Entropy Properties of Certain Record Statistics and Some Characterization Results

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Abstract. In this paper, the largest and the smallest observations are considered, at the time when a new record of either kind (upper or lower) occurs based on a sequence of independent random variables with identical continuous distributions. We prove that sequences of the residual or past entropy of the current records characterizes F in the family of continuous distributions. The exponential and the Frechet distributions are characterized through maximizing Shannon entropies of these statistics under some constraint.

1 Introduction

Let $\{X_i, \ i \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables each distributed according to cumulative distribution function (cdf) F with probability density function (pdf) f. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous defi-

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nition can be given for lower record values. For more details about record values, see Arnold *et al.* (1998) and references therein. Here we consider the largest and the smallest observations, respectively, at the time when a new record of either kind (upper or lower) occurs. In the literature, they are called the n-th *current upper record* and the n-th *current lower record*, respectively, of the X_n sequence when the n-th record of any kind (either an upper or a lower) is observed. In this paper, for the convenience of notations, we denote them by R_n^l and R_n^s, respectively ($R_0^s = R_0^l = X_1$). For $n \geq 1$, the interval (R_n^s, R_n^l) will be called the *record coverage*, and the difference $R_n = R_n^l - R_n^s$ could be called the n-th *record range*. Houchens (1984) introduced the current records in his Ph.D thesis and derived the joint pdf of (R_n^s, R_n^l). For $n \geq 1$, the marginal densities of R_n^s and R_n^l are given by (see Arnold *et al.* 1998, p. 276)

$$f_{R_n^s}(x) = 2^n f(x)\{1 - F(x) \sum_{j=0}^{n-1} \frac{[-\log F(x)]^j}{j!}\} \qquad (1)$$

and

$$f_{R_n^l}(x) = 2^n f(x)\{1 - \bar{F}(x) \sum_{j=0}^{n-1} \frac{[-\log \bar{F}(x)]^j}{j!}\}, \qquad (2)$$

respectively, where $\bar{F}(x)$ is the survival function of X.

Recently, some authors have been interested in studying current records and their applications, see for example, Basak (2000), Ahmadi and Balakrishnan (2005), Ahmadi and Fashandi (2008, 2009) and Raqab (2009). The current records can be used in general sequential methods for model choice and outlier detection involving the record range, see Basak (2000).

If X is a random variable having an absolutely continuous cdf $F(t)$ and pdf $f(t)$, then the basic uncertainty measure for distribution F is defined as

$$H(X) = - \int_{-\infty}^{+\infty} f(x) \log f(x) dx,$$

provided the integral exists. In the literature, $H(X)$ is commonly referred to as the *entropy* of X or *Shannon information* measure and used as a tool to determine the amount of information contained in X regarding its parent distribution. A little research has been done based on entropy properties of the endpoints of record coverage; see Ahmadi and Fashandi (2008, 2009). In this paper, we will present some characterization results regarding the exponential and

the Frechet distributions based on entropy property of the current records. We also give some characterization results on the basis of the residual and the past entropies of the current records.

The rest of this paper is as follows: In Section 2, we prove that sequences of the residual or past entropy of the current records characterizes F in the family of continuous distributions. Section 3 gives some results on maximum and minimum entropy related to current records. The exponential and the Frechet distributions are characterized under some conditions. The exponential distribution is one of the commonly used models in reliability applications, and the Frechet distribution is defined for nonnegative random variables and is the limiting distribution for the maxima of random variables. In that context the Frechet distribution is also termed a Type II extreme value distribution, and is well suited as a general measurement model in engineering.

