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Estimation of Parameters for an Extended Generalized Half Logistic Distribution Based on Complete and Censored Data

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Abstract. This paper considers an Extended *Generalized Half Logistic* distribution. We derive some properties of this distribution and then we discuss estimation of the distribution parameters by the methods of moments, maximum likelihood and the new method of minimum spacing distance estimator based on complete data. Also, maximum likelihood equations for estimating the parameters based on Type-I and Type-II censored data are given. In addition, the asymptotic variance and covariance of the estimators are given. We then evaluate the properties of maximum likelihood estimation (MLE) through the mean squared error, relative absolute bias and relative error. Furthermore, the asymptotic confidence intervals of the estimators are presented. Finally, simulation results are carried out to study the precision of the MLEs for the parameters involved.

Key words and phrases: Extended generalized half logistic distribution, Fisher information matrix, maximum likelihood estimation, minimum spacing distance estimator, type-I and type-II censored data

1 Introduction

In most life-testing experiments, we cannot continue the experiment until the last failure is observed based on cost and time considerations. So, the experiment is usually terminated when either a prefixed censoring time t arrives (Type-I censoring scheme) or when the rth failure is observed (Type-II censoring scheme); see, for example, Harter and Balakrishnan [11].

Balakrishnan [2] considers half logistic probability models obtained as the distribution of the absolute value of the standard logistic. Some key references about the half logistic distribution include Balakrishnan and Aggarwala [3], Balakrishnan and Wong [8] and Balakrishnan and Chan [4].

Balakrishnan and Puthenpura [6] obtained the best linear unbiased estimator of location and scale parameters of the half logistic distribution through linear functions of order statistics. Balakrishnan and Wong [7] obtained approximate maximum likelihood estimates for the location and scale parameters of the half logistic distribution with Type-II right-censoring. The half logistic distribution has not received much attention from researchers in terms of generalization. A generalized (Type-II) version of logistic distribution was considered and some interesting properties of the distribution were derived by Balakrishnan and Hossain [5]. The generalized versions of the half logistic distribution namely Type-I and Type-II were considered along with point estimation of scale parameters and estimation of stress strength reliability based on complete sample by Ramakrishnan [13].

In the next section, we present an extended generalized half logistic distribution and derive some of its properties. In Section 3, we discuss different estimating methods of parameters of this distribution for complete data. Then, in Section 4, we derive maximum likelihood equations for estimating the parameters based on censored data (Type I and Type II censored data). In Section 5, asymptotic confidence intervals of the estimators are presented. In Section 6, asymptotic variance covariance matrix of the estimators are given. In Section 7, simulation studies and properties of maximum likelihood estimators are given. Finally, we state some results in Section 8.

2 An extended generalized half logistic distribution

The probability density function (PDF) of an extended generalized half logistic distribution with the parameters μ, σ, λ and θ is defined by

$$f(x) = \frac{\theta(1+\lambda)^{\theta} e^{-\theta(\frac{x-\mu}{\sigma})}}{\sigma(1+\lambda e^{-\frac{x-\mu}{\sigma}})^{\theta+1}}, \qquad x > \mu, \tag{1}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$ and $\theta > 0$. The corresponding cumulative distribution function (CDF), the hazard rate function and survival function are

$$F(x) = 1 - \frac{(1+\lambda)^{\theta} e^{-\theta(\frac{x-\mu}{\sigma})}}{(1+\lambda e^{-\frac{x-\mu}{\sigma}})^{\theta}}, \qquad x > \mu, \tag{2}$$

$$\lambda(x) = \frac{\theta}{\sigma(1+\lambda e^{-\frac{x-\mu}{\sigma}})}, \tag{3}$$

$$S(x) = \frac{(1+\lambda)^{\theta} e^{-\theta(\frac{x-\mu}{\sigma})}}{(1+\lambda e^{-\frac{x-\mu}{\sigma}})^{\theta}}, \qquad x > \mu, \tag{4}$$

respectively. If $\mu = 0$, $\sigma = 1$ and $\lambda = 1$, then the extended generalized half logistic distribution reduces to the generalized half logistic distribution with the parameter θ proposed by Arora et al. [1]. Figure 1 illustrates the pdf and hazard rate function of the extended generalized half logistic distribution for $\mu = 0$, $\sigma = 1$ for some values of λ and θ. The first two moments of the extended generalized half logistic distribution are given by

$$\begin{aligned} E(X) &= \sigma\left((1+\lambda)\Gamma(\theta)\frac{{}_2F_1\Big(\{1,1\};1+\theta;-\lambda\Big)}{\Gamma(1+\theta)}\right) + \mu, \\ E(X^2) &= \sigma^2\left(\frac{2}{\theta^2}(1+\lambda)^{\theta}{}_3F_2\Big(\{\theta,\theta,\theta\};\{1+\theta,1+\theta\};-\lambda\Big)\right) + \mu^2 \\ &\quad +2\sigma\mu\left((1+\lambda)\Gamma(\theta)\frac{{}_2F_1\Big(\{1,1\};1+\theta;-\lambda\Big)}{\Gamma(1+\theta)}\right), \end{aligned}$$

