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A Probability Space based on Interval Random Variables

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Abstract. This paper considers an extension of probability space based on interval random variables. In this approach, first, a notion of interval random variable is introduced. Then, based on a family of continuous distribution functions with interval parameters, a concept of probability space of an interval random variable is proposed. Then, the mean and variance of an interval random variable are introduced. The presented theoretical results will be illustrated with some lemmas. Some numerical examples will be used to show the performance of the proposed method.

Keywords. Interval mean, interval parameter, interval random variable, probability space, variance of an interval random variable.

MSC: 62G10, 03E72.

1 Introduction

Modelling real world problems typically involves processing uncertainty of two distinct types. These are uncertainties arising from a lack of knowledge relating to concepts and uncertainties due to inherent vagueness in concepts themselves which, in the sense of classical logic, may be well defined. Traditionally, the above can be modelled in terms of probability theory and uncertainty theory respectively which are quite distinct

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theoretical foundations for reasoning and decision making in situations of uncertainty. However, there are many situations where there is insufficient information regarding vague concepts, i.e. in practical studies; however, there are many practical problems that require dealing with observations that represent inherently imprecise, uncertain, or linguistic characteristics. In such cases, closed intervals may be more effective in encoding observations rather than precise ones. For instance, if one reports a perceived length of an object in a perceptual study, imprecise responses such as "between 10 and 15" may reflect the perceived length better than real-valued responses. Moreover, it is frequently difficult to assume that the parameter, for which the distribution of a random variable is determined, has a precise value, and so on. To produce suitable probability theory dealing with imprecise information, so we need to model the imprecise information and extend the usual probability space to imprecise environments. This suggests the need for a theory of the probability of closed intervals random variables.

The topic of probability theory with imprecise information has been studied by some authors. Below is a brief review of some studies relevant to the present work. A fuzzy event is a fuzzy set whose membership function is Borel measurable and its probability is defined by Zadeh [24] as the expected value of the membership function characterizing the fuzzy set. Yager [21] introduced a methodology for obtaining a precise fuzzy measure of the probability of a fuzzy event in the face of probabilistic uncertainty on the base elements. Klement [10] has suggested a modification of Yager's definition which leads to a piecewise continuous fuzzy subset. Yager [22] provided an appropriate interpretation for Klements modification and used it to provide an alternative definition for a fuzzy probability of a fuzzy event. Heilpern [9] studied the fuzzy subsets of the space of all probability measures which the probability of fuzzy event is equal to a fuzzy probability. Baldwin et al. [1] introduced the probability of a fuzzy event using Mass assignment theory techniques for processing uncertainty together with the t-norm definition of conditional probabilities. Toth [20] redesigned some definitions of a probability of a fuzzy event based on the operational viewpoint of f-set theory and on some concepts of operational statistics. Plasecki [15] defined the probability of fuzzy events as a denumerable additivity measure. He also defined a notion of conditional probability of fuzzy events, complete fuzzy repartition and independent fuzzy events by means of the probability measure. Stein [19] discussed the treatment of fuzzy probabilities in setting of fuzzy variables and joint possibility distributions. Smets [18] proposed some axioms to justify the natural definition of the probability of a fuzzy event initially given by Zadeh [24]. They are based (1) on the postulate that the sum of the conditional probability of a fuzzy event and of its complement given any fuzzy event adds to one or (2) on soft independence for orthogonal sets with independent constitutive elements. Cheng et al. [4] extended the fuzzy probability of a fuzzy event from a fuzzy algebra to the fuzzy σ-algebra generated by it. Grzegorzewski [7] generalized the notion of independence of events and the concept of conditional probability on the intuitionistic fuzzy events. Chinag et al. [5] considered fuzzy probabilities constructed over fuzzy topological spaces. They also extend a notion of product fuzzy topological space with product fuzzy topological space. Mesiar [14] transformed probability measures on intuitionistic fuzzy events that were axiomatically characterized by Riečan [16] as interval-valued fuzzy sets. En-lin et al. [6], based on the interval probability, studied the second kind of fuzzy random problem and provided some definitions of fuzzy probability random variable and its distribution function, distribution sequence, fuzzy math expectation, fuzzy variance and so on.

The purpose of this paper is to provide a novel method of constructing a probability space based on interval random variables induced by a family of parametric continuous distribution functions. Then, a concepts of cumulative distribution function, density function, mean and variance of an interval random variable are derived as a main result.

