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Generalized Family of Estimators for Imputing Scrambled Responses

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Abstract. When there is a high correlation between the study and auxiliary variables, the rank of the auxiliary variable also correlates with the study variable. Then, the use of the rank as an additional auxiliary variable may be helpful to increase the efficiency of the estimator of the mean or the total number of the population. In the present study, we propose two generalized families of estimators for imputing the scrambling responses by using the variance and the rank of the auxiliary variable. Expressions for the bias and the mean squared error are obtained up to the first order of approximation. A numerical study is carried out to observe the performance of estimators.

Keywords. Higher order moments, Imputation, Missing data, Scrambled responses, Sensitive attribute.

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1 Introduction

In most of socio-economic surveys, missing value is a common problem that happens for many reasons, such as asking sensitive or embarrassing questions and unavailability of the respondent. Situations like asking a female 'how many times have you induced abortions?', questions related to income, tax evasion, etc., can be considered as sensitive questions. Initially, for these situations, Warner (1965) introduced the randomized response technique (RRT) to handle the problem of non-response relevant to the sensitive issues of the society. A randomized device was used to collect the responses from the respondents and protected their privacy despite of getting direct response. After Warner, various randomized response models were used for estimating the proportion of the nominal variable in the population(see e.g., Greenberg *et al.* (1969), Folsom *et al.* (1973), Mangat and Singh (1990), Gupta *et al.* (2010)).

Greenberg *et al.* (1971), Eichhorn and Hayre (1983), Mangat and Singh (1990), Gupta *et al.* (2013) considered the simple and multistage scrambling models (MSM) for estimating the mean of sensitive variables. Chaudhuri and Adhikary (1990) introduced a scrambling response model, where the j-th respondent in the sample should select two cards randomly and independently. The card S_{1j} is selected from a box containing m cards with a known mean (θ_1) and a known variance (σ_1^2). The second card S_{2j} is selected from t cards with a known mean (θ_2) and a known variance (σ_2^2). Let y_j be the actual status of the j-th respondent in the sample (s). The j-th respondent reports the scrambled value/response z_j as

$$z_j = S_{1j}y_j + S_{2j}. \tag{1.1}$$

In spite of using RRT, missing observations can occur due to reasons aside from sensitive issues. Then, how can one handle such incomplete data sets? In such circumstances, imputation is one of the most reliable methods to sort out such a problem and to build reliable data sets for a valid inference about the stated population. Many imputation techniques have been developed and considered the problem in different ways (see e.g., Rubin (1976), Heitjan and Basu (1996), Ahmed *et al.* (2006), Singh *et al.* (2010), Mohamed *et al.* (2017)). In Figure 1, we illustrate the imputation procedure for imputing the missing values.

Mohamed *et al.* (2017) considers an RRT, in which the j-th respondent is asked to select two cards (say S_1 and S_2) from two decks of cards (say Δ_1 and Δ_2), respectively.

The j-th respondent can report the scrambled response as

$$z_j = \frac{S_1 y_j + S_2 - \theta_2}{\theta_1}. \tag{1.2}$$

Let E_1 and V_1 be the expected value and variance with respect to the randomization device, respectively. We remark that all $E_1(S_1) = \theta_1$, $E_1(S_2) = \theta_2$, $V_1(S_1) = \sigma_1^2$, and $V_1(S_2) = \sigma_2^2$ are known, and we have $V_1(z_j) = \frac{\sigma_1^2 y_j^2 + \sigma_2^2}{\theta_1^2} = C_{\theta_1}^2 y_j^2 + (\frac{\theta_2}{\theta_1})^2 C_{\theta_2}^2$, where $C_{\theta_1}^2 = (\frac{\sigma_1}{\theta_1})^2$ and $C_{\theta_2}^2 = (\frac{\sigma_2}{\theta_2})^2$.

Let s be a simple random sample of size n drawn from the population (Ω) having N units, and let r be the total number of respondents in the subset A of the sample s that belongs to the sensitive characteristics with the help of the mentioned randomization scheme. Here, $(n - r)$ are those who belong to $A^{'}$, the subset of s, who refuse to answer the question, therefore, $s = A \cup A^{'}$. It is also known that $\bar{z}_r = \frac{1}{r} \sum_{j=1}^{r} z_j$ is the sample mean of the scrambling response obtained from the group A. Then, we have the lemma 1.1 as follows.

Lemma 1.1. *The variance of $\bar{z}_r$ is given by*

$$V(\bar{z}_r) \cong \left(\frac{1}{r} - \frac{1}{N}\right) S_y^2 + \frac{1}{r}\left[C_{\theta_1}^2 \bar{Y}^2 \left\{1 + \frac{(N-1)}{N} C_y^2\right\} + \left(\frac{\theta_2}{\theta_1}\right)^2 C_{\theta_2}^2\right], \tag{1.3}$$

where S_y^2, C_y^2, $C_{\theta_1}^2$ and other parameters are defined in Appendix.

Proof. See Mohamed *et al.* (2017) for proof. □

The rest of the article is outlined as follows. Two generalized families of estimators for imputing scrambling response are proposed by using higher order moments along with the rank of the auxiliary variable. Theoretical comparison of the proposed generalized imputation methods over the mean method of imputation is considered in Section 3. For evaluating the relative performance of the proposed generalized families, a numerical study is conducted for various choices of the constants in Section 4. Finally, Section 5 concludes the article.

2 Proposed Estimators

In survey sampling, it is a common practice to incorporate known auxiliary information in the estimation stage of population characteristics for improving efficiency. The use of the auxiliary information can increase the precision of the estimators both at estimation and design stages. The traditional ratio, product and classical regression estimators are used, when there is a high correlation between the study and the auxiliary variables. If the correlation between the study variable and the auxiliary variable is sufficiently large, then the rank of the auxiliary variable is also correlated with the study variable. The inclusion of the rank of a variable may help to improve the efficiency of the estimators. Let $\bar{y}$, $\bar{x}$ and $\bar{r}$ be the sample means and $\bar{Y}$, $\bar{X}$ and $\bar{R}$ be the population means of the study variable, the auxiliary variable, and the rank of the auxiliary variable, respectively. Let $s_y^2 = \sum_{j=1}^{n}(y_j - \bar{y})^2/(n-1)$, $s_{xn}^2 = \sum_{j=1}^{n}(x_j - \bar{x})^2/(n-1)$ and $s_r^2 = \sum_{j=1}^{n}(x_i - \bar{r})^2/(n-1)$ be the unbiased sample variances corresponding to the population variances $S_y^2 = \sum_{j=1}^{N}(y_i - \bar{Y})^2/(N-1)$, $S_x^2 = \sum_{j=1}^{N}(x_j - \bar{X})^2/(N-1)$, and $S_r^2 = \sum_{j=1}^{N}(r_j - \bar{R})^2/(N-1)$ of y, x, and r, respectively. We propose two generalized families of estimators for the imputation by using higher order moments and rank of the auxiliary variable. We consider ratio and regression types of estimators in the following subsections.

2.1 Generalized Ratio Type Estimators

The generalized ratio method of imputation for imputing scrambling response is

$$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right)^{g_1}\left(\frac{s_{xn}^2}{s_{xr}^2}\right)^{g_2}\left(\frac{\bar{r}_n}{\bar{r}_r}\right)^{g_3} - \frac{r\bar{z}_r}{(n-r)} & \text{if} j\epsilon A'. \end{cases} \tag{2.1}$$

The point estimator for the finite population mean is defined as

$$\begin{aligned} \bar{y}_{S.G_R} =& \frac{1}{n}\sum\nolimits_{j\epsilon s} z_j \\ =& \frac{1}{n}\Big[\sum\nolimits_{j\epsilon A} z_j + \sum\nolimits_{j\epsilon A'}\left\{\frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right)^{g_1}\left(\frac{s_{xn}^2}{s_{xr}^2}\right)^{g_2}\left(\frac{\bar{r}_n}{\bar{r}_r}\right)^{g_3} - \frac{r\bar{z}_r}{(n-r)}\right\}\Big] \\ =& \bar{z}_r\left(\frac{\bar{x}_n}{\bar{x}_r}\right)^{g_1}\left(\frac{s_n^2}{s_r^2}\right)^{g_2}\left(\frac{\bar{r}_n}{\bar{r}_r}\right)^{g_3}, \end{aligned} \tag{2.2}$$

where g_1, g_2 and g_3 are unknown constants and their values are to be determined by minimizing the mean squared error and ($\bar{x}_n$ and $\bar{x}_r$), ($s_{x_n}^2$ and $s_{x_r}^2$), and ($\bar{r}_n$ and $\bar{r}_r$) are the mean, variance and rank of the auxiliary variable for n and r units, respectively. We define the $\bar{y}_{s.G_R}$ in the form of error terms given in the Appendix as

$$\begin{aligned}
\bar{y}_{S.G_R} &= \bar{Y}(1+e_0)(1+e_2)^{g_1}(1+e_1)^{-g_1}(1+e_4)^{g_2}(1+e_3)^{-g_2}(1+e_7)^{g_3}(1+e_6)^{-g_3} \\
&= \bar{Y}\Big[\frac{g_1(g_1+1)}{2}e_1^2 + \frac{g_1(g_1-1)}{2}e_2^2 + \frac{g_3(g_3+1)}{2} + \frac{g_3(g_3-1)}{2}e_7^2 \\
&\quad + \frac{g_2(g_2+1)}{2}e_3^2 + \frac{g_2(g_2-1)}{2}e_4^2 - (g_1 e_0 e_1 + g_2 e_0 e_3 + g_3 e_0 e_6 \\
&\quad + g_1 g_2 e_1 e_3 + g_1 g_3 e_1 e_6 + g_2 g_3 e_3 e_6) - (g_1 e_2^2 + g_2 e_4^2 + g_3 e_7^2)\Big]. \qquad (2.3)
\end{aligned}$$

