three classes of tests are considered, the first class being the classical tests based on the empirical distribution function, while the other class utilizes the empirical characteristic function and the final class utilizes the Kullback-Leibler information function. All methods are extended to cover the case of generalized three-parameter Birnbaum-Saunders distributions.

Keywords. Birnbaum-Saunders, Entropy, Monte-Carlo Methods, Test of Birnbaum-Saunders, Test Power.

MSC: 62E15; 62G30.

Saeed Darijani (Saeed_darijani@yahoo.com)
Corresponding Author: Hojatollah Zakerzadeh (hzaker@yazd.ac.ir)
Hamzeh Torabi (htorabi@yazd.ac.ir)

1 Introduction

The well-known Birnbaum-Saunders (BS) distribution, introduced by Birnbaum and Saunders (1969a,b), has recently considered as a popular model in the area of reliability and life distributions. An important motivation of considering BS distribution is that it can be used to model the lifetime of materials exposed to a cyclic stress pattern where ultimate failure is due to the growth of a dominant crack. The main assumptions of deriving BS distribution are based on the cumulative damage process, see Owen (2006) for more details. However, these assumptions do not always hold and nobody can guarantee that. In this situation, robust estimation methods for computing parameter estimates and quantiles Dupuis and Mills (1998) or model selection or goodness-of-fit tests may be used. Even though the parameter estimation for the BS distribution was considered by researchers (Xu and Tang, 2009; Cysneiros *et al.*, 2008; Lemonte *et al.*, 2007; Leiva *et al.*, 2008; Sanhueza *et al.*, 2008), goodness-of-fit testing for this distribution has been done only in Meintanis (2010).

Here, some tests of fit for the two-parameter BS distribution and for its generalizations with extra shape parameters have been considered. We concentrate consistent procedures for any arbitrary deviations from the BS distribution. To make a good comparison with a more recent test based on the Kullback-Leibler information function of the data, we use a test based on the empirical characteristic function of the data and the classical test statistics which is computed via the empirical distribution function. The proposed tests can also test the normality of transformed data. This transformation was originally suggested by Chen and Balakrishnan (1995) and was used by Meintanis (2009) to perform well.

The rest of the paper is organized as follows. Section 2 reviews the BS distribution and some of its properties. Section 3 is devoted to the extension of the methods to certain generalized BS distribution. This section also reviews the goodness-of-fit test for BS distribution. In Section 5, we propose a new statistics test for BS distribution based on normality test. The proposed procedures are compared via Monte Carlo techniques and real-data examples in Section 6 and Section 7, respectively. Some concluding remarks are finally given in Section 8.

2 The Birnbaum-Saunders Distribution

Obtained on the basis of the standard normal distribution, a random variable (r.v.) X has the BS distribution with shape parameter $\gamma > 0$ and scale parameter $\beta > 0$ if it admits the following stochastic representation

$$X \stackrel{d}{=} \frac{\beta}{4}\left[\gamma Z_1 + \sqrt{(\gamma Z_1)^2 + 4}\right]^2, \tag{2.1}$$

where $\stackrel{d}{=}$ means equal in distribution, $Z_1 \sim N(0,1)$. Hereafter, we will write $X \sim BS(\gamma, \beta)$. In this distribution, the parameter β is the median. It can be easily shown that

$$Z_1 \stackrel{d}{=} \frac{1}{\gamma}\left[\sqrt{\frac{X}{\beta}} - \sqrt{\frac{\beta}{X}}\right]. \tag{2.2}$$

The probability density function (*PDF*) and the cumulative distribution function (*CDF*) of X is

$$f_0(x;\gamma,\beta) = \phi\left(a(x;\gamma,\beta)\right) A(x;\gamma,\beta) \qquad x > 0, \gamma > 0, \beta > 0,$$

and

$$F_0(x;\gamma,\beta) = \Phi\left(a(x;\gamma,\beta)\right), \tag{2.3}$$

respectively, where $\phi(.)$ is the standard normal density, $\Phi(.)$ standard normal CDF,

$$a(x;\gamma,\beta) = \frac{1}{\gamma}\left[\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right], \text{ and } A(x;\gamma,\beta) = \frac{\partial a(x;\gamma,\beta)}{\partial x} = \frac{x+\beta}{2\gamma\sqrt{\beta x^3}}.$$

3 Generalized Birnbaum-Saunders Distributions

Instead of stochastic relation (2.2), Owen (2006) derived a generalized form of BS distribution (GBS) which is obtained by solving the equation $\sqrt{\beta} Z X^{\kappa} - X + \beta = 0$, where $Z \sim N(0, \gamma^2)$. This new relation is used to generate a random number from GBS distribution. Owen showed that the CDF of GBS can be written as

$$F(x;\gamma,\beta,\kappa) = \Phi\left[\frac{1}{\gamma}\left(\frac{x^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{x^{\kappa}}\right)\right], \tag{3.1}$$

$\gamma, \beta > 0$ and $0 < \kappa < 1$. As an especial case, $\kappa = 1/2$ corresponds to the original two-parameter BS distribution. It can be easily seen that although the maximum likelihood estimator (MLE) of the parameter γ has a closed form $\hat{\gamma} = \sqrt{(n\beta)^{-1}\sum_{j=1}^{n}\omega_j^2}$, the MLEs of (β, κ) are found numerically (Owen, 2006) from the equations

$$\sum_{j=1}^{n}(\beta - X_j)\nu_j - \sum_{j=1}^{n}\log X_j + \frac{n\sum_{j=1}^{n}\omega_j^2 X_j^{-1}\log X_j}{\sum_{j=1}^{n}\omega_j^2} = 0,$$

$$\kappa\sum_{j=1}^{n}\nu_j + \frac{n\sum_{j=1}^{n}\omega_j X_j^{-1}}{\sum_{j=1}^{n}\omega_j^2} = 0,$$

where $\omega_j = (X_j - \beta)/X_j^{\kappa}$ and $\nu_j = ((1-\kappa)X_j + \beta\kappa)^{-1}$.