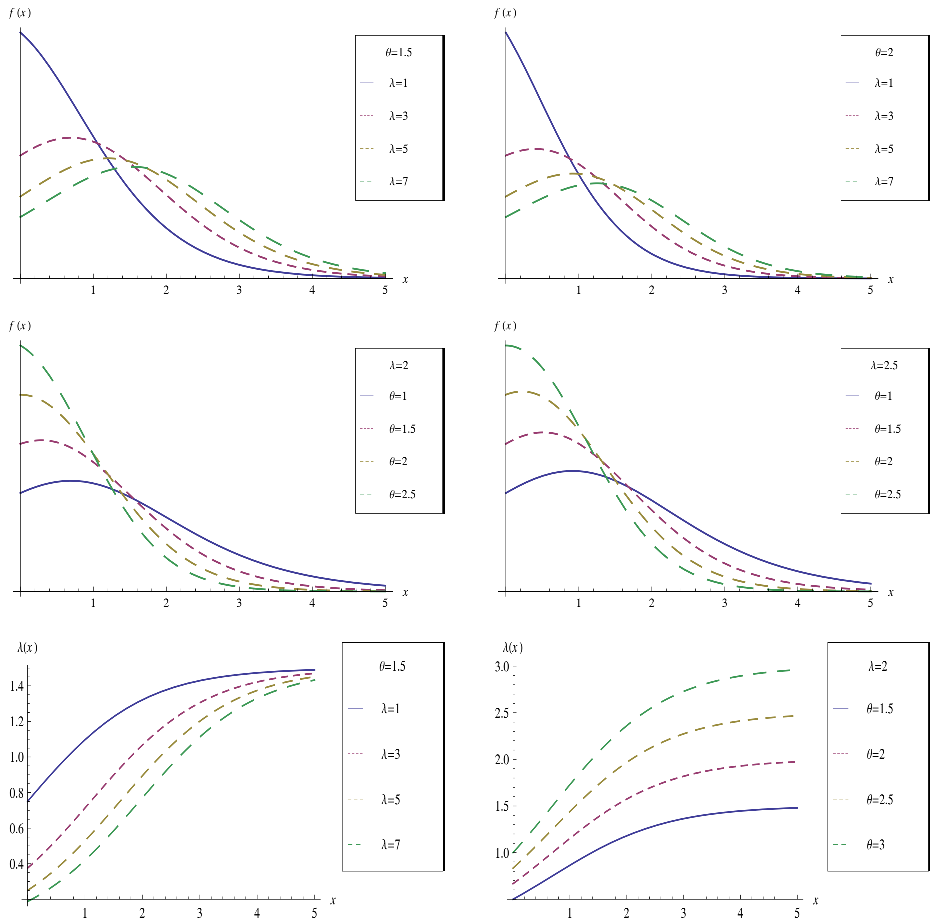


Figure 1: The pdf and hazard rate function for $\mu = 0$ and $\sigma = 1$.

where ${}_2F_1(\{a,b\};c;z)$ and ${}_3F_2(\{a,b\};c;z)$ are well-known functions which are defined for all $z \in \mathbb{R} - \{1\}$; in the special case for $|\lambda| < 1$, we have

$$ {}_2F_1\Big(\{1,1\};1+\theta;-\lambda\Big) = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k}{(1+\theta)_k}\frac{(-\lambda)^k}{k!}, $$

$$ {}_3F_2\Big(\{\theta,\theta,\theta\};\{1+\theta,1+\theta\};-\lambda\Big) = \sum_{k=0}^{\infty} \frac{(\theta)_k(\theta)_k(\theta)_k}{(1+\theta)_k(1+\theta)_k}\frac{(-\lambda)^k}{k!}, $$

in which

$$ (.)_k = \frac{\Gamma(.+k)}{\Gamma(.)}. $$

The mean and the variance for the extended generalized half logistic distribution are calculated using the Mathematica 7.0 software and summarized in Table 1 for some values of θ and λ.

Table 1: Mean, variance of the parameters of the extended generalized half logistic distribution for $\mu = 0$, $\sigma = 1$ and some values of θ and λ.

	λ	1	3	5	7
$\theta = 1.5$	$E(X)$	0.986	1.388	1.660	1.868
	Var(X)	0.699	1.004	1.193	1.327
$\theta = 2$	$E(X)$	0.772	1.131	1.380	1.573
	Var(X)	0.438	0.687	0.853	0.975

3 Different methods for estimating

In this section, estimating by the methods of moments, maximum likelihood and the new method of minimum spacing distance estimator are discussed.

3.1 Maximum likelihood estimation

The maximum likelihood is one of the most important and widely used methods in statistics. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. Furthermore, maximum likelihood estimators are consistent and asymptotically normally distributed under some certain regularity conditions, see for more details, Casella and Berger [9], Page 516.

3.1.1 Point estimates based on complete data

In order to determine the maximum likelihood estimations of the parameters μ, σ, λ and θ based on an i.i.d sample $X_1, ..., X_n$, the likelihood function in the complete sample case is

$$L(\mu, \sigma, \theta, \lambda) = \prod_{i=1}^{n} f(x_i)$$

if all $x_i \geq \mu$. Let $\min(x_1, ..., x_n) = x_{1:n}$, then we can write $L(\mu, \sigma, \theta, \lambda) = \prod_{i=1}^{n} f(x_i)$ if $x_{1:n} \geq \mu$. For arbitrary and fixed σ, θ and λ, it is clear that $L(\mu, \sigma, \theta, \lambda)$ is an increasing function in μ, so it is maximized at $\hat{\mu} = x_{1:n}$, i.e. $L(\mu, \sigma, \theta, \lambda) \leq L(\hat{\mu}, \sigma, \theta, \lambda)$ for all σ, θ and λ.

