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# On the Ratio of Rice Random Variables

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**Abstract.** The ratio of independent random variables arises in many applied problems. In this article, the distribution of the ratio $X/Y$ is studied, when X and Y are independent Rice random variables. Ratios of such random variable have extensive applications in the analysis of noises of communication systems. The exact forms of probability density function (PDF), cumulative distribution function (CDF) and the existing moments have been derived in terms of several special functions. The delta method is used to approximate moments. As a special case, we have obtained the PDF and CDF of the ratio of independent Rayleigh random variables.

## 1 Introduction

For given random variables $X$ and $Y$, the distribution of the ratio $X/Y$ arises in a wide range of natural phenomena of interest, such as in engineering, hydrology, medicine, number theory, psychology, etc. More specifically, Mendelian inheritance ratios in genetics, mass

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to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics are exactly of this type. The distribution of the ratio random variables (RRV), has been extensively investigated by many authors specially when $X$ and $Y$ are independent and belong to the same family. Various methods have been compared and reviewed by [4],[7],[8],[9],[10]. In this paper, we will derive the exact distribution of $X/Y$ when $X$ and $Y$ are independent random variables (RVs) having the Rice distributions with parameters $(\sigma, \nu)$ and $(\lambda, u)$, respectively.

The Rice distribution is well known and of common use in engineering, specially in signal processing and communication theory. Some usual situations in which the Rice ratio random variable (RRRV) appear are as follows. In the case that $X$ and $Y$ represent the random noises of two signals, studying the distribution of the quotient $|X/Y|$ is always of interest. For example in communication theory it may represent the relative strength of two different signals and in MRI, it may represent the quality of images. Moreover, because of the important concept of moments of RVs as magnitude of power and energy in physical and engineering sciences, the possible moments of the RRRV have been also obtained. Some applications of Rice distribution and ratio RV may be found in [5],[6],[12],[13],[14], and references therein.

If $X$ has a Rice distribution with parameters $(s, r)$, then the PDF of $X$is as follows

$$f_X(x) = \frac{x}{s^2}\exp\left\{-\frac{(x^2+r^2)}{2s^2}\right\} I_0\left(\frac{xr}{s^2}\right), \qquad x > 0, s > 0, r \geq 0,$$

where $x$ is the signal amplitude, $I_0(.)$ is the modified Bessel function of the first kind of order 0, $2s^2$ is the average fading-scatter component and $r^2$ is the line-of-sight (LOS) power component. The Local Mean Power is defined as $\Omega = 2s^2 + r^2$ which equals $E[X^2]$, and the Rice factor $K$ of the envelope is defined as the ratio of the signal power to the scattered power, i.e., $K = r^2/2s^2$. When $K$ goes to zero, the channel statistic follows Rayleigh's distribution [5], [6] and [12], whereas if $K$ goes to infinity, the channel becomes a non-fading channel.

This paper is organized as follows. In Section 2, some notation, preliminaries and special functions are mentioned. The exact expressions for the PDF and CDF of the RRRV are derived in Section 3. Section 4 deals with calculating the moments of the ratio random

variables.

## 2 Notation and Preliminaries

In this section, we first recall some special mathematical functions, which will be used repeatedly in the next sections. The modified Bessel function of first kind of order $\nu$, is

$$I_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{(k!)\Gamma(\nu+k+1)}\,.$$

The generalized hypergeometric function is denoted by

$${}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!},$$

the Gauss hypergeometric function is

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}\,,$$

and the Kummer confluent hypergeometric function is

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!},$$

where $(a)_k$, $(b)_k$ represent Pochhammer's symbol given by

$$(a)_k = a(a+1)\cdots(a+k-1).$$

The parabolic cylinder function is

$$D_\nu(z) = 2^{\nu/2}e^{-z^2/4}\Psi(-\frac{1}{2}\nu,\frac{1}{2};\frac{1}{2}z^2),$$

where $\Psi(a,c;z)$ represents the confluent hypergeometric function given by

$$\begin{aligned}\Psi(a,c;z) &= \Gamma\left[\begin{array}{c}1-c\\1+a-c\end{array}\right]{}_1F_1(a;c;z)\\ &\quad +\Gamma\left[\begin{array}{c}c-1\\a\end{array}\right]2^{1-c}{}_1F_1(1+a-c;2-c;z)\,,\end{aligned}$$

in which

$$\Gamma\left[\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array}\right] = \frac{\prod_{i=1}^{m}\Gamma(a_i)}{\prod_{j=1}^{n}\Gamma(b_j)}.$$