As an alternative way of extending BS distribution to the three-parameter families, we consider the generalized BS distribution with CDF

$$F(x;\gamma,\beta,\kappa) = \Phi\left[\frac{1}{\gamma}\left\{\left(\frac{x}{\beta}\right)^{\kappa} - \left(\frac{\beta}{x}\right)^{\kappa}\right\}\right]. \tag{3.2}$$

Again, the two-parameter BS distribution is an especial case of (3.2) with $\kappa = 1/2$. Moreover, random numbers from this distribution can be easily obtained as $X \stackrel{d}{=} \beta(1 + 2Z^2 + 2Z\sqrt{1+Z^2})$ where $Z \sim N(0, \gamma^2/4)$. To compute the ML estimates of unknown parameters involved in (3.2), we construct the log-likelihood function, omitting additive constants, as

$$\ell(\gamma,\beta,\kappa) = n\log\left(\frac{\kappa}{\gamma}\right) - \sum_{j=1}^{n}\log X_j + \sum_{j=1}^{n}\omega_j - \frac{1}{2\gamma^2}\sum_{j=1}^{n}(\omega_j^2 - 4), \tag{3.3}$$

where $\omega_j = (X_j/\beta)^{\kappa} + (\beta/X_j)^{\kappa}$. Maximizing (3.3) over γ leads to the estimator

$$\hat{\gamma} = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(\omega_j^2 - 4)}. \tag{3.4}$$

Now, by substituting (3.4) in (3.3), we obtain average profile likelihood as

$$\bar{\ell}(\gamma,\beta,\kappa) = \log\left(\frac{\kappa}{\gamma}\right) + \frac{1}{n}\sum_{j=1}^{n}\omega_j.$$

So, the MLEs of β and κ are found by solving the following equations which are obtained by computing the derivatives of $\bar{\ell}$ with respect to β and κ, respectively,

$$\sum_{j=1}^{n}\left(\omega_j - \frac{\hat{\gamma}_n^2}{\omega_j}\right)\nu_j = 0, \quad \text{and} \quad \kappa\sum_{j=1}^{n}\left(\omega_j - \frac{\hat{\gamma}_n^2}{\omega_j}\right)\log\left(\frac{X_j}{\beta}\right) = n\hat{\gamma}_n^2, \tag{3.5}$$

where $\nu_j = (X_j/\beta)^{\kappa} - (\beta/X_j)^{\kappa}$. However, the equations in Eq. (3.5) always have a solution when $\kappa \to 0$. It means that there is a critical problem to obtain the estimate of the parameter via maximizing of the profile likelihood. *i.e*, we are unable to compute (γ, β) estimation from the profile likelihood for any κ.

4 Classical and Characteristic Function Statistics Test for Birnbaum-Saunders Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous density function, $f(x)$, with positive support. Our hypothesis of interest is

$$H_0: \ f(x) = f_0(x) \ \ vs \ \ H_1: \ f(x) \neq f_0(x).$$

Denote the order statistics of the random sample by $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The classical test statistic is defined as some measure of CDF distance $F_n(x) - F_0(x, \hat{\gamma}, \hat{\beta})$ where $F_n(x) = n^{-1}\sum_{i=1}^{n}\{\#X_i \leq t\}$ represents the empirical CDF, $F_0(., \hat{\gamma}, \hat{\beta})$ is the CDF fitted to the data in which $(\hat{\gamma}, \hat{\beta})$ denotes an efficient estimator of (γ, β) (the MLEs) introduced in (2.3).

In particular, if we put $\hat{\mathrm{U}}_i = F_0(X_{(i)}, \hat{\gamma}, \hat{\beta})$, the popular Kolmogorov-Smirnov (KS) statistic is defined by

$$KS = \max\{D^+, D^-\},$$

where $\mathrm{D}^- = \max_{1\leq i\leq n}\left[\frac{i}{n} - \hat{\mathrm{U}}_i\right]$ and $\mathrm{D}^+ = \max_{1\leq i\leq n}\left[\hat{\mathrm{U}}_i - \frac{i-1}{n}\right]$. Also, the Cramer-von Mises (CM) and Anderson-Darling (AD) test statistics tests can be respectively obtained as

$$\mathrm{CM} = \frac{1}{12n} + \sum_{i=1}^{n}\left(\hat{\mathrm{U}}_i - \frac{2i-1}{2n}\right)^2,$$

$$\mathrm{AD} = -n - \frac{1}{n}\sum_{i=1}^{n}\Big((2i-1)\log\hat{\mathrm{U}}_i + (2(n-i)+1\log(1-\hat{\mathrm{U}}_i)\Big).$$

Since the characteristic function (CF), $\varphi(t) = E(e^{itX})$, is uniquely corresponded to the law of the random variable X, an alternative goodness-of-fit statistic can be defined as a functional of $\varphi_n(t) - \hat{\varphi}(t)$, where $\varphi_n(t) = n^{-1}\sum_{j=1}^{n} e^{itX_j}$ is the empirical CF and $\hat{\varphi}(t) = \varphi_0(t;\hat{\gamma},\hat{\beta})$ is the fitted CF. As a result, Meintanis (2010) introduced a test statistic via using a parametric transformation which renders the original observations approximately normally distributed. The Meintanis statistics is

$$\begin{aligned} T_\eta &= n\int_{-\infty}^{\infty} |\varphi_n(t) - e^{-t^2/2}| e^{-\eta t^2}\mathrm{dt} \\ &= \sqrt{\frac{\pi}{\eta}}\frac{1}{n}\sum_{j,k=1}^{n} e^{-(Z_j - Z_k)^2/4\eta} + n\sqrt{\frac{\pi}{1+\eta}} - 2\sqrt{\frac{2\pi}{1+2\eta}}\sum_{j=1}^{n} e^{-Z_j^2/(2+4\eta)}, \end{aligned}$$

with $\eta > 0$, where replacing $\varphi_n(t)$ by $n^{-1}\sum_{j=1}^{n} e^{itZ_j}$, $\varphi(t)$ by $e^{-t^2/2}$ and Z_j is obtained by the following steps:

i: Efficiently estimate (γ, β) by $(\hat{\gamma}, \hat{\beta})$.