The log-likelihood function in this complete sample case is

$$\begin{aligned} l(\hat{\mu}, \sigma, \theta, \lambda) &= n \log \theta - n \log \sigma + n\theta \log(1+\lambda) - \theta \sum_{i=1}^{n} \frac{x_i - \hat{\mu}}{\sigma} \\ &\quad -(\theta+1) \sum_{i=1}^{n} \log(1 + \lambda e^{-\frac{x_i - \hat{\mu}}{\sigma}}) \end{aligned} \tag{5}$$

The log-likelihood equations yield

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\theta}{\sigma} \sum_{i=1}^{n} \frac{x_i - \hat{\mu}}{\sigma} - (\theta+1) \frac{\lambda}{\sigma^2} \sum_{i=1}^{n} \frac{e^{-\frac{x_i - \hat{\mu}}{\sigma}}}{1 + \lambda e^{-\frac{x_i - \hat{\mu}}{\sigma}}} = 0, \tag{6}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + n \log(1+\lambda) - \sum_{i=1}^{n} \frac{x_i - \hat{\mu}}{\sigma} - \sum_{i=1}^{n} \log(1 + \lambda e^{-\frac{x_i - \hat{\mu}}{\sigma}}) = 0, \tag{7}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n\theta}{1+\lambda} - (\theta+1) \sum_{i=1}^{n} \frac{e^{-\frac{x_i - \hat{\mu}}{\sigma}}}{1 + \lambda\, e^{-\frac{x_i - \hat{\mu}}{\sigma}}} = 0. \tag{8}$$

From log-likelihood equations, it can be obtained a closed-form expression for the MLE of θ as

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \frac{x_i - \hat{\mu}}{\hat{\sigma}} + \sum_{i=1}^{n} \log(1 + \hat{\lambda} e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}) - n \log(1 + \hat{\lambda})},$$

and by substituting of $\hat{\theta}$ in the Equation (6), we have

$$\begin{aligned} \frac{\partial l}{\partial \sigma} &= -n\hat{\sigma} + \Big(\frac{n}{\sum_{i=1}^{n} \frac{x_i - \hat{\mu}}{\hat{\sigma}} + \sum_{i=1}^{n} \log(1 + \hat{\lambda} e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}) - n \log(1 + \hat{\lambda})} \Big) \\ &\quad \times \Big(\sum_{i=1}^{n} (x_i - \hat{\mu}) - \hat{\lambda} \sum_{i=1}^{n} \frac{e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}}{1 + \hat{\lambda} e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}} \Big) - \hat{\lambda} \sum_{i=1}^{n} \frac{e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}}{1 + \hat{\lambda} e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}}, \end{aligned}$$

also by substituting of $\hat{\theta}$ in the last likelihood equation for λ, we have

$$\frac{\partial l}{\partial \lambda} = \Big(\frac{n}{1 + \hat{\lambda}} - \sum_{i=1}^{n} \frac{e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}}{1 + \hat{\lambda} e^{-\frac{x_i - \hat{\mu}}{\hat{\sigma}}}} \Big)$$

$$\times\Big(\frac{n}{\sum_{i=1}^{n}\frac{x_i-\hat{\mu}}{\hat{\sigma}}+\sum_{i=1}^{n}\log(1+\hat{\lambda}e^{-\frac{x_i-\hat{\mu}}{\hat{\sigma}}})-n\log(1+\hat{\lambda})}\Big)$$
$$-\sum_{i=1}^{n}\frac{e^{-\frac{x_i-\hat{\mu}}{\hat{\sigma}}}}{1+\hat{\lambda}e^{-\frac{x_i-\hat{\mu}}{\hat{\sigma}}}}=0.$$

Numerical methods must be applied for simultaneously solving the nonlinear equations to obtain $\hat{\sigma}$ and $\hat{\lambda}$. Then $\hat{\theta}$ will be calculated easily. The required numerical evaluations were implemented using the R Software through the package (stats4), command `mle` with the L-BFGS-B method.

Henceforth, in the following sections, we consider $\mu = 0$ and $\sigma = 1$, i.e., instead of the extended generalized half logistic distribution with the parameters μ, σ, θ and λ we deal with the extended generalized half logistic distribution with the parameters θ and λ, but in each case, one can consider the extended form, easily.

3.2 Method of moment estimators

In this subsection we provide moment method estimators (MME) of the parameters of the extended generalized half logistic distribution. Let $X_1, \ldots, X_n$ be a random sample from the pdf in the Equation (1) with the parameters θ and λ. Define

$$\imath_1 := (1+\lambda)\Gamma(\theta)\frac{{}_2F_1\Big(\{1,1\};1+\theta;-\lambda\Big)}{\Gamma(1+\theta)}-\frac{1}{n}\sum_{i=1}^{n}X_i,$$
$$\imath_2 := \frac{2}{\theta^2}(1+\lambda)^{\theta}{}_3F_2\Big(\{\theta,\theta,\theta\};\{1+\theta,1+\theta\};-\lambda\Big)-\frac{1}{n}\sum_{i=1}^{n}X_i^2.$$

The MME of the parameters are obtained by minimizing $\imath_1^2+\imath_2^2$ with respect to the unknown parameters. In this paper, the required numerical evaluations were implemented using the R Software through the command `nlminb`.

3.3 Minimum spacing distance estimator

In this subsection we provide the minimum spacing distance estimator (MSDE) of the extended generalized half logistic distribution. Let $X_1, \ldots, X_n$ be a random sample from continuous function $F_{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in$

$\mathbf{\Theta} \subset R^k$ with support on R. Let the order statistics be denoted by $Y_1, \ldots, Y_n$. Define

$$D_i(\boldsymbol{\theta}) = F_{\boldsymbol{\theta}}(Y_i) - F_{\boldsymbol{\theta}}(Y_{i-1}), \qquad i = 1, \ldots, n+1, \tag{9}$$

where $F_{\boldsymbol{\theta}}(Y_0) = 0$ and $F_{\boldsymbol{\theta}}(Y_{n+1}) = 1$. The MSDE of $\boldsymbol{\theta}$ obtain the estimators by minimizing of

$$T(\boldsymbol{\theta}) = \sum_{i=1}^{n+1} h\left(D_i(\boldsymbol{\theta}), 1/n\right),$$

in which $h(x, y)$ is a appropriate distance. Some choices of $h(x, y)$ are $|x - y|$ and $\exp\{x - y\} - (x - y) - 1$, which are called "absolute" and "linex" distance, respectively. These estimators, called "minimum spacing absolute distance estimator" (MSADE) and "minimum spacing linex distance estimator" (MSLDE).