A well-known representation for the CDF of Rice random variable is as

$$F_X(x) = 1 - Q_1(r/s, x/s),$$

where $Q_M(\alpha,\beta)$ is defined by

$$Q_M(\alpha,\beta) = e^{-(\alpha^2+\beta^2)/2}\sum_{k=1-M}^{\infty}\left(\frac{\alpha}{\beta}\right)^k I_k(\alpha\beta),$$

and $I_k(.)$ denotes the modified Bessel function of first kind of order $k$. Also we have

$$E(X^k) = (2s^2)^{k/2}e^{-\frac{r^2}{2s^2}}\Gamma\left(1+\frac{k}{2}\right){}_1F_1\left(1+\frac{k}{2};1;\frac{r^2}{2s^2}\right). \qquad (1)$$

The following lemmas are of frequent use.

**Lemma 2.1.** *(Equation(2.15.20.7), [11], vol. 2). For $Re(p) > 0$, $Re(\alpha+\mu+\nu) > 0$,*

$$\int_0^{\infty} x^{\alpha-1}e^{-px^2}I_\mu(bx)I_\nu(cx)dx = \frac{b^\mu c^\nu p^{-(\alpha+\mu+\nu)/2}}{2^{\mu+\nu+1}\Gamma(\nu+1)}\sum_{k=0}^{\infty}\Gamma\left[\begin{array}{c} k+(\alpha+\mu+\nu)/2 \\ \mu+k+1 \end{array}\right]\frac{1}{k!} \times \left(\frac{b}{2p}\right)^{2k}{}_2F_1(-k,-\mu-k;\nu+1;\frac{c^2}{b^2}).$$

**Lemma 2.2.** *(Equation(2.15.5.4), [11], vol. 2). For $Re(p) > 0$, $Re(\alpha+\nu) > 0$; $|argc| < \pi$,*

$$\int_0^{\infty} x^{\alpha-1}e^{-px^2}I_\nu(cx)dx = 2^{-\nu-1}c^\nu p^{-\frac{(\alpha+\nu)}{2}}\Gamma\left[\begin{array}{c} \frac{(\alpha+\nu)}{2} \\ \nu+1 \end{array}\right]{}_1F_1\left(\frac{\alpha+\nu}{2};\nu+1;\frac{c^2}{4p}\right).$$

**Lemma 2.3.** *(Equation(2.21.1.15), [11], vol. 3). For* $Re(\alpha) > 0$, $Re(a - \alpha + \rho) > 0$; $Re(b - \alpha + \rho) > 0$, $|arg\ \omega| < \pi$, $|arg\ z| < \pi$,

$$\int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho}\, {}_2F_1(a,b,c;-\omega x)dx =$$

$$z^{\alpha-\rho}B(\alpha,\rho-\alpha){}_3F_2(a,b,\alpha;c,\alpha-\rho+1;\omega z)$$

$$+\omega^{\rho-\alpha}\Gamma\left[\begin{array}{c} c, a-\alpha+\rho, b-\alpha+\rho, \alpha-\rho \\ a, b, c-\alpha+\rho \end{array}\right]$$

$${}_3F_2(a-\alpha+\rho, b-\alpha+\rho, \rho; c-\alpha+\rho, \rho-\alpha+1; \omega z).$$

See [3] and [11] for more properties and details.

## 3 The Ratio of Rice Random Variables

In this section, the explicit expressions for the CDF and PDF of $X/Y$ are derived in terms of the Gauss hypergeometric function. The ratio of Rayleigh RVs is also considered as a special case.