ii: Compute $\hat{\mathrm{U}}_j = F_0(X_{(j)}; \hat{\gamma}, \hat{\beta})$.

iii: Compute $\mathrm{Y}_j = \Phi^{-1}(\hat{\mathrm{U}}_j)$ and then $\bar{\mathrm{Y}} = n^{-1}\sum_{i=1}^{n} \mathrm{Y}_i$, and $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(\mathrm{Y}_i - \bar{\mathrm{Y}})^2$, and

$$Z_j = \frac{\mathrm{Y}_j - \bar{\mathrm{Y}}}{S}, \quad j = 1, \ldots, n.$$

5 New Test Statistics for Birnbaum-Saunders Distribution

As a measure of the divergence between two PDF $f(x)$ and $f_0(x)$, the Kullback-Leibler (K-L) information function is defined as

$$\begin{aligned} I(f, f_0) &= \int_{-\infty}^{\infty} f(x)\log\left(\frac{f(x)}{f_0(x)}\right)\mathrm{dx} \\ &= -H(f) - \int_{-\infty}^{\infty} f(x)\log\left(f_0(x)\right)\mathrm{dx}. \end{aligned}$$

where $H(f)$ is the entropy of the random variable X, obtained by Shannon (1948), evaluated by

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log\left(f(x)\right)\mathrm{dx}.$$

Having the property that $I(f, f_0) \geq 0$, it can be expressed as a new goodness-of-fit statistic in the form of a function of $\widehat{I(f, f_0)} = -\widehat{H(f)} - \int_{-\infty}^{\infty}\widehat{f(x)}\log\left(f_0(x)\right)\mathrm{dx}$, where $\widehat{H(f)}$ is a non-parametric estimation of $H(f)$ and $\int_{-\infty}^{\infty}\widehat{f(x)}\log\left(f_0(x)\right)\mathrm{dx}$ is the fitted $\int_{-\infty}^{\infty} f(x)\log\left(f_0(x)\right)\mathrm{dx}$ incorporating estimates of the parameters. However, for the classical, the CF-based and K-L tests alike, the distribution of each test statistic depends on the distribution being tested, the estimators $\hat{\gamma}$ and $\hat{\beta}$ employed as well as on the unknown true value of these parameters. Moreover, the asymptotic null distribution is highly non-standard and the calculation of large-sample percentage points, if feasible at all, is by itself an arduous task. In order to remove this drawback, we propose (refer to Chen and Balakrishnan (1995) and Meintanis (2009)) to efficiently estimate the parameters of the BSD based on the original data and apply a parametric transformation which renders the original observations approximately normally distributed. Then, a goodness-of-fit test for normality is computed based on the transformed observations. The advantage of reducing the problem of testing the null hypothesis H_0 to a normality test lies in the fact that for normality testing, efficient estimators are readily available, and that critical points are independent of these estimators and the true parameter values.

In particular, we can go through the following steps:

1. **Estimation**: Estimate (γ, β) via $(\hat{\gamma}, \hat{\beta})$, efficiently.

2. **Transformation**: In this step, the data is transformed to approximately normally distributed under the null hypothesis H_0 by computing $\mathrm{Y}_i = \left[\sqrt{T_i/\hat{\beta}} - \sqrt{\hat{\beta}/T_i}\right]/\hat{\gamma}$.

3. **Testing:** Finally, compute $Z_i = \frac{\mathrm{Y}_i - \bar{\mathrm{Y}}}{S}$ for all $i = 1, \ldots, n$, where $\bar{\mathrm{Y}} = n^{-1}\sum_{i=1}^{n}\mathrm{Y}_i$ and $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(\mathrm{Y}_i - \bar{\mathrm{Y}})^2$. This step will bring us to the classical tests for normality incorporating the standardized observations Z_j, and the standard normal CDF, $\Phi(.)$.

By replacing $H(f)$ by $\widehat{H(f)}$ and f_0 by the density function of the standard normal distribution, $e^{-x^2/2}\Big/\sqrt{2\pi}$, we can obtain the test statistic as

$$T = -\widehat{H(f)} - \int_{-\infty}^{\infty} f(x)\log\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)\mathrm{dx}.$$