2 Characterization Results

First, we recall the following lemma which is an extension of the classical Müntz-Szász Theorem and is used in the proofs of the results in this paper. The well-known Müntz-Szász Theorem asserts that the sequence of powers $\{x^{n_j}, j \geq 1\}$ is complete on (a, b), where $a \geq 0$, if and only if

$$\sum_{j=1}^{+\infty} n_j^{-1} = +\infty, \text{ where } 0 < n_1 < n_2 < \cdots. \tag{3}$$

Hwang and Lin (1984) extended the Müntz-Szász Theorem for $\{f^{n_j}(x), j \geq 1\}$, where $f(x)$ is absolutely continuous and monotone on (a, b):

Lemma 2.1.   *(Hwang and Lin, 1984) Let $f(x)$ be a function absolutely continuous on (a, b) with $f(a)f(b) \geq 0$, and let its derivative satisfy $f'(x) \neq 0$ a.e. on (a, b). Then under the assumption (3) the sequence $\{f^{n_j}(x), j \geq 1\}$ is complete on (a, b) if and only if the function $f(x)$ is monotone on (a, b).*

The question, "does the Shannon information characterize the parent distribution?" was addressed by Baratpour *et al.* (2007, 2008), in which the authors investigated this problem based on Shannon's and Rényi's entropies of order statistics and usual records. Ahmadi

(2009) extended these results to the case of progressive Type-II censored order statistics. Ahmadi and Fashandi (2009) proved that the equality of the entropy of the endpoints of record coverage is a characteristic property of symmetric distribution. In this section we study this problem on the basis of residual and past entropy properties of current records.

2.1   Characterization Based on Residual Entropy

In life testing experiments, frequently, one has information about the current age of the systems under consideration. Then $H(X)$ is not suitable in such situations and it should be modified to take the current age into account. Accordingly, Ebrahimi (1996) introduced a measure of uncertainty, known as residual entropy, which is defined as

$$H(X;t) = - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx.$$

If $H(X;t)$ is decreasing in t then it is said that F have the property of *decreasing uncertainty in residual life* (DURL). Ahmadi and Balakrishnan (2005) have shown that the DURL property is preserved by current upper records R_n^l. Now, we show that the parent distribution can be identified uniquely by the residual entropy properties of the current records.

Theorem 2.1.1.   *Let X and Y be two random variables with absolutely continuous cdfs $F(.)$ and $G(.)$ and pdfs $f(.)$ and $g(.)$, respectively having finite entropies. In addition, assume that there exists non-negative constant t_0 such that $F(t_0) = G(t_0)$. Then the following two statements are equivalent:*
(i) X is identical in distribution with Y,
(ii) $H(R_n^l(X), t_0) = H(R_n^l(Y), t_0)$, for $n \geq 1$,
where $R_n^l(X)$ and $R_n^l(Y)$ are the n-th current upper records of X and Y, respectively.

Proof. The first part $[(i) \Rightarrow (ii)]$ is obvious. We will prove that the second part, $[(ii) \Rightarrow (i)]$, holds. It can be easily shown that

$$H(X;t) = 1 - E[\log r_X(X)|X > t], \tag{4}$$

where $r_X(x) = f(x)/\bar{F}(x)$ is the hazard rate function of X. From Lemma 3 of Ahmadi and Balakrishnan (2005)

$$r_{R_n^l}(t) = r_X(t)k_n(F(t)), \tag{5}$$

where

$$k_n(1-u) = \frac{2^n(1 - u\sum_{j=0}^{n-1}\{-\log u\}^j/j!)}{2^n - u\sum_{j=0}^{n-1}(2^n - 2^j)\{-\log u\}^j/j!}. \tag{6}$$

From (4) and (5) we have

$$\begin{aligned}
1 - H(R_n^l(X); t) &= E[\log r_{R_n^l}(R_n^l)|R_n^l > t] \\
&= E[\log\left(r_X(R_n^l)k_n(F(R_n^l))\right)|R_n^l > t] \\
&= E\left[\log\left(r_X(F^{-1}(U_n^l))k_n(U_n^l)\right)|U_n^l > F(t)\right] \\
&= \int_{F(t)}^1 \log\left[r_X(F^{-1}(u))k_n(u)\right]\frac{f_{U_n^l}(u)}{\bar{F}_{U_n^l}(F(t))}du, \tag{7}
\end{aligned}$$

where U_n^l is the n-th current upper record from Uniform(0, 1), and $f_{U_n^l}(u)$ and $\bar{F}_{U_n^l}(u)$ are the pdf and the survival function of U_n^l, respectively.

