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Beta-Linear Failure Rate Distribution and its Applications

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Abstract. We introduce in this paper a new four-parameter generalized version of the linear failure rate distribution which is called Beta-linear failure rate distribution. The new distribution is quite flexible and can be used effectively in modeling survival data and reliability problems. It can have a constant, decreasing, increasing and bathtub-shaped failure rate function depending on its parameters. It includes some well-known lifetime distributions as special sub-models. We provide a comprehensive account of the mathematical properties of the new distribution. In particular, A closed form expressions for the probability density, cumulative distribution and hazard rate functions of this new distribution is given. Also, the rth order moment of this distribution is derived. We discuss the maximum likelihood estimation of the unknown parameters of the new model for complete data and obtain an expression for the Fisher information matrix. In the end, to show the flexibility of the new distribution and illustrative purposes, an application using a real data set is presented.

Keywords. Beta distribution, hazard function, linear failure rate distribution, maximum likelihood estimation, moments, simulation.

MSC: 60E05, 62F12.

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1 Introduction

The linear failure rate (LFR) distribution with parameters $a \geq 0$ and $b \geq 0$, $(a + b > 0)$ which is denoted by LFR(a, b), has the cumulative distribution function (cdf)

$$G(x) = 1 - \exp(-ax - \frac{b}{2}x^2), \qquad x > 0, \tag{1}$$

and probability density function (pdf)

$$\mathrm{g}(x) = (a + bx)\exp(-ax - \frac{b}{2}x^2), \qquad x > 0.$$

Note that if $b = 0$ and $a \neq 0$, then the LFR distribution is reduced to the exponential distribution with parameter a (Exp(a)), and if $a = 0$ and $b \neq 0$ then we can obtain the Rayleigh distribution with parameter b (Rayleigh(b)). A basic structural properties of LFR(a, b) is that it is the distribution of minimum of two independent random variables X_1 and X_2 having Exp(a) and Rayleigh(b) distributions, respectively (Sen and Bhattacharyya, 1995).

If G denotes the cdf of a random variable, then a generalized class of distributions can be defined by

$$F(x) = I_{G(x)}(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1}(1 - t)^{\beta-1} dt, \tag{2}$$

for $\alpha > 0$ and $\beta > 0$, where $I_y(\alpha, \beta) = \frac{B_y(\alpha,\beta)}{B(\alpha,\beta)}$ is the incomplete beta function ratio and $B_y(\alpha, \beta) = \int_0^y t^{\alpha-1}(1-t)^{\beta-1}dt$ is the incomplete beta function.

Many authors considered various forms of G and studied their properties: Eugene et al. (2002) (Beta Normal distribution), Nadarajah and Kotz (2004) (Beta Gumbel distribution), Nadarajah and Gupta (2004) and Barreto-Souza et al. (2011) (Beta Fréchet distribution), Famoye et al. (2005), Lee et al. (2007) and Cordeiro et al. (2011) (Beta Weibull distribution), Nadarajah and Kotz (2006) (Beta exponential distribution), Akinsete et al. (2008) (Beta Pareto distribution), Silva et al. (2010) (Beta modified Weibull distribution), Barreto-Souza et al. (2010) (Beta generalized exponential distribution), Khan (2010) (Beta inverse Wibull distribution), Mahmoudi (2011) (Beta generalized Pareto distribution), Cordeiro et al. (2013b) (Beta-exponentiated Weibull distribution), Cordeiro et al. (2013c) (Beta-Weibull geometric distribution), Singla et al. (2012) (Beta generalized Weibull distribution), Cordeiro et al. (2013a) (Beta generalized gamma distribution) Jafari et al. (2014) (Beta-Gompertz distribution) and Cintra et al. (2014) (Beta generalized normal distribution).