Under the null hypothesis, H_0, f is the density function of the standard normal distribution, the test statistic is obtained by

$$T = -\widehat{H(f)} + \log\,\sqrt{2\pi e}.$$

It is obvious that if the sample comes from a non-normal distribution, the value of T gets large. On the other hand, if we consider the following monotone transformation

$$KL = \exp(-T) = \exp(\widehat{H(f)} - \log\,\sqrt{2\pi e}) = \frac{\exp(\widehat{H(f)})}{\sqrt{2\pi e}},$$

the small values of KL represent the non-normal distribution of the sample. As another test of normality, Vasicek (1975) proposed the following test statistics

$$KL_{mn} = \frac{\exp\{HV_{mn}\}}{\sqrt{2\pi e}},$$

where

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}(Z_{(i+m)} - Z_{(i-m)})\right\},$$

is Vasicek entropy estimator such that the window size m is a positive integer smaller than $n/2$, $Z_{(i)} = Z_{(1)}$ if $i < 1$, $Z_{(i)} = Z_{(n)}$ if $i > n$ and $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are order statistics obtained from a random sample of size n. For the population entropy $H(f)$, Vasicek also showed that HV_{mn} is consistent. For testing normality, Van Es (1992) also proposed the following test statistics, based on improved or modified versions of the Vasicek entropy estimator as

$$TVE_{mn} = \frac{\exp\{HVE_{mn}\}}{\sqrt{2\pi e}},$$

where

$$HVE_{mn} = \frac{1}{n-m}\sum_{i=1}^{n-m}\left\{\frac{n+1}{m}(Z_{(i+m)} - Z_{(i)})\right\} + \sum_{k=m}^{n}\frac{1}{k} + \log(m) - \log(n+1).$$

An alternative test statistic of normality was proposed by Correa (1995). He considered a sample entropy as

$$HC_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}\left(Z_{(j)}-\bar{Z}_{(i)}\right)(j-i)}{n\sum_{j=i-m}^{i+m}\left(Z_{(j)}-\bar{Z}_{(i)}\right)^2}\right),$$

where

$$\bar{Z}_{(}i) = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} Z_{(j)}.$$

Therefore, the test statistic based on HC_{mn} is obtained by

$$TC_{mn} = \frac{\exp\{HC_{mn}\}}{\sqrt{2\pi e}}.$$

As an extension of the test statistic based on the sample entropy, Ebrahimi *et al.* (1994) proposed a modified sample entropy as

$$HE_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{c_i m}(Z_{(i+m)} - Z_{(i-m)})\right\},$$

where

$$c_i = \begin{cases} 1+\frac{i-1}{m} & \text{if } 1 \le i \le m \\ 2 & \text{if } m+1 \le i \le n-m \\ 1+\frac{n-i}{m} & \text{if } n-m+1 \le i \le 2 \end{cases}$$

They showed that $HE_{mn} \xrightarrow{Pr} H(f)$ as $n, m \longrightarrow \infty$ in such a way that $m/n \longrightarrow 0$. The following test statistic based on HE_{mn} for testing of normality can be considered

$$TE_{mn} = \frac{\exp\{HE_{mn}\}}{\sqrt{2\pi e}}.$$

By considering $nh\widehat{f(Z_i)} = \sum_{j=1}^{n} k\left(\frac{Z_i - Z_j}{h}\right)$, Alizadeh Noughabi (2010) defined an estimator of entropy given by

$$HA_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{f(\widehat{Z_{(i+m)}}) + f(\widehat{Z_{(i-m)}})}{2}\right\},$$

where the kernel function is chosen to be the standard normal density function and the bandwidth $h = 1.06sn^{1/5}$, where s is the sample standard deviation. Also $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. He concluded that $HA_{mn} \xrightarrow{Pr} H(f)$ as $n, m \longrightarrow \infty$ such that $m/n \longrightarrow 0$.

Here for testing normality, we propose the following test statistic, based on HA_{mn} as

$$TA_{mn} = \frac{\exp\{HA_{mn}\}}{\sqrt{2\pi e}}.$$

To perform the tests, an efficient estimator of the parameters of the BS distribution is needed. Let $\bar{X}_n = n^{-1}\sum_{j=1}^{n} X_j$, $\bar{H}_n = (n^{-}1\sum_{j=1}^{n} X_j^{-1})^{-1}$ and $W(\beta) = \left[n^{-1}\sum_{j=1}^{n}(\beta + X_j)^{-1}\right]$. Then, the efficient estimator of β can be found by solving the equation

$$\beta^2 - \beta\left[2\bar{H}_n + W(\beta)\right] + \bar{H}_n\left[\bar{X}_n + W(\beta)\right] = 0, \tag{5.1}$$

and the estimator of γ is found as

$$\hat{\gamma}_n = \sqrt{\frac{\bar{X}_n}{\hat{\beta}_n} + \frac{\hat{\beta}_n}{\bar{X}_n} - 2}.$$

Due to the existence and uniqueness problems of classical moment estimation, Ng *et al.* (2003) proposed the modified moment estimators as

$$\hat{\beta} = \sqrt{\bar{X}_n\bar{H}_n}, \qquad \hat{\gamma}_n = \sqrt{\bar{X}_n\sqrt{\frac{\bar{X}_n}{\bar{H}_n}} - 1}.$$

By recalling that β is the median of $BS(\gamma, \beta)$, we have also tried a mixed quantile-likelihood estimator where $med_{1\le j\le n}X_j$ (the sample median) and $\hat{\gamma}_n$ is found by replacing β by $\hat{\beta}$. Each test statistic was implemented by employing all three methods of estimation. The results however did not vary considerably among estimation methods and therefore in the next section results will be reported only for the tests based on the MLEs.

6 Simulation Study

In this section, we compare the power of the proposed test with other tests under several alternative distributions and for sample sizes $n = 10$ and 20. We show that our proposed test achieves higher power compared with the tests based on TV_{mn}, TC_{mn}, TE_{mn}, TVE_{mn} and TA_{mn} statistics.