The survival function of U_n^l is given by

$$\bar{F}_{U_n^l}(x) = (1-x)\{2^n - (1-x)\sum_{j=0}^{n-1}(2^n - 2^j)(-\log(1-x))^j/j!\}.$$

Now suppose, for two random variables X and Y with cdfs F and G, that there exists non-negative constant t_0 such that $F(t_0) = G(t_0) = a$ and $H(R_n^l(X), t_0) = H(R_n^l(Y), t_0)$. Then from (7), we get

$$\int_a^1 \log\left(\frac{r_X(F^{-1}(u))}{r_Y(G^{-1}(u))}\right)f_{U_n^l}(u)du = 0. \tag{8}$$

From (2) and (8) we have,

$$\int_0^{1-a}\log\left(\frac{r_X(F^{-1}(1-u))}{r_Y(G^{-1}(1-u))}\right)[1-u\sum_{j=0}^{n-1}\frac{(-\log u)^j}{j!}]du = 0, \text{ for all } n \geq 1.$$

Also we have,

$$\int_0^{1-a}\log\left(\frac{r_X(F^{-1}(1-u))}{r_Y(G^{-1}(1-u))}\right)[1 - u\sum_{j=0}^{n}\frac{(-\log u)^j}{j!}]du = 0.$$

Thus we conclude that,

$$\int_0^{1-a}\log\left(\frac{r_X(F^{-1}(1-u))}{r_Y(G^{-1}(1-u))}\right)u(-\log u)^ndu = 0. \tag{9}$$

Appealing to a change in variable $u = (1-a)y$, (9) can be expressed as

$$\int_0^1 \log\left(\frac{r_X(F^{-1}(1-(1-a)y))}{r_Y(G^{-1}(1-(1-a)y))}\right) y[-\log((1-a)y)]^n dy = 0. \quad (10)$$

If (10) holds for $n \geq 1$, then from Lemma 2.1, it follows that

$$r_X(F^{-1}(1-(1-a)y)) = r_Y(G^{-1}(1-(1-a)y)), \quad \text{for all} \quad y \in (0,1).$$

This implies that $F^{-1}(x) - G^{-1}(x)$ is a constant on $(0,1)$. By condition $F(t_0) = G(t_0)$, it follows $F^{-1}(u) = G^{-1}(u)$, for $0 < u < 1$.
$\square$

Remark 2.1.1. Theorem 2.1.1 also holds for current lower records.

Remark 2.1.2. Let $\{X_i, \ i \geq 1\}$ be a sequence of non-negative iid random variables from a continuous cdf F, taking $t_0 = 0$ in Theorem 2.1.1 we conclude that sequences of $H(R_n^l)$ [or $H(R_n^s)$] characterizes F in the family of continuous distributions. So, Theorems 3 and 4 of Ahmadi and Fashandi (2009) are obtained as special cases.

2.2 Characterization Based on Past Entropy

Let X denote the lifetime of an item with pdf f and cdf F, then the past entropy of an item is defined as (see, for example, Crescenzo and Longobardi, 2002)

$$H^*(X;t) = -\int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(x)}{F(t)}\right) dx.$$

It can be shown that

$$H^*(X;t) = 1 - E[\log \tilde{r}_X(X)|X < t], \quad (11)$$

where $\tilde{r}_X(x) = f(x)/F(x)$ is the reversed hazard rate function of X. If $H^*(X;t)$ is increasing in t then it is said that F have the property of *increasing uncertainty in past life* (IUPL). Kundu *et al.* (2009) have studied some properties of order statistics related to IUPL class. Ahmadi and Balakrishnan (2005) have shown that the IUPL property is preserved by current lower records R_n^s. We have the following results for the past entropy of the current records.

