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Characterizations of Certain Marshall-Olkin Generalized Distributions

G.G. Hamedani

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI, USA

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Abstract. Several characterizations of Marshall-Olkin generalized distributions, introduced by Gui (2013) and by Al-Saiari *et al*. (2014), are presented. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function.

Keywords. Marshall-Olkin generalized distributions, Characterization, Truncated moment, Hazard function

MSC: Primary 60E05; Secondary 60E10.

1 Introduction

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Various characterizations of distributions have been established in many different directions in the literature. In this short note, several characterizations of Marshall-Olkin Power Log-Normal (M-OPLN) distribution,

G.G. Hamedani (gholamhoss.hamedani@marquette.edu)

introduced by Gui (2013), and Marshall-Olkin Extended Burr Type XII (M-OEBXII) distribution, introduced by Al-Saiari *et al*. (2014), are presented in two directions. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function. Our characterizations (*i*) will employ an interesting result due to Glänzel (1987) (Theorem 2.1 of Section 2 below). The advantage of these type of characterizations is that cumulative distribution function F need not have a closed form and is given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equations.

The cdf (cumulative distribution function) $F(x)$ and pdf (probability density function) $f(x)$ of M-OPLN distribution are given, respectively, by

$$F(x) = F(x;\mu,\sigma,p,\alpha) = \frac{1-\left[\Phi\left(\frac{\mu-\ln x}{\sigma}\right)\right]^{p}}{1-(1-\alpha)\left[\Phi\left(\frac{\mu-\ln x}{\sigma}\right)\right]^{p}}, \quad x \geq 0, \tag{1.1}$$

and

$$f(x) = f(x;\mu,\sigma,p,\alpha) = \frac{p\alpha\phi\left(\frac{\mu-\ln x}{\sigma}\right)\left[\Phi\left(\frac{\mu-\ln x}{\sigma}\right)\right]^{p}}{x\sigma\left(1-(1-\alpha)\left[\Phi\left(\frac{\mu-\ln x}{\sigma}\right)\right]^{p}\right)^{2}}, \quad x > 0, \tag{1.2}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $p > 0$, and $\alpha > 0$ are parameters and $\Phi(x)$ and $\phi(x)$ are cdf and pdf of the standard normal distribution. For further properties and the domain of applicability of M-OPLN distribution, we refer the interested reader to Gui (2013).

The cdf $F(x)$ and pdf $f(x)$ of M-OEBXII distribution are given, respectively, by

$$F(x) = F(x;\alpha,c,k) = \frac{1-(1+x^{c})^{-k}}{1-(1-\alpha)(1+x^{c})^{-k}}, \quad x \geq 0, \tag{1.3}$$

and

$$f(x) = f(x;\alpha,c,k) = \frac{\alpha c k x^{c-1}(1+x^{c})^{-(k+1)}}{\left[1-(1-\alpha)(1+x^{c})^{-k}\right]^{2}}, \quad x > 0, \tag{1.4}$$

where α , c and k are all positive parameters. For further properties and the domain of applicability of M-OEBXII distribution, we refer the interested reader to Al-Saiari *et al*. (2014).

The presentation of the content of this work is as follows. In Section 2, we present our characterization results based on truncated moments. Section 3 is devoted to characterization of M-OEBXII distribution in terms of the hazard function.

2 Characterizations Based on Truncated Moments

In this section, we present characterizations of M-OPLN and M-OEBXII distributions in terms of a simple relationship between two truncated moments. As mentioned in the Introduction, our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 2.1, below). The advantage of the characterizations given here is that cdf F need not have a closed form and are given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation.

Theorem 2.1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a given probability space and let* $H = [a, b]$ *be an interval for some* $a < b$ $(a = -\infty\ ,\ b = \infty$ *might as well be allowed*$)$. *Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* h *and* g *be two real functions defined on* H *such that*

$$\frac{E\left[g(X) \mid X \geq x\right]}{E\left[h(X) \mid X \geq x\right]} = \eta(x), \quad x \in H, \tag{2.1}$$

is defined with some real function η. *Assume that* $h, g \in C^1(H)$, $\eta \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H. *Finally, assume that the equation* $\eta h = g$ *has no real solution in the interior of* H. *Then* F *is uniquely determined by the functions* h, g *and* η, *particularly*

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, h(u) - g(u)} \right| \exp\left(-s(u)\right)\, du,$$

where the function s *is a solution of the differential equation* $s' = \frac{\eta' h}{\eta h - g}$ *and* C *is a constant, chosen to make* $\int_H dF = 1$.

We would like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 2.1 and let $g_n \to g$ and $h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F. Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x) = \frac{E\left[g(X) \mid X \geq x\right]}{E\left[h(X) \mid X \geq x\right]}. \tag{2.2}$$

Remark 1. (a) In Theorem 2.1 the interval H need not be closed since the condition is only on the interior of H. (b) Clearly, Theorem 2.1 can be stated in terms of two functions g and η by taking $h(x) \equiv 1$, which will reduce the condition given in Theorem 2.1 to $E[g(X) \mid X \geq x] = \eta(x)$. However, adding an extra function will give a lot more flexibility, as far as its application is concerned.

