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Approximating the Distributions of Singular Quadratic Expressions and their Ratios

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Abstract. Noncentral indefinite quadratic expressions in possibly non-singular normal vectors are represented in terms of the difference of two positive definite quadratic forms and an independently distributed linear combination of standard normal random variables. This result also applies to quadratic forms in singular normal vectors for which no general representation is currently available. The distribution of the positive definite quadratic forms involved in the representations is approximated by means of gamma-type distributions. We are also considering general ratios of quadratic forms, as well as ratios whose denominator involves an idempotent matrix and ratios for which the quadratic form in the denominator is positive definite. Additionally, an approximation to the density of ratios of quadratic expressions in singular normal vectors is being proposed. The results are applied to the Durbin-Watson statistic and Burg's estimator, both of which are expressible as ratios of quadratic forms.

Keywords. Burg's estimator, density approximation, Durbin-Watson statistic, indefinite quadratic expressions, quadratic forms, simulations, singular Gaussian vectors.

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1 Introduction

Numerous results are available in connection with the distribution of ratios of quadratic forms in normal random variables. However, to our knowledge, the case of quadratic forms and quadratic expressions in possibly singular normal vectors and their ratios has yet to be fully developed. In particular, when dealing with quadratic forms in singular normal vectors, it has been assumed in the literature that the rank of the matrix of the quadratic form is greater than or equal to that of the covariance matrix of the singular normal vector. This is the case for instance, in Representation 3.1a.5 in [21] and Equation (1) in [38] which do not involve a linear term. Such a term is present in the general representation given in Equation (3).

This paper provides a methodology that yields very accurate approximations to the density and distribution functions of *any* quadratic form or expression in singular normal vectors. Such quadratic forms are involved for instance in singular linear models as pointed out in [29], in least-squares estimators as discussed in [13] and in genetic studies in connection with genome scans and the determination of haplotype frequencies as explained in [38]. It should be noted that the computational routines that are currently available for determining the distribution of quadratic forms do not address the singular case. Conditions for the independence of quadratic expressions and quadratic forms in singular normal vectors are discussed for example in [36] and [27]. The latter as well as [13] also provide necessary and sufficient conditions for such quadratic forms to follow a chi-square distribution.

One of the first papers that extended the study of quadratic forms to the study of their ratios is due to [30]. The notion of mixture distributions was utilized to obtain convergent series expansions for the distribution of a positive definite quadratic form as well as that of certain ratios thereof; the distribution function of the ratio of two independent central positive definite quadratic forms in normal variables was obtained as a double infinite series of beta distributions. A mixture representation is utilized in [2] to derive the moments of the ratios. Inequalities applying to ratios of quadratic forms in independent normal variates were obtained in [16].

Ratios of independent quadratic forms involving chi-squares having even degrees of freedom are considered in [3]. An inversion formula for the distribution of ratios of linear combinations of chi-square random variables is derived in [10]. An expressions for the moments of the ratios of certain quadratic forms as well as conditions for their existence is

provided in [19]. Other results on the moments of ratios of quadratic forms may be found in [18], [15], [34] and [31].

The moments of the quantity Q_1/Q_2 with $Q_1 = \sum a_i X_i + \sum c_i Z_i$ and $Q_2 = \sum b_i Y_i + \sum d_i Z_i$ where X_i, Y_i, Z_i are mutually independent chi-square random variables, are derived in [5]; a representation of the moments about the origin of the ratio Q_1/Q_2 was obtained in closed form in [24]. Representations of the distribution function of ratios of sums of gamma random variables were derived in [25] and [26]. The fractional moments of certain quadratic expressions were obtained in [20]. Some versions of Cochran's theorem for generalized quadratic expressions are discussed in [41].

Numerous estimators and test statistics can be expressed as ratios of quadratic forms. For example, the ratio of the mean square successive differences to the variance is studied in [40]; a statistic involved in a two-stage test is considered in [39]; test statistics having this structure are derived in connection with a two-way analysis of variance for stationary periodic time series in [37]; certain ratios used in time series analysis were investigated in [7] and [23]; and test statistics related to some general linear models are considered in [17].

Ratios of quadratic forms that are connected to certain analysis of variance problems such as the determination of the effects of inequality of variance and of correlation between errors in the two-way classification, are considered in [4]. Another example is the sample circular serial correlation coefficient associated with a first order Gaussian autoregressive process, X_t, which, in [42], was taken to be an estimator of the parameter ρ in the stochastic difference equation $X_t = \rho X_{t-1} + U_t$ where the $U_t^{'}$ s are independent standard normal variables. The first few terms for the series expansions of the first and second moments of this serial correlation coefficient are derived in [32]. An approximation to the distribution of the ratio of two quadratic forms in connection with time series valued designs is discussed in [37]. A statistic whose structure is a ratio of two sums of gamma variables for the problem of testing the equality of two gamma populations with common shape parameter is derived in [33]. The sample serial correlation coefficient as defined in [1] and discussed in [28] as well as the sample innovation cross-correlation function for an ARMA time series whose asymptotic distribution was derived in [22], also have such a structure. Certain ratios of quadratic forms are investigated in [17] in the context of the general linear model. An approximation to the null distribution of the Durbin-Watson statistic which tests for autoregressive disturbances in a linear regression model

with a lagged dependent variable is considered in [14].

A decomposition of noncentral indefinite quadratic expressions in possibly singular Gaussian vectors is provided in Section 2. It should be noted that only real Gaussian vectors are being considered in this paper. The moments of such quadratic expressions are determined from a certain recursive relationship involving their cumulants in Section 3. Approximations to the distribution of quadratic expressions by means of gamma-type distributions and polynomially adjusted density functions are introduced in Section 4. Ratios of quadratic forms are discussed in Section 5. More specifically, ratios whose distribution can be determined from that of the difference of positive definite quadratic forms and ratios involving idempotent or positive definite matrices in their denominators are being considered, and suitable approaches are proposed for approximating their distributions. Several illustrative examples are provided including applications to the Durbin-Watson statistic and Burg's estimator. Section 6 focuses on the case of ratios of quadratic expressions in singular normal vectors.

2 A Decomposition for Indefinite Quadratic Expressions

A decomposition of noncentral indefinite quadratic expressions in possibly singular normal vectors is given in terms of the difference of two positive definite quadratic forms and an independently distributed normal random variable. Their moments are determined from a certain recursive relationship involving their cumulants.