Theorem 2.2.1. *Under the assumptions of Theorem 2.1.1, the following two statements are equivalent:*

(i) X is identical in distribution with Y,

(ii) $H^*(R_n^s(X), t_0) = H^*(R_n^s(Y), t_0)$, for $n \geq 1$,

where $R_n^s(X)$ and $R_n^s(Y)$ are the n-th current lower records of X and Y, respectively.

Proof. The first part $[(i) \Rightarrow (ii)]$ is obvious. For $(ii) \Rightarrow (i)$, from Lemma 5 of Ahmadi and Balakrishnan (2005)

$$\tilde{r}_{R_n^s}(t) = \tilde{r}_X(t)k_n(1 - F(t)), \tag{12}$$

where $k_n(.)$ is defined in (6). Similar to the proceeding of the proof of Theorem 2.1.1, and using (11) and (12) it can be shown that

$$1 - H^*(R_n^s(X); t) = \int_0^{F(t)} \log\left[\tilde{r}_X(F^{-1}(u))k_n(1 - u)\right] \frac{f_{U_n^s}(u)}{F_{U_n^s}(F(t))} du,$$

where U_n^s is the n-th current lower record from Uniform(0, 1). The rest of the proof is similar to that of Theorem 2.1.1. □

Remark 2.2.1. Theorem 2.2.1 also holds for current upper records.

3 Characterization Based on Maximum and Minimum Entropy

The problem of finding which pdf, among those satisfying certain conditions, has maximum entropy has been considered extensively in the literature. In this section, we are interested in finding the parent distribution, whose current records have maximum entropy as well as characterizing the exponential and the Frechet distributions based on current records. First, for fixed positive real values $m_i, i = 1, ..., 4$, and for all $x \geq 0$, let us consider some classes of univariate distributions as follows:

$$\mathcal{C}_1 = \{F : F(x) = 1 - \exp[-\alpha_1(\theta)h_1(x)], \ h_1'(x) \geq m_1 > 0\},$$

$$\mathcal{C}_2 = \{F : F(x) = 1 - \exp[-\alpha_2(\theta)h_2(x)], \ 0 < h_2'(x) \leq m_2\},$$

$$\mathcal{C}_3 = \{F : F(x) = \exp[-\frac{\alpha_3(\theta)}{h_3(x)}], \ h_3'(x) \geq m_3 > 0\}$$

and

$$\mathcal{C}_4 = \{F : F(x) = \exp[-\frac{\alpha_4(\theta)}{h_4(x)}], \ 0 < h_4'(x) \leq m_4\},$$

where for $i = 1, ..., 4$, $\alpha_i(\theta)$ and $h_i(x)$ are non-negative real functions, of θ and x alone, respectively, moreover $h_i(x)$ is strictly increasing. Indeed, $\mathcal{C}_i$'s are subclasses of exponential family. The constraint in the cases of $\mathcal{C}_1$ and $\mathcal{C}_2$, are equivalent to the fact that the hazard rate function is bounded below in $\mathcal{C}_1$ and is bounded in $\mathcal{C}_2$. It is well-known that the hazard rate function is an important characteristic for the analysis of reliability data.

3.1 Exponential Distribution

Let X be a random variable having the exponential distribution with mean $1/\lambda$. The exponential distribution is the simplest and most important distribution in reliability studies, and is applied in a wide variety of statistical procedures, especially in life testing problems. See Balakrishnan and Basu (1995) for some research on this distribution. The entropy of lower bound of record coverage in the exponential model is given by (see, Ahmadi and Fashandi, 2008)

$$H(R_n^s) = H(U_n^s) - \log \lambda + B(n), \tag{13}$$

where

$$B(n) = -E(\log(1 - U_n^s)) = 2^n \sum_{i=1}^{+\infty} \frac{(i+2)^{-n}}{i(i+1)}. \tag{14}$$

We have the following theorem regarding the maximum entropy of current records in $\mathcal{C}_1$.