In this article, we propose a new four-parameter distribution, referred to as the beta linear failure rate (BLFR) distribution. The main reasons for introducing this new distribution are: (1) The quality of procedures used in statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort over the years has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. In fact, the statistics literature is filled with hundreds of continuous univariate distributions. However, in recent years, applications from the environmental, financial, biomedical sciences, engineering and economics have further shown that data sets following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions, a significant progress has been made toward the generalization of some well-known distributions.
(2) The LFR distribution is also known as the linear exponential distribution, containing the exponential and Rayleigh distributions as special cases, is a well-known distribution for modeling lifetime data in reliability and medical studies. It is also models phenomena with increasing failure rate. The LFR distribution does not provide a reasonable parametric fit for modeling phenomenon with decreasing, non linear increasing, or non-monotone failure rates such as the bathtub shape, which are common in firm ware reliability modeling and biological studies (Lai et al., 2001; Zhang et al., 2005).
(3) The BLFR distribution has greater tail flexibility than the LFR distribution. The most realistic hazard rate is bathtub-shaped. This occurs in most real-life systems. Such hazard rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. A state-of-the-art survey on the class of such distributions can be found in Nadarajah (2009). Thus the BLFR distribution which contains this type of hazard rate is a reasonable model to fit these data.
(4) The new proposed four-parameter distribution contains many flexible lifetime distributions as special sub-models. These models are: the Beta exponential (BE), Beta Rayleigh (BR), generalized linear failure rate (GLFR) and LFR distributions, among others.

The reminder of the paper is organized as follows: In Section 2, we define the BLFR distribution and investigate some properties of the distribution in this Section. Some of these properties are the limit behavior and shapes of the pdf and hazard rate function of the BLFR distribution. Section 3 provides a general expansion for the moments of the BLFR distribution. In Section 4, we discuss the maximum likelihood estimation (MLE) and calculate the elements of the observed information matrix. Application of the BLFR distribution is given in the Section 5. A simulation study is performed in Section 6. Finally, Section 7 concludes the paper.

2 Definition of the BLFR Distribution and Some Special Cases

We now introduce the BLFR distribution by taking $G(x)$ in (2) to be the cdf of the LFR distribution. Hence, the cdf and pdf of the BLFR are given by

$$F(x) = I_{1-\exp(-ax-\frac{b}{2}x^2)}(\alpha,\beta) = \int_0^{1-\exp(-ax-\frac{b}{2}x^2)} t^{\alpha-1}(1-t)^{\beta-1}dt,$$

and

$$f(x) = \frac{a+bx}{B(\alpha,\beta)}\left(1-\exp(-ax-\frac{b}{2}x^2)\right)^{\alpha-1}\exp(-a\beta x-\frac{b\beta}{2}x^2), \quad (3)$$

respectively. We use the notation $X \sim \text{BLFR}(a,b,\alpha,\beta)$. The hazard rate function of BLFR distribution is given by

$$h(x) = \frac{a+bx}{B(\alpha,\beta)-B_{G(x)}(\alpha,\beta)}(1-\exp(-ax-\frac{b}{2}x^2))^{\alpha-1}\exp(-a\beta x-\frac{b\beta}{2}x^2).$$

2.1 Special cases of the BLFR distribution

1. If $\beta = 1$, then we get the GLFR distribution (GLFR(a,b,α)) which is introduced by Sarhan and Kundu (2009).
2. If $\beta = 1$ and $b = 0$, then we get the generalized exponential distribution (GE) (Gupta and Kundu, 1999).
3. If $\beta = 1$ and $a = 0$, then we get two-parameter Burr X distribution which is introduced by Surles and Padgett (2005) and also is known as generalized Rayleigh distribution (GR) (Kundu and Raqab, 2005).
4. If $\alpha = 1$, then (3) reduces to LFR distribution LFR$(a\beta, b\beta)$.

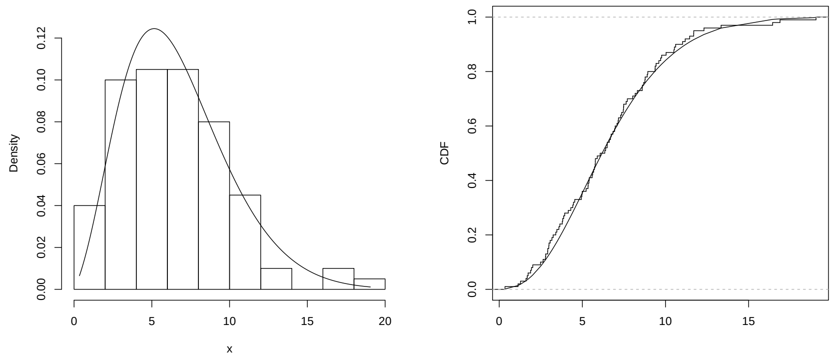

Figure 1: The histogram of a generated data set with size 100 and the exact pdf of BLFR (left) and the empirical cdf and exact cdf (right).