Let $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X}+\mathbf{a}'\mathbf{X}+d$ be a quadratic expression in a possibly singular normal vector $\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma \geq 0$, A is a $p \times p$ real symmetric matrix, $\mathbf{a}$ is a p-dimensional vector and d is a scalar constant. Let the rank of Σ be $r \leq p$; we make use of the spectral decomposition theorem to express Σ as UWU' where W is a diagonal matrix whose first r diagonal elements are positive, the remaining diagonal elements, if any, being equal to zero. Next, we let $B^*_{p\times p} = UW^{1/2}$ and remove the $p-r$ last columns of B^*, which are null vectors, to obtain the matrix $B_{p\times r}$. Then, it can be verified that $\Sigma = BB'$. When Σ is nonsingular, B is taken to be the symmetric square root of Σ, that is, $\Sigma^{1/2}$ whose dimension is $p \times p$. On expressing $\mathbf{X}$ as $\boldsymbol{\mu} + B\,\mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}_r(\mathbf{0}, I)$,

$r \leq p$, one has

$$\begin{aligned} Q^*(\mathbf{X}) &\equiv Q^*(\mathbf{Z}) = (\boldsymbol{\mu} + B\mathbf{Z})'A(\boldsymbol{\mu} + B\mathbf{Z}) + \mathbf{a}'(\boldsymbol{\mu} + B\mathbf{Z}) + d \\ &= \boldsymbol{\mu}'A\boldsymbol{\mu} + 2\boldsymbol{\mu}'A\,B\mathbf{Z} + \mathbf{Z}'B'AB\mathbf{Z} + \mathbf{a}'B\mathbf{Z} + \mathbf{a}'\boldsymbol{\mu} + d. \end{aligned}$$

Let P be an $r \times r$ orthogonal matrix such that $P'B'ABP = \mathcal{D}iag(\lambda_1, \ldots, \lambda_r)$, where $\lambda_1, \ldots, \lambda_{r_1}$ are the positive eigenvalues of $B'AB$, $\lambda_{r_1+1} = \cdots = \lambda_{r_1+\theta} = 0$ and $\lambda_{r_1+\theta+1}, \ldots, \lambda_r$ are the negative eigenvalues of $B'AB$, $\mathbf{m}' = (m_1, \ldots, m_r) = \mathbf{a}'BP$, $\mathbf{b}^{*\prime} = (b_1^*, \ldots, b_r^*) = \boldsymbol{\mu}'ABP$, d is a scalar constant, $B'AB \neq O$ and $c_1 = \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d$. Then, on letting $\mathbf{W} = P'\mathbf{Z}$ and noting that $\mathbf{W} = (W_1, \ldots, W_r)' \sim \mathcal{N}_r(\mathbf{0}, I_r)$, one has

$$\begin{aligned} & Q^*(\mathbf{X}) \\ &\equiv Q^*(\mathbf{W}) = 2\mathbf{b}^{*\prime}\mathbf{W} + \mathbf{W}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{W} + \mathbf{m}'\mathbf{W} + c_1 \\ &= (\mathbf{m}' + 2\mathbf{b}^{*\prime})\mathbf{W} + \mathbf{W}'\mathcal{D}iag(\lambda_1, \ldots, \lambda_r)\mathbf{W} + c_1 \\ &= 2\sum_{j=1}^{r}(\frac{1}{2}m_j + b_j^*)W_j + \sum_{j=1}^{r}\lambda_j W_j^2 + c_1 \\ &= 2\sum_{j=1}^{r_1} n_j W_j + \sum_{j=1}^{r_1}\lambda_j W_j^2 + 2\sum_{j=r_1+\theta+1}^{r} n_j W_j - \sum_{j=r_1+\theta+1}^{r}|\lambda_j| W_j^2 \\ &\qquad\qquad +2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + c_1 \\ &= \sum_{j=1}^{r_1}\lambda_j\Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r}|\lambda_j|\Big(W_j + \frac{n_j}{\lambda_j}\Big)^2 + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j \\ &\qquad\qquad +\Big(c_1 - \sum_{j=1}^{r_1}\frac{n_j^2}{\lambda_j} - \sum_{j=r_1+\theta+1}^{r}\frac{n_j^2}{\lambda_j}\Big) \\ &\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + 2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1 \\ &\equiv Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1, \end{aligned} \tag{2.1}$$

where $Q_1(\mathbf{W}^+)$ and $Q_2(\mathbf{W}^-)$ are positive definite quadratic forms with $\mathbf{W}^+ = (W_1 + n_1/\lambda_1, \ldots, W_{r_1} + n_{r_1}/\lambda_{r_1})' \sim \mathcal{N}_{r_1}(\boldsymbol{\nu}_1, I)$, $\boldsymbol{\nu}_1 = (n_1/\lambda_1, \ldots, n_{r_1}/\lambda_{r_1})'$, $\mathbf{W}^- = (W_{r_1+\theta+1} + n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, W_r + n_r/\lambda_r)' \sim \mathcal{N}_{r-r_1-\theta}$, $(\boldsymbol{\nu}_2, I)$, $\boldsymbol{\nu}_2 = (n_{r_1+\theta+1}/\lambda_{r_1+\theta+1}, \ldots, n_r/\lambda_r)'$, θ being number of null eigenvalues, $n_j = \frac{1}{2}m_j + b_j^*$, $\kappa_1 = \Big(c_1 - \sum_{j=1}^{r_1} n_j^2/\lambda_j - \sum_{j=r_1+\theta+1}^{r} n_j^2/\lambda_j\Big)$

and $T_1 = (2\sum_{j=r_1+1}^{r_1+\theta} n_j W_j + \kappa_1) \sim \mathcal{N}(\kappa_1\,, 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2)$. Note that $T_1 = \kappa_1$ whenever $rank(A\Sigma) = rank(\Sigma)$.

When $\boldsymbol{\mu} = \mathbf{0}$, a central quadratic expression can be represented as follows:

$$\begin{aligned} Q^*(\mathbf{X}) &= \sum_{j=1}^{r} \lambda_j W_j^2 + \sum_{j=1}^{r} m_j W_j + d \\ &\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + \sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^* \\ &\equiv Q_1(\mathbf{W}_1^+) - Q_2(\mathbf{W}_1^-) + T_1^*, \end{aligned} \tag{2.2}$$

where $Q_1(\mathbf{W}_1^+)$ and $Q_2(\mathbf{W}_1^-)$ are positive definite quadratic forms with $\mathbf{W}_1^+ = (W_1 + m_1/(2\lambda_1), \ldots, W_{r_1} + m_{r_1}/(2\lambda_{r_1}))' \sim \mathcal{N}_{r_1}(\boldsymbol{\mu}_1, I)$, $\boldsymbol{\mu}_1 = (m_1/(2\lambda_1), \ldots, m_{r_1}/(2\lambda_{r_1}))'$, $\mathbf{W}_1^- = (W_{r_1+\theta+1} + m_{r_1+\theta+1}/(2\lambda_{r_1+\theta+1}), \ldots, W_r + m_r/(2\lambda_r))' \sim \mathcal{N}_{r-r_1-\theta}(\boldsymbol{\mu}_2, I)$, $\boldsymbol{\mu}_2 = (m_{r_1+\theta+1}/(2\lambda_{r_1+\theta+1}), \ldots, m_r/(2\lambda_r))'$, $\kappa_1^* = \Big(d - \sum_{j=1}^{r_1} m_j^2 \Big/ (4\lambda_j) - \sum_{j=r_1+\theta+1}^{r} m_j^2/\,(4\lambda_j)\Big)$ and $T_1^* = (\sum_{j=r_1+1}^{r_1+\theta} m_j W_j + \kappa_1^*) \sim \mathcal{N}(\kappa_1^*\,,\ \sum_{j=r_1+1}^{r_1+\theta} m_j^2)$.