The bias and mean squared error of $\bar{y}_{S.G_R}$ are given by

$$\begin{aligned}
\text{Bias}(\bar{y}_{S.G_R}) &= \Big[\theta_{rN}\frac{g_1(g_1+1)}{2} + \theta_{nN}\frac{g_1(g_1-1)}{2}\Big]\bar{Y}C_x^2 + \Big[\theta_{rN}\frac{g_3(g_3+1)}{2} + \theta_{nN}\frac{g_3(g_3-1)}{2}\Big]\bar{Y}C_r^2 \\
&+ \Big[\theta_{rN}\frac{g_2(g_2+1)}{2} + \theta_{nN}\frac{g_2(g_2-1)}{2}\Big]\bar{Y}(\lambda_{040}-1) \\
&- \theta_{rn}\bar{Y}\Big[g_1\rho_{xy}C_yC_x + g_2C_y\lambda_{120} + g_3\rho_{ry}C_yC_r + g_1g_2C_x\lambda_{030} + g_1g_3\rho_{xr}C_xC_r + g_2g_3C_r\lambda_{003}\Big] \\
&- \theta_{nN}\bar{Y}\Big[g_1C_x^2 + g_2(\lambda_{040}-1) + g_3C_r^2\Big], \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
MSE(\bar{y}_{S.G_R}) &\cong \theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\Big[S_y^2 \\
&\quad + g_1^2Sx^2 + g_2^2S_x^4(\lambda_{040}-1) + g_3^2S_r^2 - 2\Big(g_1S_{xy} + g_2S_x^2S_y\lambda_{120} \\
&\quad + g_3S_{ry} - g_1g2S_x^3\lambda_{030} - g_1g_3S_{xr} - g_2g_3S_rS_x^2\lambda_{003}\Big)\Big]. \qquad (2.5)
\end{aligned}$$

The optimum values of g_1, g_2, and g_3 are obtained respectively, as

$$g_{1(opt.)} = \frac{w_1}{t_1} = G_1, \quad g_{2(opt.)} = \frac{w_2}{t_2} = G_2, \quad \text{and} \quad g_{3(opt.)} = \frac{w_3}{t_3} = G_3, \qquad (2.6)$$

where

$$
\begin{aligned}
w_2 &= S_x^2 S_y^2 \lambda_{120}(S_x^2 S_r^2 - S_{xr}^2) - S_{xy} S_x \lambda_{030}(S_x^2 S_r^2 - S_{xr}^2) - (S_{ry} S_x^2 - S_{xy} S_{rx}) \\
&\quad (S_x^2 S_r \lambda_{003} - S_x S_{xr} \lambda_{030}), \\
t_2 &= (S_x^2 S_r^2 - S_{xr}^2)(\lambda_{040} - 1) S_x^4 - (S_r S_x^2 \lambda_{003} - S_{rx} S_x^3 \lambda_{030})(S_r S_x^2 \lambda_{003} - S_{rx} S_x \lambda_{030}), \\
w_3 &= S_{ry} S_x^2 t_2 - S_{xr} S_{xy} t_2 - w_2 (S_r S_x^2 \lambda_{003} - S_{rx} S_x^3 \lambda_{030}), \\
t_3 &= t_2 (S_x^2 S_r^2 - S_{xr}^2), \\
w_1 &= S_{xy} t_2 t_3 - S_{xr} t_2 w_3 - w_2 t_3 S_x^3 \lambda_{030} \text{ and } t_1 = t_2 t_3 S_x^2.
\end{aligned}
$$

Substituting the optimum values of g_q for $q = 1, 2, 3$ in (2.5), we get the minimum mean squared error of $\bar{y}_{S.G_R}$ as

$$
\begin{aligned}
MSE(\bar{y}_{S.G_R})_{min.} \cong\ & \theta_{n,N} S_y^2 + \frac{1}{r}\left[\bar{Y}^2 C_{\theta_1}^2 \left\{1 + \frac{(N-1)}{N} C_y^2\right\} + \frac{\theta_2^2}{\theta_1^2} C_{\theta_2}^2\right] \\
& + \theta_{r,n}\Big[S_y^2 + G_1^2 S_x^2 + G_2^2 S_x^4(\lambda_{040} - 1) + G_3^2 S_r^2 \\
& - 2G_1 S_{xy} - 2G_2 S_x^2 S_y \lambda_{120} - 2G_3 S_{ry} + 2G_1 G_2 S_x^3 \lambda_{030} \\
& + 2G_1 G_3 S_{xr} + 2G_2 G_3 S_r S_x^2 \lambda_{003}\Big]. \qquad (2.7)
\end{aligned}
$$

Some members of the proposed family of ratio type estimators for different choices of g_q for $q = 1, 2, 3$ are given in Table 1. The biases and mean squared errors of $\bar{y}_{S.k}$ for $(k = 2, 3, \cdots, 8)$ up to the first order approximation are given in Tables 2 and 3, respectively. Similarly, mean squared error of the $\bar{y}_{s.k}$ are given in Table 3.

2.2 Generalized Difference Method of Imputation

The generalized regression method of imputation with known regression coefficients is given by

$$
\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + d_1\beta_1(x_i - \bar{x}_r) + d_2\beta_2\Big[\frac{n(x_j - \bar{x}_n)^2}{(n-1)} + \frac{n\sum_{j\epsilon A}(x_j - \bar{x}_n)^2}{(n-r)(n-1)} & \\ -\frac{n\sum_{j\epsilon A}(x_j - \bar{x}_r)^2}{(n-r)(r-1)}\Big] + d_3\beta_3(r_j - \bar{r}_r) & \text{if } j\epsilon A'. \end{cases} \qquad (2.8)
$$

Table 1: Some members of the proposed ratio type estimators.

k	g_1	g_2	g_3	$\bar{y}_{S.k}$	**Imputation Method**
1	0	0	0	$\bar{y}_r.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r & \text{if } j\epsilon A' \end{cases}$
2	1	0	0	$\bar{z}_r \frac{\bar{x}_n}{\bar{x}_r}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
3	0	1	0	$\bar{z}_r \frac{s_n^2}{s_r^2}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{s_n^2}{s_r^2}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
4	0	0	1	$\bar{z}_r \frac{\bar{r}_n}{\bar{r}_r}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{r}_n}{\bar{r}_r}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
5	1	1	0	$\bar{z}_r \frac{\bar{x}_n}{\bar{x}_r}\frac{s_n^2}{s_r^2}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right)\left(\frac{s_n^2}{s_r^2}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
6	1	0	1	$\bar{z}_r \frac{\bar{x}_n}{\bar{x}_r}\frac{\bar{r}_n}{\bar{r}_r}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right)\left(\frac{\bar{r}_n}{\bar{r}_r}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
7	0	1	1	$\bar{z}_r \frac{s_n^2}{s_r^2}\frac{\bar{r}_n}{\bar{r}_r}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{s_n^2}{s_r^2}\right)\left(\frac{\bar{r}_n}{\bar{r}_r}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$
8	1	1	1	$\bar{z}_r \frac{\bar{x}_n}{\bar{x}_r}\frac{s_n^2}{s_r^2}\frac{\bar{r}_n}{\bar{r}_r}.$	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \frac{n\bar{z}_r}{(n-r)}\left(\frac{\bar{x}_n}{\bar{x}_r}\right)\left(\frac{s_n^2}{s_r^2}\right)\left(\frac{\bar{r}_n}{\bar{r}_r}\right) - \frac{r\bar{z}_r}{(n-r)} & \text{if } j\epsilon A' \end{cases}$

