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Power Function Distribution Characterized by Dual Generalized Order Statistics

Mahdi Tavangar

Department of Statistics, University of Isfahan, Iran.

Abstract. The dual generalized order statistics is a unified model which contains the well known decreasingly ordered random data such as (reversed ordered) order statistics and lower record values. In the present paper, some characterization results on the power function distribution based on the properties of dual generalized order statistics are provided. The results are proved without any restriction on the parameters of the model of dual GOS.

Keywords. Generalized Pareto distributions, lower Pfeifer records, lower record values, mean inactivity time function, order statistics.

MSC: 62E10, 62H05.

1 Introduction

In the literature, models of ordered random variables have been extensively studied. In testing the strength of materials, reliability analysis, lifetime studies, etc., the realizations of the experiment arise in non-decreasing order and therefore we need to consider several models of ascendingly ordered random variables. Theoretically, many of these models are contained in the model of generalized order statistics (GOS). The concept of GOS has been introduced by Kamps (1995). He showed that usual order statistics, sequential order statistics, upper record values, progressively Type-II right censored order statistics and some other

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Mahdi Tavangar (✉)(m.tavangar@stat.ui.ac.ir)
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ordered random variables can be considered as special cases of the GOS. We refer the reader to Kamps (1995) for a comprehensive study on the properties of GOS.

Although the GOS contain many useful models of ordered random variables, the random variables that are decreasingly ordered can not be integrated into this framework. Burkschat et al. (2003) introduced so-called *dual* GOS as a systematic approach to some models of descendingly ordered random variables. The random variables $U_d(1,n,\tilde{m},k)$, $U_d(2,n,\tilde{m},k),\ldots,U_d(n,n,\tilde{m},k)$ are called *uniform* dual GOS if their joint density function is given by

$$\begin{aligned}
&f_{U_d(1,n,\tilde{m},k),U_d(2,n,\tilde{m},k),\ldots,U_d(n,n,\tilde{m},k)}(u_1,u_2,\ldots,u_n)\\
&\qquad = c_{n-1}\left\{\prod_{j=1}^{n-1} u_j^{m_j}\right\} u_n^{\gamma_n-1}, \quad 1 \geq u_1 \geq u_2 \geq \cdots \geq u_n > 0,
\end{aligned}$$

where $n \in \mathbb{N}$, $k > 0$ and $m_1, m_2, \ldots, m_{n-1} \in \mathbb{R}$ such that $\gamma_r = k + \sum_{j=r}^{n-1}(m_j+1) > 0$ for all $r \in \{1,2,\ldots,n-1\}$, $c_{n-1} = \prod_{j=1}^{n}\gamma_j$, and $\widetilde{m} = (m_1,m_2,\ldots,m_{n-1})$, if $n \geq 2$ ($\widetilde{m} \in \mathbb{R}$ arbitrary, if $n=1$). The quantile transformation $X_d(r,n,\tilde{m},k) = F^{-1}(U_d(r,n,\tilde{m},k))$, $r = 1,2,\ldots,n$, yields dual GOS based on an arbitrary cumulative distribution function (cdf) F, where $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$, is called the quantile (or generalized inverse) function. The joint density function of the first r dual GOS based on an absolutely continuous cdf F is then given by

$$\begin{aligned}
&f_{X_d(1,n,\tilde{m},k),\ldots,X_d(r,n,\tilde{m},k)}(x_1,\ldots,x_r)\\
&\qquad = c_{r-1}\left\{\prod_{j=1}^{r-1}\{F(x_j)\}^{m_j} f(x_j)\right\}\{F(x_r)\}^{\gamma_r-1} f(x_r), \qquad (1)\\
&\qquad x_1 \geq x_2 \geq \cdots \geq x_r.
\end{aligned}$$

Dual GOS represent a unification of models of decreasingly ordered random variables, e.g., reversed ordered order statistics, lower records, lower k-records, and lower Pfeifer records.

The aim of the present paper is to give various characterization results on the power function distribution based on dual GOS. We prove the results without any restriction on the parameters of the model of dual GOS. The rest of this paper is organized as follows. In Section 2, we present a characterization result on the power function distribution based on the independence of a normalized spacing between dual GOS

$X_d(r, n, \tilde{m}, k) - X_d(s, n, \tilde{m}, k)$ and $X_d(r, n, \tilde{m}, k)$ where $1 \leq r < s \leq n$. In Section 3, we prove a characterization based on the identical distributions of some time dependent functions of dual GOS.

Throughout the paper, for any random variable W, f_W and F_W denote the probability density function and the cdf of W, respectively.