When $\mathbf{a} = \mathbf{0}$ and $d = 0$, one has a quadratic form in a possibly singular normal vector whose representation is

$$\begin{aligned} Q(\mathbf{X}) &= \mathbf{X}'A\mathbf{X} = 2\sum_{j=1}^{r} b_j^* Z_j + \sum_{j=1}^{r} \lambda_j Z_j^2 + c^* \\ &= \sum_{j=1}^{r_1} \lambda_j \Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 - \sum_{j=r_1+\theta+1}^{r} |\lambda_j| \Big(Z_j + \frac{b_j^*}{\lambda_j}\Big)^2 \\ &\qquad\qquad +2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + \kappa^* \\ &\equiv Q_1(\mathbf{W}_1) - Q_2(\mathbf{W}_2) + T^*, \end{aligned} \tag{2.3}$$

where $Q_1(\mathbf{W}_1)$ and $Q_2(\mathbf{W}_2)$ are positive definite quadratic forms with $\mathbf{W}_1 = (W_1, \ldots, W_{r_1})'$, $\mathbf{W}_2 = (W_{r_1+\theta+1}, \ldots, W_r)'$ and $W_j = Z_j + b_j^*/\lambda_j$, $c^* = \boldsymbol{\mu}'A\boldsymbol{\mu}$, $\kappa^* = \Big(c^* - \sum_{j=1}^{r_1} b_j^{*2}/\lambda_j - \sum_{j=r_1+\theta+1}^{r} b_j^{*2}/\lambda_j\Big)$ and $T^* = 2\sum_{j=r_1+1}^{r_1+\theta} b_j^* Z_j + \kappa^* \sim \mathcal{N}(\kappa^*\,, 4\sum_{j=r_1+1}^{r_1+\theta} b_j^{*2})$. It should be pointed out that the representation (2.3) is new as it was previously assumed in the statistical literature that the rank of A is greater than or equal to that of Σ, which is quite restrictive; see for instance Representation 3.1a.5 in [21] and Equation (1) in [38].

The representation of a quadratic expression given in Equation (2.1) also holds for indefinite quadratic expressions in nonsingular normal vectors. In that case, $r = p$ in the derivation and the quadratic expression is also expressed as the difference of independently distributed linear combinations of non-central chi-square random variables having one degree of freedom each (or equivalently the difference of two positive definite quadratic forms) plus an independently distributed linear combination of standard normal random variable. In particular, a noncentral indefinite quadratic form in a nonsingular normal vector can be represented as the difference of two positive definite quadratic forms. When $\boldsymbol{\mu} = \mathbf{0}$, the representations involve central chi-square random variables. When the matrix A is positive semidefinite, so is Q, and then, $Q \sim Q_1$. Moreover, if A is not symmetric, it suffices to replace this matrix by $(A+A')/2$ in a quadratic form. Accordingly, it is assumed without any loss of generality that the matrices associated with all the quadratic forms appearing in this paper are symmetric.

3 Cumulants and Moments of Quadratic Expressions

Expressions for the characteristic function and the cumulant generating function of a quadratic expression in central normal vectors are for instance available from [8]. Representation of the cumulants of quadratic forms and quadratic expressions in nonsingular normal vectors, which are useful for estimating the parameters of the density approximants, are provided in this section.

Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, $A = A'$, $\mathbf{a}$ be a $p-$dimensional constant vector, d be a scalar constant, $Q^*(\mathbf{X}) = \mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ and $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$; then, the h^{th} cumulants of $Q^*(\mathbf{X})$ and $Q(\mathbf{X})$, are respectively

$$\begin{aligned} k^*(h) &= 2^{h-1}h!\Big\{\frac{\operatorname{tr}(A\Sigma)^h}{h} + \frac{1}{4}\mathbf{a}'(\Sigma A)^{h-2}\Sigma\mathbf{a} + \boldsymbol{\mu}'(A\Sigma)^{h-1}A\boldsymbol{\mu} \\ &\qquad + \mathbf{a}'(\Sigma A)^{h-1}A\boldsymbol{\mu}\Big\}, \qquad \text{for} \quad h \geq 2 \\ &= \operatorname{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} + \mathbf{a}'\boldsymbol{\mu} + d, \qquad \text{for} \quad h = 1\,; \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} k(h) &= 2^{h-1}h!\Big\{\frac{\operatorname{tr}(A\Sigma)^h}{h} + \boldsymbol{\mu}'(A\Sigma)^{h-1}A\boldsymbol{\mu}\Big\}, \qquad \text{for} \quad h \geq 2 \\ &= \operatorname{tr}(A\Sigma) + \boldsymbol{\mu}'A\boldsymbol{\mu} \quad \text{for} \quad h = 1. \end{aligned} \tag{3.2}$$

These expressions are derived in [21]. Alternatively, the h^{th} cumulant of $Q(\mathbf{X}) = \mathbf{X}'A\mathbf{X}$ can be expressed as

$$k(h) \;=\; 2^{h-1}h!\sum_{j=1}^{p}\lambda_j^h(b_j^2 + 1/h) = 2^{h-1}(h-1)!\;\theta_h \qquad (3.3)$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, $\text{tr}(\cdot)$ denotes the trace of $(\cdot)$, $\mathbf{b}' = (b_1, \ldots, b_p) = (P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'$ and $\theta_h = \sum_{j=1}^{p}\lambda_j^h(h\,b_j^2 + 1)$, $h = 1, 2, \ldots$. Note that $\text{tr}(A\Sigma)^h = \sum_{j=1}^{p}\lambda_j^h$.

In the case of a quadratic expression in a singular normal vector whose associated covariance matrix Σ has rank $r < p$, the h^{th} cumulant is

$$\begin{aligned} k^*(h) \;&=\; 2^{h-1}h!\Big\{(1/h)\text{tr}(B'AB)^h + (1/4)\mathbf{a}'B(B'AB)^{h-2}B'\mathbf{a} \\ &\quad + \boldsymbol{\mu}'AB(B'AB)^{h-2}B'A\boldsymbol{\mu} + \mathbf{a}'B(B'AB)^{h-2}B'A\boldsymbol{\mu}\Big\} \\ &=\; 2^{h-1}h!\Big\{(1/h)\text{tr}(A\Sigma)^h + (1/4)\mathbf{a}'(\Sigma A)^{h-2}\Sigma\mathbf{a} \\ &\qquad +\boldsymbol{\mu}'(A\Sigma)^{h-1}A\boldsymbol{\mu} + \mathbf{a}'(\Sigma A)^{h-1}\boldsymbol{\mu}\Big\}\,, \end{aligned} \qquad (3.4)$$

where $\Sigma = BB'$, B being of dimension $p \times r$, and by assumption $B'AB \neq O$.