The point estimator of the population mean is given by

$$\begin{aligned} \bar{y}_{S.G_{Re}} &= \frac{1}{n}\sum\nolimits_{j\epsilon s} z_j \\ &= \frac{1}{n}\Big[\sum\nolimits_{j\epsilon A} z_j + \sum\nolimits_{j\epsilon A'} \Big\{\bar{z}_r + d_1\beta_1(x_i - \bar{x}_r) + d_2\beta_2\Big[\frac{n(x_j - \bar{x}_n)^2}{(n-1)} \end{aligned}$$

$$+\frac{n\sum_{j\epsilon A}(x_j-\bar{x}_n)^2}{(n-r)(n-1)}-\frac{n\sum_{j\epsilon A}(x_j-\bar{x}_r)^2}{(n-r)(n-1)}\Bigg]+d_3\beta_3(r_j-\bar{r}_r)\Big\}\Big]$$
$$=\bar{z}_r+d_1\beta_1(\bar{x}_n-\bar{x}_r)+d_2\beta_2(s^2_{xn}-s^2_{xr})+d_3\beta_3(\bar{r}_n-\bar{r}_r), \quad (2.9)$$

where d_1, d_2 and d_3 are the unknown constants and their values are to be determined by minimizing the mean squared error. β_1, β_2 and β_3 are the population constants which are supposed to be known in advance.

In terms of errors (see Appendix), we have

$$\bar{y}_{S.G_{Re}} = \bar{Y}(1+e_0)+d_1\beta_1\bar{X}(e_2-e_1)+d_2\beta_2S^2_x(e_4-e_3)+d_3\beta_3\bar{R}(e_7-e_6),$$

and

$$MSE(\bar{y}_{S.G_{Re}}) \cong \theta_{n,N}S^2_y+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}\left\{1+\frac{(N-1)C^2_y}{N}\right\}+\frac{\theta^2_2}{\theta^2_1}C^2_{\theta_2}\right]+\theta_{r,n}\Big[S^2_y$$
$$+d^2_1\beta^2_1S^2_x+d^2_2\beta^2_2S^4_x(\lambda_{040}-1)+d^2_3\beta^2_3S^2_r-2\Big(d_1\beta_1S_{xy}+d_2\beta_2S^2_xS_y\lambda_{120}$$
$$+d_3\beta_3S_{ry}-d_1d_2\beta_1\beta_2S^3_x\lambda_{030}-d_1d_3\beta_1\beta_3S_{xr}-d_2d_3\beta_2\beta_3S_rS^2_x\lambda_{003}\Big)\Big]. \quad (2.10)$$

Table 2: Bias of k^{th}-ratio estimator.

k	Bias$(\bar{y}_{S.k})$
2	$\theta_{r,n}\bar{Y}\Big(C^2_x-\rho_{xy}C_xC_y\Big)$.
3	$\theta_{r,n}\bar{Y}\Big(\lambda_{040}-1-C_y\lambda_{120}\Big)$.
4	$\theta_{r,n}\bar{Y}\Big(C^2_r-\rho_{ry}C_rC_y\Big)$.
5	$\theta_{r,n}\bar{Y}\Big[C^2_x+(\lambda_{040}-1)-C_x\lambda_{030}-C_y\lambda_{120}-\rho_{xy}C_xC_y\Big]$.
6	$\theta_{r,n}\bar{Y}\left(C^2_x+C^2_r-\rho_{xy}C_xC_y-\rho_{ry}C_rC_y\right)$.
7	$\theta_{r,n}\bar{Y}\Big[C^2_r+(\lambda_{040}-1)-C_r\lambda_{003}-C_y\lambda_{120}-\rho_{ry}C_rC_y\Big]$.
8	$\theta_{rn}\bar{Y}\Big[C^2_x+(\lambda_{040}-1)+C^2_r-C_x\lambda_{030}-C_r\lambda_{003}-\rho_{ry}C_y-C_y\lambda_{120}+\rho_{xr}C_xC_r\Big]$.

Table 3: Mean squared error of the proposed estimators.

k	$MSE(\bar{y}_{s.k})$
1	$\theta_{r,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right].$
2	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\left(S_y^2+R^2S_x^2-2RS_{xy}\right).$
3	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\left(S_y^2+\bar{Y}^2(\lambda_{040}-1)-2\bar{Y}S_y\lambda_{120}\right).$
4	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\left(S_y^2+R'^2S_r^2-2R'S_{ry}\right).$
5	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\Big[S_y^2+\bar{Y}^2(\lambda_{040}-1)+R^2S_x^2-$ $2RS_{xy}-2R\bar{Y}S_y\lambda_{120}+2R\bar{Y}S_x\lambda_{030}\Big].$
6	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\Big(S_y^2+R^2S_x^2+R'^2S_r^2-2RS_{xy}-$ $2R'S_{ry}+2RR'S_{xr}\Big).$
7	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\Big(S_y^2+\bar{Y}^2(\lambda_{040}-1)+R'^2S_r^2-$ $2R'S_{ry}-2\bar{Y}S_y\lambda_{120}+2R'\bar{Y}S_r\lambda_{003}\Big).$
8	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\Big(S_y^2+R^2S_x^2+R'^2S_r^2-2RS_{xy}-$ $2\bar{Y}S_y\lambda_{120}-2R'S_{ry}+2R\bar{Y}S_x\lambda_{030}+2RR'S_{xr}+2R'\bar{Y}S_r\lambda_{003}\Big).$

The optimum values of d_1, d_2 and d_3 are obtained by minimizing (2.10), respectively, by

$$d_{1(opt.)}=\frac{b_1}{b_{11}}=F_1,\quad d_{2(opt.)}=\frac{b_2}{b_{22}}=F_2,\quad \text{and}\quad d_{3(opt.)}=\frac{b_3}{b_{33}}=F_3, \tag{2.11}$$

where

$$b_1 \;=\; S_x^2\Big\{\Big(\lambda_{120}S_xS_y\lambda_{030}+S_{xy}(\lambda_{003}^2-\lambda_{040}+1)\Big)S_r^2$$

$$
\begin{aligned}
& -\lambda_{003}\Big(S_{ry}S_x\lambda_{030}+S_{xr}S_y\lambda_{120}\Big)S_r+S_{ry}S_{xr}(\lambda_{040}-1)\Big\},\\
b_{11} &= S_{xy}\Big\{S_r(\lambda_{003}^2+\lambda_{030}^2-\lambda_{040}+1)S_x^2-2S_rS_xS_{xr}\lambda_{030}\lambda_{003}+S_{xr}^2(\lambda_{040}-1)\Big\},\\
b_2 &= S_x(\lambda_{040}-1)^2\Big\{S_x^2S_r\Big(S_rS_y\lambda_{120}-S_{ry}\lambda_{003}\Big)-\lambda_{030}(S_rS_{xy}-S_{ry}S_{xr})\\
& +S_{xr}\Big(S_rS_{xy}\lambda_{003}-S_{xr}S_y\lambda_{120}\Big)\Big\},\\
b_{22} &= S_x^2S_y\lambda_{120}\Big\{S_r^2(-\lambda_{003}^2-\lambda_{030}^2+\lambda_{040}-1)+2S_rS_xS_{xr}\lambda_{030}\lambda_{003}-S_{xr}^2(\lambda_{040}-1)\Big\},\\
b_3 &= S_r^3S_x^2\lambda_{003}\lambda_{120}-S_r^3S_x^2S_{xy}\lambda_{003}\lambda_{030}+S_r^2S_{ry}S_x^2\lambda_{003}^2--S_r^2S_xS_{xr}S_y\lambda_{030}\lambda_{120}\\
& -S_r^2S_{ry}S_x^2\lambda_{040}+S_r^2S_{ry}S_x^2+S_r^2S_{xr}S_{xy}\lambda_{040}-S_r^2S_{xr}S_{xy},\\
b_{33} &= S_r^2S_{ry}S_x^2\lambda_{003}^2+S_r^2S_{ry}S_x^2\lambda_{030}^2-\lambda_{040}S_x^2S_{ry}S_r^2\\
& -2S_rS_xS_{xr}S_{ry}\lambda_{030}\lambda_{003}+S_r^2S_{xy}S_x^2+S_{ry}S_{rx}(\lambda_{040}-1).
\end{aligned}
$$

Substituting the optimum values of d_q for $q = 1, 2, 3$ in (2.10), the minimum mean squared error of $\bar{y}_{S.G_R}$ is obtained by

$$
\begin{aligned}
MSE(\bar{y}_{S.G_{Re}})_{min.} \cong\ & \theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)}{N}C_y^2\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]\\
& +\theta_{r,n}\Big[S_y^2+F_1^2S_x^2+F_2^2S_x^4(\lambda_{040}-1)+F_3^2S_r^2-2F_1S_{xy}\\
& -2F_2S_x^2S_y\lambda_{120}-2F_3S_{ry}+2F_1F_2S_x^3\lambda_{030}+2F_1F_3S_{xr}\\
& +2F_2F_3S_rS_x^2\lambda_{003}\Big]. \qquad (2.12)
\end{aligned}
$$

The possible members of the generalized difference family are given in Table 4. The $MSE(\bar{y}_{S.w})$ for the difference estimators is given in Table 5. The optimum values of β_t's are obtained by minimizing the mean squared error equations and they are given in Table 6. The minimum MSE equations of $(\bar{y}_{S.w})$ are given in Table 7. If the marginal distributions of x and rank of x are normal, then λ_{003} will be zero. Hence, under the assumption of normality, the last term in $MSE(\bar{y}_{S.w(min.)})$ vanishes.