2 Characterizations Involving Dependency Assumptions

Let X be a lifetime (non-negative) random variable with survival function $\bar{F}$. The random variable X is said to have generalized Pareto distribution (GPD) with parameter vector (a, b, β), if its survival function is given by

$$\bar{F}(x) = \left(\frac{b}{a(x-\beta)+b} \right)^{\frac{1}{a}+1}, \quad x \geq \beta.$$

where $a > -1$, $b > 0$ and $\beta \in \mathbb{R}$. This family of distributions includes, depending on the values of a, three distributions; in the case where $a \to 0$, the distribution is exponential, for $a > 0$, it is Pareto, and for $-1 < a < 0$, it is a rescaled beta model. Note that for $-1 < a < 0$ the distribution is bounded above. Recently, Asadi and Bayramoglu (2006), Tavangar and Asadi (2007, 2008, 2010), Hashemi and Asadi (2007) and Hashemi et al. (2010) provided some characterization results on the GPD based on GOS or its specialized versions. Applications of the GPD have been extensively investigated in the literature. It is successfully applied and widely used in a number of statistical problems related to finance, insurance, hydrological frequency analysis and other areas.

In reliability theory and survival analysis to study the lifetime properties of a component (or any other living organism) there are several measures such as the mean residual life function and the mean inactivity time (or mean past lifetime) function. Let T be a lifetime random variable with continuous cdf F and survival function $\bar{F} = 1 - F$. The mean residual life function $m(t)$ and mean inactivity time function $m^*(t)$ are defined as $m(t) = E(T - t \mid T > t)$ and $m^*(t) = E(t - T \mid T \leq t)$. We refer the reader to Kotz and Shanbhag (1980) and Nanda et al. (2003) for some results regarding these measures.

In the following result, we provide a characterization of the GPD. The proof is similar to the proof of Theorem 3.4 of Asadi et al. (2001), so it will be omitted.

Theorem 2.1. *Let $\bar{F}$ be a continuous survival function on (β, ∞), $\beta \in \mathbb{R}$, where we define it so that $\bar{F}(x) = 1 - F(x)$, $x \in (\beta, \infty)$, with F as a cdf concentrated on (β, ∞) such that the corresponding distribution has a finite mean. Let $\theta : (\beta, \infty) \to \mathbb{R}_+$. Then there exist a point $x_0 \in (\beta, \infty)$ with $\bar{F}(x_0) > 0$ and a sequence $\{x_n : n = 1, 2, ...\}$ of points lying in (x_0, ∞) such that it converges to x_0 and*

$$\bar{F}(x_n + \theta(x_n)y) = \bar{F}(x_n)\bar{F}(y + \beta), \quad n = 0, 1, 2, ...; \; y > 0, \tag{2}$$

if and only if the distribution corresponding to $\bar{F}$ is GPD with parameter vector $(.,.,\beta)$, and $\theta(x) = m(x)/m(\beta)$, where $m(x)$ is the mean residual life function corresponding to $\bar{F}$.

Remark 2.2. Eq. (2) has a probabilistic interpretation as follows. Let X be a continuous random variable on (β, ∞). Then Eq. (2) is equivalent to $P\{X - x_n > \theta(x_n)y \mid X > x_n\} = P\{X - \beta > y\}$, $n = 0, 1, 2, ...; y > 0$. Hence Theorem 2.1 provide a characterization of the power function distribution based on the equality in distribution of the residual life relative to an accelerated life model and the original life.

We now begin by establishing a general theorem, namely, Theorem 2.3, on characterization of the power function distribution; we refer to this theorem in our derivations of the main results.

Theorem 2.3. *Let F be a continuous cdf concentrated on $(0, \beta)$ and $\theta^* : (0, \beta) \to \mathbb{R}_+$. Then there exist a point $x_0 \in (0, \beta)$ with $F(x_0) > 0$, and a sequence $\{x_n : n = 1, 2, ...\}$ of points lying in $(0, x_0)$ such that it converges to x_0, and*

$$F(x_n - \theta^*(x_n)y) = F(x_n)F(\beta - y), \quad n = 0, 1, 2, ...; \; y > 0, \tag{3}$$

if and only if F is a (rescaled) power function distribution with parameter vector (α, β) of the form $F(x) = (\frac{x}{\beta})^\alpha$, $x \in (0, \beta)$, for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.

Proof. The 'if' part of the theorem is trivial and hence it is sufficient if we prove the 'only if' part. Let Eq. (3) hold. Then we have

$$\int_0^\infty F(x_n - \theta^*(x_n)y)dy = F(x_n) \int_0^\infty F(\beta - y)dy,$$

which, in turn, implies that

$$\int_0^{x_n/\theta^*(x_n)} F(x_n - \theta^*(x_n)y)dy = F(x_n) \int_0^\beta F(\beta - y)dy.$$