In general, the moments of a random variable can be obtained from its cumulants by means of a recursive relationship given in [35], which can be deduced for instance from Theorem 3.2b.2 of [21]. Accordingly, the h^{th} moment of $Q^*(\mathbf{X})$ can be obtained as follows:

$$\mu_h^* \;=\; \sum_{i=0}^{h-1}\frac{(h-1)!}{(h-1-i)!\,i!}\,k^*(h-i)\,\mu_i^*\,, \qquad (3.5)$$

where $k^*(h)$ is as specified in (3.4).

4 Approximate Distributions for Quadratic Expressions

Since the representations of indefinite quadratic expressions involve $Q = Q_1 - Q_2$ where Q_1 and Q_2 are independently distributed positive definite quadratic forms, some approximations to the density function of Q are proposed in Sections 4.1 and 4.2.

Letting $Q(\mathbf{X}) = Q_1(\mathbf{X}_1) - Q_2(\mathbf{X}_2)$, $f_1(q_1)\,\mathcal{I}_{(\tau_1,\infty)}(q_1)$, $f_2(q_2)\,\mathcal{I}_{(\tau_2,\infty)}(q_2)$ and $h(q)\,\mathcal{I}_{\Re}(q)$ respectively denote the approximate densities of $Q_1(\mathbf{X}_1) > 0$, $Q_2(\mathbf{X}_2) > 0$ and $Q(\mathbf{X})$ where $\mathbf{X}' = (\mathbf{X}_1'\,,\,\mathbf{X}_2')$ and $\mathbf{X}_1$ and $\mathbf{X}_2$ are independently distributed, $\mathcal{I}_{\mathcal{A}}(.)$ being the indicator function with respect to the set $\mathcal{A}$, an approximation to density function of the indefinite quadratic form $Q(\mathbf{X})$ can be obtained as follows via the transformation of variables technique:

$$h(q) \;= h_n(q)\,\mathcal{I}_{(-\infty\,,\,\tau_1-\tau_2)}(q) + h_p(q)\,\mathcal{I}_{[\tau_1-\tau_2\,,\,\infty)}(q), \tag{4.1}$$

where

$$h_p(q) = \int_{q+\tau_2}^{\infty} f_1(y) f_2(y-q)\,\mathrm{d}y \tag{4.2}$$

and

$$h_n(q) = \int_{\tau_1}^{\infty} f_1(y) f_2(y-q)\,\mathrm{d}y\,. \tag{4.3}$$

In Equation (2.1), a quadratic expression is represented as the difference of two positive definite quadratic forms plus T_1, an independently distributed normal random variable. One can make use of Equation (4.1) to obtain an approximation to the distribution of $Q = Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-)$. Then, on noting that Q and T_1 are independently distributed, one has that their joint density function is $f(q,t) = h(q)\,\eta(t)$ where $\eta(t)$ is the density function of T_1. In order to determine an approximation to the distribution of $V = Q + T_1$, it suffices to apply transformation of variables technique. Letting $U = T_1$, the joint density function of U and V is $g(v\,,\,u) = f(v-u\,,\,u)|J|$ where J is the Jacobian of the inverse transformation. Thus, the density function of V is

$$s(v) = \int_{-\infty}^{\infty} g(v,u)\,\mathrm{d}u\,. \tag{4.4}$$

Approximations to the distribution of quadratic expressions by means of gamma-type distributions are discussed in the next subsection. As explained in Section 4.2, one can improve upon such approximations by resorting to polynomial adjustments.

4.1 Approximations Based on Gamma-Type Distributions

Gamma-type approximations are appropriate to approximate the density function of noncentral quadratic forms. First, let us consider the

gamma distribution whose density function is given by

$$\psi(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}\mathcal{I}_{[0,\infty)}(x)\,, \text{ for } \alpha > 0 \text{ and } \beta > 0. \tag{4.5}$$

Let the raw moments of a noncentral quadratic form be denoted by μ_j, $j = 1, 2, \ldots$; then, a gamma approximation can obtained by equating the first two moments associated with (4.5) to μ_1 and μ_2, respectively, and evaluating α and β. In this case, $\alpha\beta = \mu_1$ and $\alpha(\alpha+1)\beta^2 = \mu_2$, which yields

$$\alpha = \frac{\mu_1^2}{\mu_2 - \mu_1^2}\,, \quad \beta = \frac{\mu_2}{\mu_1} - \mu_1.$$

A generalized gamma density function can be expressed in the following form:

$$\psi_1(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,x^{\alpha\,\gamma-1}e^{-(x/\beta)^{\gamma}}\;\mathcal{I}_{[0,\infty)}(x) \tag{4.6}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Denoting its moments by m_j, $j = 0, 1, \ldots$, one has,

$$m_j = \frac{\beta^j\,\Gamma(\alpha + j/\gamma)}{\Gamma(\alpha)}\,. \tag{4.7}$$

When $\psi_1(x)$ is used to approximate the distribution of a noncentral quadratic form, the parameters α, β and γ are determined by solving simultaneously the equations

$$\mu_j = m_j \quad \text{for } j = 1, 2, 3, \tag{4.8}$$

which are nonlinear.

A four-parameter gamma or shifted generalized gamma density function is given by

$$\psi_2(x) = \frac{\gamma}{\beta^{\alpha\,\gamma}\Gamma(\alpha)}\,(x-\tau)^{\alpha\,\gamma-1}e^{-(\frac{x-\tau}{\beta})^{\gamma}}\;\mathcal{I}_{[\tau,\infty)}(x) \tag{4.9}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\tau \in \Re$.