This paper is organized as follows: In Section 2, first some preliminaries about closed intervals are reviewed and a distance between closed intervals is introduced. Some concepts of the classical probability space is also recalled in this section. In Section 3, a concept of interval random variable is introduced. In Section 4, a method is proposed to construct the probability of an event based on a family of parametric continuous distribution function with interval parameters. In the same section, a concept of mean and variance of an interval random variable is proposed. Some basic properties of the proposed methods are also put into investigation in this section. Moreover, some numerical examples are provided to clarify the discussions in this article and give a possible applications in Section 5. Finally, a brief conclusion and some proposals for further study conclude the paper.

2 Preliminaries

2.1 closed intervals

The arithmetic operators on closed intervals are basic content in interval mathematics. Let's have a look at the operations of closed interval [13]. Let $I = [a_1, a_2]$ and $J = [b_1, b_2]$ be two closed intervals, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Some of the main operations of closed interval we will use in this paper are given as follows (for more see [8]):

- Addition: $I \oplus J = [a_1 + b_1, a_2 + b_2]$,
- Multiplication:

$$I \otimes J = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}],$$

In this paper, the set of all compact intervals in $\mathbb{R}$ is denoted by $\mathcal{C}(\mathbb{R})$.

Definition 2.1. Let $I, J \in \mathcal{C}(\mathbb{R})$. The distance between two fuzzy numbers I and J is defined as follows

$$d(I, J) = \int_0^1 (I_\alpha - J_\alpha)^2 d\alpha,$$

where $I_\alpha = (1-\alpha)I^L + \alpha I^U$ for all $\alpha \in [0, 1]$.

It is easy to verify that $d : \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\mathbb{R}) \to [0, \infty)$ has the following properties.

Lemma 2.1. *For three closed intervals I, J and K,*

- $d(I, J) = 0$ *if and only if* $I = J$,
- $d(I, J) = d(J, I)$,
- $d(I, K) \leq d(I, J) + d(J, K)$.

We will use the proposed distance to define the variance of an interval random variable in Section 4.

2.2 The Classical Probability Space

A probability space is denoted by the standard notation $(\Omega, \mathcal{A}, P)$ where: Ω is a sample space, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and P is a function from $\mathcal{A}$ to $[0, 1]$ with $P(\Omega) = 1$ and such that if $E_1, E_2, \ldots \in \mathcal{A}$ are disjoint then $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. A set of probability measures

$P_{\boldsymbol{\theta}}$ on $(\Omega, \mathcal{A})$ indexed by a parameter $\boldsymbol{\theta} \in \Theta$ is said to be a parametric family if and only if $\Theta \subset \mathbb{R}^p$ for some fixed positive integer p and each $P_{\boldsymbol{\theta}}$ is a known probability measure when $\boldsymbol{\theta}$ is known. The set Θ is called the p-dimensional parameter space.

In statistical inference, the data set is viewed as a realization or observation of a random variable X defined on a probability space $(\Omega, \mathcal{A}, P)$ related to the random experiment. Now, let $X : \Omega \to \mathbb{R}$ be a random variable, where $\mathbb{R}$ is equipped with the σ-algebra $\mathcal{B}(\mathbb{R})$, the set of all Borel subsets of $\mathbb{R}$. Then the probability measure induced by X, $P_X : \mathcal{B}(\mathbb{R}) \to [0, 1]$, is defined as follows:

$$\forall\ A \in \mathcal{B}(\mathbb{R}), \quad P_X(A) = P\{X \in A\} = \int_A dP(\omega). \tag{1}$$

In particular, if $A = (-\infty, x]$, then the cumulative distribution function of X, for each $x \in \mathbb{R}$, is obtained as follows:

$$\begin{aligned} F_X(x) &= P\{X \leq x\} \\ &= \int_{\{\omega\in\Omega | X(\omega)\leq x\}} dP(\omega). \end{aligned} \tag{2}$$

Moreover, the probability of $A = (a, b]$ can be obtained by $P(a < X \leq b) = F_X(b) - F_X(a)$.