From the fact that

$$m^*(t) = \frac{\int_0^t F(x)dx}{F(t)},$$

it follows that $\theta^*(x) = m^*(x)/m^*(\beta)$. Let X be a random variable with cdf F and let $Y = 2\beta - X$. Then for all $x \in (0, \beta]$, $m^*(x) = m(2\beta - x)$, where $m(x)$ is the mean residual life function corresponding to Y, and hence $\theta^*(x) = \theta(2\beta - x)$, where $\theta(y) = m(y)/m(\beta)$, $y > \beta$. It is easy to observe that Eq. (3) is equivalent to

$$S(t_n + \theta(t_n)y) = S(t_n)S(y + \beta), \quad y > 0, \tag{4}$$

where $t_n = 2\beta - x_n$, $n = 0, 1, 2, ...$, and $S(y)$ is the survival function of Y. Obviously $S(t_0) > 0$ and the sequence $\{t_n : n = 1, 2, ...\}$ lyes in (t_0, ∞) and it converges to t_0. Appealing now to Theorem 2.1, it follows that Y is distributed as GPD with parameter vector (a, b, β). Consequently, F belongs to the class $\mathcal{H} = \{H_{a,b,\beta} :\ a > -1, b > 0, \beta \in \mathbb{R}\}$ of distributions with

$$H_{a,b,\beta}(x) = S(2\beta - x) = \left(\frac{b}{a(\beta - x) + b}\right)^{\frac{1}{a}+1}, x \leq \beta.$$

Under the assumption $F(0) = 0$, we have $F(x) = (\frac{x}{\beta})^{\alpha}$, $x \in (0, \beta)$, with $\alpha = -(1 + \frac{1}{a})$ and $a\beta + b = 0$. Thus, we have the theorem. $\square$

Remark 2.4. The probabilistic interpretation of Eq. (3) is as follows. Let X be a continuous random variable on $(0, \beta)$. Eq. (3) can be rewritten as $P\{x_n - X > \theta^*(x_n)y \mid X \leq x_n\} = P\{\beta - X > y\}$, $n = 0, 1, 2, ...$; $y > 0$. Hence, in Theorem 2.3, we have obtained a characterization result based on the equality in distribution of the inactivity time relative to an accelerated life model and the original life.

Before stating the main result of this section, we obtain a representation for the joint density function of two dual GOS. For two consecutive dual GOS, one can easily observe that

$$\begin{aligned} &f_{U_d(r,n,\tilde{m},k),U_d(r+1,n,\tilde{m},k)}(u_r, u_{r+1}) \\ &\qquad = c_r g_{r-1}(u_r) u_r^{m_r} u_{r+1}^{\gamma_{r+1}-1},\ 1 \geq u_r \geq u_{r+1} > 0, \end{aligned} \tag{5}$$

where

$$g_{r-1}(u) = \int_{\mathcal{A}} \prod_{j=1}^{r-1} u_j^{m_j} du_1 du_2 ... du_{r-1},$$

and $\mathcal{A} = \{(u_1, ..., u_{r-1}) : 1 \geq u_1 \geq u_2 \geq \cdots \geq u_{r-1} \geq u > 0\}$. Using (5), one can also obtain the marginal density function of rth dual GOS as follows,

$$f_{X_d(r,n,\tilde{m},k)}(x_r) = c_{r-1} g_{r-1}(F(x_r))\{F(x_r)\}^{\gamma_r - 1} f(x_r), \ -\infty < x_r < \infty. \quad (6)$$

Now assume that $r + 2 \leq s \leq n$. Using (1), we have

$$f_{U_d(r,n,\tilde{m},k),U_d(s,n,\tilde{m},k)}(u_r, u_s) = c_{s-1} u_r^{m_r} g_{r-1}(u_r) u_s^{\gamma_s - 1} \times \int_{\mathcal{B}} \prod_{j=r+1}^{s-1} u_j^{m_j} du_{r+1} du_{r+2} ... du_{s-1}, \quad (7)$$

where $\mathcal{B} = \{(u_{r+1}, ..., u_{s-1}) : \ u_r \geq u_{r+1} \geq u_{r+2} \geq \cdots \geq u_{s-1} \geq u_s\}$. Upon making the transformation $u_j = u_r v_j$, $j = r+1, r+2, ..., s-1$, we obtain

$$\begin{aligned} \int_{\mathcal{B}} \prod_{j=r+1}^{s-1} u_j^{m_j} du_{r+1} du_{r+2} ... du_{s-1} &= \int_{\mathcal{C}} u_r^{\sum_{j=r+1}^{s-1}(m_j+1)} \prod_{j=r+1}^{s-1} v_j^{m_j} dv_{r+1} dv_{r+2} ... dv_{s-1} \\ &= u_r^{\gamma_{r+1} - \gamma_s} \Psi_{r+1,s-1}\left(\frac{u_s}{u_r}\right), \end{aligned} \quad (8)$$

where $\mathcal{C} = \{(v_{r+1}, ..., v_{s-1}) : 1 \geq v_{r+1} \geq v_{r+2} \geq \cdots \geq v_{s-1} \geq \frac{u_s}{u_r}\}$,

$$\Psi_{r+1,s-1}(u) = \int_{\mathcal{D}} \prod_{j=r+1}^{s-1} v_j^{m_j} dv_{r+1} dv_{r+2} ... dv_{s-1}, \ 1 \leq r < s \leq n,$$