Letting $\tau_1 = \tau_2 = 0$ in Equation (4.1) and making use of gamma approximations as specified by (4.5), the density function of $Q(\mathbf{X}) = Q_1(\mathbf{X}) - Q_2(\mathbf{X})$ is given by $h_n(q)\,\mathcal{I}_{(-\infty\,,0)}(q) + h_p(q)\,\mathcal{I}_{[0\,,\infty)}(q)$, with

$$\begin{aligned} h_n(q) &= \int_0^\infty f_1(y)\, f_2(y-q)\, \mathrm{d}y \\ &= \frac{\beta_1^{-\alpha_1}\, \beta_2^{-\alpha_2}}{\Gamma(\alpha_2)}\, e^{q/\beta_2}\, \vartheta^{-(\alpha_1+\alpha_2)/2}\, e^{-\vartheta\, q/2}\, (-q)^{(\alpha_1+\alpha_2-2)/2} \\ &\quad \times W_{(\alpha_2-\alpha_1)/2,(1-\alpha_1-\alpha_2)/2}(-\vartheta q) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} h_p(q) &= \int_q^\infty f_1(y)\, f_2(y-q)\, \mathrm{d}y \\ &= \frac{\beta_1^{-\alpha_1}\, \beta_2^{-\alpha_2}}{\Gamma(\alpha_1)}\, e^{q/\beta_2}\, \vartheta^{-(\alpha_1+\alpha_2)/2}\, e^{-\vartheta\, q/2}\, q^{(\alpha_1+\alpha_2-2)/2} \\ &\quad \times W_{(\alpha_1-\alpha_2)/2,(1-\alpha_1-\alpha_2)/2}(\vartheta q) \end{aligned} \tag{4.11}$$

where $\vartheta = \frac{\beta_1+\beta_2}{\beta_1\beta_2}$, $\vartheta q \neq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $(1-\alpha_1-\alpha_2)$ is not a negative integer or zero and $W(\cdot)$ denotes the Whittaker function, which on making use of some identities given in Sections 9.220 and 9.210.1 of [9] can be expressed as follows:

$$\begin{aligned} W_{l,\mu}(z) &= \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-l)}\, z^{\mu+\frac{1}{2}}\, e^{-z/2}\, {}_1F_1\left(\mu - l + \frac{1}{2}; 2\mu+1; z\right) \\ &+ \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-l)}\, z^{-\mu+\frac{1}{2}}\, e^{-z/2}\, {}_1F_1\left(-\mu - l + \frac{1}{2}; -2\mu+1; z\right) \end{aligned}$$

where ${}_1F_1(a,b,z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b)\, z^k}{\Gamma(a)\,\Gamma(b+k)\, k!}$.

Consider the singular quadratic expression $Q^*(\mathbf{X})$ and its decomposition, that is, $Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-) + T_1$. The approximate density function of $Q = Q_1(\mathbf{W}^+) - Q_2(\mathbf{W}^-)$ is as specified in (4.10) and (4.11) while $T_1 \sim \mathcal{N}(\kappa_1\,,\, 4\, \sum_{j=r_1+1}^{r_1+\theta} n_j^2)$ is distributed independently of $Q_1(\mathbf{W}^+)$ and $Q_2(\ \mathbf{W}^-)$. The density function of T_1 being $\eta(t) = (1/(\sqrt{(}2\pi\sigma^2))$ $e^{-(t-\kappa_1)^2/(2\sigma^2)}$ where $\sigma^2 = 4\sum_{j=r_1+1}^{r_1+\theta} n_j^2$, it follows from Equation (4.4)

that the approximate density function of $V = Q + T_1$ is

$$\begin{aligned} s(v) &= \int_{-\infty}^{\infty} g(v\,,\,u)\mathrm{d}u = \int_{-\infty}^{\infty} h(v-u)\eta(u)\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \Big(h_n(v-u)\,\mathcal{I}_{(-\infty,0)}(v-u)\eta(u) \\ &\qquad\qquad + h_p(v-u)\,\mathcal{I}_{(0,\infty)}(v-u)\eta(u)\Big)\mathrm{d}u \\ &= \int_{v}^{\infty} h_n(v-u)\,\eta(u)\,\mathrm{d}u + \int_{-\infty}^{v} h_p(v-u)\,\eta(u)\,\mathrm{d}u \\ &\equiv (s_n(v) + s_p(v))\mathcal{I}_{\Re}(v) \end{aligned} \tag{4.12}$$

with

$$\begin{aligned} s_n(v) &= \int_{v}^{\infty} h_n(v-u)\,\eta(u)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \Bigg\{ \frac{1}{\sqrt{\pi}k!\Gamma(\alpha_2)} 2^{\frac{k}{2}-2} e^{-(v-\kappa_1)^2/(2\sigma^2)}\,\beta_1^{\alpha_2}\vartheta^k\beta_2^{\alpha_1-2}b^{-a}\,\sigma^{k-2} \\ &\quad \times \Bigg(\frac{1}{\Gamma(1-\alpha_2)^2\Gamma(k-a+2)} 2^{\frac{a}{2}}\beta_2\Gamma(k-\alpha_2+1)\Gamma(1-a) \\ &\quad \times\Gamma(2-a)\Bigg(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right)\,{}_1F_1\left(\frac{1}{2}(k+a);\frac{1}{2};\zeta\right) \\ &\quad -2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k+a+1}{2}\right)\,{}_1F_1\Big(\frac{k+a+1}{2};\frac{3}{2};\zeta\Big)\Bigg) \\ &\quad \times(\vartheta\sigma)^a + \frac{1}{\beta_1\Gamma(\alpha_1)^2\Gamma(k+a)}\sqrt{2}\beta_2\sigma\Gamma(k+\alpha_1) \\ &\quad \times\Gamma(a-1)\Gamma(a)\left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\zeta\right)\right. \\ &\quad \left. -2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\zeta\right)\right) \\ &\quad +\frac{1}{\Gamma(\alpha_1)^2\Gamma(k+a)}2\sigma\Gamma(k+\alpha_1)\Gamma(a-1)\Gamma(a) \\ &\quad \times\left(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right)\,{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\zeta\right)\,\sqrt{2}\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\right. \\ &\quad \left. -\Gamma\left(\frac{k}{2}+1\right)\,{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\zeta\right)\right\} \end{aligned} \tag{4.13}$$

and

$$
\begin{aligned}
s_p(v) &= \int_{-\infty}^{v} h_p(v-u)\,\eta(u)\,\mathrm{d}u \\
&= \sum_{k=0}^{\infty}\left\{\frac{\left(\sqrt{\pi}\sigma^2 k!\Gamma(\alpha_1)^2\right)^{-1} 2^{\frac{k}{2}-2} e^{-\frac{(v-\kappa_1)^2}{2\sigma^2}} \beta_1^{-\alpha_1}\beta_2^{-\alpha_2-2}}{(\Gamma(1-\alpha_2)\Gamma(k-a+2)\Gamma(\alpha_2))}\right. \\
&\quad \times\left(\frac{1}{\Gamma(1-\alpha_1)\Gamma(k+a)} 2^{\frac{a}{2}}\beta_2\sigma^{k+a}\Gamma(k+\alpha_1)\Gamma(1-a)\,\Gamma(a)\right. \\
&\quad \times\left(\sqrt{2}\beta_2\sigma\Gamma\left(\frac{1}{2}(k+a)\right)\,{}_1F_1\left(\frac{1}{2}(k+a);\frac{1}{2};\zeta\right)\right. \\
&\quad +2\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{1}{2}(k+a+1)\right) \\
&\quad \times{}_1F_1\left(\frac{1}{2}(k+a+1);\frac{3}{2};\zeta\right)(-\vartheta)^k + \left(2\beta_2\left(\frac{1}{\vartheta}\right)^{a-1}\sigma\,(\vartheta\sigma)^k\right. \\
&\quad \times\Gamma(\alpha_1)\Gamma(k-\alpha_2+1)\Gamma(2-a)\Gamma(a-1)\Big(\beta_2\sigma\Gamma\left(\frac{k+1}{2}\right) \\
&\quad \times{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\zeta\right)+\sqrt{2}\left(\sigma^2+v\beta_2-\beta_2\kappa_1\right)\Gamma\left(\frac{k}{2}+1\right) \\
&\quad \left.\left.\left.\left.\times{}_1F_1\left(\frac{k+2}{2};\frac{3}{2};\zeta\right)\right)\right)\right\}
\end{aligned}
$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $a = \alpha_1 + \alpha_2$, $b = \beta_1 + \beta_2$, $\vartheta = (\beta_1+\beta_2)/(\beta_1\beta_2)$, $\zeta = (\sigma^2+v\beta_2-\beta_2\kappa_1)^2/(2\beta^2\sigma^2)$, and it is assumed that α_1 and α_2 are not positive integers and that $(1-\alpha_1-\alpha_2)$ is not a negative integer or zero.