Theorem 3.1.1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous non-negative random variables from the cdf F with pdf f and entropy $H(X) < \infty$. In addition, assume that F belong to $\mathcal{C}_1$. Then the following two statements are equivalent:*
(i) $h_1(x) = m_1 x$,
(ii) the n-th current lower record of the distribution F has maximum entropy in $\mathcal{C}_1$.

Proof. Without loss of generality, let $\alpha_1(\theta) = \theta$.
(i) $\Rightarrow$ (ii): For computing $H(R_n^s)$, we find

$$f(F^{-1}(t)) = \theta(1 - t)h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1 - t)\right)].$$

Then

$$E\left[\log f(F^{-1}(U_n^s))\right]$$

$$= \log\theta + E[\log(1-U_n^s)] + E[(\log\left(h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-U_n^s)\right)]\right)]$$

$$= \log\theta - B(n) + E[(\log\left(h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-U_n^s)\right)]\right)]$$

$$\geq \log m_1\theta - B(n).$$

The second equality is obtained by (14). So by part (i) of Lemma 3.2 of Ahmadi and Fashandi (2008), in this case:

$$H(R_n^s) = H(U_n^s) - E\left[\log f(F^{-1}(U_n^s))\right] \leq H(U_n^s) - \log m_1\theta + B(n).$$

From (13) and Remark 2.1.2, $H(U_n^s) - \log m_1\theta + B(n)$ is the entropy of the n-th current lower record from exponential distribution with mean $\frac{1}{m_1\theta}$.

(ii) $\Rightarrow$ (i): Suppose that the n-th current lower record of the distribution F has maximum entropy in $\mathcal{C}_1$. Considering the proof of (i) $\Rightarrow$ (ii), we conclude that

$$E(\log\left(h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-U_n^s)\right)]\right)) = \log m_1.$$

Then from (1), we have, for $n \geq 1$

$$\int_0^1 \log\left(\frac{1}{m_1}h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-x)\right)]\right)[1-x\sum_{j=0}^{n-1}\frac{(-\log x)^j}{j!}]dx = 0.$$

By the same line as in the proof of Theorem 2.1.1 we get, for $n \geq 1$

$$\int_0^1 \log\left(\frac{1}{m_1}h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-x)\right)]\right)x(-\log x)^n dx = 0.$$

From Lemma 2.1, we conclude that

$$\log\left(\frac{1}{m_1}h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-x)\right)]\right) = 0, \ \forall x \in (0,\ 1).$$

So

$$h_1'[h_1^{-1}\left(-\frac{1}{\theta}\log(1-x)\right)] = m_1, \ \forall x \in (0,\ 1).$$

By differentiating the inverse function, we get

$$\frac{d}{dx} h_1^{-1}\left(-\frac{1}{\theta}\log(1-x)\right) = \frac{1}{m_1\theta(1-x)}.$$

Straightforward computations lead to $h_1(x) = m_1 x$. $\qquad\qquad\Box$

We have the next result concerning the minimum entropy in $\mathcal{C}_2$. The proof is similar to that of Theorem 3.1.1 and is omitted.

Theorem 3.1.2. *Assume that the conditions of Theorem 3.1.1 hold and let F belong to $\mathcal{C}_2$. Then the following two statements are equivalent:*

(i) $h_2(x) = m_2 x$,

(ii) the n-th current lower record of the distribution F has minimum entropy in $\mathcal{C}_2$.

Remark 3.1.1. Actually, Theorems 3.1.1 and 3.1.2 characterize the exponential distribution based on the entropy of current records under some constraint.

3.2 Frechet Distribution

Let X be a random variable having the Frechet distribution with cdf $F(x) = e^{-\theta/x}$, $x > 0$. The Frechet distribution is becoming increasingly important in engineering statistics as a suitable model to represent phenomena with unusually large maximum observation. For more details about this distribution and its application, see, for instance, Kotz and Nadarajah (2000). In this case, we find $f(F^{-1}(t)) = \frac{t}{\theta}(-\log t)^2$. To obtain $H(R_n^s)$, we need the following expressions (see, Ahmadi and Fashandi, 2008)

$$E(\log U_n^s) = -\frac{n+2}{2}. \tag{15}$$

$$E[\log(-\log U_n^s)] = 2^n \sum_{j=n+1}^{+\infty} \frac{\psi(j)}{2^j} - \log 2, \tag{16}$$

where $\psi(.)$ is the digamma function.