This method was originally explored by Torabi [14] and it has been used quite successfully for the extended generalized half logistic distribution.

In the extended generalized half logistic distribution, we have

$$D_i(\boldsymbol{\theta}) = (1+\lambda)^{\theta}\Big(\frac{e^{-\theta Y_{i-1}}}{(1+\lambda e^{-Y_{i-1}})^{\theta}} - \frac{e^{-\theta Y_i}}{(1+\lambda e^{-Y_i})^{\theta}}\Big)$$

where $\boldsymbol{\theta} := (\theta, \lambda)$. The MSADE and MSLDE of parameters can be obtained by minimizing

$$T(\boldsymbol{\theta}) = \sum_{i=1}^{n+1} \left|D_i(\boldsymbol{\theta}) - 1/n\right|,$$

and

$$T(\theta) = \sum_{i=1}^{n+1} \left[\exp\left\{D_i(\boldsymbol{\theta}) - 1/n\right\} - (D_i(\boldsymbol{\theta}) - 1/n) - 1\right],$$

with respect to $\boldsymbol{\theta}$, respectively.

We simulate n =80, 100, 120, 150 and 200 times the extended generalized half logistic distribution for $\theta = 1.3$ and $\lambda = 2$. For each sample size, we compute the MLE's, MME's, MSADE's and MSLDE's of the parameters. We repeat this process 1000 times and compute the average estimate (AE) and MSE. The results are reported in Table 2.

Table 2: Estimated AE and MSE of MLE, MME, MSADE and MSLDE of parameters based on 1000 simulations of the extended generalized half logistic distribution for $\theta = 1.3$ and $\lambda = 2$ with n= 80, 100, 120, 150 and 200.

		MLE		MME		MSADE		MSLDE	
n		AE	MSE	AE	MSE	AE	MSE	AE	MSE
80	θ	1.461	0.292	1.468	0.292	1.338	0.369	1.210	0.090
	λ	2.608	3.786	2.625	3.627	2.235	4.040	2.003	0.001
100	θ	1.442	0.182	1.448	0.168	1.341	0.333	1.224	0.075
	λ	2.548	2.698	2.567	2.394	2.230	4.015	2.002	0.001
120	θ	1.421	0.142	1.432	0.142	1.342	0.233	1.228	0.070
	λ	2.458	2.003	2.497	2.020	2.248	3.973	2.002	0.001
150	θ	1.367	0.078	1.373	0.079	1.315	0.130	1.218	0.071
	λ	2.273	1.131	2.296	1.130	2.109	1.850	2.003	0.001
200	θ	1.363	0.066	1.368	0.068	1.312	0.090	1.225	0.065
	λ	2.251	0.932	2.270	0.953	2.099	1.336	2.002	0.001

Comparing the performance of all the estimators, it is observed that for all methods, the MSE's decrease as the sample size increases. Note that, the performances of the MSLDE's are the best as far as the MSE is concerned, but after this method, the MLE's and the MME's performances are considerable. Considering all the points, we recommend to use the MSLDE for estimating of parameters.

3.4 Applications

In this subsection we fit the extended generalized half logistic model to one real data set. Data set is given by Hinkley [12] on the thirty successive values of March precipitation (in inches) in Minneapolis/St Pau. The data set consists of 30 observations.

The TTT plot for the Hinkley data in Figure 2 shows an increasing hazard rate function and therefore, indicates the appropriateness of the extended generalized half logistic distribution to fit these data.

Here we will fit the extended generalized half logistic model to this real data set and will show that the extended generalized half logistic distribution is more flexible in analyzing of the data than the generalized half logistic distribution proposed by Arora et al. [1].

In order to compare the models, two criteria are usually used: Akaike Information Criterion(AIC) and BIC (Bayesian Information

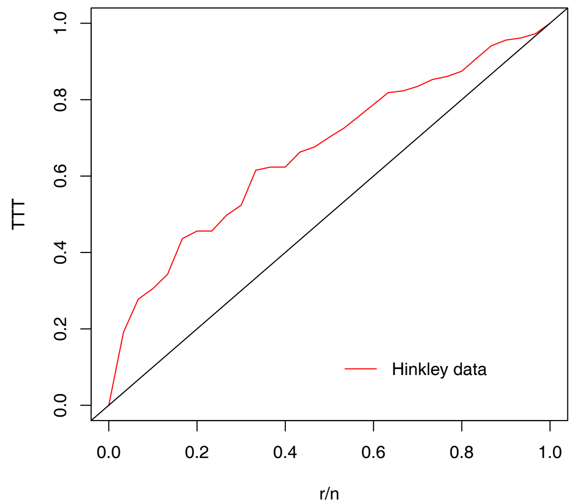


Figure 2: TTT plot for the Hinkley data

Criterion) which are defined as follows:

$$AIC = -2\log \hat{L} + 2k, \quad BIC = -2\log \hat{L} + k\log(n),$$

where k is the number of free parameters in the model and n is the sample size. For fitting a data set, the best model is a model with the smallest value of AIC and BIC statistics.

We can also perform formal goodness-of-fit tests in order to verify which distribution fits better to these data. We apply Kolmogorov-Smirnov (K-S) statistics and the p-value from the chi-square goodness of fit test, where small values of K-S statistic and large values of p-value for models indicate that these models could be chosen as the best model to fit the data.

The K-S statistic and the corresponding p-value evaluations were implemented using the R software through the command `ks.test`.

We first estimate the unknown parameter from the generalized half logistic distribution based on this data set; $\hat{\theta} = 0.828$. Then, we fit the generalized half logistic distribution with the parameter θ to the data set and perform the one sample Kolmogorov-Smirnov (K-S) test. The K-S statistic between the empirical distribution function and the fitted generalized half logistic distribution function is 0.1819 and the corresponding p-value is 0.2739. Also AIC and BIC are 86.869 and 88.271, respectively.