3 Theoretical Comparisons

In this section, we consider the theoretical comparison of the proposed ratio family of estimators with the mean method of imputation.

For the ratio estimator, we have

(i) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.2})$, if

$$2\rho_{ry}C_y - C_x > 0. \tag{3.1}$$

(ii) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.3})$, if

$$\bar{Y}^2(\lambda_{04} - 1) - 2\bar{Y}S_y\lambda_{12} > 0. \tag{3.2}$$

Table 4: Some members of the proposed and existing regression type estimators.

w	d_1	d_2	d_3	$\bar{y}_{S.w}$	Imputation Procedure
1	0	0	0	$\bar{z}_r$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r & \text{if } j\epsilon A' \end{cases}$
2	1	0	0	$\bar{z}_r + \beta_1(\bar{x}_n - \bar{x}_r)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_1(x_i - \bar{x}_r). & \text{if } j\epsilon A' \end{cases}$
3	0	1	0	$\bar{z}_r + \beta_2(s_n^2 - s_r^2)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_2\Big[\frac{n(x_j-\bar{x}_n)^2}{(n-1)} + \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_n)^2}{(n-r)(n-1)} & \\ -\frac{n\sum_{j\epsilon A}(x_j-\bar{x}_r)^2}{(n-r)(n-1)}\Big] & \text{if } j\epsilon A' \end{cases}$
4	0	0	1	$\bar{z}_r + \beta_3(\bar{r}_n - \bar{r}_r)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_3(r_j - \bar{r}_r). & \text{if } j\epsilon A' \end{cases}$
5	1	1	0	$\bar{z}_r + \beta_1(\bar{x}_n - \bar{x}_r) + \beta_2(s_n^2 - s_r^2)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_1(x_i - \bar{x}_r) & \\ +\beta_2\Big[\frac{n(x_j-\bar{x}_n)^2}{(n-1)} + \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_n)^2}{(n-r)(n-1)} & \\ -\frac{n\sum_{j\epsilon A}(x_j-\bar{x}_r)^2}{(n-r)(n-1)}\Big] & \text{if } j\epsilon A' \end{cases}$
6	1	0	1	$\bar{z}_r + \beta_1(\bar{x}_n - \bar{x}_r) + \beta_3(r_n - r_r)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_1(x_i - \bar{x}_r) + \beta_3(r_j - \bar{r}_r) & \text{if } j\epsilon A' \end{cases}$

7	1	1	0	$\bar{z}_r + \beta_3(\bar{r}_n - \bar{r}_r) + \beta_2(s_n^2 - s_r^2)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_3(r_i - \bar{r}_r) + \beta_2\left[\frac{n(x_j-\bar{x}_n)^2}{(n-1)} + \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_n)^2}{(n-r)(n-1)} - \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_r)^2}{(n-r)(n-1)}\right] & \text{if } j\epsilon A' \end{cases}$
8	1	1	1	$\bar{z}_r + \beta_3(\bar{x}_n - \bar{x}_r) + \beta_2(s_n^2 - s_r^2) + \beta_3(r_n - r_r)$.	$\hat{z}_j = \begin{cases} z_j & \text{if } j\epsilon A \\ \bar{z}_r + \beta_1(x_i - \bar{x}_r) + \beta_2\left[\frac{n(x_j-\bar{x}_n)^2}{(n-1)} + \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_n)^2}{(n-r)(n-1)} - \frac{n\sum_{j\epsilon A}(x_j-\bar{x}_r)^2}{(n-r)(n-1)}\right] + \beta_3(r_j - \bar{r}_r). & \text{if } j\epsilon A' \end{cases}$

(iii) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.4})$, if

$$2\rho_{ry}C_y - C_r > 0. \tag{3.3}$$

(iv) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.5})$, if

$$\rho_{xy} + \frac{1}{C_y}\left\{\lambda_{030} - \frac{C_x}{2}\left(1 - \frac{(\lambda_{040} - 1)}{C_x^2}\right)\right\} - \frac{\lambda_{120}}{C_x} > 0. \tag{3.4}$$

(v) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.6})$, if

$$\rho_{xr} - \frac{1}{2C_xC_r}\left[C_x^2 + C_r^2 - 2C_y\left(\rho_{xy}C_x + \rho_{ry}C_r\right)\right] > 0. \tag{3.5}$$

(vi) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.7})$, if

$$\rho_{ry} + \frac{1}{C_y}\left[\lambda_{003} - \frac{C_r}{2}\left(1 - \frac{(\lambda_{040} - 1)}{C_r^2}\right)\right] - \frac{\lambda_{120}}{C_r} > 0. \tag{3.6}$$

(vii) $MSE(\bar{y}_{S.1}) > MSE(\bar{y}_{S.8})$, if

$$\rho_{xr} - C_y\left(\frac{C_x^{'} + C_r^{'}}{2} - \rho_{xy}^{'} + \rho_{ry}^{'} + \lambda_{120}^{'} - \lambda_{030}^{'} - \lambda_{003}^{'}\right) > 0, \tag{3.7}$$

where

$$C_x^{'} = C_x/(C_rC_y),\ C_r^{'} = C_r/(C_xC_y),\ \rho_{xy}^{'} = \rho ry/(C_rC_y),\ \rho_{wy}^{'} = \rho wy/(C_xC_y),$$
$$\lambda_{120}^{'} = \lambda_{120}/(C_xC_r)\lambda_{030}^{'} = \lambda_{030}/(C_rC_y),\ \lambda_{003}^{'} = \lambda_{003}/(C_xC_y).$$

The proposed generalized ratio family of imputing the scrambling response is more efficient than the simple mean imputation method if conditions (3.1) - (3.7) are satisfied.

Table 5: Mean squared error equation of difference estimators.

w	$MSE(\bar{y}_{S.w})$
1	$\theta_{r,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right].$
2	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left(S_y^2 + \beta_1^2S_x^2 - 2\beta_1S_{xy}\right).$
3	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left[S_y^2 + \beta_2^2S_x^4(\lambda_{040} - 1) - 2\beta_2S_x^2S_y\lambda_{120}\right].$
4	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left(S_y^2 + \beta_3^2S_r^2 - 2\beta_3S_{ry}\right).$
5	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left[S_y^2 + \beta_1^2S_x^2 + \beta_2^2S_x^4(\lambda_{040} - 1) - 2\beta_2S_x^2S_y\lambda_{120} - 2\beta_1S_{xy} + 2\beta_1\beta_2S_x^3\lambda_{030}\right].$
6	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)}{N}C_y^2\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left[S_y^2 + \beta_1S_x^2 + \beta_3S_x^2 - 2\beta_1S_{xy} - 2\beta_3S_{ry} + 2\beta_1\beta_2S_{xr}\right].$
7	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left[S_y^2 + \beta_2^2S_x^4(\lambda_{040} - 1) + \beta_3^2S_r^2 - 2\beta_2S_x^2S_y\lambda_{120} - 2\beta_3S_{ry} + 2\beta_2\beta_3S_x^2S_r\lambda_{003}\right].$
8	$\theta_{n,N}S_y^2 + \frac{1}{r}\left[\bar{Y}^2C_{\theta_1}\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right] + \theta_{r,n}\left[S_y^2 + \beta_1^2S_x^2 + \beta_2^2S_x^4(\lambda_{040} - 1) + \beta_3^2S_r^2 - 2\left(\beta_1S_{xy} + \beta_2S_x^2S_y\lambda_{120} + \beta_3S_{ry} - \beta_1\beta_2S_x^3\lambda_{030} - \beta_1\beta_3S_{xr} - \beta_2\beta_3S_rS_x^2\lambda_{003}\right)\right].$

4 Numerical Study

We use the data set by Murthy (967). for this data set, we have y as the number of workers, and x as the Capital fixed in (000) rupees. In addition, we have:

$N = 80$, $n = 60$, $\bar{Y} = 1126.00$, $\bar{X} = 285.10$, $S_y = 845.61$, $S_x = 270.43$, $\lambda_{040} = 1.0892$, $\lambda_{120} = 2.8306$, $\lambda_{120} = 2.8306$, $\rho_{xy} = 0.9884$, $\rho_{ry} = 0.8851$, $\rho_{xr} = 0.9201$.