So, using (15), (16) and Lemma 3.2 of Ahmadi and Fashandi (2008), the entropy of the n-th lower current record is

$$H(R_n^s) = H(U_n^s) + \log(4\theta) + \frac{n+2}{2} - 2^n \sum_{j=n+1}^{\infty} \frac{\psi(j)}{2^{j-1}}. \tag{17}$$

We have the next result regarding the maximum entropy in $\mathcal{C}_3$.

Theorem 3.2.1. *Assume that the conditions of Theorem 3.1.1 hold and let F belong to $\mathcal{C}_3$. Then the following two statements are equivalent:*

(i) $h_3(x) = m_3 x$,

(ii) the n-th current lower record of the distribution F has maximum entropy in $\mathcal{C}_3$.

Proof. Without loss of generality, let $\alpha_3(\theta) = \theta$.

(i) $\Rightarrow$ (ii): We find $f(F^{-1}(t)) = \frac{t}{\theta}(-\log t)^2 h_3'[h_3^{-1}\left(\frac{-\theta}{\log t}\right)]$. Then

$$
\begin{aligned}
E\left[\log f(F^{-1}(U_n^s))\right] &= -\log 4\theta - \frac{n+2}{2} + 2^n \sum_{j=n+1}^{\infty} \frac{\psi(j)}{2^{j-1}} \\
&\quad + E\left(\log\left(h_3'[h_3^{-1}\left(-\frac{\theta}{\log U_n^s}\right)]\right)\right) \\
&\geq -\log\left(\frac{4\theta}{m_3}\right) - \frac{n+2}{2} + 2^n \sum_{j=n+1}^{\infty} \frac{\psi(j)}{2^{j-1}}.
\end{aligned}
$$

The first equality is obtained by (15) and (16). So by part (i) of Lemma 3.2 of Ahmadi and Fashandi (2008), in this case:

$$
H(R_n^s) \leq H(U_n^s) + \log\left(\frac{4\theta}{m_3}\right) + \frac{n+2}{2} - 2^n \sum_{j=n+1}^{\infty} \frac{\psi(j)}{2^{j-1}}.
$$

From (17) and Remark 2.1.2, the right hand side of the above inequality is the entropy of the n-th current lower record coming from the Frechet distribution with cdf $F(x) = e^{-\frac{\theta}{m_3 x}}$.

(ii) $\Rightarrow$ (i): Suppose that the n-th current lower record of the distribution F has maximum entropy in $\mathcal{C}_3$. So by noting the proof of (i) $\Rightarrow$ (ii), we get

$$
E\left(\log\left(h_3'[h_3^{-1}\left(-\frac{\theta}{\log U_n^s}\right)]\right)\right) = \log m_3.
$$

The rest of the proof is similar to that of Theorem 3.1.1. $\square$

The n-th current lower record of the distribution F has minimum entropy in $\mathcal{C}_4$, whenever $h_4(x) = m_4 x$. This is stated in the following theorem. The proof is similar to the proof of Theorem 3.2.1.

Theorem 3.2.2.   *Assume that the conditions of Theorem 3.1.1 hold and let F belong to $\mathcal{C}_4$. Then the following two statements are equivalent:*

(i) $h_4(x) = m_4 x$,

(ii) the n-th current lower record of the distribution F has minimum entropy in $\mathcal{C}_4$.

Remark 3.2.1.   *In fact, Theorems 3.2.1 and 3.2.2 characterize the Frechet distribution based on the entropy of current records under some constraint.*

Remark 3.2.2.   *The results of Section 3 also hold based on the Shannon information of the upper bounds of record coverage.*

Acknowledgments

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