For $\mu = 0$ and $\sigma = 1$, we try to fit the extended generalized half logistic distribution with the parameters θ and λ to this data set. The maximum likelihood estimations of parameters θ and λ from the extended generalized half logistic distribution based on this data set yield $\hat{\theta} = 1.856$ and $\hat{\lambda} = 7.065$. The K-S statistic between the empirical distribution function and the fitted extended generalized half logistic distribution function is 0.0914 and the corresponding p-value is 0.9637 and if μ and σ are unknown, then p-value is 1. Also AIC and BIC are 83.969 and 86.772, respectively. Hence, it is clear that the extended generalized half logistic distribution fits quite well to this data set and is better than the generalized half logistic distribution with the parameter θ.

The plots of the estimated survival function or Kaplan-Meier curve (right plot) and estimated densities of distributions (left plot) fitted to the data set are given in Figure 3. This figure shows that the extended generalized half logistic distribution gives a better fit than the generalized half logistic distribution.

4 Point estimates based on censored data

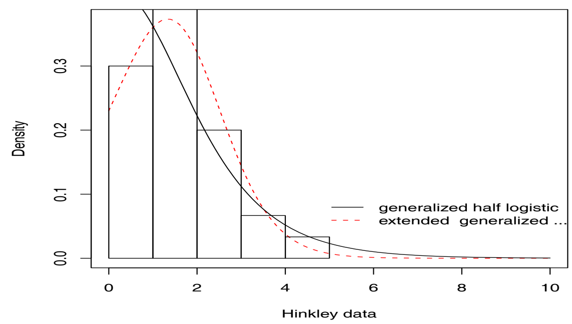
In this section, we just determine the maximum likelihood estimations of the parameters of the extended generalized half logistic distribution based on Type-I and Type-II censored data.

In the case of Type-I censored data, let t denote a pre-fixed censoring time and $x_{1:n}, ..., x_{r:n}$ the ordered observed failure times. The log-likelihood is

$$\begin{aligned} l(\theta,\lambda) &= r\log(\theta) + r\theta\log(1+\lambda) \\ &\quad -\theta\sum_{i=1}^{r} x_{i:n} - (\theta+1)\sum_{i=1}^{r}\log(1+\lambda e^{-x_{i:n}}) \\ &\quad +\theta(n-r)\log(1+\lambda) \\ &\quad -\theta(n-r)t - \theta(n-r)\log(1+\lambda e^{-t}). \end{aligned} \tag{10}$$

The likelihood equations system is

$$\begin{aligned} \frac{\partial l}{\partial \theta} &= \frac{r}{\theta} + r\log(1+\lambda) - \sum_{i=1}^{r} x_{i:n} - \sum_{i=1}^{r}\log(1+\lambda e^{-x_{i:n}}) \\ &\quad +(n-r)\log(1+\lambda) - (n-r)t - (n-r)\log(1+\lambda e^{-t}) \\ &= 0, \end{aligned} \tag{11}$$



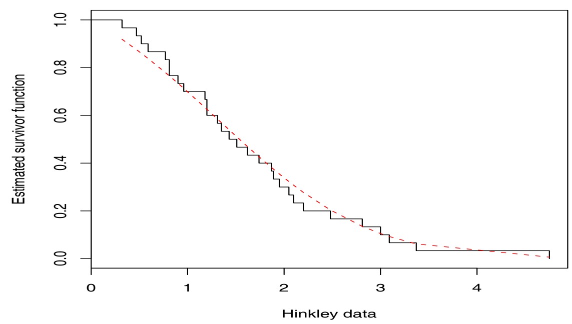


Figure 3: Fitted pdf and survival function of the extended generalized half logistic and the generalized half logistic distributions.

$$\frac{\partial l}{\partial \lambda} = \frac{r\theta}{1+\lambda} - (\theta+1)\sum_{i=1}^{r}\frac{e^{-x_{i:n}}}{1+\lambda e^{-x_{i:n}}} + \frac{\theta(n-r)}{1+\lambda} - \frac{\theta(n-r)e^{-t}}{1+\lambda e^{-t}} = 0. \tag{12}$$

From the Equation (11), the maximum likelihood estimation of θ is expressed by $\hat{\theta} = \frac{r}{a}$ where,

$$a = \sum_{i=1}^{r} x_{i:n} + \sum_{i=1}^{r}\log(1+\hat{\lambda}e^{-x_{i:n}}) - n\log(1+\hat{\lambda}) + (n-r)\Big(t + \log(1+\hat{\lambda}e^{-t})\Big).$$

Consequently, by substituting of $\hat{\theta}$ in the Equation (12), the system of likelihood equations are reduced to one nonlinear equation as follows:

$$\left(\frac{r}{1+\hat{\lambda}}-\sum_{i=1}^{r}\frac{e^{-x_{i:n}}}{1+\hat{\lambda}e^{-x_{i:n}}}+\frac{(n-r)}{1+\hat{\lambda}}-\frac{(n-r)e^{-t}}{1+\hat{\lambda}e^{-t}}\right)\frac{r}{a}-\sum_{i=1}^{r}\frac{e^{-x_{i:n}}}{1+\hat{\lambda}e^{-x_{i:n}}}=0.$$

Since the closed form solution to this nonlinear likelihood equation are very hard to obtain, numerical methods are applied for solving the nonlinear equation to obtain $\hat{\lambda}$. Then, $\hat{\theta}$ is calculated easily.

For Type-II censoring, the difference lies in the fact that in Type I censoring, the time of testing is fixed and the number of failures r is a random variable, whereas, in this case, the likelihood function is similar to (10) with replacing t by $x_{r:n}$. The subsequent results follow easily with suitable adjustments.