5. If $b = 0$, then we get the BE distribution (BE(a, α, β)) which is introduced by Nadarajah and Kotz (2006).
6. If $a = 0$, then we get the BR distribution (BR(b, α, β)) which is defined by Akinsete and Lowe (2009) and is a special case of beta Weibull distribution (Famoye et al., 2005).
7. If the random variable X has BLFR distribution, then the random variable $Y = 1 - \exp(-aX - \frac{b}{2}X^2)$, satisfies the beta distribution with parameters α and β. Therefore, $T = aX + \frac{b}{2}X^2$ satisfies the BE distribution with parameters 1, α and β (BE$(1, \alpha, \beta)$).
8. If $\alpha = i$ and $\beta = n - i$, where i and n are positive integer values, then the $f(x)$ is the pdf of ith order statistic of LFR distribution.

The following result helps in simulating data from the BLFR distribution: If Y follows Beta distribution with parameters α and β, then

$$X = G^{-1}(Y) = \begin{cases} \frac{-a+\sqrt{a^2-2b\log(1-Y)}}{b} & \text{if} \quad a \geq 0, b > 0 \\ -\frac{\log(1-Y)}{a} & \text{if} \quad a > 0, b = 0, \end{cases} \tag{4}$$

follows BLFR distribution with parameters a, b, α, and β.

For checking the consistency of the simulating data set form BLFR distribution, the histogram for a generated data set with size 100 and the exact pdf of BLFR$(0.2, 0.1, 2, 0.3)$, are displayed in Fig 1 (left). Also, the empirical cdf and the exact cdf is given in Fig 1 (right).

2.2 Properties of the BLFR Distribution

In this section, limiting behavior of pdf and hazard rate function of the BLFR distribution and their shapes are studied.

2.2.1 Mathematical Behaviour of the pdf

Theorem 2.1. *Let $f(x)$ be the pdf of the BLFR distribution. The limiting behaviour of $f(x)$ for different values of its parameters is given bellow:*
i. If $\alpha = 1$ then $\lim_{x\to 0} f(x) = a\beta$.
ii. If $\alpha > 1$ then $\lim_{x\to 0} f(x) = 0$.
iii. If $0 < \alpha < 1$ then $\lim_{x\to 0} f(x) = \infty$.
iv. $\lim_{x\to\infty} f(x) = 0$.

Proof. The proof of parts (i)-(iii) are obvious. For part (iv), we have

$$0 \le (1 - \exp(-ax - \frac{b}{2}x^2))^{\alpha-1} < 1.$$

Therefore,

$$0 < f(x) < \frac{a + bx}{B(\alpha, \beta)}\exp(-a\beta x - \frac{b\beta}{2}x^2)\ .$$

It can be easily shown that $\lim_{x\to\infty}(a+bx)\exp(-a\beta x - \frac{b\beta}{2}x^2) = 0.$ and the proof is completed. □

Theorem 2.2. *Let $f(x)$ denotes the pdf of the BLFR distribution.*
(a) Let $b = 0$, we have:
i. If $\alpha > 1$, the pdf $f(x)$ is unimodal and the mode is given by $x_0 = a^{-1}\log(\frac{\alpha+\beta-1}{\beta})$.
ii. If $0 < \alpha < 1$, then the pdf $f(x)$ is decreasing.
(b) Let $b > 0$, we have:
i. If $\alpha > 1$, the pdf $f(x)$ is log-concave and hence unimodal. The mode is the solution of the equation $\frac{b}{2bz+a^2} + (\alpha - 1)\frac{\exp(-z)}{1-\exp(-z)} - \beta = 0$.
ii. If $0 < \alpha < 1$, then the pdf $f(x)$ may be either decreasing and unimodal.