**Theorem 3.1.** *Suppose that* $X$ *and* $Y$ *are independent Rice random variables with parameters* $(\sigma, \nu)$ *and* $(\lambda, u)$*, respectively. The CDF of the ratio random variable* $T = X/Y$ *is*

$$F_T(t) = 1 - \left\{ \frac{\sigma^2 e^{-\left(\frac{\lambda^2\nu^2+u^2\sigma^2}{2\sigma^2\lambda^2}\right)}}{t^2\lambda^2+\sigma^2} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu^2}{2\sigma^2}\right)^k}{k!} \right.$$

$$\left. \times \left[ \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{u\sigma^2}{t^2\lambda^2+\sigma^2} \right)^{2j} {}_2F_1\left( -j, -j; k+1; \frac{\nu^2 t^2 \lambda^4}{u^2\sigma^2} \right) \right] \right\}. \quad (2)$$

**Proof.** The CDF, $F_T(t)$ can be expressed as

$$\begin{aligned} F_T(t) &= Pr(X/Y \le t) \\ &= \int_0^\infty F_X(ty) f_Y(y) dy \\ &= \int_0^\infty \{1 - Q_1(\nu/\sigma, ty/\sigma)\} \frac{y}{\lambda^2} e^{-(y^2+u^2)/2\lambda^2} I_0\left(\frac{yu}{\lambda^2}\right) dy. \quad (3) \end{aligned}$$

Substituting $Q_1(\nu/\sigma, ty/\sigma)$ in (3), gives

$$F_T(t) = 1 - \left\{ \frac{e^{-\left(\frac{\lambda^2\nu^2+u^2\sigma^2}{2\sigma^2\lambda^2}\right)}}{\lambda^2} \sum_{k=0}^{\infty} \left(\frac{\nu}{t}\right)^k \left[\int_0^{\infty} y^{-k+1} e^{-\left(\frac{\lambda^2t^2+\sigma^2}{2\sigma^2\lambda^2}\right)y^2} I_o\left(\frac{u}{\lambda^2}y\right) I_k\left(\frac{\nu t}{\sigma^2}y\right) dy\right] \right\}. \quad (4)$$

Thus we get (2) by using Lemma 2.1 in (4). □

**Corollary 3.2.** *Suppose that $X$ and $Y$ are independent Rayleigh random variables with parameters $\sigma$ and $\lambda$, respectively. The CDF of the ratio random variable $T = X/Y$ can be expressed as*

$$F_T(t) = 1 - \frac{\sigma^2}{t^2\lambda^2 + \sigma^2}, \qquad t \geq 0.$$

**Proof.** Take $\nu = u = 0$ in (2). □

**Theorem 3.3.** *Suppose that $X$ and $Y$ are independent Rice random variables with parameters $(\sigma, \nu)$ and $(\lambda, u)$, respectively. The PDF of the ratio random variable $T = X/Y$ is*

$$f_T(t) = \frac{2t\sigma^2\lambda^2 e^{-\left(\frac{\nu^2\lambda^2+u^2\sigma^2}{2\sigma^2\lambda^2}\right)}}{(t^2\lambda^2+\sigma^2)^2} \sum_{k=0}^{\infty} \frac{k+1}{k!} \left(\frac{u\sigma^2}{t^2\lambda^2+\sigma^2}\right)^{2k} {}_2F_1\left(-k, -k; 1; \frac{\nu^2t^2\lambda^4}{u^2\sigma^4}\right). \quad (5)$$

**Proof.** The PDF $f_T(t) = \int_0^{\infty} y f_X(ty) f_Y(y) dy$ can be written as

$$f_T(t) = \int_0^{\infty} y \frac{ty}{\sigma^2} e^{-\frac{t^2y^2+\nu^2}{2\sigma^2}} I_0\left(\frac{ty\nu}{\sigma^2}\right) \frac{y}{\lambda^2} e^{-\frac{y^2+u^2}{2\lambda^2}} I_0\left(\frac{yu}{\lambda^2}\right) dy.$$

The result now follows by using the Lemma 2.1. □

**Corollary 3.4.** *Suppose that $X$ and $Y$ are independent Rayleigh random variables with parameters $\sigma$ and $\lambda$, respectively. The PDF of the ratio random variable $T = X/Y$ can be expressed as*

$$f_T(t) = \frac{2t\sigma^2\lambda^2}{(t^2\lambda^2+\sigma^2)^2}, \qquad t \geq 0.$$

**Proof.** The result immediately follows by taking $\nu = u = 0$ in (5). □

**Remark 3.5.** One may suggest to use the theory of transformation to obtain the distribution of $T$. But this approach leads to the same results which are given in this article. For instance, if we define

$$T = \frac{X}{Y} \quad , \qquad V = Y$$

and work through the jacobian, then the PDF of $T$ will be obtained from the integral $f_T(t) = \int_0^\infty v f_X(tv) f_Y(v) dv$. Solving this integral also needs to use the special functions and gives (5). □

## 4 Moments of the Ratio Random Variable

In the sequel, we shall use the independence of $X$ and $Y$ several times for computing the moments of the ratio random variable. The results obtained are expressed in terms of confluent hypergeometric functions. The delta method has also been used for approximation and simple representations of the moments.