and $\mathcal{D} = \{(v_{r+1}, ..., v_{s-1}) : \ 1 \geq v_{r+1} \geq v_{r+2} \geq \cdots \geq v_{s-1} \geq u\}$. Let us define $\Psi_{r+1,r}(u) \equiv 1$. Substituting (8) in (7), we obtain the joint density function of $X_d(r, n, \tilde{m}, k)$ and $X_d(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$, as follows,

$$\begin{aligned} f_{X_d(r,n,\tilde{m},k),X_d(s,n,\tilde{m},k)}(x_r, x_s) &= c_{s-1}\{F(x_r)\}^{\gamma_r - \gamma_s - 1} g_{r-1}(F(x_r)) f(x_r) \\ &\times \{F(x_s)\}^{\gamma_s - 1} \Psi_{r+1,s-1}\left(\frac{F(x_s)}{F(x_r)}\right) f(x_s), \ x_r > x_s. \end{aligned} \quad (9)$$

Now we are ready to give the following theorem.

Theorem 2.5. *Let $X_d(1,n,\tilde{m},k), X_d(2,n,\tilde{m},k), ..., X_d(n,n,\tilde{m},k)$ be the dual GOS based on an absolutely continuous cdf F which is concentrated on $(0,\beta)$ and $\theta^* : (0,\beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Assume that F is strictly increasing on $(0,\beta)$ and also F has continuous derivative on this interval. Then the random variable $\{X_d(r,n,\tilde{m},k) - X_d(s,n,\tilde{m},k)\}/\theta^*(X_d(r, n,\tilde{m},k))$, $1 \le r < s \le n$, is independent of $X_d(r,n,\tilde{m},k)$ if and only if F is a (rescaled) power function distribution with parameter vector (α,β), for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.*

Proof. Note that $\theta^*(x)$ is a continuous function and hence $\theta^*(X_d(r,n,\tilde{m}, k))$ is a random variable. We first prove the 'only if' part of the theorem. From the assumption of the theorem, we have for all $x, y \in (0,\beta)$,

$$P\{X_d(r,n,\tilde{m},k) - y\theta^*(X_d(r,n,\tilde{m},k)) \le X_d(s,n,\tilde{m},k), X_d(r,n,\tilde{m},k) \le x\}$$
$$= P\{X_d(r,n,\tilde{m},k) \le x\}G(y), \tag{10}$$

where

$$G(y) = P\{X_d(r,n,\tilde{m},k) - y\theta^*(X_d(r,n,\tilde{m},k)) \le X_d(s,n,\tilde{m},k)\}.$$

From (6) and (9), we get, respectively,

$$P\{X_d(r,n,\tilde{m},k) \le x\} \quad = \quad c_{r-1}\int_0^x g_{r-1}(F(u))\{F(u)\}^{\gamma_r-1}dF(u)$$

and

$$\begin{aligned}
&P\{X_d(r,n,\tilde{m},k) - y\theta^*(X_d(r,n,\tilde{m},k)) \le X_d(s,n,\tilde{m},k), X_d(r,n,\tilde{m},k) \le x\}\\
&= c_{s-1}\int_0^x \int_{u-y\theta^*(u)}^{u} \{F(u)\}^{\gamma_r-\gamma_s-1} g_{r-1}(F(u))\\
&\quad \times \{F(v)\}^{\gamma_s-1}\Psi_{r+1,s-1}\left(\frac{F(v)}{F(u)}\right) dF(v)dF(u)\\
&= c_{s-1}\int_0^x \{F(u)\}^{\gamma_r-\gamma_s-1} g_{r-1}(F(u))\\
&\quad \times \left\{\int_{\frac{F(u-y\theta^*(u))}{F(u)}}^{1} \{F(u)\}^{\gamma_s} z^{\gamma_s-1}\Psi_{r+1,s-1}(z)dz\right\} dF(u)\\
&= c_{s-1}\int_0^x \{F(u)\}^{\gamma_r-1} g_{r-1}(F(u))\\
&\quad \times \left\{\int_{\frac{F(u-y\theta^*(u))}{F(u)}}^{1} z^{\gamma_s-1}\Psi_{r+1,s-1}(z)dz\right\} dF(u).
\end{aligned}$$

Substituting these equations in (10), we have

$$c_{s-1}\int_0^x \{F(u)\}^{\gamma_r-1} g_{r-1}(F(u)) \left\{ \int_{\frac{F(u-y\theta^*(u))}{F(u)}}^1 z^{\gamma_s-1}\Psi_{r+1,s-1}(z)dz \right\} dF(u)$$
$$= c_{r-1}G(y)\int_0^x g_{r-1}(F(u))\{F(u)\}^{\gamma_r-1}dF(u).$$