5 Interval estimates

Because the MLE of the model parameters do not have any closed form expressions, it is not possible to derive their distributions, and therefore the corresponding exact confidence intervals (CI). Hence, we will discuss here the asymptotic confidence intervals.

For large sample size n, the derivation of the asymptotic confidence intervals for the parameters θ and λ under complete and censored data will be based on the pivotal quantities $(\hat{\theta}-E(\hat{\theta}))/\sqrt{V(\hat{\theta})}$ and $(\hat{\lambda}-E(\hat{\lambda}))/\sqrt{V(\hat{\lambda})}$. The maximum likelihood estimates, under appropriate regularity conditions (A1-A4 conditions stated in Casella and Berger [9] Page 516) are consistent and asymptotically normally distributed. It can be easily shown that all regularity conditions are satisfied for the extended generalized half logistic distribution. Therefore, we end up with the asymptotic two-sided $100(1-\alpha)\%$ CI of the form $\hat{\theta}\pm z_{\alpha/2}\sqrt{V(\hat{\theta})}$ and $\hat{\lambda}\pm z_{\alpha/2}\sqrt{V(\hat{\lambda})}$, where z_p is the p-th upper percentile of the standard normal distribution. Here, $V(\hat{\theta})$ and $V(\hat{\lambda})$ are the diagonal elements of the inverse of the observed Fisher information matrix presented in next section.

6 Asymptotic variances and covariances of estimates

The asymptotic variances and covariances of maximum likelihood estimators are given by the elements of the inverse of the Fisher information matrix

$$I_{ij}(\boldsymbol{\theta}) = E\{-\partial^2 l/\partial\theta_i\partial\theta_j\}, \qquad i,j = 1,2,$$

where $\boldsymbol{\theta} = (\theta, \lambda)$, $\theta_1 = \theta$ and $\theta_2 = \lambda$.

Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, we give the Fisher information matrix for the MLE, which is obtained by dropping the expectation operator E, see Cohen [10].

6.1 Complete data

The Fisher information matrix is the symmetric matrix and its elements based on complete data can be shown to be

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{-n}{\theta^2}, \tag{13}$$

$$\frac{\partial^2 l}{\partial\theta\partial\lambda} = \frac{n}{1+\lambda} - \sum_{i=1}^{n} \frac{e^{-x_i}}{1+\lambda e^{-x_i}}, \tag{14}$$

$$\frac{\partial^2 l}{\partial\lambda^2} = \frac{-n\theta}{(1+\lambda)^2} + (\theta+1)\sum_{i=1}^{n} \frac{e^{-2x_i}}{(1+\lambda e^{-x_i})^2}. \tag{15}$$

Consequently, the maximum likelihood estimators of θ and λ have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix and then substituting θ and λ by $\hat{\theta}$ and $\hat{\lambda}$.

6.2 Censored data

In the case of Type-I censored data, the Fisher information matrix is the symmetric matrix and its elements are given by

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{-r}{\theta^2}, \tag{16}$$

$$\frac{\partial^2 l}{\partial\theta\partial\lambda} = \frac{r}{1+\lambda} - \sum_{i=1}^{r} \frac{e^{-x_{i:n}}}{1+\lambda e^{-x_{i:n}}} + \frac{(n-r)}{1+\lambda} - \frac{(n-r)e^{-t}}{1+\lambda e^{-t}}, \tag{17}$$

$$\frac{\partial^2 l}{\partial\lambda^2} = \frac{-r\theta}{(1+\lambda)^2} + (\theta+1)\sum_{i=1}^{r} \frac{e^{-2x_{i:n}}}{(1+\lambda e^{-x_{i:n}})^2} - \frac{\theta(n-r)}{(1+\lambda)^2} + \frac{\theta(n-r)e^{-2t}}{(1+\lambda e^{-t})^2}. \tag{18}$$

For Type-II censoring, the Fisher information matrix elements is similar to Type-I with replacing t by $x_{r:n}$. Hence, the maximum likelihood estimators of θ and λ have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix and finally, substituting θ and λ by $\hat{\theta}$ and $\hat{\lambda}$.

7 Simulation study

In order to obtain the MLEs of θ and λ and study the properties of their estimates through the mean squared error (MSE), relative absolute bias (RAB) and relative error (RE) a simulation procedure of the generalized half logistic distribution under complete and censored data is described below:

7.1 Complete data

For given values of n, λ and θ,
Step 1. we generate a random sample of size n from the Uniform $(0, 1)$ distribution;

Step 2. then the observations $x_1, ..., x_n$ from the generalized half logistic distribution are calculated as follows

$$x_i = -\log\left(\frac{(1-u_i)^{\frac{1}{\theta}}}{(1+\lambda)-\lambda(1-u_i)^{\frac{1}{\theta}}}\right) \qquad i = 1, ..., n$$

Step 3. Based on the observations given in Step 2 we can get the MLEs of parameters.

Step 4. Repeat Steps 1-3 r times representing r different samples. The value of r has been taken to be 1000.

Step 5. If $\hat{\psi_{kl}}$ is a MLE of $\psi_l, l = 1, 2$ (where ψ_l is a general notation that can be replaced by θ and λ i.e. $\psi_1 \equiv \theta$ and $\psi_2 \equiv \lambda$), based on sample $k, k = 1, ..., r$ then the average estimate (AE), MSE, RAB and RE of $\hat{\psi_l}$ over the r samples are given, respectively, by

$\bar{\hat{\psi_l}} = \frac{1}{r}\sum\limits_{k=1}^{r}\hat{\psi_{kl}},$

$MSE(\hat{\psi}_l) = \frac{1}{r} \sum_{k=1}^{r} (\hat{\psi_{kl}} - \psi_l)^2,$

$RAB(\hat{\psi}_l) = \frac{|\bar{\hat{\psi}}_l - \psi_l|}{\psi_l},$

$RE(\hat{\psi}_l) = \frac{\sqrt{MSE(\hat{\psi}_l)}}{\psi_l}.$

Step 6. From Step 5 compute $\bar{\hat{\theta}}$, $\bar{\hat{\lambda}}$, $MSE(\hat{\theta})$, $MSE(\hat{\lambda})$, $RAB(\hat{\theta})$, $RAB(\hat{\lambda})$, $RE(\hat{\theta})$ and $RE(\hat{\lambda})$.