The percentage of relative efficiencies (PREs) are obtained as

$$PRE(.) = \frac{Var(\bar{y}_{S.M})}{MSE(\bar{y}_{S.G_R}) \text{ or } MSE(\bar{y}_{S.G_{Re}})} \times 100, \tag{4.1}$$

$$PRE(k) = \frac{Var(\bar{y}_{S.M})}{MSE(\bar{y}_{S.k})} \times 100, \tag{4.2}$$

$$PRE(w) = \frac{Var(\bar{y}_{S.M})}{MSE(\bar{y}_{S.w})} \times 100, \tag{4.3}$$

where $k, w = 2, \cdots, 8$. For the ratio of the mean of two scrambling variables and the coefficients of variation of the first and the second scrambling cards, we use

$$\theta_2^2/\theta_1^2 = 0.5, 1.0, 1.5 \quad \text{and} \quad C_{\theta_1}^2 = C_{\theta_2}^2 = 0.1, 0.2. \tag{4.4}$$

Based on the mentioned data set, the PREs of the generalized estimators and their special cases are shown in Tables 8, 10, and 11 over the different values of constants at various response rates. It is clearly noticed that the proposed family outperforms according to the mean method of imputation. One more thing which we wish to remark in Tables 8, 10, and 11 is that the proposed generalized method of imputation has considerably greater efficiency at low response rates.

The summary statistics of the results are given in Table 9. Note that the maximum value of PRE(R) is 1127.1130% for the generalized ratio estimator, and for generalized regression estimator is 1069.7540%. This shows that the proposed estimators are quite efficient.

5 Conclusion

In the present study, we suggested two generalized families of estimators for imputing scrambled responses by utilizing higher order moments along with known mean of the ranks of the auxiliary variable. Based on numerical findings, it is shown out that the proposed generalized methods are more efficient as compared to their counterpart. Thus, we recommend the use of the proposed generalized methods for efficiently estimating the finite population mean of the sensitive variable.

The current work can easily be extended to the generalized exponential, exponential-ratio and exponential-product type estimators in different sampling schemes. This article is a part of an ongoing research that will appear in forthcoming articles.

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Table 6: Optimum values of $\boldsymbol{\beta_t}$ for $t = 1, 2,$ and 3.

w	**Optimum Values**
2	$\beta_{1(opt.)} = \frac{S_{xy}}{S_x^2}$.
3	$\beta_{2(opt.)} = \frac{S_y\lambda_{120}}{S_x^2(\lambda_{40}-1)}$.
4	$\beta_{3(opt.)} = \frac{S_{ry}}{S_r^2}$.
5	$\beta_{1(opt.)} = \frac{S_{xy}(\lambda_{040}-1)-S_xS_y\lambda_{030}}{S_x^2(\lambda_{040}-1-\lambda_{030}^2)}$ and $\beta_{2(opt.)} = \frac{S_yS_x\lambda_{120}-S_{xy}\lambda_{030}}{S_x^2(\lambda_{040}-1-\lambda_{030}^2)}$.
6	$\beta_{1(opt.)} = \frac{S_r^2S_{yx}-S_{xr}S_{yr}}{S_r^2S_x^2-S_{xr}^2}$, and $\beta_{3(opt.)} = \frac{S_{yr}S_x^2-S_{xr}S_{yx}}{S_x^2S_r^2-S_{xr}^2}$.
7	$\beta_{2(opt.)} = \frac{S_{ry}\lambda_{003}-S_rS_y\lambda_{120}}{S_rS_x^2(\lambda_{003}^2-(\lambda_{040}-1))}$ and $\beta_{3(opt.)} = \frac{S_yS_r\lambda_{120}\lambda_{003}-S_{ry}(\lambda_{040}-1)}{S_r^2(\lambda_{003}^2-(\lambda_{040}-1))}$.
8	$\beta_{1(opt.)} = \frac{w_1}{t_1} = B_1$, $\beta_{2(opt.)} = \frac{w_2}{t_2} = B_2$, and $\beta_{3(opt.)} = \frac{w_3}{t_3} = B_3$.

Table 7: Mean square error equations of the proposed estimators.

w	$MSE(\bar{y}_{S.w(min.)})$
1	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right].$
2	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\rho_{xy}^2\right).$
3	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\frac{\lambda_{120}^2}{(\lambda_{040}-1)^2}\right).$
4	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\rho_{ry}^2\right).$
5	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\rho_{xy}^2-\frac{\left(\lambda_{120}-\rho_{xy}\lambda_{030}\right)^2}{\lambda_{040}-1-\lambda_{030}^2}\right).$
6	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\frac{\rho_{xy}^2S_r^2+\rho_{yr}^2-2\rho_{yr}\rho_{xr}\rho_{xy}}{1-\rho_{xr}^2}\right).$
7	$\theta_{n,N}S_y^2+\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)C_y^2}{N}\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}S_y^2\left(1-\rho_{ry}^2-\frac{\lambda_{120}^2}{\lambda_{040}-1}\right).$
8	$\theta_{n,N}S_y^2++\frac{1}{r}\left[\bar{Y}^2C_{\theta_1}^2\left\{1+\frac{(N-1)}{N}C_y^2\right\}+\frac{\theta_2^2}{\theta_1^2}C_{\theta_2}^2\right]+\theta_{r,n}\Big[S_y^2+B_1^2S_x^2+B_2^2S_x^4(\lambda_{040}-1)+B_3^2S_r^2-2B_1S_{xy}-2B_2S_x^2S_y\lambda_{120}-2B_3S_{ry}+2B_1B_2S_x^3\lambda_{030}+2B_1B_3S_{xr}+2B_2B_3S_rS_x^2\lambda_{003}\Big].$

Table 8: PRE(.) of generalized ratio and regression estimator.

r	C_{θ_1}	C_{θ_2}	θ_2^2/θ_1^2	RR	PRE(R)	PRE(Re)
10	0.1	0.1	0.5	6552.5720	1127.1130	1069.7540
			1.0	6552.5730	1127.1110	1069.7530
			1.5	6552.5740	1127.1090	1069.7510
		0.2	0.5	1069.7530	1127.1110	1069.7530
			1.0	6552.5730	1127.1060	1069.7480
			1.5	6552.5760	1127.0970	1069.7400
	0.2	0.1	0.5	6552.5810	673.1071	653.5542
			1.0	7013.1150	673.1067	653.5538
			1.5	7013.1170	673.1060	653.5532
		0.2	0.5	7013.1150	673.1067	653.5538
			1.0	7013.1180	673.1050	653.5522
			1.5	7013.1230	673.1023	653.5497
20	0.1	0.1	0.5	2819.2100	461.3460	636.1122
			1.0	2819.2110	461.3458	636.1118
			1.5	2819.2110	461.3455	636.1110
		0.2	0.5	2819.2110	654.5258	636.1118
			1.0	2819.2120	654.5239	636.1099
			1.5	2819.2150	654.5207	636.1069
	0.2	0.1	0.5	3049.4820	461.3460	452.8047
			1.0	3049.4820	461.3458	452.8045
			1.5	3049.4830	461.3455	452.8041
		0.2	0.5	3049.4820	461.3458	452.8045
			1.0	3049.4840	461.3450	452.8037
			1.5	3049.4860	461.3437	452.8024
30	0.1	0.1	0.5	1574.7570	413.8498	407.1786
			1.0	1574.7570	413.8496	407.1784
			1.5	1574.7570	413.8493	407.1781
		0.2	0.5	1574.7570	413.8496	407.1784
			1.0	1574.7580	413.8488	407.1777
			1.5	1574.7600	413.8474	407.1763
	0.2	0.1	0.5	1728.2710	323.6291	319.8946
			1.0	1728.2710	323.6290	319.8945
			1.5	1728.2710	323.6288	319.8944
		0.2	0.5	1728.2710	323.6290	319.8945