Proof. (a). Let $b = 0$. Suppose $\eta(x) = -\frac{\partial}{\partial x}\log(f(x))$. Then $\eta(x) = a(1-\alpha)\frac{\exp(-ax)}{1-\exp(-ax)} + a\beta$.
(i) If $\alpha > 1$, then $\eta'(x) > 0$, thus $f(x)$ is log-concave and hence unimodal. the mode is given by $x_0 = a^{-1}\log(\frac{\alpha+\beta-1}{\beta})$. (ii) If $0 < \alpha < 1$, then $\eta(x) > 0$ and hence the pdf $f(x)$ is decreasing.
(b) Let $b > 0$. Consider $z = ax + \frac{b}{2}x^2 = \frac{b}{2}(x + \frac{a}{b})^2 - \frac{a^2}{2b}$. It implies that $z > 0$ for $x > 0$ and also, it is increasing with respect to x. We have $x = \frac{1}{b}\sqrt{2bz + a^2} - \frac{a}{b}$. Now, rewriting the pdf of BLFR distribution as function of z, $\xi(z)$ say, we obtain

$$\xi(z) = f(\sqrt{2bz + a^2} - \frac{a}{b}) = \frac{\sqrt{2bz + a^2}}{B(\alpha, \beta)}(1 - \exp(-z))^{\alpha-1}\exp(-\beta z)\,.$$

If $\eta(z) = -\frac{\partial}{\partial z}\log(\xi(z))$ then

$$\eta'(z) = -\frac{\partial^2}{\partial z^2}\log(\xi(z)) = 2b^2\left(2bz + a^2\right)^{-2} + (\alpha - 1)\frac{\exp(-z)}{(1 - \exp(-z))^2},$$

(i) If $\eta'(z) > 0$ then $f(x)$ is log-concave and hence is unimodal. Thus the pdf $f(x)$ is unimodal if $\alpha > 1$. The mode is the solution of the equation $\frac{b}{2bz+a^2} + (\alpha - 1)\frac{\exp(-z)}{1-\exp(-z)} - \beta = 0$.
(ii) If $0 < \alpha < 1$, then $f(x)$ may be either decreasing and unimodal. Unfortunately, it is not possible to determine analytically (theoretically) the behaviour of the pdf. □

2.2.2 Mathematical Behaviour of the Hazard Rate Function

Theorem 2.3. *Let $h(x)$ be the hazard rate function of the BLFR distribution. The limiting behaviour of $h(x)$ for different values of its parameters is given bellow:*
(i) If $b > 0$ then

$$\lim_{x\to 0^+} h(x) = \begin{cases} +\infty & \alpha < 1 \\ a\beta & \alpha = 1 \\ 0 & \alpha > 1, \end{cases} \qquad \lim_{x\to +\infty} h(x) = \infty.$$

(ii) If $b = 0, a > 0$ then

$$\lim_{x\to 0^+} h(x) = \begin{cases} +\infty & 0 < \alpha < 1 \\ a\beta & \alpha = 1 \\ 0 & \alpha > 1, \end{cases} \qquad \lim_{x\to +\infty} h(x) = \begin{cases} 0 & 0 < \alpha < 1 \\ a\beta & \alpha = 1 \\ +\infty & \alpha > 1. \end{cases}$$

Proof. The proof is obvious and is omitted. □

Theorem 2.4. *Let $h(x)$ be the hazard rate function of the BLFR distribution. Consider the following cases:*
(a) Consider $b = 0$, we have
i. If $\alpha > 1$, then BLFR distribution has an increasing hazard rate function.
ii. If $0 < \alpha < 1$, then the hazard rate function of the BLFR distribution is decreasing.
(b) Consider $b > 0$, we have
i. If $\alpha > 1$, then BLFR distribution has an increasing hazard rate function.
ii. If $0 < \alpha < 1$, then the hazard rate function of the BLFR distribution is bathtub-shaped.

Proof. (a) Let $b = 0$. Then, $\eta'(x) = -\frac{\partial^2}{\partial x^2}\log(f(x)) = a(\alpha-1)\frac{\exp(-ax)}{(1-\exp(-ax))^2}$.
i. If $\alpha > 1$, then $\eta'(x) > 0$ and hence $h(x)$ is an increasing function of x using a theorem given by Glaser (1980).
ii. If $0 < \alpha < 1$, then $\eta'(x) < 0$ and hence $h(x)$ is a decreasing function of x. If $\alpha = 1$, then $h(x)$ is a linear function of x.
(b) Consider $b > 0$, then we have