Note that under the assumptions of the theorem, two integrands in the last equation are continuous on $(0, \beta)$, and F is strictly increasing and differentiable on $(0, \beta)$. By fundamental theorem of calculus, the probability density function f corresponding to F equals the derivative of F. Hence $f(x)$ and $g_{r-1}(F(x))$ must be positive on $(0, \beta)$. Therefore after differentiating the last equality with respect to x, we get

$$G(y) = \frac{c_{s-1}}{c_{r-1}} \int_{\frac{F(x-y\theta^*(x))}{F(x)}}^1 z^{\gamma_s-1}\Psi_{r+1,s-1}(z)dz.$$

This implies that the fraction $F(x - y\theta^*(x))/F(x)$ does not depend on x and is only a function of y, say, $\phi(y)$. (Note that the right hand side of the last equation is a strictly decreasing function of the lower integral bound.) By considering the limits as $x \to \beta^-$ and using the continuity of F and θ^*, we conclude that $\phi(y) = F(\beta - y)$. Now the result follows from Theorem 2.3. The 'if' part of the theorem is easy to verify and hence is omitted. The proof is complete. $\square$

In the following, we describe some applications of Theorem 2.5 to order statistics and lower k-record values.

Corollary 2.6. *Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the order statistics based on a random sample of size n from an absolutely continuous cdf F which is concentrated on $(0, \beta)$ and $\theta^* : (0, \beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Assume that F is strictly increasing on $(0, \beta)$ and also F has continuous derivative on this interval. Then the random variable $\{X_{s:n} - X_{r:n}\}/\theta^*(X_{s:n})$, $1 \leq r < s \leq 1$, and $X_{s:n}$ are independent if and only if F is a (rescaled) power function distribution with parameter vector (α, β), for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.*

Remark 2.7. For power function distribution, it can be shown that $\theta^*(t) = t/\beta$. Hence we obtain a characterization result based on the independence of $X_{r:n}/X_{s:n}$ and $X_{s:n}$ that can be used to construct goodness-of-fit tests.

Corollary 2.8. *Let $Z_1^{(k)}, Z_2^{(k)}, ..., Z_n^{(k)}$ be the first n lower k-records corresponding to a sequence of independent and identically distributed random variables with an absolutely continuous cdf F which is concentrated on $(0, \beta)$ and $\theta^* : (0, \beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Assume that F is strictly increasing on $(0, \beta)$ and also F has continuous derivative on this interval. Then the random variable $\{Z_m^{(k)} - Z_n^{(k)}\}/\theta^*(Z_m^{(k)})$, $n \geq m+1$, is independent of $Z_m^{(k)}$ if and only if F is a (rescaled) power function distribution with parameter vector (α, β), for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.*

Remark 2.9. Let F^{γ_i}, $i = 1, 2, ...$, be the underlying cdf until the ith lower record occurs, where F is an absolutely continuous cdf. Then the lower Pfeifer records are included in the model of dual GOS (see Burkschat et al., 2003) and hence, from Theorem 2.5, we obtain a characterization of the power function distribution based on the lower Pfeifer records.

3 Characterizations Based on Distributional Relationships

In this section, we prove some characterization results based on the conditional random variable

$$[X_d(s, n, \tilde{m}, k) \mid X_d(r+1, n, \tilde{m}, k) \leq t < X_d(r, n, \tilde{m}, k)], \quad 1 \leq r < s \leq n.$$

First, in the following lemma, we show that this conditional random variable can be considered as a dual GOS based on a truncated distribution.

Lemma 3.1. *Let $X_d(1, n, \tilde{m}, k), X_d(2, n, \tilde{m}, k), ..., X_d(n, n, \tilde{m}, k)$ denote the dual GOS based on any continuous cdf F. Then for each $1 \leq r < s \leq n$,*

$$[X_d(s, n, \tilde{m}, k) \mid X_d(r+1, n, \tilde{m}, k) \leq t < X_d(r, n, \tilde{m}, k)] \stackrel{d}{=} X_d^*(s-r, n-r, \tilde{\mu}_r, k),$$

where $X_d^*(s-r, \gamma_{r+1}, \tilde{\mu}_r, k)$ is the $(s-r)$th dual GOS based on the cdf F truncated from the right at t, $\tilde{\mu}_r = (m_{r+1}, ..., m_{n-1})$, and $\stackrel{d}{=}$ stands for the equality in distribution.