3 Interval Random Variables

As it was mentioned in Introduction, in practical studies, we may come across the interval data rather than precise ones. Such situations often occur in humanities, especially in psychology, social studies, management, etc. In this section, a notion of interval random variable induced by a family of parametric continuous distribution functions is proposed. It is noticeable that one of the most popular notions of a fuzzy random variable is introduced by Kwakernaak [11, 12]. Here, the Kwakernaak's definition is applied to introduce the notion of interval random variable.

Definition 3.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. An interval-valued function $\overline{X} : \Omega \to \mathcal{C}(\mathbb{R})$, where $\overline{X}(\omega) = [X^L(\omega), X^U(\omega)]$, is called an interval random variable (**I.R.V.**) if $X^L : \Omega \to \mathbb{R}$ and $X^U : \Omega \to \mathbb{R}$ are real-valued random variables.

The following property follows easily from this definition.

Proposition 3.1. *Let $\overline{X} : \Omega \to \mathcal{C}(\mathbb{R})$ be an interval-valued function. Then $\overline{X}$ is an* ***I.R.V.*** *if and only if, for each $\alpha \in [0,1]$, $\overline{X}_\alpha : \Omega \to \mathbb{R}$ defined by $\overline{X}_\alpha(\omega) = (1-\alpha)X^L(\omega) + \alpha X^U(\omega)$, for all $\omega \in \Omega$, is a random variable.*

Here we consider a special case of **I.R.V.**s as the member of a family of a classical continuous parametric population based on Wu [23] and Chachi et al. [3].

Definition 3.2. Assume $\{F_{\boldsymbol{\theta}} : \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p) \in \mathbb{R}^p\}$ is a family of continuous c.d.f. An **I.R.V.** $\overline{X}$ is said to have the same distribution as F with a vector of interval parameters $\overline{\boldsymbol{\theta}} = (\overline{\theta}_1, \overline{\theta}_2, \ldots, \overline{\theta}_p) \in (\mathcal{C}(\mathbb{R}))^p$, denoted by $\overline{X} \sim F_{\overline{\boldsymbol{\theta}}}$, if at each level of $\alpha \in [0,1]$, $\overline{X}_\alpha \sim F_{\overline{\boldsymbol{\theta}}(\alpha)}$ where $\overline{\boldsymbol{\theta}}(\alpha) = (\overline{\theta}_1(\alpha), \ldots, \overline{\theta}_p(\alpha))$ and $\overline{\theta}_j(\alpha) \in \overline{\theta}_j = [\theta_j^L, \theta_j^U]$ for any $j = 1, 2, \ldots, p$.

Remark 3.1. It should be noted that for an *I.R.V.* $\overline{X}$, if $\alpha_1, \alpha_2 \in [0,1]$ satisfy $\alpha_1 < \alpha_2$ then, essentially, $\overline{X}_{\alpha_1}$ and $\overline{X}_{\alpha_2}$ should satisfy this property that $\overline{X}_{\alpha_2}$ is "stochastically greater than" $\overline{X}_{\alpha_1}$. Therefore, to have the relation $\overline{X} \sim F_{\overline{\boldsymbol{\theta}}}$, the distribution $F_{\overline{\boldsymbol{\theta}}}$ has to satisfy $F_{\overline{\boldsymbol{\theta}}(\alpha_1)}(x) \geq F_{\overline{\boldsymbol{\theta}}(\alpha_2)}(x)$ for all $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 < \alpha_2$ and all $x \in \mathbb{R}$. To clarify the discussion how to indicate $\overline{\boldsymbol{\theta}}$, in a family of distribution with interval parameters, consider the following examples:

Example 3.1. Assume that $\overline{X} \sim F_{\overline{\lambda}}$, $\overline{\lambda} \in \mathcal{C}(0,\infty)$ where $F_\lambda(x) = 1 - e^{-\lambda x}$, $x > 0$. Let $\overline{\lambda}(\alpha) = \alpha\lambda^L + (1-\alpha)\lambda^U = \overline{\lambda}_{1-\alpha}$ for all $\alpha \in [0,1]$. Then, we can easily observe that $F_{\overline{\lambda}(\alpha_1)}(x) \geq F_{\overline{\lambda}(\alpha_2)}(x)$ for all $x \in \mathbb{R}$ and for every $\alpha_1 < \alpha_2$. So $\overline{X} \sim F_{\overline{\lambda}}$ if $\overline{X}_\alpha \sim F_{\overline{\lambda}_{1-\alpha}}$ for all $\alpha \in [0,1]$. Hence, in particular, $X^L \sim F_{\lambda^U}$ and $X^U \sim F_{\lambda^L}$. Now, let $F_\lambda(x) = 1 - e^{\frac{-x}{\lambda}}$, $x > 0$. In this case, if $\overline{X}_\alpha \sim F_{\overline{\lambda}(\alpha)}$ where $\overline{\lambda}(\alpha) = \overline{\lambda}_\alpha$, then it is easy to verify that $F_{\overline{\lambda}(\alpha_1)}(x) \geq F_{\overline{\lambda}(\alpha_2)}(x)$ for all $x \in \mathbb{R}$ and for every $\alpha_1 < \alpha_2$. In this case, note that $X^L \sim F_{\lambda^L}$ and $X^U \sim F_{\lambda^U}$.

Example 3.2. Let $\overline{\boldsymbol{\theta}} = (\overline{\mu}, \overline{\sigma}) \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}(0,\infty)$ where $\overline{\mu}(\alpha) = \overline{\mu}_\alpha = (1-\alpha)\mu^L + \alpha\mu^U$, $\overline{\sigma}(\alpha) = \overline{\sigma}_\alpha = (1-\alpha)\sqrt{(\sigma^2)^L} + \alpha\sqrt{(\sigma^2)^U}$ and $F_{\mu,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\exp(\frac{-(x-\mu)^2}{2\sigma^2})$. Then, it is easy to verify that $F_{\overline{\boldsymbol{\theta}}(1-\alpha_1)}(x) \geq F_{\overline{\boldsymbol{\theta}}(1-\alpha_2)}(x)$, where $\overline{\boldsymbol{\theta}}(1-\alpha) = (\overline{\mu}_{1-\alpha}, \overline{\sigma}_{1-\alpha})$, for all $x \in \mathbb{R}$ and for every $\alpha_1 < \alpha_2$. So $\overline{X}$ has the normal distribution with interval parameters $\overline{\mu}$ and $\overline{\sigma^2}$ if $\overline{X}_\alpha \sim F_{\overline{\mu}_{1-\alpha}, \overline{\sigma^2}_{1-\alpha}}$ and therefore $X^L \sim F_{\mu^U,(\sigma^2)^U}$ and $X^U \sim F_{\mu^L,(\sigma^2)^L}$.

Example 3.3. Let $\overline{X} \sim F_{\overline{\boldsymbol{\theta}}}$ where $\overline{\boldsymbol{\theta}} = (\overline{\beta}, \overline{\eta}) \in \mathcal{C}(0,\infty) \times \mathcal{C}(0,\infty)$ with the c.d.f. $F_{\boldsymbol{\theta}}(x) = 1 - e^{-(\frac{x}{\eta})^{\beta}}$, $\boldsymbol{\theta} = (\beta, \eta)$. Then, we easily have:

$$F_{X^L}(x) = \begin{cases} 1 - e^{-(\frac{x}{\eta^L})^{\beta^L}} & x \leq \eta^U, \\ 1 - e^{-(\frac{x}{\eta^L})^{\beta^U}} & x > \eta^U, \end{cases}$$

and

$$F_{X^U}(x) = \begin{cases} 1 - e^{-(\frac{x}{\eta^U})^{\beta^U}} & x \leq \eta^U, \\ 1 - e^{-(\frac{x}{\eta^L})^{\beta^L}} & x > \eta^U. \end{cases}$$

4 Probability Space Induced by I.R.V.s

This section constructs a probability measure for an **I.R.V.** To do this, the concept of c.d.f for an **I.R.V.** is extended firstly to interval environment. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$ be a probability space and $\overline{X} : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be an **I.R.V**. For a given $x \in \mathbb{R}$, let

$$F^L(x) = \inf_{\alpha \in [0,1]} F_{\overline{X}_\alpha}(x) = F_{X^U}(x),$$

$$F^U(x) = \sup_{\alpha \in [0,1]} F_{\overline{X}_\alpha}(x) = F_{X^L}(x).$$

As one can imagine naturally, the value of c.d.f. of $\overline{X}$ at $x \in \mathbb{R}$ should belong to the interval $[F^L(x), F^U(x)]$. On the other hand, since every convex linear combination of F^L and F^U is also a cumulative distribution function; therefore, (in particular) the mean of lower and upper bounds F^L and F^U, i.e.

$$\overline{F}_{\overline{X}}(x) = \int_0^1 ((1-\alpha)F^L(x) + \alpha F^U(x))d\alpha = \frac{F_{X^L}(x) + F_{X^U}(x)}{2}, \; x \in \mathbb{R}, \tag{3}$$

is also a c.d.f. In the sequel, we will denote by $\overline{F}_{\overline{X}}(x)$ the c.d.f. of $\overline{X}$ at $x \in \mathbb{R}$.