7.2 Type-I censoring

Based on n, t, λ and θ,
Step 1. we generate a random sample of size n from the Uniform $(0, 1)$ distribution, and obtain the order statistics $(u_{1:n}, ..., u_{n:n})$;

Step 2. Find r such that $u_{r:n} < 1 - \frac{(1+\lambda)^\theta e^{-\theta t}}{(1+\lambda e^{-t})^\theta} < u_{r+1:n}$, For $1 \leq i \leq r$ the ordered observations $x_{1:n}, ..., x_{r:n}$ are given by

$$x_{i:n} = -\log\left(\frac{(1 - u_{i:n})^{\frac{1}{\theta}}}{(1 + \lambda) - \lambda(1 - u_{i:n})^{\frac{1}{\theta}}}\right),$$

Step 3. Based on the ordered observations given in Step 2 we can get the MLEs of parameters.

Then we will do Steps 4-6 exactly along the lines of Subsection 5.1.

7.3 Type-II censoring

Based on n, r, λ and θ,
Step 1. we generate a random sample of size n from the Uniform $(0, 1)$ distribution, and obtain the order statistics $(u_{1:r}, ..., u_{r:r})$;

Step 2. For $1 \leq i \leq r$, the ordered observations $x_{1:n}, ..., x_{r:n}$ are given by

$$x_{i:n} = -\log\left(\frac{(1 - u_{i:n})^{\frac{1}{\theta}}}{(1 + \lambda) - \lambda(1 - u_{i:n})^{\frac{1}{\theta}}}\right)$$

Step 3. Based on the ordered observations given in Step 2 we can get the MLEs of parameters.

Then we will do Steps 4-6 exactly along the lines of Subsection 5.1.

8 Numerical results

All simulation results are summarized in Tables 3 - 11. Table 3 gives the MSE, RAB and RE of the estimators under complete data. Table 4 and 5 gives the MSE, RAB and RE of the estimators under Type-I and Type-II censored data, respectively. The asymptotic variances and covariances of the estimators based on complete, Type-I and Type-II censored data are displayed in Tables 6 - 8 respectively. Based on complete, Type-I and Type-II censored data the approximated confidence limits at 95% and 99% for the parameters are presented in Tables 9 - 11.

Table 3. The MSE, RAB and RE of the parameters θ and λ under complete data based on 1000 simulations. Population parameter values: $\theta = 1.3$, $\lambda = 2$.

n	$\bar{\hat{\theta}}$ / $\bar{\hat{\lambda}}$	$MSE(\hat{\theta})$ / $MSE(\hat{\lambda})$	$RAB(\hat{\theta})$ / $RAB(\hat{\lambda})$	$RE(\hat{\theta})$ / $RE(\hat{\lambda})$
100	1.426	0.182	0.097	0.329
	2.488	2.554	0.244	0.799
150	1.388	0.094	0.068	0.236
	2.321	1.240	0.161	0.557
200	1.360	0.066	0.046	0.198
	2.233	0.946	0.117	0.486
250	1.344	0.047	0.034	0.166
	2.176	0.641	0.088	0.400
300	1.334	0.038	0.026	0.151
	2.128	0.530	0.064	0.364
350	1.332	0.029	0.025	0.131
	2.118	0.416	0.059	0.323
400	1.330	0.028	0.022	0.130
	2.104	0.387	0.052	0.311
450	1.323	0.022	0.018	0.115
	2.096	0.305	0.048	0.276
500	1.319	0.021	0.015	0.114
	2.062	0.282	0.030	0.267

Table 4. The MSE, RAB and RE of the parameters θ and λ under Type-I censoring based on 1000 simulations. Population parameter values: $\theta = 1.3$, $\lambda = 2$ and censoring time $t = 1.5$.

n	$\hat{\theta}$ $\bar{\hat{\lambda}}$	$MSE(\hat{\theta})$ $MSE(\hat{\lambda})$	$RAB(\hat{\theta})$ $RAB(\hat{\lambda})$	$RE(\hat{\theta})$ $RE(\hat{\lambda})$
100	2.394	7.680	0.841	2.132
	5.571	79.181	1.875	4.450
150	1.935	4.010	0.489	1.539
	4.085	42.371	1.042	3.255
200	1.685	2.121	0.296	1.120
	3.247	21.569	0.623	2.322
250	1.574	1.125	0.211	0.816
	2.863	10.929	0.432	1.653
300	1.528	0.807	0.176	0.691
	2.735	8.718	0.366	1.476
350	1.518	0.741	0.167	0.662
	2.696	7.629	0.348	1.381
400	1.450	0.509	0.116	0.549
	2.486	5.050	0.243	1.124
450	1.452	0.417	0.117	0.497
	2.477	4.097	0.238	1.012
500	1.413	0.278	0.087	0.406
	2.367	2.725	0.184	0.825

Table 5. The MSE, RAB and RE of the parameters θ and λ under Type-II censoring based on 1000 simulations. Population parameter values: $\theta = 1.3$, $\lambda = 2$.