Proof. First we prove the result in the case where the underling distribution is uniform $U(0,1)$. Let $U_d(1,n,\tilde{m},k), U_d(2,n,\tilde{m},k), ..., U_d(n,n,\tilde{m},k)$ be uniform dual GOS and $U_d^*(s-r,n-r,\tilde{\mu}_r,k)$ denote the $(s-r)$th dual GOS based on the cdf $G_t(x)=\frac{x}{t},\ \ 0\leq x<t$. In view of (1), one can write

$$\begin{aligned} &P\{U_d(s-r,n-r,\tilde{\mu}_r,k)>x\} \\ &\qquad =\frac{c_{s-1}}{c_{r-1}}\int_{\mathcal{E}} u_{s-r}^{\gamma_s-1}\left\{\prod_{j=1}^{s-r-1} u_j^{m_{r+j}}\right\}du_1du_2...du_{s-r}, \end{aligned} \tag{11}$$

for each $x\in[0,1)$ where $\mathcal{E}=\{(u_1,...,u_{s-r}):1\geq u_1\geq u_2\geq\cdots\geq u_{s-r}>x\}$. Then it is readily seen that

$$\begin{aligned} &P\{U_d^*(s-r,n-r,\tilde{\mu}_r,k)>x\} \\ &\qquad =\frac{c_{s-1}}{c_{r-1}}\int_{\mathcal{F}} u_{s-r}^{\gamma_s-1}\left\{\prod_{j=1}^{s-r-1} u_j^{m_{r+j}}\right\}du_1du_2...du_{s-r}, \end{aligned} \tag{12}$$

where $\mathcal{F}=\{(u_1,...,u_{s-r}):1\geq u_1\geq u_2\geq\cdots\geq u_{s-r}>\frac{x}{t}\}$. From (5) and the relation

$$g_r(t)=\int_t^1 u_r^{m_r}g_{r-1}(u_r)du_r,$$

we obtain

$$P\{U_d(r+1,n,\tilde{m},k)\leq t<U_d(r,n,\tilde{m},k)\}=c_{r-1}t^{\gamma_{r+1}}g_r(t). \tag{13}$$

Hence, in the case $s=r+1$, we have

$$\begin{aligned} &P\{U_d(r+1,n,\tilde{m},k)>x\mid U_d(r+1,n,\tilde{m},k)\leq t<U_d(r,n,\tilde{m},k)\} \\ &\qquad =P\{U_d^*(1,\gamma_{r+1},\tilde{\mu}_r,k)>x\} \\ &\qquad =1-\left(\frac{x}{t}\right)^{\gamma_{r+1}},\ 0\leq x<t. \end{aligned}$$

Now suppose that $r+2\leq s\leq n$. Using (1), we have

$$\begin{aligned} &P\{x<U_d(s,n,\tilde{m},k)\leq U_d(r+1,n,\tilde{m},k)\leq t<U_d(r,n,\tilde{m},k)\} \\ &\qquad =c_{s-1}g_r(t)\int_{\mathcal{G}} u_s^{\gamma_s-1}\left\{\prod_{j=r+1}^{s-1} u_j^{m_j}\right\}du_{r+1}du_{r+2}...du_s \\ &\qquad =c_{s-1}t^{\gamma_{r+1}}g_r(t)\int_{\mathcal{F}} z_{s-r}^{\gamma_s-1}\left\{\prod_{j=1}^{s-r-1} z_j^{m_{r+j}}\right\}dz_1dz_2...dz_{s-r}, \end{aligned}$$

where $\mathcal{G} = \{(u_{r+1}, ..., u_s) : t > u_{r+1} \geq u_{r+2} \geq \cdots \geq u_s > x\}$, and in the last equality we make use of the transformation $u_{r+j} = tz_j, \quad j = 1, 2, ..., s-r$. It follows that

$$P\{U_d(s, n, \tilde{m}, k) > x \mid U_d(r+1, n, \tilde{m}, k) \leq t < U_d(r, n, \tilde{m}, k)\}$$
$$= \frac{c_{s-1}}{c_{r-1}} \int_{\mathcal{F}} z_{s-r}^{\gamma_s - 1} \left\{ \prod_{j=1}^{s-r-1} z_j^{m_{r+j}} \right\} dz_1 dz_2 ... dz_{s-r}. \tag{14}$$

From (12) and (14), we conclude that the result of the lemma is true for uniform dual GOS. It follows from the quantile transformation and the relation

$$X_d^*(s-r, n-r, \tilde{\mu}_r, k) \stackrel{d}{=} F^{-1}(U_d^*(s-r, n-r, \tilde{\mu}_r, k))$$

that the result is also true for dual GOS based on an arbitrary continuous cdf F. This completes the proof of the result. $\square$

Remark 3.2. Let the support of the absolutely continuous cdf F is $(\alpha(F), \omega(F))$ and F is strictly increasing on this interval. Taking a clue from Eq. (13) and using the density function of a dual GOS in (6), one could obtain the following relation between distributions of $X_d(r, n, \tilde{m}, k)$ and $X_d(r+1, n, \tilde{m}, k)$:

$$F_{X_d(r+1,n,\tilde{m},k)}(x) - F_{X_d(r,n,\tilde{m},k)}(x)$$
$$= \frac{1}{\gamma_{r+1}} \frac{F(x)}{f(x)} f_{X_d(r+1,n,\tilde{m},k)}(x), \quad x \in (\alpha(F), \omega(F)). \tag{15}$$

Similar recurrence relations for distributions of order statistics can be found in Balasubramanian et al. (1992).