**Theorem 4.1.** *Suppose that $X$ and $Y$ are Rice random variables with parameters $(\sigma, \nu)$ and $(\lambda, u)$, respectively. A representation for the k-th moment of the ratio random variable $T = X/Y$, for $-2 < k < 2$, is as follows*

$$\begin{aligned} E(T^k) = \frac{\sigma^2}{\lambda^2} e^{-\frac{\nu^2\lambda^2 + u^2\sigma^2}{2\sigma^2\lambda^2}} \sum_{j=0}^{\infty} \frac{j+1}{j!} \left(\frac{u\sigma^2}{\lambda^2}\right)^{2j} \\ \times \quad \left\{ \left(\frac{\sigma^2}{\lambda^2}\right)^{k/2-2j-1} B(1 + \frac{k}{2}, 2j - \frac{k}{2} + 1) \right. \\ \times \quad {}_3F_2\left(-j, -j; 1 + \frac{k}{2}; 1; \frac{k}{2} - 2j; -\frac{\nu^2\lambda^2}{u^2\sigma^2}\right) \\ + \quad \left(-\frac{\nu^2\lambda^4}{u^2\sigma^4}\right)^{2j-\frac{k}{2}+1} \Gamma \left[ \begin{matrix} 1, j - \frac{k}{2} + 1, j - \frac{k}{2} + 1, \frac{k}{2} - 2j - 1 \\ -j, -j, 2j - \frac{k}{2} + 2 \end{matrix} \right] \quad (6) \\ \times \quad \left. {}_3F_2\left(j - \frac{k}{2} + 1, j - \frac{k}{2} + 1, 2j + 2; 2j - \frac{k}{2} + 1, 2j - \frac{k}{2} + 2; -\frac{\nu^2\lambda^2}{u^2\sigma^2}\right) \right\}. \end{aligned}$$

**Proof.** By definition

$$\begin{aligned} E(T^k) &= \int_0^\infty t^k f_T(t)dt \\ &= 2\sigma^2\lambda^2 e^{-\frac{\nu^2\lambda^2+u^2\lambda^2}{2\sigma^2\lambda^2}} \sum_{j=0}^{\infty} \frac{j+1}{j!}(u\sigma^2)^{2j} \\ &\quad \int_0^\infty \frac{t^{k+1}}{(t^2\lambda^2+\sigma^2)^{2j+2}}\ {}_2F_1\left(-j,-j;1;\frac{\nu^2\lambda^4}{u^2\sigma^4}t^2\right)dt. \end{aligned}$$

Now, the desired result follows by using Lemma 2.3. □
In the following theorem, we give an alternative representation for $E(T^k)$, which is easier to handle than (6).

**Theorem 4.2.** *Suppose that $X$ and $Y$ are independent Rice random variables with parameters $(\sigma,\nu)$ and $(\lambda,u)$, respectively. A representation for the k-th moment of the ratio random variable $T = X/Y$, $-2 < k < 2$, can be expressed by*

$$\begin{aligned} E(T^k) &= \frac{\left(\frac{\sigma}{\lambda}\right)^k \Gamma\left(\frac{2+k}{2}\right)\Gamma\left(\frac{2-k}{2}\right)}{e^{\frac{\nu^2\lambda^2+u^2\sigma^2}{2\sigma^2\lambda^2}}} \\ &\quad {}_1F_1\left(\frac{2+k}{2};1;\frac{\nu^2}{2\sigma^2}\right){}_1F_1\left(\frac{2-k}{2};1;\frac{u^2}{2\lambda^2}\right). \end{aligned} \tag{7}$$

**Proof.** Using the independency of $X$ and $Y$, the expected ratio can be written as

$$E(T^k) = E\left(\frac{X^k}{Y^k}\right) = E\left(X^k\right)E\left(\frac{1}{Y^k}\right),$$

in which

$$E\left(\frac{1}{Y^k}\right) = \int_0^\infty \frac{1}{y^k}\frac{y}{\lambda^2} exp\left\{-\frac{(y^2+u^2)}{2\lambda^2}\right\} I_0\left(\frac{uy}{\lambda^2}\right)dy. \tag{8}$$