$$\eta'(z) = -\frac{\partial^2}{\partial z^2}\log \xi(z) = 2b^2\left(2bz + a^2\right)^{-2} + (\alpha - 1)\frac{\exp(-z)}{\left(1 - \exp(-z)\right)^2},$$

i. If $\alpha > 1$, then $\eta'(z) > 0$ for $z > 0$ and hence $h(x)$ is an increasing function of x.
ii. If $0 < \alpha < 1$, then there exist x_0 such that $\eta'(x) < 0$ for $x < x_0$ and $\eta'(x) > 0$ for $x > x_0$. Hence $h(x)$ is a bathtub-shaped hazard rate function. Note that in this case $\lim_{z\to 0^+}\eta'(z) = -\infty$ and $\lim_{z\to\infty}\eta'(z) = 0^+$, which shows that the hazard function is bathtub. It is not possible to determine analytically (theoretically) the parameter values which correspond to the bathtub-shaped hazard rate function for the BLFR distribution. The graphic analysis indicates (numerically) that the hazard rate function is bathtub-shaped when $b > 0$ and $0 < \alpha < 1$. □

Plots of pdf and hazard rate function of the BLFR distribution for different values of it's parameters are given in Fig. 2 and Fig. 3, respectively.

3 Some Extensions and Moments

Here, we present some representations of cdf, pdf, and the survival function of BLFR distribution. The mathematical relation given below will be useful in this section. If β is a positive real non-integer and $|z| < 1$, then $(1-z)^{\beta-1} = \sum_{j=0}^{\infty} w_j z^j$, and if β is a positive real integer, then the upper of the this summation stops at $\beta - 1$, where $w_j = \frac{(-1)^j\Gamma(\beta)}{\Gamma(\beta-j)\Gamma(j+1)}$.
1. We can express (2) as a mixture of cdf of generalized LFR distributions as follows:

$$F(x) = \sum_{j=0}^{\infty} p_j (G(x))^{\alpha+j} = \sum_{j=0}^{\infty} p_j G_j(x),$$

where $p_j = \frac{(-1)^j\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta-j)\Gamma(j+1)(\alpha+j)}$ and $G_j(x) = (G(x))^{\alpha+j}$ is cdf of a random variable which has a generalized LFR distribution with parameters a, b, and $\alpha + j$.

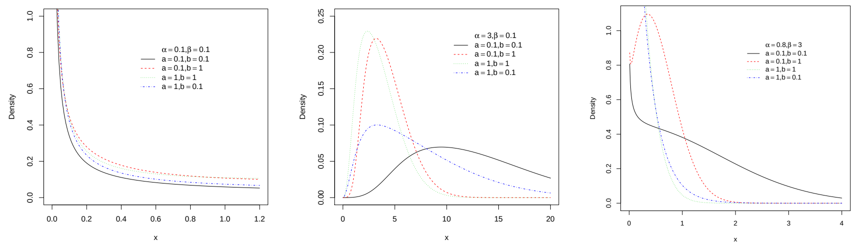

Figure 2: Plots of pdf of the BLFR distribution for selected parameters.

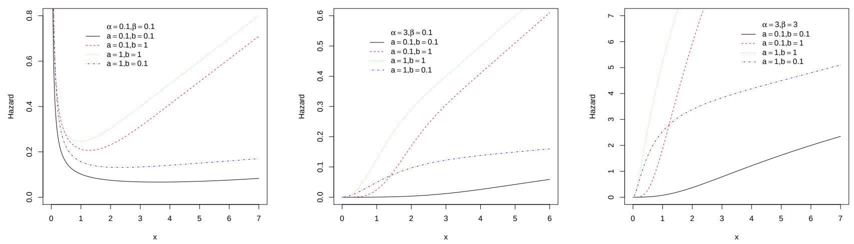

Figure 3: Plots of hazard rate function of the BLFR distribution for selected parameters.

2. We can express (3) as a mixture of pdf of generalized LFR distributions as follows:

$$f(x) = \sum_{j=0}^{\infty} p_j(\alpha + j) \mathrm{g}(x) (G(x))^{\alpha + j - 1} = \sum_{j=0}^{\infty} p_j \mathrm{g}_j(x),$$

where $\mathrm{g}_j(x)$ is pdf of a random variable which has a generalized LFR distribution with parameters a, b, and $\alpha + j$.