6.1 Critical Values

Since the distributions of the entropy estimates HV_{mn}, HVE_{mn}, HE_{mn}, HC_{mn} and HA_{mn} cannot be directly obtained under the null hypothesis, the Monte Carlo method is conducted to compute critical values $TV_{mn,\alpha}$, $TVE_{mn,\alpha}$, $TE_{mn,\alpha}$, $TC_{mn,\alpha}$ and $TA_{mn,\alpha}$ of the test statistics TV_{mn}, TVE_{mn}, TE_{mn}, TC_{mn} and TA_{mn}, respectively, for the most common significance level $\alpha = 0.05$. More precisely, for each test statistic $T_{mn} \in \{TV_{mn}, TVE_{mn}, TE_{mn}, TC_{mn}, TA_{mn}\}$, we calculate its sample values $t_{mn,1}, t_{mn,2}, \ldots, t_{mn,10{,}000}$ in 10,000 simulated random samples of size n from the Birnbaum-Saunders distribution with $\gamma = 1$ and $\beta = 1$. Since $\alpha = 0.05 = 500/10,000$, we evaluated the 500th order statistic $t_{mn,(500)}$ and specified the critical value $T_{mn,0.05}$ from the equation $T_{mn,0.05} = t_{mn,(500)}$. The critical values obtained in this manner for the statistics TV_{mn}-TA_{mn} and sample sizes $5 \leq n \leq 50$ are presented in Tables 1 to 5. All calculations of our study are done by using a program written in R software which is available from the authors upon request.

Table 1: Critical values of the TV_{mn} statistic for $\alpha = 0.05$.

	m							
n	1	2	3	4	5	6	7	8
5	0.163	0.193						
6	0.198	0.236	0.225					
7	0.230	0.266	0.258					
8	0.265	0.304	0.297	0.281				
9	0.294	0.336	0.329	0.314				
10	0.315	0.365	0.359	0.343	0.326			
15	0.397	0.459	0.461	0.452	0.439	0.422	0.405	
20	0.450	0.523	0.533	0.528	0.516	0.503	0.489	
25	0.491	0.570	0.582	0.580	0.574	0.564	0.552	
30	0.517	0.600	0.617	0.619	0.614	0.604	0.596	
40	0.556	0.644	0.668	0.674	0.673	0.669	0.663	
50	0.579	0.673	0.700	0.708	0.711	0.709	0.706	
75	0.616	0.716	0.747	0.759	0.765	0.767	0.767	0.766

Table 2: Critical values of the TE_{mn} statistic for $\alpha = 0.05$.

	m							
n	1	2	3	4	5	6	7	8
5	0.214	0.281						
6	0.256	0.326	0.349					
7	0.286	0.360	0.378					
8	0.319	0.392	0.412	0.420				
9	0.344	0.421	0.440	0.450				
10	0.359	0.439	0.463	0.471	0.477			
15	0.443	0.528	0.553	0.560	0.566	0.567	0.568	
20	0.484	0.574	0.603	0.615	0.621	0.623	0.625	0.627
25	0.518	0.615	0.647	0.660	0.667	0.671	0.673	0.676
30	0.541	0.638	0.669	0.685	0.694	0.698	0.702	0.704
40	0.570	0.671	0.707	0.725	0.736	0.743	0.749	0.753
50	0.598	0.702	0.737	0.755	0.767	0.774	0.780	0.785
75	0.631	0.740	0.777	0.797	0.809	0.818	0.824	0.829

Table 3: Critical values of the TVE_{mn} statistic for $\alpha = 0.05$.

	m							
n	1	2	3	4	5	6	7	8
5	0.323	0.356						
6	0.364	0.393	0.393					
7	0.388	0.411	0.409					
8	0.420	0.448	0.445	0.442				
9	0.448	0.466	0.457	0.454				
10	0.465	0.491	0.481	0.472	0.473			
15	0.550	0.570	0.557	0.545	0.537	0.531	0.530	
20	0.596	0.616	0.601	0.587	0.574	0.565	0.558	0.555
25	0.637	0.652	0.639	0.625	0.613	0.603	0.594	0.587
30	0.659	0.676	0.667	0.654	0.641	0.629	0.619	0.611
40	0.703	0.714	0.704	0.691	0.679	0.668	0.658	0.649
50	0.734	0.748	0.738	0.726	0.713	0.703	0.692	0.684
75	0.782	0.797	0.787	0.777	0.766	0.754	0.744	0.737

Table 4: Critical values of the TC_{mn} statistic for $\alpha = 0.05$.

	m							
n	1	2	3	4	5	6	7	8
5	0.200	0.245						
6	0.237	0.286	0.281					
7	0.274	0.335	0.330					
8	0.309	0.365	0.366	0.353				
9	0.339	0.404	0.408	0.387				
10	0.363	0.424	0.424	0.411	0.395			
15	0.454	0.536	0.538	0.526	0.514	0.501	0.486	
20	0.515	0.606	0.614	0.604	0.594	0.582	0.569	0.556
25	0.556	0.648	0.659	0.655	0.646	0.637	0.626	0.615
30	0.576	0.680	0.695	0.696	0.690	0.682	0.673	0.663
40	0.624	0.725	0.745	0.746	0.742	0.737	0.731	0.725
50	0.647	0.757	0.776	0.779	0.779	0.776	0.771	0.767
75	0.687	0.802	0.826	0.831	0.831	0.830	0.828	0.826

Table 5: Critical values of the TA_{mn} statistic for $\alpha = 0.05$.

	m							
n	1	2	3	4	5	6	7	8
5	0.213	0.334						
6	0.250	0.368	0.448					
7	0.288	0.405	0.477					
8	0.313	0.433	0.502	0.567				
9	0.332	0.453	0.520	0.576				
10	0.358	0.474	0.539	0.589	0.643			
15	0.439	0.557	0.614	0.661	0.699	0.738	0.779	
20	0.489	0.607	0.659	0.698	0.733	0.765	0.796	0.828
25	0.514	0.631	0.686	0.722	0.754	0.783	0.810	0.837
30	0.538	0.656	0.708	0.742	0.772	0.797	0.823	0.846
40	0.572	0.688	0.739	0.774	0.802	0.824	0.846	0.867
50	0.593	0.713	0.761	0.791	0.816	0.838	0.858	0.876
75	0.629	0.743	0.790	0.819	0.841	0.860	0.876	0.891