By using Lemma 2.2, the integral (8) reduces to

$$E\left(\frac{1}{Y^k}\right) = \frac{e^{-\frac{u^2}{2\lambda^2}}}{(2\lambda^2)^{k/2}}\Gamma\left(\frac{-k+2}{2}\right){}_1F_1\left(\frac{-k+2}{2};1;\frac{u^2}{2\lambda^2}\right). \tag{9}$$

The desired result now follows by multiplying (1) and (9). □

**Remark 4.3.** Formulas (6) and (7), display the exact forms for calculating $E(T)$, which have been expressed in terms of confluent hypergeometric functions. Indeed, as suggested by the referee(s) we can use the delta-method to approximate the first and second moments of the ratio $T = X/Y$. In details, by taking $\mu_X = E(X)$, $\mu_Y = E(Y)$, and following example 5.5.27, pages 244-245 in [1],

$$E(T) \approx \frac{\mu_X}{\mu_Y} = \frac{{}_1F_1(\frac{3}{2}, 1; \frac{\nu^2}{2\sigma^2})}{{}_1F_1(\frac{3}{2}, 1; \frac{u^2}{2\lambda^2})} \left(\frac{\sigma}{\lambda}\right) \; e^{\frac{u^2}{2\lambda^2} - \frac{\nu^2}{2\sigma^2}}.$$

For approximating $Var(T)$, first we recall that $E[X^2] = 2\sigma^2 + \nu^2$ and $E[Y^2] = 2\lambda^2 + u^2$. Now,

$$Var\left(\frac{X}{Y}\right) \approx \left(\frac{\mu_X^2}{\mu_Y^2}\right)\left(\frac{Var(X)}{\mu_X^2} + \frac{Var(y)}{\mu_Y^2}\right),$$

which involves confluent hypergeometric functions, but in simpler forms. □

**Remark 4.4.** The numerical computation of the obtained results in this paper entails calculation of the special functions, their sums and integrals, which have been tabulated and available in determined books and computer algebra packages, see [2],[3],[11] for more details. □

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## References

[1] Casella, G. and Berger, L. B. (2002), Statistical Inference. Duxbury, USA.

[2] Derek, R. (2002), Advanced Mathematical Methods with Maple. Cambridge University Press.

[3] Gradshteyn, I. S. and Ryzhik, I. M. (2000), Tables of Integrals, Series, and Products. San Diego: Academic Press.

[4] Greay, R. C. (1930), The frequency distribution of the quotient of two normal variates. Journal of the Royal Statistical Society, **93**, 442-446.

[5] Helstrom, C. (1997), Computing the distribution of sums of random sine waves and the rayleigh-distributed random variables by saddle-point integration. IEEE Trans. Commun., **45**(11), 1487-1494.

[6] Karagiannidis, G. K. and Kotsopoulos, S. A. (2001), On the distribution of the weighted sum of L independent rician and nakagami envelopes in the presence of AWGN. Journal of Communication and Networks, **3**(2), 112-119.

[7] Marsaglia, G. (1965), Ratios of normal variables and ratios of sums of uniform variables. JASA., **60**, 193-204.

[8] Marsaglia, G. (2006), Ratios of normal variables. Journal of Statistical Software, **16**(4), 1-10.

[9] Nadarajah, S. (2006), Quotient of laplace and gumbel random variables. Mathematical Problems in Engineering, 1-7.

[10] Pearson, K. (1910), On the constants of index-distributions as deduced from the like constants for the components of the ratio with special reference to the opsonic index. Biometrika, **7**(4), 531-541.

[11] Prudnikov, A. P., Brychkov, Y. A., and Marichev, O. I. (1986), Integrals and Series, Vol 1-3, Gordon and Breach, Amesterdam: Science Publishers.

[12] Rice, S. (1974), Probability distributions for noise plus several sin waves the problem of computation. IEEE Trans. Commun., 851-853.

[13] Salo, J., EI-Sallabi, H. M., and Vainikainen, P. (2006), The distribution of the product of independent rayleigh random variables. IEEE Transactions on Antennas and Propagation, **54**(2), 639-643.

[14] Withers, C. S. and Nadarajah, S. (2008), MGFs for rayleigh random variables. Wireless Personal Communications, **46**(4), 463-468.