4.2 Polynomially Adjusted Density Functions

In this section, the approximate densities are adjusted with polynomials whose coefficients are such that the first n moments of the approximation coincide with the first n moments of a given quadratic form.

In order to approximate the density function of a noncentral indefinite quadratic form $Q(\mathbf{X}) = Q_1(\mathbf{X}) - Q_2(\mathbf{X})$, one should first approximate the density functions of the two positive definite quadratic forms, $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$. According to Equation (3.5), the moments of the positive definite quadratic form $Q_1(\mathbf{X})$ denoted by $\mu_j^{(1)}$, $j = 1, 2, \ldots$, can be obtained recursively from its cumulants. Then, on the basis of

the first n moments of $Q_1(\mathbf{X})$, a density approximation of the following form is assumed for $Q_1(\mathbf{X})$:

$$f_n(x) = \varphi(x) \sum_{j=0}^{n} \xi_j x^j \tag{4.14}$$

where $\varphi(x)$ is an initial density approximant referred to as base density function, which could be for instance, a gamma, generalized gamma or shifted generalized gamma density.

In order to determine the polynomial coefficients, ξ_j, we equate the h^{th} moment of $Q_1(\mathbf{X})$ to the h^{th} moment of the approximate distribution specified by $f_n(x)$, that is,

$$\begin{aligned} \mu_h^{(1)} &= \int_{\tau_1}^{\infty} x^h \varphi(x) \sum_{j=0}^{n} \xi_j x^j \,\mathrm{d}x = \sum_{j=0}^{n} \xi_j \int_{\tau_1}^{\infty} x^{h+j} \varphi(x) \,\mathrm{d}x \\ &= \sum_{j=0}^{n} \xi_j \; m_{h+j}, \qquad h = 0, 1, \ldots, n, \end{aligned}$$

where m_{h+j} is the $(h+j)^{\text{th}}$ moment determined from $\varphi(x)$. For the generalized gamma, m_j is given by Equation (4.7). This leads to a linear system of $(n+1)$ equations in $(n+1)$ unknowns whose solution is

$$\begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{bmatrix}^{-1} \begin{bmatrix} \mu_0^{(1)} \\ \mu_1^{(1)} \\ \vdots \\ \mu_n^{(1)} \end{bmatrix}.$$

The resulting representation of the density function of $Q_1(\mathbf{X})$ will be referred to as a polynomially adjusted density approximant, which can be readily evaluated. As long as higher moments are available and the calculations can be carried out with sufficient precision, more accurate approximations can be obtained by making use of additional moments.

The density function of $Q_2(\mathbf{X})$ can be approximated by making use of the same procedure. An approximate density for $Q_1(\mathbf{X}) - Q_2(\mathbf{X})$ can then be obtained via Equation (4.1).

Example 4.1. Consider the singular quadratic expression $Q^*(\mathbf{X}) =$

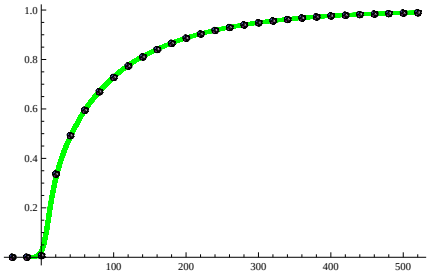

Figure 1: Approximate cdf of U_1 (dots) and simulated cdf

$\mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d$ where $\mathbf{X} \sim \mathcal{N}_5(\boldsymbol{\mu}, \Sigma)$,

$$A = \begin{pmatrix} 4 & 4 & 1 & 2 & 1 \\ 4 & 4 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 5 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{a}' = (1, 2, 3, 4, 5)$ and $d = 6$. The rank of Σ being 4, the matrix B is obtained as explained in Section 2. The eigenvalues of $B'AB$ are $\lambda_1 = 76.8865$, $\lambda_2 = 0.9121$, $\lambda_3 = 0$ and $\lambda_4 = -0.79856$. Referring to the representation specified by Equation (2.2), $r = 4$, $r_1 = 2$, $\theta = 1$, $\boldsymbol{\mu}_1 = (-0.0509184, -0.946023)'$, $\mu_2 = -1.42496$, $\kappa_1^* = 6.30293$ and $\sum_{j=r_1+1}^{r_1+\theta} m_j^2 = 20.7175$. The approximate density function was then determined from Equation (4.12). Figure 1 indicates that the gamma approximation and the simulated distribution (based on 1,000,000 replications) are in close agreement.

5 The Distribution of Ratios of Quadratic Forms

5.1 The Distribution of Ratios Expressed in terms of that of Indefinite Quadratic Forms

Let $R = Q_1(\mathbf{X})/Q_2(\mathbf{X}) = \mathbf{X}'A\mathbf{X}/\mathbf{X}'B\mathbf{X}$ where the matrices of A and B can be indefinite, the rank of B being at least one and $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$; then, one has

$$\Pr(R \le t_0) = \Pr\left(\frac{\mathbf{X}'A\mathbf{X}}{\mathbf{X}'B\mathbf{X}} \le t_0\right) = \Pr\left(\mathbf{X}'(A - t_0 B)\mathbf{X} \le 0\right). \quad (5.1)$$

Table 1: Two polynomially-adjusted approximations ($d = 10$) to the distribution function of D evaluated at certain percentage points (Sim. %) obtained by simulation

CDF	Sim. %	Gam. Poly	G. Gam. Poly
0.01	1.36069	0.010435	0.010420*
0.05	1.64792	0.050280	0.050377*
0.10	1.80977	0.099761	0.099770*
0.50	2.39014	0.495934	0.495953*
0.90	2.93742	0.902156	0.902100*
0.95	3.07679	0.952783	0.952781*
0.99	3.31005	0.991466	0.991457*

On letting $U = \mathbf{X}'(A - t_0 B)\mathbf{X}$, U can be re-expressed as a difference of two positive quadratic forms plus a constant or a linear combination of independently distributed standard normal random variables, as explained in Section 2. This approach is illustrated by the next example which involves the Durbin-Watson statistic.