Remark 3.3. Let conditions of Remark 3.2 hold. It is known that the cdf of $X(r, n, \tilde{m}, k)$ or $X_d(r, n, \tilde{m}, k)$, for any fixed $r = 1, 2, ..., n$, uniquely identifies F; though there is not any closed form for F in terms of the cdf of (dual) GOS. The following representation shows that F can be recovered (in a closed form) from the distributions of two adjacent dual GOS:

$$F(x) = \exp\left\{ -\frac{1}{\gamma_{r+1}} \int_x^{\omega(F)} \frac{f_{X_d(r+1,n,\tilde{m},k)}(t)}{F_{X_d(r+1,n,\tilde{m},k)}(t) - F_{X_d(r,n,\tilde{m},k)}(t)} \, dt \right\},$$
$$x \in (\alpha(F), \omega(F)).$$

In the next result, we provide a characterization of the power function distribution based on the equality in distribution of dual GOS.

Theorem 3.4. *Let $X_d(1,n,\tilde{m},k), X_d(2,n,\tilde{m},k), ..., X_d(n,n,\tilde{m},k)$ denote the dual GOS based on an absolutely continuous cdf F which is concentrated on $(0,\beta)$ and $\theta^* : (0,\beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Let also there exist a point $t_0 \in (0,\beta)$ with $F(t_0) > 0$, and a sequence $\{t_i : i = 1,2,...\}$ of points lying in $(0,t_0)$ such that it converges to t_0. Then, for some $1 \le r < s \le n$,*

$$\left[\frac{t_i - X_d(s,n,\tilde{m},k)}{\theta^*(t_i)} \;\middle|\; X_d(r+1,n,\tilde{m},k) \le t_i < X_d(r,n,\tilde{m},k)\right]$$
$$\stackrel{d}{=} \beta - X_d(s-r, n-r, \tilde{\mu}_r, k),\ i = 0,1,2,..., \qquad (16)$$

if and only if F is a (rescaled) power function distribution with parameter vector (α,β), for some $\alpha > 0$, $\tilde{\mu}_r = (m_{r+1},...,m_{n-1})$, $\gamma_{r+1} = k + \sum_{r+1}^{n-1}(m_j+1)$ and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.

Proof. We first prove the 'only if' part of the theorem. Using Lemma 3.1, for $x > t$, we have

$$\begin{aligned} &P\{X_d(s,n,\tilde{m},k) > x \mid X_d(r+1,n,\tilde{m},k) \le t < X_d(r,n,\tilde{m},k)\} \\ &\qquad = \frac{c_{s-1}}{c_{r-1}} \int_{\mathcal{K}} u_{s-r}^{\gamma_s - 1} \left\{\prod_{j=1}^{s-r-1} u_j^{m_{r+j}}\right\} du_1 du_2 ... du_{s-r}, \\ &\qquad = H\left(\frac{F(x)}{F(t)}\right), \end{aligned}$$

where $\mathcal{K} = \{(u_1,...,u_{s-r}) : 1 \ge u_1 \ge u_2 \ge \cdots \ge u_{s-r} > \frac{F(x)}{F(t)}\}$ and

$$H(x) = \frac{c_{s-1}}{c_{r-1}} \int_{\mathcal{E}} u_{s-r}^{\gamma_s - 1} \left\{\prod_{j=1}^{s-r-1} u_j^{m_{r+j}}\right\} du_1 du_2 ... du_{s-r}.$$

Thus for each $i = 1,2,...,$

$$P\left\{\frac{t_i - X_d(s,n,\tilde{m},k)}{\theta^*(t_i)} \le x \;\middle|\; X_d(r+1,n,\tilde{m},k) \le t_i < X_d(r,n,\tilde{m},k)\right\}$$
$$= H\left(\frac{F(t_i - \theta^*(t_i)x)}{F(t_i)}\right). \qquad (17)$$

On the other hand, using (11), we have

$$P\{\beta - X_d(s-r, n-r, \tilde{\mu}_r, k) \leq x\} = H(F(\beta - x)). \tag{18}$$

Note that $H(x)$ is a strictly decreasing function of x. If (16) holds, then from (17) and (18) we have, for every $x \in (0, \beta)$ and $i = 0, 1, 2, ...,$

$$\frac{F(t_i - \theta^*(t_i)x)}{F(t_i)} = F(\beta - x),$$

which in turn, in view of Theorem 2.3, implies that F is a (rescaled) power function distribution. The 'if' part of the theorem is straightforward and hence is omitted. The proof is complete. □

We obtain the following corollaries from Theorem 3.4.