3. The kth moment of BLFR distribution can be expressed as a mixture of the kth moment of generalized LFR distributions as follows:

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x)\,dx = \int_0^\infty x^k \sum_{j=0}^{\infty} p_j(\alpha + j)\mathrm{g}(x)(G(x))^{\alpha + j - 1} dx \\ &= \sum_{j=0}^{\infty} p_j \int_0^\infty x^k g_j(x)\,dx = \sum_{j=0}^{\infty} p_j E(X_j^k), \end{aligned}$$

where $g_j(x)$ is pdf of a random variable X_j which has a generalized LFR distribution with parameters a, b, and $\alpha + j$.

4 Estimation and Inference

Consider $X_1, \ldots X_n$ is a random sample from BLFR distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, \alpha, \beta)$ can be written as

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= \sum_{i=1}^{n} \log(a + bx_i) - n\log(\Gamma(\alpha)) - n\log(\Gamma(\beta)) \\ &\quad + n\log(\Gamma(\alpha+\beta)) + (\alpha-1)\sum_{i=1}^{n}\log(1-\exp(t_i)) + \beta\sum_{i=1}^{n} t_i, \quad (5)\end{aligned}$$

where $t_i = -ax_i - \frac{b}{2}x_i^2$. The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5). The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_a(\boldsymbol{\theta}) = \frac{\partial}{\partial a}\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{1}{a+bx_i} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i\exp(t_i)}{1-\exp(t_i)} - \beta\sum_{i=1}^{n}x_i,$$

$$U_b(\boldsymbol{\theta}) = \frac{\partial}{\partial b}\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{x_i}{a+bx_i} + \frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x_i^2\exp(t_i)}{1-\exp(t_i)} - \frac{\beta}{2}\sum_{i=1}^{n}x_i^2,$$

$$U_\alpha(\boldsymbol{\theta}) = \frac{\partial}{\partial \alpha}\ell(\boldsymbol{\theta}) = -n\psi(\alpha) + n\psi(\alpha+\beta) + \sum_{i=1}^{n}\log(1-\exp(t_i)),$$

$$U_\beta(\boldsymbol{\theta}) = \frac{\partial}{\partial \beta}\ell(\boldsymbol{\theta}) = -n\psi(\beta) + n\psi(\alpha+\beta) + \sum_{i=1}^{n}t_i.$$

where $\psi(.)$ is the digamma function.

The MLE's of parameters (a, b, α, β), say $(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})$, are the simultaneous solutions of of the equations $U(\boldsymbol{\theta}) = \mathbf{0}$. Maximization of Equation (5) can be performed by using well-established routines like nlmb or optimize functions in the R statistical package. Our numerical calculations showed that the surface of Equation (5) was smooth. The routines were able to locate the maximum of the likelihood surface for a wide range of starting values. However, to ease computations, it is useful to have reasonable starting values. These can be obtained, for example, by the method of moments.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, asymptotically

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \sim N_4(\mathbf{0}, I(\boldsymbol{\theta})^{-1}),$$

where $I(\boldsymbol{\theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\theta})$ is replaced by $J(\hat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\hat{\boldsymbol{\theta}}$. For constructing tests of hypothesis and confidence region we can use from this result. An asymptotic confidence interval with confidence level $1-\gamma$ for each parameter θ_i, is given by

$$\left(\hat{\theta}_i - z_{\gamma/2}\sqrt{J^{\hat{\theta}_i}}\ ,\ \hat{\theta}_i + z_{\gamma/2}\sqrt{J^{\hat{\theta}_i}}\right),$$

where $J^{\hat{\theta}_i}$ is the ith diagonal element of $J(\hat{\boldsymbol{\theta}})$ and z_γ is the upper γ point of standard normal distribution.

5 Application of BLFR to Real Data Set

In this section, we provide a data analysis to see how the new model works in practice. This data set is given by Aarset (1987) and consists of times to first failure of fifty devices. The data is given by 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Here, we fit BLFR, GLFR, LFR, GR, GE, Rayleigh and exponential models to the above data set. We use the MLE to estimate the model parameters and calculate the standard errors of the MLE's, respectively. The MLE's of the parameters (with std.), the maximized log-likelihood, the Kolmogorov-Smirnov (K-S) statistic with its respective p-value, the AIC (Akaike Information Criterion), AICC and BIC (Bayesian Information Criterion) for the BLFR, GLFR, LFR, GR, GE, Rayleigh and exponential models are given in Table 1.