			1.0	1728.2720	323.6286	319.8941
			1.5	1728.2740	323.6279	319.8934
40	0.1	0.1	0.5	952.5297	268.0099	265.6797
			1.0	265.6796	268.0098	952.5299
			1.5	952.5302	268.0097	265.6795
		0.2	0.5	952.5299	268.0098	265.6796
			1.0	952.5306	268.0095	265.6793
			1.5	952.5319	268.0089	265.6787
	0.2	0.1	0.5	1067.6650	226.9002	225.4070
			1.0	1067.6660	226.9001	225.4069
			1.5	1067.6660	226.9001	225.4068
		0.2	0.5	1067.6660	226.9001	225.4069
			1.0	1067.6660	226.8999	225.4067
			1.5	1067.6680	226.8996	225.4064
50	0.1	0.1	0.5	579.1935	170.1783	169.5572
			1.0	579.1937	170.1783	169.5571
			1.5	579.1939	170.1783	169.5571
		0.2	0.5	579.1937	170.1783	169.5571
			1.0	579.1943	170.1782	169.5570
			1.5	579.1953	170.1780	169.5568
	0.2	0.1	0.5	671.3021	155.2310	154.7847
			1.0	671.3022	155.2310	154.7847
			1.5	671.3025	155.2310	154.7847
		0.2	0.5	671.3022	155.2310	154.7847
			1.0	671.3028	155.2309	154.7846
			1.5	671.3038	155.2308	154.7845

Table 9: Descriptive statistics of the results.

r	**Freq.**	**Mean**	**Std.**	**Min.**	**Med.**	**Max.**
			PRE(R)			
10	12	900.1067	237.0949	673.1023	*900.1021*	1127.1130
20	12	509.6400	87.3680	461.3437	461.3458	654.5258
30	12	368.7389	47.1160	323.6279	368.7383	413.8498
40	12	247.4548	21.4688	226.8996	247.4546	268.0099
50	12	162.7046	7.8060	155.2308	162.7045	170.1783
			PRE(Re)			
10	12	861.6513	217.3518	653.5497	861.6471	1069.7540
20	12	544.4573	95.7288	452.8024	544.4558	636.1122
30	12	363.5361	45.5824	319.8934	363.5355	407.1786
40	12	302.7806	205.5982	225.4064	245.5429	952.5299
50	12	162.1709	7.7146	154.7845	162.1708	169.5572

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Appendix

For evaluating the bias and variance of the estimators, we define following useful terms. Let $e_0 = \frac{\bar{z}_r}{\bar{Y}} - 1, e_1 = \frac{\bar{x}_r}{\bar{X}} - 1, e_2 = \frac{\bar{x}_n}{\bar{X}} - 1, e_3 = \frac{s^2_{x(r)}}{S^2_x} - 1, e_4 = \frac{s^2_{x(n)}}{S^2_x} - 1, e_5 = \frac{s^2_{xz(r)}}{S^2_{xy}} - 1,$
$e_6 = \frac{\bar{r}_r}{\bar{R}} - 1, e_7 = \frac{\bar{r}_n}{\bar{R}} - 1, E(e_i) = 0, (i = 0, 1, 2, 3, 4, 5, 6, 7),$

To the first order approximation, we have

$$E\left(e_0^2\right) = \theta_{r,N}C_y^2 + \frac{1}{r}\left[C^2_{\theta_1}\left\{1 + \frac{(N-1)C_y^2}{N}\right\} + \frac{\theta_2^2}{\bar{Y}^2\theta_1^2}C^2_{\theta_2}\right],$$
$$E\left(e_1^2\right) = \theta_{r,N}C_x^2, E\left(e_2^2\right) = \theta_{n,N}C_x^2, E\left(e_3^2\right) = \theta_{r,N}\left(\lambda_{040} - 1\right), E\left(e_4^2\right) = \theta_{n,N}\left(\lambda_{040} - 1\right),$$
$$E\left(e_6^2\right) = \theta_{r,N}C_r^2, E\left(e_7^2\right) = \theta_{n,N}C_r^2, E\left(e_0e_1\right) = \theta_{r,N}\rho_{xy}C_xC_y, E\left(e_0e_2\right) = \theta_{n,N}\rho_{xy}C_xC_y,$$
$$E\left(e_0e_3\right) = \theta_{r,N}C_y\lambda_{120}, E\left(e_0e_4\right) = \theta_{n,N}C_y\lambda_{120}, E\left(e_0e_7\right) = \theta_{n,N}\rho_{ry}C_rC_y,$$
$$E\left(e_0e_6\right) = \theta_{r,N}\rho_{ry}C_rC_y, E\left(e_1e_2\right) = \theta_{n,N}{C_x}^2, E\left(e_1e_3\right) = \theta_{r,N}C_x\lambda_{030}, E\left(e_1e_4\right) = \theta_{n,N}C_x\lambda_{030},$$
$$E\left(e_1e_5\right) = \theta_{r,N}\rho_{xy}^{-1}C_x\lambda_{120}, E\left(e_1e_6\right) = \theta_{r,N}\rho_{xr}C_rC_x, E\left(e_1e_7\right) = \theta_{n,N}\rho_{xr}C_rC_x,$$
$$E\left(e_2e_3\right) = \theta_{n,N}C_x\lambda_{030}, E\left(e_2e_4\right) = \theta_{n,N}C_x\lambda_{030}, E\left(e_2e_5\right) = \theta_{n,N}\rho_{xy}^{-1}C_x\lambda_{120},$$
$$E\left(e_2e_6\right) = \theta_{n,N}\rho_{xr}C_rC_x, E\left(e_2e_7\right) = \theta_{n,N}\rho_{xr}C_xC_r, E\left(e_3e_4\right) = \theta_{n,N}\left(\lambda_{040} - 1\right),$$
$$E\left(e_3e_6\right) = \theta_{r,N}C_r\lambda_{003}, E\left(e_3e_7\right) = \theta_{n,N}C_r\lambda_{003}, E\left(e_4e_6\right) = \theta_{n,N}C_r\lambda_{003}, E\left(e_4e_7\right) = \theta_{n,N}C_r\lambda_{003},$$
$$E\left(e_6e_7\right) = \theta_{n,N}{C_r}^2,$$

where

$$\bar{\tau} = \frac{1}{N}\sum_{j=1}^{N}\tau_j, C^2_\tau = \frac{\sigma^2_\tau}{\bar{\tau}^2}, \rho_{\tau\psi} = \frac{S_{\tau\psi}}{S_\tau S_\psi}, C^2_{\theta_1} = \left(\frac{\sigma_1}{\theta_1}\right)^2, C^2_{\theta_2} = \left(\frac{\sigma_2}{\theta_2}\right)^2, R = \frac{\bar{Y}}{\bar{X}},$$
$$R' = \frac{\bar{Y}}{\bar{R}}S_{\tau\psi} = \frac{1}{N-1}\sum_{j=1}^{N}(\tau_j - \bar{\tau})(\psi_j - \bar{\psi}), \theta_{r,N} = \left(\frac{1}{r} - \frac{1}{N}\right), \theta_{n,N} = \left(\frac{1}{n} - \frac{1}{N}\right),$$
$$\theta_{r,n} = \left(\frac{1}{r} - \frac{1}{n}\right), \mu_{abc} = \frac{1}{N-1}\sum_{j=1}^{N}(y_j - \bar{Y})^a(x_j - \bar{X})^b(r_j - \bar{R})^c, \lambda_{abc} = \frac{\mu_{abc}}{\mu_{200}^{a/2}\mu_{020}^{b/2}\mu_{002}^{c/2}},$$

where $\tau = R, X, Y$ and $\psi = R, X, Y$.

Table 10: P.R.E(i) of the special cases of generalized ratio method of imputation.