Corollary 3.5. *Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the order statistics based on a random sample of size n from an absolutely continuous cdf F which is concentrated on $(0, \beta)$ and $\theta^* : (0, \beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Let also there exist a point $t_0 \in (0, \beta)$ with $F(t_0) > 0$, and a sequence $\{t_i : i = 1, 2, ...\}$ of points lying in $(0, t_0)$ such that it converges to t_0. Then, for some $1 \leq r < s \leq n$,*

$$\left[\frac{t_i - X_{r:n}}{\theta^*(t_i)} \,\middle|\, X_{s-1:n} < t_i \leq X_{s:n}\right] \stackrel{d}{=} \beta - X_{r:s}, \quad i = 0, 1, 2, ...,$$

if and only if F is a (rescaled) power function distribution with parameter vector (α, β), for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.

Remark 3.6. The k-out-of-n structures are used in reliability engineering to increase the reliability of the systems. It is well known that, in a k-out-of-n system with n components, the life length of the system is the $(n-k+1)$th ordered lifetime in the system. The conditional random variable $[t - X_{r:n} \mid X_{s-1:n} \leq t < X_{s:n}]$ is known as the inactivity time of an $(n-s+2)$-out-of-n system given that the system had failed, but the sth $(1 \leq r < s \leq n)$ component is working at time $t \geq 0$. In engineering reliability, a knowledge of $[t - X_{r:n} \mid X_{s-1:n} \leq t < X_{s:n}]$ may help the engineer to initiate preventive maintenance or a replacement of the whole system at some reasonable epoch (Zhao et al., 2008). Therefore, the characterization result in Corollary 3.5 is based on the equality in distribution of a normalized inactivity time of an $(n-s+2)$-out-of-n system and the time that has elapsed since the failure of an $(s-r+1)$-out-of-n system.

Corollary 3.7. *Let $Z_1^{(k)}, Z_2^{(k)}, ..., Z_n^{(k)}$ be the first n lower k-records corresponding to a sequence of independent and identically distributed random variables with an absolutely continuous cdf F which is concentrated on $(0, \beta)$ and $\theta^* : (0, \beta) \to \mathbb{R}_+$ be a continuous function with $\theta^*(\beta) = 1$, where β is a positive real number. Assume that F is strictly increasing on $(0, \beta)$. Then for some $n \geq m + 1$,*

$$\left[\frac{t_i - Z_n^{(k)}}{\theta^*(t_i)} \middle| \; Z_{m+1}^{(k)} < t_i \leq Z_m^{(k)} \right] \stackrel{d}{=} \beta - Z_{n-m}^{(k)}, \quad i = 0, 1, 2, ...$$

if and only if F is a (rescaled) power function distribution with parameter vector (α, β), for some $\alpha > 0$, and $\theta^*(x) = m^*(x)/m^*(\beta)$, where $m^*(x)$ is the mean inactivity time corresponding to F.

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References

Asadi, M., Bayramoglu, I. (2006), The mean residual life function of a k-out-of-n structure at the system level. IEEE Transactions on Reliability, **55**, 314–318.

Asadi, M., Rao, C. R., and Shanbhag, D. N. (2001), Some unified characterization results on generalized Pareto distributions. Journal of Statistical Planning and Inference, **93**, 29–50.

Balasubramanian, K., Balakrishnan, N., and Malik, H. J. (1992), Operator methods in order statistics. Australian Journal of Statistics, **34**, 489–496.

Burkschat, M., Cramer, E., and Kamps, U. (2003), Dual generalized order statistics. Metron, vol. LXI, 13–26.

Hashemi, M. and Asadi, M. (2007), Some characterization results on generalized Pareto distribution based on progressive Type-II right censoring. Journal of the Iranian Statistical Society, **6** (2), 99–110.

Hashemi, M., Tavangar, M., and Asadi, M. (2010), Some properties of the residual lifetime of progressively Type-II right censored order statistics. Statistics and Probability Letters, **80**, 848–859.

Kamps, U. (1995), A Concept of Generalized Order Statistics. Stuttgart: B.G. Teubner.

Kotz, S. and Shanbhag, D. N. (1980), Some new approaches to probability distributions. Advances in Applied Probability, **12**, 903–921.

Nanda, A. K., Singh, H., Misra, N., and Paul, P. (2003), Reliability properties of reversed residual lifetime. Communications in Statistics: Theory and Methods, **32**, 2031–2042.

Tavangar, M. and Asadi, M. (2007), Generalized Pareto distributions characterized by generalized order statistics. Communications in Statistics: Theory and Methods, **36** (7), 1333–1341.

Tavangar, M. and Asadi, M. (2008), On a characterization of generalized Pareto distribution based on generalized order statistics. Communications in Statistics: Theory and Methods, **37** (9), 1347–1352.

Tavangar, M. and Asadi, M. (2010), Some new characterization results on exponential and related distributions. Bulletin of the Iranian Mathematical Society, **36** (1), 257–272.

Zhao, P., Li, X., and Balakrishnan, N. (2008), Conditional ordering of k-out-of-n systems with independent but nonidentical components. Journal of Applied Probability, **45**, 1113–1125.