We can perform formal goodness-of-fit tests in order to verify which distribution fits better to the first data. We apply the Anderson-Darling (AD) and Cramér-von Mises (CM) tests. In general, the smaller the values of AD and CM, the better the fit to the data. For this data set, the values of AD and CM statistics for fitted distributions are given in Table 1.

The empirical scaled TTT transform Aarset (1987) can be used to identify the shape of the hazard function. The TTT plot for this data in Fig. 4 shows a bathtub-shaped hazard rate function and indicates the appropriateness of the BLFR distribution to fit this data. The histogram of data with the fitted pdf's of BLFR, GLFR and LFR are displayed in Fig. 4.

Table 1: MLE's (s.e.) of the fitted distribution, K-S with it p-value p-value, $-2\log(L)$, AIC, AICC, BIC, AD, CM, LR statistic with its p-value corresponds to times to first failure.

	Distribution						
	BLFR	**GLFR**	**LFR**	**GR**	**GE**	**Rayleigh**	**Exp.**
$\hat{a}$	0.3347	0.5327	—	0.3520	0.7798	—	—
(s.e.)	(0.1432)	(0.1145)	—	(0.0559)	(0.1351)	—	—
$\hat{b}$	0.1243	—	—	—	—	—	—
(s.e.)	(0.0722)	—	—	—	—	—	—
$\hat{\alpha}$	0.0172	0.0038	0.0136	—	0.0187	—	0.0219
(s.e.)	(0.0354)	(0.0030)	(0.0038)	—	(0.0036)	—	(0.0031)
$\hat{\beta}$	0.0035	0.0003	0.0002	0.0003	—	0.0006	—
(s.e.)	(0.0025)	(8e-5)	(1e-4)	(8e-5)	—	(9e-5)	—
-2log L	460.8	466.3	476.1	469.1	480.0	528.1	482.2
AIC	468.8	472.3	480.1	473.1	484.0	530.1	484.2
AICC	469.6	472.8	480.4	473.4	484.2	530.2	484.3
BIC	476.4	478.0	484.0	477.0	487.8	532.0	486.1
K-S	0.1554	0.1830	0.1768	0.2009	0.2042	0.2621	0.1911
p-value	0.1786	0.0703	0.0877	0.0353	0.0309	0.0021	0.0519
AD	1.749	2.4890	4.0346	3.0923	3.2530	13.3205	3.6505
CM	0.3574	0.4959	0.5443	0.6111	0.6472	0.8728	0.6006
LR	—	5.5	15.30	8.3	19.2	67.3	21.4
p-value	—	0.019	0.0005	0.0158	6.7e-5	1.6e-14	8.6e-5

The results for this data set show that the BLFR distribution yields the best fit among the GLFR, LFR, GR, GE, Rayleigh and exponential distributions. For this data, the K-S test statistic takes the smallest value with the largest value of its respective p-value for BLFR distribution. Also this conclusion is confirmed from the values of the AIC, AICC and BIC for the fitted models given in Table 1 and the plots of the pdf's in Fig. 4.

Using the likelihood ratio (LR) test, we test BLFR distribution (alternative hypothesis) versus other sub-models of BLFR distribution (null hypothesis) The value of the LR test statistics and the corresponding p-values are given in Table 1. These values show that the BLFR model is superior to its sub-models in terms of model fitting for this real data set at level 0.05.

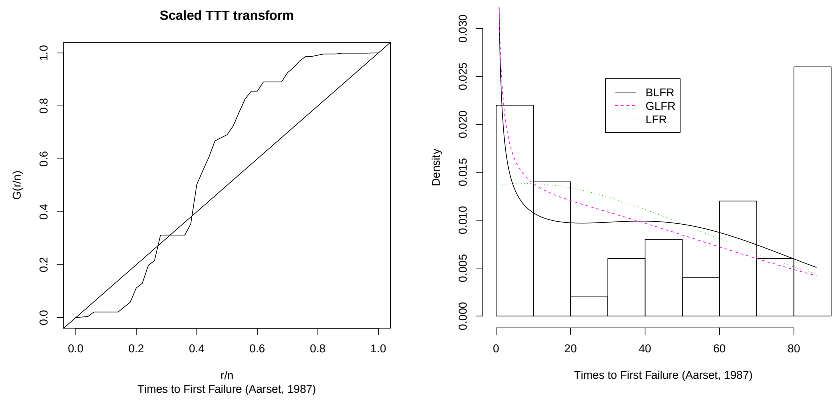

Figure 4: Plots of the empirical scaled TTT transform (left), and the histogram of data with the fitted pdf's (right).