Now, for any $A \in \mathcal{B}(\mathbb{R})$, let

$$\overline{P}(A) := \int_A d\overline{F}_{\overline{X}} = \frac{P(X^L \in A) + P(X^U \in A)}{2}. \tag{4}$$

Therefore, if $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$ is a probability space and $\overline{X} : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be an **I.R.V**, then it is readily seen that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \overline{P})$ is also a probability space. As it is seen, $\overline{P}$ has the following properties:

(i) For two events A and B if $A \subseteq B$ then $\overline{P}(A) \leq \overline{P}(B)$.

(ii) For an event A, $\overline{P}(A^c) = 1 - \overline{P}(A)$, where A^c is the complement of A.

(iii) For two events A and B, $\overline{P}(A \cup B) = \overline{P}(A) + \overline{P}(B) - \overline{P}(A \cap B)$.

(iv) For two events A and B, $\overline{P}(A - B) = \overline{P}(A) - \overline{P}(A \cap B)$.

(v) If $\overline{A}_n \downarrow \widetilde{A}$ then $\overline{P}(A_n) \downarrow \overline{P}(A)$.

(v) Suppose f is a continuous and strictly decreasing function on $\mathbb{R}$. If $\overline{A}_n \downarrow \overline{A}$, then $\overline{P}(f(A_n)) \downarrow \overline{P}(f(A))$.

Remark 4.1. Assume $\overline{X}$ is an **I.R.V.** on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $A = (a, b]$. Then it is easy to verify that the probability of $(a, b]$ can be obtained by $\overline{P}((a, b]) = \overline{F}_{\overline{X}}(b) - \overline{F}_{\overline{X}}(a)$. In addition, the probability density function (p.d.f) of $\overline{X}$ is given by

$$\overline{f}_{\overline{X}}(x) = \frac{d}{dx}\overline{F}_{\overline{X}}(x) = \frac{f_{\overline{X}^L}(x) + f_{\overline{X}^U}(x)}{2}. \tag{5}$$

Definition 4.1. Let $\overline{X}$ is an **I.R.V.** on a probability space $(\Omega, \mathcal{A}, P)$. The mean of $\overline{X}$ is defined by $\overline{E}(\overline{X}) \in \mathcal{C}(\mathbb{R})$ where $(\overline{E}(\overline{X}))_\alpha = E(\overline{X}_\alpha)$ for any $\alpha \in [0, 1]$. In addition, the variance of an **I.R.V.** $\overline{X}$ is defined by $\overline{var}(\overline{X}) = var(d(\overline{X}, \overline{E}(\overline{X}))) = \int_0^1 var(\overline{X}_\alpha)d\alpha$, where d denotes the distance between two closed interval, introduced in Definition 2.1.

Based on interval arithmetic, it is easy to verify that the mean and variance of an **I.R.V.** $\overline{X}$ satisfy the following properties:

Lemma 4.1. *Let $\overline{X}$ is an **I.R.V.** on a probability space $(\Omega, \mathcal{A}, P)$. Then,*

(i)

$$\overline{E}(a \otimes \overline{X} \oplus b) = \begin{cases} [aE(X^L) + b, aE(X^U) + b] & a > 0, b \in \mathbb{R}, \\ [aE(X^U) + b, aE(X^L) + b] & a < 0, b \in \mathbb{R}, \end{cases}$$

(ii) $\overline{var}(a \otimes \overline{X} \oplus b) = a^2 \overline{var}(\overline{X})$, *for all $a, b \in \mathbb{R}$, where $\oplus$ and $\otimes$ denote the sum and multiply operations on $\mathcal{C}(\mathbb{R})$, respectively.*

5 Numerical Examples

In this section, some examples are provided to compute the probability of an interval, mean and variance of an **I.R.V.** based on the proposed methods.