Example 5.1. The statistic proposed by Durbin and Watson in [6], which in fact assesses whether the disturbances in the linear regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ are uncorrelated, can be expressed as $D = \hat{\boldsymbol{\epsilon}}' A^* \hat{\boldsymbol{\epsilon}} \,/\, \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}$ where $\hat{\boldsymbol{\epsilon}} = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}$ is the vector of residuals, $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ being the ordinary least-squares estimator of $\boldsymbol{\beta}$, and $A^* = (a^*_{ij})$ is a symmetric tridiagonal matrix with $a^*_{11} = a^*_{pp} = 1$; $a^*_{ii} = 2$, for $i = 2, \ldots, p-1$; $a^*_{ij} = -1$ if $|i-j| = 1$; and $a^*_{ij} = 0$ if $|i-j| \geq 2$. Assuming that the error vector is normally distributed, one has $\boldsymbol{\epsilon} \sim \mathcal{N}_p(\mathbf{0}, I)$ under the null hypothesis. Then, on writing $\hat{\boldsymbol{\epsilon}}$ as $M\mathbf{Y}$ where $M_{p\times p} = I - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = M'$ is an idempotent matrix of rank $p - k$, the test statistic can be expressed as the following ratio of quadratic forms:

$$D \;=\; \frac{\mathbf{Z}' M A^* M \mathbf{Z}}{\mathbf{Z}' M \mathbf{Z}}\,, \tag{5.2}$$

where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$; this can be seen from the fact that $M\mathbf{Y}$ and $M\mathbf{Z}$ are identically distributed singular normal vectors with mean vector $\mathbf{0}$ and covariance matrix MM'. The cumulative distribution function of

D at t_0 is

$$\Pr(D < t_0) = \Pr\Big(\mathbf{Z}'M(A^*M - t_0 I)\mathbf{Z} < 0\Big), \tag{5.3}$$

where $U_1 = \mathbf{Z}'M(A^*M - t_0 I)\mathbf{Z}$ is an indefinite quadratic from with $A = M(A^*M - t_0 I)$, $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$. One can obtain the moments and the various approximations of the density functions of U_1 from Equations (3.5) and (4.1). An 8^{th}-degree polynomially adjusted gamma cdf approximant is plotted in Figure 1.

We make use of a data set that is provided in [12]. In this case, there are $k = 5$ independent variables, $p = 18$, the observed value of D is 0.96, and the 13 non-zero eigenvalues of $M(A^*M - t_0 I)$ are those of MA^*M minus t_0. The non-zero eigenvalues of MA^*M are $3.92807, 3.82025$, 3.68089, 3.38335, 3.22043, 2.9572, 2.35303, 2.25696, 1.79483, 1.48804, 0.948635, 0.742294 and 0.378736. For instance, when $t_0 = 1.80977$, which corresponds to the 10^{th} percentile of the simulated cumulative distribution functions resulting from 1,000,000 replications, the eigenvalues of the positive definite quadratic form $Q_1(\mathbf{X})$ are 2.11817, 2.01035, 1.87099, 1.57345, 1.41053, 1.14734, 0.54-313 and 0.44706, while those of $Q_2(\mathbf{X})$ are 0.01507, 0.3218, 0.86126, 1.06761 and 1.43116.

Polynomially adjusted density functions were obtained for D with gamma and generalized gamma base density functions. The corresponding cumulative distribution functions were evaluated at certain percentiles of the distribution obtained by simulation. The results reported in Table 1 suggest that the polynomially adjusted generalized gamma approximation (G. Gam. Poly) is slightly more accurate. (The closer approximations are indicated with an asterisk.)

5.2 Denominators Involving Idempotent Matrices

Let $R = \mathbf{X}'A\mathbf{X}/\mathbf{X}'B\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, A is indefinite and B is idempotent. Then, as stated in [11], the h^{th} moment of the ratio of such quadratic forms is equal to the ratio of their h^{th} moments, that is, $E(R^h) = E[(\mathbf{X}'A\mathbf{X})^h]\ /E[(\mathbf{X}'B\mathbf{X})^h]$. As a matter of fact, the previous example involves such a ratio.

Example 5.2. In Example 5.1, M, the matrix of the quadratic form appearing in the denominator of D, is indeed idempotent. Thus, the h^{th} moment of D can be obtained as $E(\mathbf{Z}'MA^*M\mathbf{Z})^h/E(\mathbf{Z}'M\mathbf{Z})^h$, and polynomially adjusted generalized gamma density approximants as defined in Section 4.2 can be directly determined from the exact moments of D.

Table 2: Approximate cdf's based on the moments of D

CDF	Sim. %	G. Gam.	G. Gam. Poly
0.01	1.36069	0.011744	0.010365*
0.05	1.64792	0.050061*	0.050308
0.25	2.08536	0.243139	0.247947*
0.50	2.39014	0.495703	0.495807*
0.75	2.68610	0.754125	0.748325*
0.95	3.07679	0.952770	0.952614*
0.99	3.31005	0.989273*	0.991458

A polynomial adjustment of degree $d = 10$ was made. The approximate cumulative distribution functions resulting from the generalized gamma and the polynomially adjusted generalized gamma were evaluated at certain percentiles obtained from the empirical distribution generated from 1,000,000 replications. The results reported in Table 2 indicate that the proposed approximations are very accurate.

5.3 Denominators Involving Positive Definite Matrices

In this section, we consider ratios of quadratic forms for which the quadratic form in the denominator is positive definite. Accordingly, let $R = \mathbf{X}'A\mathbf{X}/\mathbf{X}'B\mathbf{X}$ where A is indefinite and B is positive definite. Then, one has the integral representation of the h^{th} moment of R given in (5.4) whenever it exists.