6 Simulations

Here, we assess the performance of the MLE's with respect to sample size n. The assessment is based on a simulation study:
1. We generate 10,000 samples of size $n = 50$, 200 from the BLFR distribution for $\alpha = (0.5, 1, 3), \beta = (0.5, 2, 3), a = (1, 2, 3)$ and $b = (1, 2, 3)$. The random sample from the BLFR distribution can be obtain using (4).
2. Compute the MLE's of the parameters $\boldsymbol{\theta} = (a, b, \alpha, \beta)$, say $\hat{\theta}_i^{[j]}$, for $j = 1, \ldots, 4$, $j = 1, \ldots, 10000$, using well-established routines like nlmb or optimize functions in the R statistical package.
3. Calculate the AE's and SD's given by

$$AE(\hat{\theta}_i) = \frac{1}{h}\sum_{j=1}^{h}\hat{\theta}_i^{[j]}, \qquad SD(\hat{\theta}_i) = \sqrt{\frac{1}{h-1}\sum_{j=1}^{h}(\hat{\theta}_i^{[j]} - \hat{\theta}_i)^2},$$

where $h = 10000$ is the number of replications. The results of simulation study for the BLFR distribution is shown in Table 2, which indicate the following results: (i) convergence has been achieved in all cases and this emphasizes the numerical stability of the MLE method. (ii) The differences between the average estimates and the true values are almost small. (iii) These results suggest that the MLE's have performed consistently. (iv) The standard errors of the MLE's decrease when the sample size increases. (v) The biases for each parameter either decrease

Table 2: The averages of the 10000 MLE's and mean of the simulated standard errors for BLFR distribution.

		AE				SD			
n	(α,β,a,b)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$sd(\hat{\alpha})$	$sd(\hat{\beta})$	$sd(\hat{a})$	$sd(\hat{b})$
50	(.5,.5,1,1)	0.497	0.714	1.623	1.705	0.151	0.843	2.505	2.153
	(.5,.5,1,2)	0.504	0.709	1.798	3.720	0.170	0.934	2.233	4.437
	(.5,.5,3,1)	0.491	0.730	3.579	1.937	0.116	0.540	6.525	2.658
	(1,2,1,3)	1.079	2.053	1.658	5.232	0.385	1.392	2.272	6.470
	(3,2,1,1)	4.308	4.237	1.222	2.635	6.246	2.982	1.654	7.820
	(3,3,3,3)	3.086	3.874	3.131	3.859	1.307	1.923	3.073	4.716
200	(.5,.5,1,1)	0.489	0.799	1.465	1.121	0.074	0.909	2.022	0.825
	(.5,.5,1,2)	0.488	0.777	1.642	2.495	0.0845	0.955	2.227	1.902
	(.5,.5,3,1)	0.494	0.737	3.762	1.036	0.057	0.498	4.427	1.038
	(1,2,1,3)	1.005	2.062	1.264	4.522	0.197	1.122	0.810	4.719
	(3,2,1,1)	3.070	4.021	0.944	1.987	1.311	2.944	0.967	3.587
	(3,3,3,3)	2.999	3.766	3.053	3.626	0.633	2.025	1.940	3.962

or increase to zero as $n \to \infty$.

7 Conclusion

We define a new model, called the BLFR distributions, which generalizes the LFR and GLFR distributions. The BLFR distributions contain the GLFR, LFR,GR, GE, Rayleigh and exponential distributions as special cases. The BLFR distribution present hazard functions with a very flexible behavior. We obtain closed form expressions for the moments. The Maximum likelihood estimation is discussed. Finally, we fitted BLFR distribution to a real data set to show the potential of the new proposed class.

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