Example 5.1. Suppose that a component/system is characterized by an exponential life $\overline{X} \sim F_{\overline{\lambda}}$ where $F_\lambda(x) = 1 - e^{-\lambda x}$. Therefore, from Eq. (3), the c.d.f. of $\overline{X}$ at $x > 0$ is given by

$$\overline{F}_{\overline{X}}(x) = 1 - \frac{e^{-\lambda^U x} + e^{-\lambda^L x}}{2}.$$

For example suppose that $\overline{\lambda} = [0.02, 0.04]$. The graph of c.d.f. of $\overline{F}_{\overline{X}}$ is shown in Fig. 1. We can also obtain $\overline{P}([8, 15]) = 0.144$. In addition, assume we want to calculate the probability that the component will fail within $A = [8, 15]$, given that it has survived at least 8 hours. So, from Eq. (4),

$$\overline{P}([8,15]|(8,\infty)) = \frac{\overline{P}([8,15] \cap (8,\infty))}{\overline{P}((8,\infty))} = \frac{\overline{F}_{\overline{X}}(15) - \overline{F}_{\overline{X}}(8)}{1 - \overline{F}_{\overline{X}}(8)} = 0.182.$$

Moreover, the mean and variance of $\overline{X}$ are obtained as $\overline{E}(\overline{X}) = [E(X^L), E(X^U)] = [25, 50]$ and $\overline{var}(\overline{X}) = \int_0^1 (\frac{1}{0.02\alpha + (1-\alpha)0.04})^2 d\alpha = 1250$.

Example 5.2. Assume that $\overline{X} \sim F_{\overline{\mu},\overline{\sigma^2}}$ where $\overline{\mu} \in \mathcal{C}(\mathbb{R})$, $\overline{\sigma} = \sqrt{\overline{\sigma^2}} \in \mathcal{C}(0,\infty)$ and $F_{\mu,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\exp(\frac{-(x-\mu)^2}{2\sigma^2})$. Then, from Eq. (3), the c.d.f. of $\overline{X}$ at $x \in \mathbb{R}$ is given by

$$\overline{F}_{\overline{X}}(x) = \frac{\Phi(\frac{x-\mu^U}{\sigma^U}) + \Phi(\frac{x-\mu^L}{\sigma^L})}{2},$$

where Φ denotes the c.d.f. of standard normal distribution. For example, suppose that $\overline{\mu} = [-1, 1]$, $\overline{\sigma^2} = [0.5, 1.5]$. The graph of p.d.f. of $\overline{f}_{\overline{X}}$ is shown in Fig. 2. In addition, based on Eq. (4), the calculations show that $\overline{P}([-1, 2]) = 0.62$.

Example 5.3. Let $\overline{X} \sim F_{\overline{\beta},\overline{\eta}}$ with the c.d.f. $F_{\beta,\eta}(x) = 1 - e^{-(\frac{x}{\eta})^\beta}$ where $\overline{\beta}, \overline{\lambda} \in \mathcal{C}(0,\infty)$. From Example 3.3, therefore, the c.d.f. of $\overline{X}$ at $x \in \mathbb{R}$ is obtained as