6.2 Power Comparison

For studying the power of the tests defined in Section 5, we also use the Monte-Carlo simulations. To facilitate comparisons in this study, we select the $BS(\gamma, 1)$ distribution which is simply denoted by $BS(\gamma)$, and seven alternative distribution considered by Meintanis (2010) listed as:

1. The exponentioal ($E(\alpha)$) distribution with density $\alpha \exp\{-\alpha x\}$.
2. The gamma ($G(\alpha, 1)$) distribution with density $x^{\gamma-1} \exp\{-x\} \big/ \Gamma(\alpha)$.
3. The Weibull ($W(\gamma, 1)$) distribution with density $\gamma x^{\gamma-1} \exp\{-x^{\gamma}\}$.
4. The inverse-Gaussian ($IG(\gamma)$) distribution with density $\gamma \exp\{\frac{-(x-\gamma)^2}{2x}\} \big/ \sqrt{2\pi x^3}$.
5. The log-normal ($LN(\gamma)$) distribution with density $(\gamma x)^{-1}(2\pi)^{-1/2} \exp\{-(\log x)^2 \big/ 2\gamma^2\}$.
6. The shifted-Pareto distribution ($P(\gamma)$) with density $\gamma \big/ (1 + x)^{1+\gamma}$.
7. The half-normal (HN) distribution with density $\sqrt{2\big/\pi} \exp\{-\frac{x^2}{2}\}$.
8. The skew-normal Birnbaum-Saunders ($SN - BS(\lambda)$) introduced by Vilca *et al.* (2011) with density $\phi\left(a(x; \gamma, \beta)\right) \Phi\left(\lambda a(x; \gamma, \beta)\right) A(x; \gamma, \beta)$, $\gamma = 1$ and $\beta = 1$.

We compute the powers of the tests based on TV_{mn}, TVE_{mn}, TE_{mn}, TC_{mn}, TA_{mn}, CF and CDF statistics by means of a Monte Carlo simulation. Under each alternative, we generated 5,000 samples of size 25, 50 and 75. We evaluated for each sample and for several values of the parameter m the statistics (TV_{mn}, TVE_{mn}, TE_{mn}, TC_{mn}, TA_{mn}) and the power of the corresponding test were estimated by the frequency of "the event the statistic is smaller than the critical value". The power estimates are given in Tables 10-8.

7 Real Data Analysis

In this section we present two real data sets.

Table 6: Monte-Carlo power estimates of the tests against $\alpha = 0.05$ (m) with $n = 25$.

	TV_{mn}	TE_{mn}	TVE_{mn}	TC_{mn}	TA_{mn}
E(1)	0.6938(2)	0.6938(2)	0.6136(3)	0.6914(2)	0.6938(2)
E(2)	1.0000(1)	1.0000(1)	1.0000(2)	1.0000(1)	1.0000(1)
G(0.5)	0.9994(2)	0.9994(2)	0.9988(3)	0.9992(2)	0.9994(2)
G(1)	0.6842(2)	0.6842(2)	0.6068(3)	0.6844(2)	0.6842(2)
G(1.5)	0.1082(2)	0.1082(2)	0.0918(2)	0.1102(2)	0.1082(2)
W(1)	0.6910(2)	0.6910(2)	0.6098(3)	0.6870(2)	0.6910(2)
W(2)	1.0000(3)	1.0000(3)	1.0000(3)	1.0000(3)	1.0000(3)
W(3)	1.0000(3)	1.0000(3)	1.0000(3)	1.0000(3)	1.0000(3)
IG(0.25)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
IG(0.5)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
LN(1)	0.1208(2)	0.1208(2)	0.1234(4)	0.1206(1)	0.1208(2)
P(4)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
P(7)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
HN	0.9978(4)	0.9978(4)	0.0128(3)	0.9978(3)	0.9978(4)
SN-BS(-2)	0.9930(1)	0.9927(1)	0.9925(1)	0.9933(1)	0.9951(1)
SN-BS(2)	0.9902(1)	0.9900(1)	0.9895(1)	0.9906(1)	0.9915(1)

7.1 Example 1

The new procedures for these models are applied to sample (iii) of the 101 aluminum coupon failure data set in Table II of Birnbaum and Saunders (1969b). We examine whether a generalized BSD provides a better fit than the classical BSD employed by Birnbaum and Saunders (1969a).

As an indication of goodness-of-fit, we report in Table 9 the value of each test statistic. (Note that we are using the modified classical test statistics KS*, CM*, and AD*). The first five lines correspond to the BSD with CDF given by (3.2) fitted by the data for different values of κ. (This distribution is referred to as GBS2 in Table 9). These values indicate an improved fit with $\kappa < 1/2$. The corresponding MLEs are $\hat{\beta}_n = 1342.04$ and $\hat{\gamma}_n = 0.061$. A further improvement in fitting these data is shown in the line of Table 9 which corresponds to the BSD with CDF given by Eq. (3.1) (referred to as GBS1 in Table 9). The same conclusion is reached by looking at the corresponding p-values, which are reported in the last line of Table 9. According to p-values we conclude that test for BSD based on Kullback-Leibler information is better than other tests for BSD. The MLEs are $\hat{\beta}_n = 1391.5$, $\hat{\gamma}_n = 5.73$, and $\hat{\kappa}_n = 0.084$.

Table 7: Monte-Carlo power estimates of the tests against $\alpha = 0.05$ (m) with $n = 50$.