Proposition 2.1. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $h(x) = \left(1 - (1-\alpha)\left[\Phi\left(\frac{\mu - \ln x}{\sigma}\right)\right]^p\right)^2$ *and* $g(x) = h(x)\Phi\left(\frac{\mu - \ln x}{\sigma}\right)$ *for* $x \in (0, \infty)$. *Then, pdf of X is* (1.2) *if and only if the function η defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{p}{p+1}\Phi\left(\frac{\mu - \ln x}{\sigma}\right), \quad x > 0. \tag{2.3}$$

Proof. Let X have pdf (1.2), then

$$(1 - F(x))\, E[h(X) \mid X \geq x] = \alpha\left[\Phi\left(\frac{\mu - \ln x}{\sigma}\right)\right]^p, \quad x > 0,$$

and

$$(1 - F(x))\, E[g(X) \mid X \geq x] = \frac{\alpha p}{p+1}\left[\Phi\left(\frac{\mu - \ln x}{\sigma}\right)\right]^{p+1}, \quad x > 0,$$

and finally,

$$\eta(x) h(x) - g(x) = -\frac{1}{p+1} g(x) < 0 \;\; for \;\; x > 0.$$

Conversely, if $\eta(x)$ is given by (2.1), then

$$s'(x) = \frac{\eta'(x) h(x)}{\eta(x) h(x) - g(x)} = \frac{p\phi\left(\frac{\mu - \ln x}{\sigma}\right)}{\sigma x \Phi\left(\frac{\mu - \ln x}{\sigma}\right)},$$

from which we obtain

$$s(x) = -\ln\left\{\Phi\left(\frac{\mu - \ln x}{\sigma}\right)^p\right\}, \;\; x > 0.$$

Now, in view of Theorem 2.1, X has cdf (1.1) and pdf (1.2). $\square$

Corollary 2.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let h be as in Proposition 2.1. Then, pdf of X is (1.2) if and only if there exist function g and η defined in Theorem 2.1 satisfying the differential equation

$$\frac{\eta'(x)\,h(x)}{\eta(x)\,h(x) - g(x)} = \frac{p\phi\left(\frac{\mu-\ln x}{\sigma}\right)}{\sigma x\Phi\left(\frac{\mu-\ln x}{\sigma}\right)}, \quad x > 0. \tag{2.4}$$

Remark 2. (*a*) The general solution of the differential equation (2.4) is

$$\eta(x) = \left[\Phi\left(\frac{\mu-\ln x}{\sigma}\right)\right]^{-p}\left[-\int \frac{p}{\sigma x}\phi\left(\frac{\mu-\ln x}{\sigma}\right)(h(x))^{-1}\,g(x)\,dx + D\right],$$

for $x > 0$, where D is a constant. One set of function (h, g, η) satisfying the above equation is given in Proposition 2.1 for $D = 0$.

(*b*) Clearly there are other triple of functions (h, g, η) satisfying the conditions of Theorem 2.1. We presented one such pair in Proposition 2.1.

Proposition 2.2. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable and let* $g(x) = \left[1-(1-\alpha)(1+x^c)^{-k}\right]^2$ *and* $h(x) = g(x)(1+x^c)^{-1}$ *for* $x \in (0, \infty)$. *Then, pdf of X is* (1.4) *if and only if the function* η *defined in Theorem 2.1 has the form*

$$\eta(x) = \frac{k+1}{k}(1+x^c), \quad x > 0.$$

Proof. It is similar to that of Proposition 2.1. □

A corollary and remarks similar to Corollary 2.1 and Remarks 2 can be stated for M-OEBXII distribution as well.

3 Characterization Based on Hazard Function

It is obvious that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{h'_F(x)}{h_F(x)} - h_F(x) = q(x),$$

where $q(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{h'_F(x)}{h_F(x)} - h_F(x) = \frac{f'(x)}{f(x)} \tag{3.1}$$

for many univariate continuous distributions (3.1) seems to be the only differential equation in terms of the hazard function. The goal of the characterization based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (3.1) . Here, we present a characterization of the of M-OEBXII model based on a nontrivial differential equation in terms of the hazard function.

Proposition 3.1. *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable. Then,* X *has* pdf (1.4) *if and only if its hazard function* h_F *for* $x > 0$ *satisfies the differential equation*

$$h_F'(x) - (c-1)\, x^{-1} h_F(x) = \frac{-c^2 k x^{2(c-1)} (1+x^c)^{-k-2} \left[(k-1)(1-\alpha) + (1+x^c)^k\right]}{\left[1-(1-\alpha)(1+x^c)^{-k}\right]^2}. \tag{3.2}$$

Proof. If X has pdf (1.4), then clearly (3.2) holds. Now, if (3.2) holds, then after multiplying both sides of (3.2) by $x^{-(c-1)}$, we arrive at

$$\frac{d}{dx}\left\{x^{-(c-1)} h_F(x)\right\} = ck \frac{d}{dx}\left\{\frac{(1+x^c)^{-1}}{1-(1-\alpha)(1+x^c)^{-k}}\right\},$$

from which we have

$$h_F(x) = \frac{f(x)}{1-F(x)} = \frac{ckx^{c-1}(1+x^c)^{-1}}{1-(1-\alpha)(1+x^c)^{-k}} = \frac{ckx^{c-1}(1+x^c)^{k-1}}{(1+x^c)^k - (1-\alpha)}. \tag{3.3}$$

Integrating both sides of (3.3) from 0 to x , we have

$$-\ln(1-F(x)) = \ln\left\{\frac{(1+x^c)^k - (1-\alpha)}{\alpha}\right\}.$$

From which we obtain

$$1-F(x) = \frac{\alpha(1+x^c)^{-k}}{1-(1-\alpha)(1+x^c)^{-k}}, x \geq 0.$$

□

Remark 3. For $k = 1$, equation (3.2) reduces to the following simple equation

$$h_F'(x) - (c-1)\, x^{-1} h_F(x) = \frac{-c^2 x^{2(c-1)}}{(x^c+\alpha)^2},\ x > 0.$$

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