r	C_{θ_1}	C_{θ_2}	θ_2^2/θ_1^2	*RR*	PRE(2)	PRE(3)	PRE(4)	PRE(5)	PRE(6)	PRE(7)	PRE(8)
10	0.1	0.1	0.5	6552.5720	784.7886	132.2070	290.4700	67.8920	343.3627	993.4518	2099.6610
			1.0	6552.5730	784.7880	132.2070	290.4700	67.8920	343.3626	993.4508	2099.6560
			1.5	6552.5740	784.7870	132.2070	290.4698	67.8920	343.3625	993.4491	2099.6480
		0.2	0.5	1069.7530	784.7880	132.2070	290.4700	67.8920	343.3626	993.4508	2099.6560
			1.0	6552.5730	784.7856	132.2070	290.4697	67.8920	343.3622	993.4468	2099.6370
			1.5	6552.5760	784.7815	132.2070	290.4693	67.8920	343.3616	993.4400	2099.6050
	0.2	0.1	0.5	6552.5810	541.3484	129.4688	258.1774	69.3543	296.0500	626.1045	907.7053
			1.0	7013.1150	541.3477	129.4687	258.1773	69.3543	296.0498	626.1036	907.7032
			1.5	7013.1170	541.3455	129.4687	258.1769	69.3544	296.0493	626.1004	907.6962
		0.2	0.5	7013.1150	541.3482	129.4688	258.1773	69.3543	296.0499	626.1042	907.7045
			1.0	7013.1180	541.3472	129.4687	258.1771	69.3544	296.0497	626.1028	907.7014
			1.5	7013.1230	541.3455	129.4687	258.1769	69.3544	296.0493	626.1004	907.6962
20	0.1	0.1	0.5	2819.2100	529.7668	129.2800	256.1702	69.4598	293.2027	610.2027	872.7863
			1.0	2819.2110	529.7665	129.2800	256.1702	69.4598	293.2026	610.2023	872.7854
			1.5	2819.2110	529.7660	129.2800	256.1701	69.4598	293.2025	610.2016	872.7839
		0.2	0.5	2819.2110	529.7665	129.2800	256.1702	69.4598	293.2026	610.2023	872.7854
			1.0	2819.2120	529.7653	129.2800	256.1700	69.4598	293.2023	610.2006	872.7818
			1.5	2819.2150	529.7633	129.2800	256.1696	69.4598	293.2018	610.1978	872.7758
	0.2	0.1	0.5	3049.4820	399.9678	126.4835	229.1476	71.0994	255.8732	440.4961	551.1603
			1.0	3049.4820	399.9676	126.4835	229.1476	71.0994	255.8732	440.4959	551.1600
			1.5	3049.4830	399.9674	126.4835	229.1475	71.0994	255.8731	440.4956	551.1595
		0.2	0.5	3049.4820	399.9676	126.4835	229.1476	71.0994	255.8732	440.4959	551.1600
			1.0	3049.4840	399.9670	126.4835	229.1474	71.0994	255.8730	440.4952	551.1588
			1.5	3049.4860	399.9660	126.4835	229.1472	71.0995	255.8726	440.4940	551.1567
30	0.1	0.1	0.5	1574.7570	365.1765	125.4285	220.1187	71.7581	243.8033	397.5078	482.0815
			1.0	1574.7570	365.1763	125.4285	220.1186	71.7581	243.8032	397.5076	482.0812
			1.5	1574.7570	365.1761	125.4285	220.1186	71.7581	243.8031	397.5073	482.0808
		0.2	0.5	1574.7570	365.1763	125.4285	220.1186	71.7581	243.8032	397.5076	482.0812
			1.0	1574.7580	365.1757	125.4285	220.1185	71.7581	243.8030	397.5068	482.0801
			1.5	1574.7600	365.1747	125.4284	220.1182	71.7581	243.8026	397.5056	482.0781
	0.2	0.1	0.5	1728.2710	295.5593	122.6580	198.8972	73.6045	216.1887	314.4188	359.9275
			1.0	1728.2710	295.5592	122.6580	198.8972	73.6046	216.1887	314.4187	359.9274
			1.5	1728.2710	295.5591	122.6580	198.8971	73.6046	216.1886	314.4185	359.9271

		0.2	0.5	1728.2710	295.5592	122.6580	198.8972	73.6046	216.1887	314.4187	359.9274
			1.0	1728.2720	295.5589	122.6580	198.8971	73.6046	216.1885	314.4183	359.9268
			1.5	1728.2740	295.5583	122.6580	198.8969	73.6046	216.1883	314.4176	359.9259
40	0.1	0.1	0.5	952.5297	250.1606	120.1321	182.1776	75.4528	195.1474	262.2386	289.9808
			1.0	265.6796	250.1606	120.1321	182.1776	75.4528	195.1474	262.2386	289.9807
			1.5	952.5302	250.1604	120.1320	182.1776	75.4528	195.1473	262.2384	289.9805
		0.2	0.5	952.5299	250.1606	120.1321	182.1776	75.4528	195.1474	262.2386	289.9807
			1.0	952.5306	250.1603	120.1320	182.1775	75.4528	195.1472	262.2382	289.9803
			1.5	952.5319	250.1598	120.1320	182.1773	75.4528	195.1470	262.2377	289.9795
	0.2	0.1	0.5	1067.6650	215.2973	117.5794	167.3474	77.5044	176.9875	223.1902	240.6734
			1.0	1067.6660	215.2972	117.5794	167.3474	77.5044	176.9875	223.1901	240.6733
			1.5	1067.6660	215.2972	117.5794	167.3474	77.5044	176.9874	223.1900	240.6732
		0.2	0.5	1067.6660	215.2972	117.5794	167.3474	77.5044	176.9875	223.1901	240.6733
			1.0	1067.6660	215.2970	117.5794	167.3473	77.5044	176.9874	223.1899	240.6731
			1.5	1067.6680	215.2968	117.5793	167.3472	77.5044	176.9872	223.1896	240.6727
50	0.1	0.1	0.5	579.1935	165.2532	112.3900	142.1945	82.3714	147.2184	168.6282	175.7402
			1.0	579.1937	165.2532	112.3900	142.1945	82.3714	147.2184	168.6282	175.7402
			1.5	579.1939	165.2531	112.3900	142.1945	82.3714	147.2183	168.6281	175.7401
		0.2	0.5	579.1937	165.2532	112.3900	142.1945	82.3714	147.2184	168.6282	175.7402
			1.0	579.1943	165.2531	112.3900	142.1945	82.3714	147.2183	168.6281	175.7400
			1.5	579.1953	165.2529	112.3900	142.1944	82.3714	147.2182	168.6279	175.7398
	0.2	0.1	0.5	671.3021	151.6734	110.5113	134.4128	84.4131	138.2608	154.1160	159.1962
			1.0	671.3022	151.6734	110.5113	134.4128	84.4132	138.2608	154.1160	159.1962
			1.5	671.3025	151.6734	110.5113	134.4127	84.4132	138.2607	154.1160	159.1961
		0.2	0.5	671.3022	151.6734	110.5113	134.4128	84.4132	138.2608	154.1160	159.1962
			1.0	671.3028	151.6733	110.5113	134.4127	84.4132	138.2607	154.1159	159.1961
			1.5	671.3038	151.6732	110.5113	134.4127	84.4132	138.2606	154.1158	159.1959

Table 11: P.R.E(i) of the special cases of generalized regression method of imputation.

r	C_{θ_1}	C_{θ_2}	θ_2^2/θ_1^2	*RR*	**PRE(2)**	**PRE(3)**	**PRE(4)**	**PRE(5)**	**PRE(6)**	**PRE(7)**	**PRE(8)**
10	0.1	0.1	0.5	6552.5720	1094.7390	251.7675	368.6937	1187.1730	1139.5290	114.4118	1127.1130
			1.0	6552.5730	1094.7380	251.7674	368.6936	1187.1720	1139.5280	114.4118	1127.1110
			1.5	6552.5740	1094.7360	251.7674	368.6934	1187.1690	1139.5260	114.4118	1127.1090
		0.2	0.5	1069.7530	1094.7380	251.7674	368.6936	1187.1720	1139.5280	114.4118	1127.1110