	TV_{mn}	TE_{mn}	TVE_{mn}	TC_{mn}	TA_{mn}
E(1)	0.9306(3)	0.9306(3)	0.8716(3)	0.9272(3)	0.9306(3)
E(2)	1.0000(1)	1.0000(1)	1.0000(2)	1.0000(1)	1.0000(1)
G(0.5)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
G(1)	0.9308(3)	0.9308(3)	0.8764(3)	0.9306(3)	0.9308(3)
G(1.5)	0.1590(3)	0.1590(3)	0.1182(3)	0.1574(3)	0.1590(3)
W(1)	0.9328(4)	0.9328(4)	0.8868(3)	0.9308(3)	0.9328(4)
W(2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
W(3)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
IG(0.25)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
IG(0.5)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
LN(1)	0.1208(2)	0.1208(2)	0.1234(4)	0.1206(1)	0.1208(2)
P(4)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
P(7)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
HN	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
SN-BS(-2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
SN-BS(2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)

Table 8: Monte-Carlo power estimates of the tests against $\alpha = 0.05$ (m) with $n = 75$.

	TV_{mn}	TE_{mn}	TVE_{mn}	TC_{mn}	TA_{mn}
E(1)	0.9874(4)	0.9874(4)	0.9686(3)	0.9864(4)	0.9874(4)
E(2)	1.0000(1)	1.0000(1)	1.0000(2)	1.0000(1)	1.0000(1)
G(0.5)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
G(1)	0.9858(4)	0.9858(4)	0.9650(4)	0.9856(3)	0.9858(4)
G(1.5)	0.1902(5)	0.1902(5)	0.1452(3)	0.1912(3)	0.1902(5)
W(1)	0.9870(6)	0.9870(6)	0.9674(4)	0.9674(4)	0.9870(6)
W(2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
W(3)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
IG(0.25)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
IG(0.5)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
P(4)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
P(7)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
LN(1)	0.1520(2)	0.1520(2)	0.1546(4)	0.1450(2)	0.1520(2)
HN	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
SN-BS(-2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)
SN-BS(2)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)	1.0000(1)

Table 9: Values of the test statistics and the corresponding p-values for Sample (iii).

$GBS2$	TV_{mn}	TE_{mn}	TVE_{mn}	TC_{mn}	TA_{mn}	AD^*	CM^*	KS^*
$\kappa = 0.60$	0.835(6)	0.831(8)	0.812(2)	0.872(4)	0.915(8)	0.705	0.106	0.752
$\kappa = 0.50$	0.847(6)	0.842(8)	0.821(2)	0.880(4)	0.921(8)	0.681	0.102	0.752
$\kappa = 0.40$	0.853(6)	0.851(8)	0.829(2)	0.886(4)	0.929(8)	0.661	0.100	0.752
$\kappa = 0.25$	0.861(6)	0.859(8)	0.836(2)	0.893(4)	0.933(8)	0.642	0.097	0.753
$\kappa = 0.10$	0.870(6)	0.865(8)	0.843(2)	0.903(4)	0.941(8)	0.632	0.095	0.752
$\kappa = 0.05$	0.878(6)	0.873(8)	0.855(2)	0.915(4)	0.950(8)	0.631	0.095	0.752
$GBS1$	0.821(5)	0.855(3)	0.845(3)	0.925(4)	0.961(8)	0.158	0.022	0.428
$p-value$	0.96	0.97	0.97	0.99	0.99	0.97	0.97	0.96

Table 10: Values of the test statistics and the corresponding p-values for the daily ozone data.

$GBS2$	TV_{mn}	TE_{mn}	TVE_{mn}	TC_{mn}	TA_{mn}	AD^*	CM^*	KS^*
$\kappa = 0.60$	0.916(2)	0.893(5)	0.919(6)	0.942(2)	0.881(3)	0.620	0.126	0.375
$\kappa = 0.50$	0.925(2)	0.899(5)	0.922(6)	0.953(2)	0.886(3)	0.615	0.131	0.372
$\kappa = 0.40$	0.933(2)	0.905(5)	0.925(6)	0.960(2)	0.890(3)	0.630	0.127	0.372
$\kappa = 0.25$	0.950(2)	0.915(5)	0.933(6)	0.967(2)	0.895(3)	0.617	0.126	0.380
$\kappa = 0.10$	0.943(2)	0.891(5)	0.927(6)	0.939(2)	0.864(3)	0.624	0.119	0.369
$\kappa = 0.05$	0.940(2)	0.830(5)	0.920(6)	0.945(2)	0.860(3)	0.623	0.117	0.368
$GBS1$	0.956(3)	0.920(2)	0.925(4)	0.960(3)	0.897(5)	0.639	0.118	0.355
$p-value$	0.98	0.98	0.96	0.99	0.99	0.95	0.96	0.95

7.2 Example 2

To illustrate the model proposed in this paper, we use the daily ozone concentrations data used by Vilca *et al.* (2011), provided by the New York State Department of Conservation. We assume that the data are uncorrelated and, therefore, a diurnal or cyclic trend analysis is not necessary. This assumption has been supported by several authors for different reasons, for example, environmental data are sometimes reported as averages and so there is no spatial-time dependence. We examine whether a BSD provides a better fit than the generalized BSD.

The MLEs for generalized BSD are $\hat{\beta}_n = 27.316$, $\hat{\gamma}_n = 0.919$, and $\hat{\kappa}_n = 0.193$. Also, the MLEs estimator for BSD are $\hat{\beta}_n = 28.023$ and $\hat{\gamma}_n = 0.982$. As an indication of goodness-of-fit, we report in Table 10 the value of each test statistic. According to p-values, we conclude test for BSD based on Kullback-Leibler information is better than other tests for BSD.