Letting $Q_1 = \mathbf{X}'A\mathbf{X}$ and $Q_2 = \mathbf{X}'B\mathbf{X}$,

$$\begin{aligned} E(R)^h &= E\left[(\mathbf{X}'A\mathbf{X})^h(\mathbf{X}'B\mathbf{X})^{-h}\right] \\ &= E\left(Q_1^h \ \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}e^{-Q_2 y}\right)\,\mathrm{d}y \\ &= \frac{1}{\Gamma(h)}\int_0^\infty y^{h-1}E\left(Q_1^h \ e^{-Q_2 y}\right)\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \frac{\mathrm{d}^h}{\mathrm{d}s^h} \left(M_{Q_1,Q_2}(s,-y) \mid_{s=0} \right) \mathrm{d}y \\
&= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \Big(\frac{\mathrm{d}^h}{\mathrm{d}s^h} \mid I - 2sA\Sigma + 2yB\Sigma \mid^{-1/2} \mid_{s=0} \Big) \mathrm{d}y \\
&= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \mid \Sigma^{-1} \mid^{1/2} \\
&\quad \times \Big(\frac{\mathrm{d}^h}{\mathrm{d}s^h} \mid \Sigma^{-1} - 2sA + 2yB \mid^{-1/2} \mid_{s=0} \Big) \mathrm{d}y
\end{aligned}
$$

where $M_{Q_1,Q_2}(s,y)$ is the joint moment generating function of $Q_1(\mathbf{X})$ and $Q_2(\mathbf{X})$.

In the next example, we determine the moments of Burg's estimator of an autoregressive parameter, as well as its approximate distribution.

Example 5.3. Burg's estimator of the parameter α in an AR(1) process is defined as $\bar{\alpha} = 2\sum_{t=2}^n x_t x_{t-1} / \sum_{t=2}^n \left(x_t^2 + x_{t-1}^2\right)$, which can be expressed as $\bar{\alpha} = \mathbf{X}'B_1\mathbf{X}/\mathbf{X}'B_0\mathbf{X}$ where

$$
B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}
$$

and $\mathbf{X} \sim N_n(\mathbf{0}, \Sigma)$, the inverse of the covariance matrix of an AR(1) process being

$$
\Sigma^{-1} = \begin{pmatrix} 1 & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -\alpha & 1+\alpha^2 & -\alpha \\ 0 & \cdots & 0 & -\alpha & 1 \end{pmatrix}.
$$

In light of Equation (5.4), the h^{th} moment of Burg's estimator is given by $E(\bar{\alpha})^h = E\left[(\mathbf{X}'B_1\mathbf{X})^h(\mathbf{X}'B_0\mathbf{X})^{-h}\right]$. Thus letting $Q_2 = \mathbf{X}'B_0\mathbf{X}$ and $Q_1 = \mathbf{X}'B_1\mathbf{X}$, one has

$$
\begin{aligned}
E(\bar{\alpha})^h &= E\left(Q_1^h \ \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} e^{-Q_2 y}\right) \mathrm{d}y \\
&= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} \mid \Sigma^{-1} \mid^{1/2} \\
&\quad \times \left(\frac{\mathrm{d}^h}{\mathrm{d}s^h} \mid \Sigma^{-1} - 2sB_1 + 2yB_0 \mid^{-1/2} |_{s=0}\right) \mathrm{d}y. \quad (5.4)
\end{aligned}
$$

In this case, the matrix $\Sigma^{-1} - 2sB_1 + 2yB_0$ is tridiagonal, which simplifies the calculations. Since the support of the distribution is finite, we approximate the distribution of R by making use of a beta distribution as base density function on the basis of the moments of Burg's estimator. The proposed methodology comprises the following steps:

1. The moments of $\bar{\alpha}$ with $n = 50$ and $\alpha = 0.5$ are determined from Equation (5.4).
2. Consider the following beta density function as base density

$$\phi(x) = \frac{1}{\mathrm{B}(\alpha, \beta)} \, x^{\alpha-1}(1-x)^{\beta-1} \, \mathcal{I}_{(0,1)}(x), \ \alpha > 0, \ \ \beta > 0,$$

where $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\,\Gamma(\beta)/\Gamma(\alpha+\beta)$.
3. The support (q, r) of y is mapped onto the interval (0,1) with the affine transformation, $x = (y-q)/(r-q)$, which implies that $y = x(r-q)+q$.
4. The transformed moments on $(0, 1)$ are determined from the binomial expansion.
5. The parameters of the beta density are evaluated as follows:

$$\alpha = -\mu_1 + \frac{(1-\mu_1)\,\mu_1^2}{\mu_2 - \mu_1^2}, \quad \beta = -1 - \alpha + \frac{(1-\mu_1)\,\mu_1}{\mu_2 - \mu_1^2}.$$

6. Approximate densities are obtained with and without polynomial adjustments using the procedure described in Section 4.2.

A polynomial adjustment of degree $d = 7$ was made. The cumulative distribution functions of the beta and the polynomially adjusted beta approximations were evaluated at certain empirical percentiles which were determined from 1,000,000 replications. The results reported in Table 3 clearly indicate that the proposed polynomially adjusted beta approximation is indeed very accurate.

Table 3: Approximate cdf's based on the moments of $\bar{\alpha}$ ($n = 50$ and $\alpha = 0.5$)

CDF	Sim. %	Beta	Beta poly
0.001	0.034102	0.000595	0.001177*
0.01	0.157475	0.008215	0.010906*
0.05	0.263371	0.048012	0.051062*
0.50	0.490013	0.508722	0.500073*
0.95	0.667366	0.944097	0.949802*
0.99	0.727433	0.986898	0.989970*
0.999	0.785393	0.998254	0.999007*

6 Ratios of Quadratic Expressions in Singular Vectors

Let $A_1 = A_1'$ and $A_2 = A_2'$ be indefinite matrices, $\mathbf{X}$ be a $p \times 1$ vector with $E(\mathbf{X}) = \boldsymbol{\mu}$, $\mathrm{Cov}(\mathbf{X}) = \Sigma \geq 0$, $\rho(\Sigma) = r \leq p$, $\Sigma = BB'$, B being a $p \times r$ matrix, $\mathbf{a}_1$ and $\mathbf{a}_2$ be p-dimensional constant vectors and d_1 and d_2 be scalar constants. Then, letting $Q_1^*(\mathbf{X}) = \mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1$ and $Q_2^*(\mathbf{X}) = \mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2$, the distribution of the ratio of quadratic expressions

$$R = \frac{\mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1}{\mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2}, \tag{6.1}$$

can be determined by noting that

$$\begin{aligned} F_R(t_0) &= \Pr(R \leq t_0) = \Pr(Q_1^*(\mathbf{X}) - t_0 Q_2^*(\mathbf{X}) \leq 0) \\ &= \Pr((\mathbf{X}'A_1\mathbf{X} + \mathbf{a}_1'\mathbf{X} + d_1) - t_0(\mathbf{X}'A_2\mathbf{X} + \mathbf{a}_2'\mathbf{X} + d_2) \leq 0) \\ &= \Pr(\mathbf{X}'A\mathbf{X} + \mathbf{a}'\mathbf{X} + d \leq 0) \end{aligned} \tag{6.2}$$

where $A = A_1 - t_0\, A_2$, $\mathbf{a}' = \mathbf{a}_1' - t_0\, \mathbf{a}_2'$ and $d = d_1 - t_0\, d_2$.

According to (2.1), the distribution of R can be obtained in terms of that of a difference of two positive quadratic forms plus a constant or an independently distributed normal random variable.

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