			1.0	6552.5730	1094.7330	251.7673	368.6932	1187.1660	1139.5230	114.4118	1127.1060
			1.5	6552.5760	1094.7250	251.7670	368.6924	1187.1560	1139.5130	114.4117	1127.0970
	0.2	0.1	0.5	6552.5810	662.1809	228.9495	313.3957	692.6601	677.2246	113.3391	673.1071
			1.0	7013.1150	662.1805	228.9495	313.3957	692.6597	677.2241	113.3391	673.1067
			1.5	7013.1170	662.1798	228.9494	313.3955	692.6590	677.2235	113.3391	673.1060
		0.2	0.5	7013.1150	662.1805	228.9495	313.3957	692.6597	677.2241	113.3391	673.1067
			1.0	7013.1180	662.1789	228.9493	313.3954	692.6579	677.2225	113.3391	673.1050
			1.5	7013.1230	662.1762	228.9491	313.3949	692.6550	677.2197	113.3391	673.1023
20	0.1	0.1	0.5	2819.2100	644.2407	227.4964	310.1177	672.9058	658.3995	113.2643	654.5263
			1.0	2819.2110	644.2403	227.4964	310.1176	672.9053	658.3990	113.2643	654.5258
			1.5	2819.2110	644.2395	227.4963	310.1175	672.9045	658.3982	113.2643	654.5250
		0.2	0.5	2819.2110	644.2403	227.4964	310.1176	672.9053	658.3990	113.2643	654.5258
			1.0	2819.2120	644.2384	227.4962	310.1173	672.9033	658.3971	113.2643	654.5239
			1.5	2819.2150	644.2353	227.4960	310.1167	672.8999	658.3938	113.2643	654.5207
	0.2	0.1	0.5	3049.4820	456.5958	207.5177	267.6513	469.7061	463.1214	112.1411	461.3460
			1.0	3049.4820	456.5956	207.5177	267.6512	469.7059	463.1212	112.1411	461.3458
			1.5	3049.4830	456.5953	207.5176	267.6511	469.7055	463.1208	112.1411	461.3455
		0.2	0.5	3049.4820	456.5956	207.5177	267.6512	469.7059	463.1212	112.1411	461.3458
			1.0	3049.4840	456.5948	207.5176	267.6510	469.7050	463.1203	112.1411	461.3450
			1.5	3049.4860	456.5935	207.5174	267.6506	469.7036	463.1190	112.1411	461.3437
30	0.1	0.1	0.5	1574.7570	321.5575	184.1950	223.5201	327.2323	324.3975	110.5603	323.6279
			1.0	1574.7570	410.1437	200.6653	254.1200	420.3479	415.2323	111.7103	413.8498
			1.5	1574.7570	410.1435	200.6652	254.1199	420.3477	415.2321	111.7103	413.8496
		0.2	0.5	1574.7570	410.1432	200.6652	254.1198	420.3474	415.2318	111.7103	413.8493
			1.0	1574.7580	410.1427	200.6651	254.1197	420.3469	415.2313	111.7103	413.8488
			1.5	1574.7600	410.1414	200.6649	254.1193	420.3455	415.2299	111.7103	413.8474
	0.2	0.1	0.5	1728.2710	321.5587	184.1952	223.5206	327.2336	324.3987	110.5603	323.6291
			1.0	1728.2710	321.5586	184.1952	223.5205	327.2335	324.3986	110.5603	323.6290
			1.5	1728.2710	321.5585	184.1952	223.5205	327.2333	324.3984	110.5603	323.6288
		0.2	0.5	1728.2710	321.5586	184.1952	223.5205	327.2335	324.3986	110.5603	323.6290
			1.0	1728.2720	321.5582	184.1951	223.5204	327.2331	324.3982	110.5603	323.6286
			1.5	1728.2740	321.5575	184.1950	223.5201	327.2323	324.3975	110.5603	323.6279
40	0.1	0.1	0.5	952.5297	266.7197	170.8465	200.5341	270.2463	268.4884	109.4873	268.0099
			1.0	265.6796	266.7196	170.8465	200.5340	270.2462	268.4883	109.4873	268.0098
			1.5	952.5302	266.7195	170.8465	200.5340	270.2461	268.4882	109.4873	268.0097
		0.2	0.5	952.5299	266.7196	170.8465	200.5340	270.2462	268.4883	109.4873	268.0098
			1.0	952.5306	266.7193	170.8464	200.5339	270.2459	268.4880	109.4873	268.0095

			1.5	952.5319	266.7187	170.8462	200.5336	270.2453	268.4874	109.4873	268.0089
	0.2	0.1	0.5	1067.6650	226.0742	158.7203	180.9197	228.3274	227.2061	108.3785	226.9002
			1.0	1067.6660	226.0742	158.7203	180.9197	228.3273	227.2060	108.3785	226.9001
			1.5	1067.6660	226.0741	158.7202	180.9196	228.3272	227.2059	108.3785	226.9001
		0.2	0.5	1067.6660	226.0742	158.7203	180.9197	228.3273	227.2060	108.3785	226.9001
			1.0	1067.6660	226.0740	158.7202	180.9196	228.3271	227.2058	108.3785	226.8999
			1.5	1067.6680	226.0736	158.7201	180.9194	228.3268	227.2055	108.3785	226.8996
50	0.1	0.1	0.5	579.1935	169.8352	137.5117	149.2074	170.7686	170.3051	106.0448	170.1783
			1.0	579.1937	169.8352	137.5117	149.2074	170.7686	170.3051	106.0448	170.1783
			1.5	579.1939	169.8351	137.5117	149.2073	170.7686	170.3051	106.0448	170.1783
		0.2	0.5	579.1937	169.8352	137.5117	149.2074	170.7686	170.3051	106.0448	170.1783
			1.0	579.1943	169.8350	137.5116	149.2073	170.7685	170.3050	106.0448	170.1782
			1.5	579.1953	169.8348	137.5116	149.2071	170.7683	170.3048	106.0448	170.1780
	0.2	0.1	0.5	671.3021	154.9846	130.7805	139.7705	155.6545	155.3220	105.1725	155.2310
			1.0	671.3022	154.9846	130.7805	139.7705	155.6545	155.3220	105.1725	155.2310
			1.5	671.3025	154.9845	130.7805	139.7705	155.6544	155.3220	105.1725	155.2310
		0.2	0.5	671.3022	154.9846	130.7805	139.7705	155.6545	155.3220	105.1725	155.2310
			1.0	671.3028	154.9845	130.7805	139.7704	155.6544	155.3219	105.1725	155.2309
			1.5	671.3038	154.9843	130.7804	139.7704	155.6542	155.3218	105.1725	155.2308

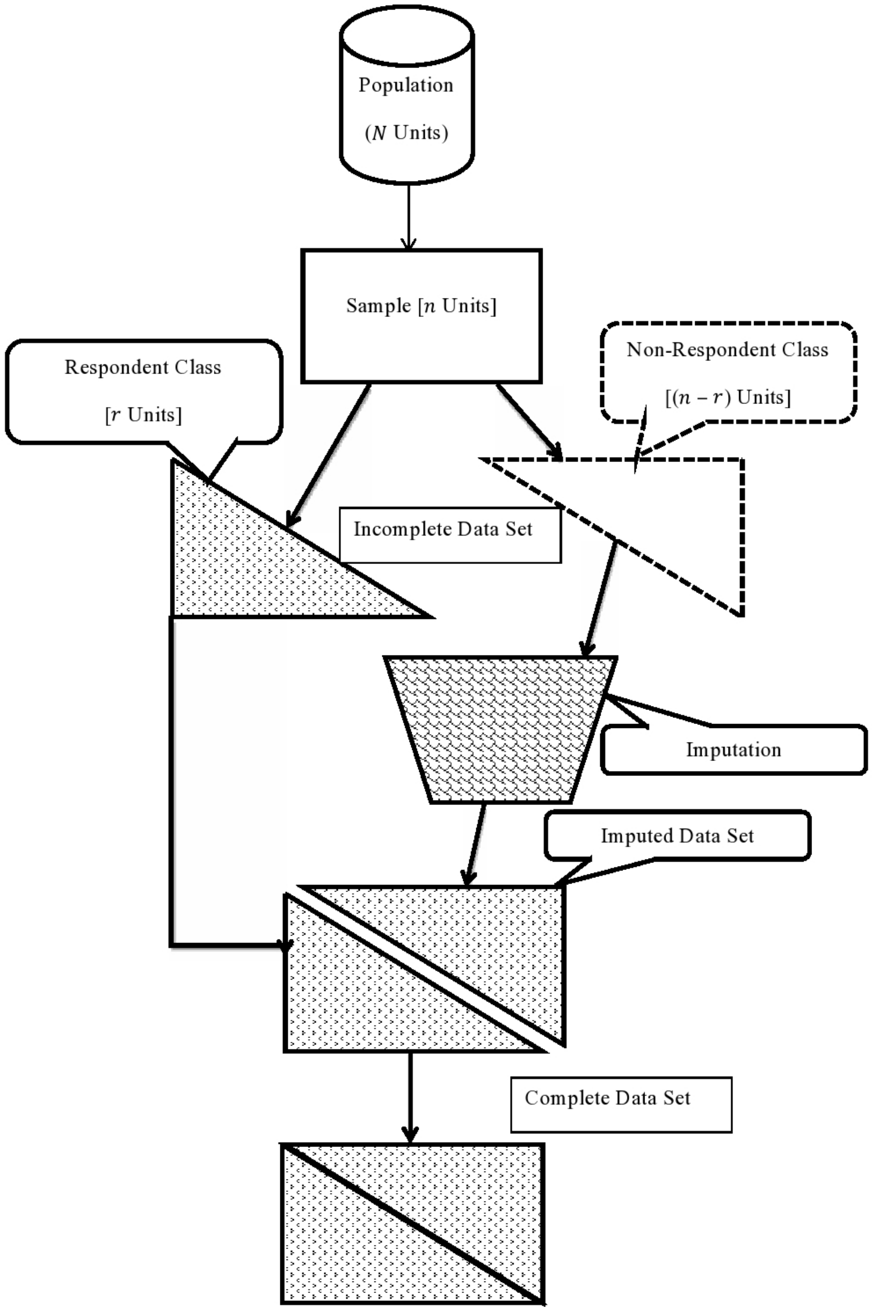

Figure 1: **Illustration of data imputation scheme.**