8 Conclusions

In this paper, we dealt with the classical and more recent goodness-of-fit tests for the BS distribution. By considering an accurate transformation incorporating parameter estimates, all proposed methods reduce to normality testing with estimated parameters. The results of our study show that the methods perform well with respect to power even with small sample sizes. Also, we have observed that under each test statistic converges relatively fast to the same asymptotic distribution regardless of the value of the shape parameter of the underlying BS distribution under the null hypothesis. An MC simulation study shows that the Anderson-Darling test and a test based on the empirical characteristic function are the most powerful for the alternatives considered. Moreover, The procedures are also extended to three parameter generalizations of the BS distribution. Finally, two real-data examples reveal that application of the methods to these extensions may result in an improved fit of certain three-parameter generalizations, compared to that obtained by the same data and the classical two-parameter BS distribution.

There are a number of avenues for future research. The BS distributions are lifetime models, in many applications their censored or truncated versions are of more interest. An extension for a test based on BS distribution is useful in such situations. Also, a straightforward extension is to consider bivariate and multivariate Birnbaum-Saunders distributions introduced by Kundu *et al.* (2010) and Kundu *et al.* (2013), respectively.

References

Alizadeh Noughabi, H. (2010), A new estimator of entropy and its application in testing normality. *Journal of Statistical Computation and Simulation*, **80**, 1151–1162.

Barros, M., Leiva, V. R., Ospina A., and Tsuyuguchi, A. (2014), Goodness-of-fit tests for the Birnbaum-Saunders distribution with censored reliability data. *IEEE Transactions on Reliability*, **63**(2), 543–554.

Birnbaum, Z. W. and Saunders, S. C. (1969a), A new family of life distributions. *Journal of Applied Probability*, **6**, 319–327.

Birnbaum, Z. W. and Saunders, S. C. (1969b), Estimation for a family of life distributions with applications to fatigue. *Journal of Applied Probability*, **6**, 328–347.

Chen, G. and Balakrishnan, N. (1995), A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, **27**, 154–161.

Correa, J. C. (1995), A new estimator of entropy. *Communications in Statistics–Theory and Methods*, **24**, 2439–2449.

Cysneiros, A., Cribari-Neto, F., and Araujo J. C. (2008), On Birnbaum–Saunders inference. *Computational Statistics and Data Analysis*, **52**, 4939–4950.

DÁgostino, C. and Stephens, M. A. (1986), *Goodness of Fit Techniques*. New York: Marcel Dekker, Inc.

Dunn, P. K. and Smyth, G. K. (1996), Randomized quantile residuals. *Journal of Computational and Graphical Statistics*, **5**, 236–244.

Dupuis, D. J. and Mills, J. E. (1998), Robust estimation of the Birnbaum–Saunders distribution. *IEEE Transactions on Reliability*, **47**, 88–95.

Ebrahimi, N., Pflughoeft, K., and Soofi, E. (1994), Two measures of sample entropy. *Statistics and Probability Letters*, **20**, 225–234.

Epps, T. W. and Pulley, L. B. (1983), A test for normality based on the empirical characteristic function procedures. *Biometrika*, **70**, 723–726.

Kundu, D., Balakrishnan, N., and Jamalizadeh, A. (2010), Bivariate Birnbaum–Saunders distribution and associated inference. *Journal of Multivariate Analysis*, **101**, 113–125.

Kundu, D., Balakrishnan, N., and Jamalizadeh, A. (2013), Generalized multivariate Birnbaum-Saunders distributions and related inferential issues. *Journal of Multivariate Analysis*, **116**, 230–244.

Leiva, V., Riquelme, M., Balakrishnan, N. and Sanhueza, A. (2008), Lifetime analysis based on the generalized Birnbaum–Saunders distribution. *Computational Statistics and Data Analysis*, **52**, 2079–2097.

Lemonte, A.J., Cribari-Neto, F., and Vasconcellos, K.L. (2007), Improved statistical inference for the two–parameter Birnbaum–Saunders distribution. *Computational Statistics and Data Analysis*, **51**, 4656–4681.

Meintanis, S. G. (2010), Inference procedures for the Birnbaum–Saunders distribution and its generalizations. *Computational Statistics and Data Analysis*, **54**, 367–373.

Meintanis, S. G., (2009), Goodness-of-fit testing by transforming to normality: Comparison between classical and characteristic function based methods. *Journal of Statistical Computation and Simulation*, **79**, 205–212.

Ng, H. K. T., Kundu, D., and Balakrishnan, N. (2003), Modified moment estimation for the two–parameter Birnbaum–Saunders distribution. *Computational Statistics and Data Analysis*, **43**, 283–298.

Owen, W. J. (2006), A new three–parameter extension to the Birnbaum–Saunders distribution. *IEEE Transactions on Reliability*, **55**, 475–479.

Sanhueza, A., Leiva, V., and Balakrishnan, N., (2008), The generalized Birnbaum–Saunders distribution and its theory, methodology and application. *Communications in Statistics–Theory and Methods*, **37**, 645–670.

Shannon, C.E. (1948), A mathematical theory of communication. *Bell System Technical Journal*, **27**, 379–423.

Van Es, B. (1992), Estimating functionals related to a density by class of statistics based on spacings. *Scandinavian Journal of Statistics*, **19**, 61–72.

Vasicek, O. (1975), A test for normality based on sample entropy. *Journal of the Royal Statistical Society series B*, **38**, 54–59.

Vilca, F., Santana, L., Leiva, V., and Balakrishnan, N. (2011), Estimation of extreme percentiles in Birnbaum–Saunders distributions. *Computational Statistics and Data Analysis*, **55**, 1665–1678.

Xu, A. and Tang, Y. (2009), Reference analysis for Birnbaum–Saunders distribution. *Computational Statistics and Data Analysis*, **5**, 185–192.