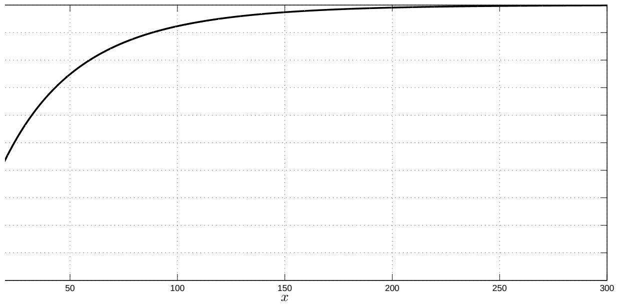

Figure 1: Cumulative distribution function in Example 5.1

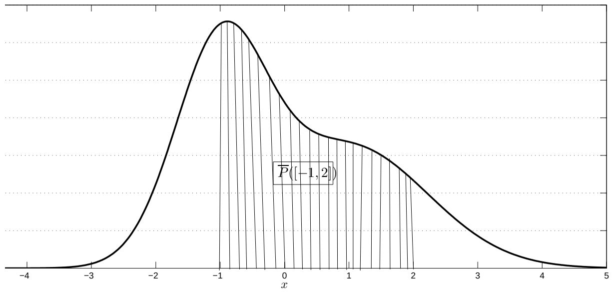

Figure 2: Probability density function in Example 5.2

$$\overline{F}_{\overline{X}}(x) = \begin{cases} 1 - \dfrac{e^{-(\frac{x}{\eta^L})^{\beta^L}} + e^{-(\frac{x}{\eta^U})^{\beta^U}}}{2} & x < \eta^U, \\ 1 - \dfrac{e^{-(\frac{x}{\eta^L})^{\beta^U}} + e^{-(\frac{x}{\eta^U})^{\beta^L}}}{2} & x > \eta^U. \end{cases}$$

For instance, assume $\overline{\beta} = [1.1, 1.3]$ and $\overline{\eta} = [3, 4]$. The graph of c.d.f. of $\overline{F}_{\overline{X}}$ is shown in Fig. 3. In addition, We have $\overline{P}([5, 15]) = 0.203$.

Remark 5.1. (**A comparison study**) It is mentioned that Buckley [2] proposed a fuzzy probability of an event for the case where a (precise) random variable X is induced from a family of distributions with fuzzy parameters [2]. However, if we apply closed intervals instead of fuzzy numbers, then the following interval probability

$$\overline{P}(A) = [\inf_{\alpha\in[0,1]} \int_A dF_{\overline{\boldsymbol{\theta}}_\alpha}, \sup_{\alpha\in[0,1]} \int_A dF_{\overline{\boldsymbol{\theta}}_\alpha}],$$

is obtained. It is noticeable that $\overline{P}$ has the following properties:

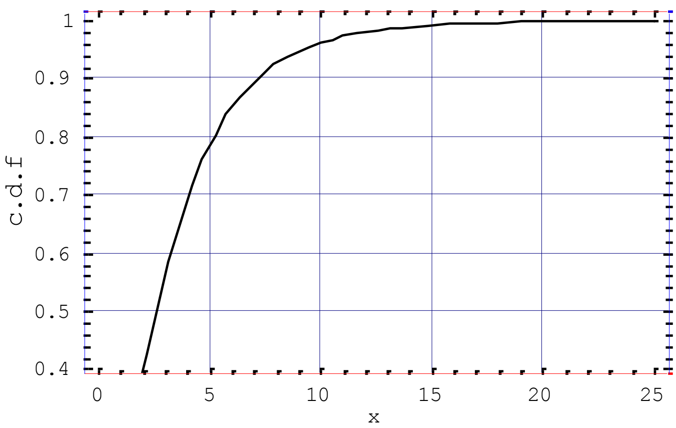

Figure 3: Cumulative density function in Example 3.3

1. If $A \cap B \neq \emptyset$, then $\overline{P}(A \cup B) \preceq \overline{P}(A) \oplus \overline{P}(B) \ominus \overline{P}(A \cap B)$, where $\preceq$ denotes the ordering introduced by Wu [23] and $\oplus$, $\ominus$ represent the sum and minus of two closed intervals (for more see [13]).
2. If $A \subseteq B$, then $\overline{P}(A) \preceq \overline{P}(B)$.
3. $0 \preceq \overline{P}(A) \preceq 1$ for all A and $\overline{P}(\emptyset) = 0$, $\overline{P}(\Omega) = 1$.
4. $\overline{P}(A) \oplus \overline{P}(A^c) \succeq 1$.

However, in this paper, we construct a (precise) probability measure based on **I.R.V.**s induced by a family of continuous parametric distribution functions.

6 Conclusion

In this paper, a method to construct a probability space based on interval random variables has been introduced. We proposed a probability of an event in the case where the interval random variable is assumed to be induced from a family of continuous parametric distribution functions. It is also shown that the proposed method satisfies all properties as in the classical case. A concept of mean and variance of an interval random variable is also proposed and their properties are also investigated. Some numerical examples are provided to clarify the discussions in this paper. The proposed method is an extension of the classical one, i.e. the proposed methods reduce to the classical probability space, mean and variance whenever all interval random variables as well as interval parameters are reduced to the precise ones.

The topic of statistical inference based on interval random variables including parameter estimation, statistical hypothesis test and Bayesian inference are some interesting topics for future studies.

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