n / r	$\hat{\theta}$ / $\bar{\hat{\lambda}}$	$MSE(\hat{\theta})$ / $MSE(\hat{\lambda})$	$RAB(\hat{\theta})$ / $RAB(\hat{\lambda})$	$RE(\hat{\theta})$ / $RE(\hat{\lambda})$
100	1.641	1.888	0.262	1.057
85	3.146	19.109	0.573	2.186
150	1.467	0.312	0.129	0.429
135	2.607	3.891	0.303	0.986
200	1.426	0.200	0.097	0.344
175	2.442	2.489	0.221	0.789
250	1.380	0.099	0.062	0.243
225	2.278	1.266	0.139	0.563
300	1.352	0.061	0.040	0.189
275	2.191	0.766	0.095	0.437
350	1.346	0.047	0.035	0.167
325	2.155	0.585	0.078	0.382
400	1.339	0.038	0.030	0.150
375	2.137	0.500	0.068	0.354
450	1.336	0.034	0.028	0.141
425	2.132	0.439	0.066	0.331
500	1.324	0.027	0.019	0.127
475	2.084	0.361	0.042	0.300

Table 6. Asymptotic variances and covariances of estimates under complete data. Population parameter values: $\theta = 1.3$, $\lambda = 2$.

n	$Var(\hat{\theta})$	$Var(\hat{\lambda})$	$Cov(\hat{\theta}, \hat{\lambda})$
100	0.102	1.421	0.344
150	0.071	1.015	0.242
200	0.056	0.902	0.204
250	0.043	0.709	0.160
300	0.037	0.415	0.111
350	0.026	0.408	0.092
400	0.024	0.364	0.086
450	0.020	0.324	0.072
500	0.017	0.273	0.061

Table 7. Asymptotic variances and covariances of estimates under Type-I censoring. Population parameter values: $\theta = 1.3$, $\lambda = 2$ and censoring time $t = 1.5$.

n	$Var(\hat{\theta})$	$Var(\hat{\lambda})$	$Cov(\hat{\theta}, \hat{\lambda})$
100	0.467	3.710	1.284
150	0.392	2.871	1.036
200	0.291	2.805	0.884
250	0.254	2.340	0.755
300	0.202	2.153	0.645
350	0.136	1.258	0.400
400	0.133	1.014	0.359
450	0.099	0.892	0.290
500	0.083	0.851	0.258

Table 8. Asymptotic variances and covariances of estimates under Type-II censoring. Population parameter values: $\theta = 1.3$, $\lambda = 2$.

$n(r)$	$Var(\hat{\theta})$	$Var(\hat{\lambda})$	$Cov(\hat{\theta}, \hat{\lambda})$
100 (85)	0.132	2.232	0.512
150 (135)	0.123	1.381	0.388
200 (175)	0.079	0.914	0.253
250 (225)	0.069	0.757	0.214
300 (275)	0.051	0.663	0.172
350 (325)	0.046	0.524	0.149
400 (375)	0.036	0.523	0.127
450 (425)	0.029	0.343	0.093
500 (475)	0.022	0.268	0.071

Table 9. Interval estimation for parameters based on complete data.

n	parameters	95%	99%
100	θ	(1.159, 1.558)	(1.097, 1.620)
	λ	(0, 5.019)	(0, 5.894)
150	θ	(1.270, 1.547)	(1.226, 1.591)
	λ	(0.469, 4.448)	(0, 5.074)
200	θ	(1.327, 1.549)	(1.292, 1.583)
	λ	(0.981, 4.516)	(0.425, 5.072)
250	θ	(1.263,1.433)	(1.236,1.459)
	λ	(1.041, 3.818)	(0.604, 4.254)
300	θ	(1.261,1.400)	(1.239,1.422)
	λ	(1.023, 2.650)	(0.767, 2.905)
350	θ	(1.257,1.358)	(1.242,1.373)
	λ	(1.491, 3.091)	(1.239, 3.343)
400	θ	(1.309,1.406)	(1.294,1.422)
	λ	(1.602, 3.028)	(1.378, 3.252)
450	θ	(1.291,1.369)	(1.279,1.382)
	λ	(1.787, 3.059)	(1.587, 3.259)
500	θ	(1.233,1.299)	(1.223,1.310)
	λ	(1.636, 2.706)	(1.468, 2.874)

Table 10. Interval estimation for parameters based on Type-I censoring.

n	parameters	95%	99%
100	θ	(0.308, 2.140)	(0.020, 2.428)
	λ	(0, 8.824)	(0, 11.109)
150	θ	(0.587, 2.124)	(0.346, 2.366)
	λ	(0, 7.394)	(0, 9.163)
200	θ	(0.681, 1.823)	(0.501, 2.003)
	λ	(0, 7.426)	(0, 9.154)
250	θ	(0.781, 1.777)	(0.625, 1.934)
	λ	(0, 6.504)	(0, 7.945)
300	θ	(0.829, 1.619)	(0.705,1.744)
	λ	(0, 6.236)	(0, 7.562)
350	θ	(0.941,1.465)	(0.859,1.548)
	λ	(0, 4.230)	(0, 5.004)
400	θ	(1.019,1.542)	(0.937, 1.624)
	λ	(0, 3.608)	(0, 4.232)
450	θ	(0.954,1.341)	(0.893,1.402)
	λ	(0, 3.270)	(0, 3.819)
500	θ	(0.981,1.304)	(0.930,1.355)
	λ	(0.072, 3.408)	(0, 3.932)

Table 11. Interval estimation for parameters based on Type-II censoring.

n	parameters	95%	99%
100	θ	(0.897, 1.418)	(0.815, 1.500)
85	λ	(0, 6.468)	(0, 7.843)
150	θ	(1.147, 1.631)	(1.071, 1.707)
135	λ	(0, 4.833)	(0, 5.683)
200	θ	(1.124, 1.434)	(1.075, 1.483)
175	λ	(0.103, 3.685)	(0, 4.247)
250	θ	(1.246, 1.515)	(1.203, 1.557)
225	λ	(0.612, 3.581)	(0.146, 4.047)
300	θ	(0.879, 1.765)	(0.740, 1.904)
275	λ	(0.523, 3.714)	(0.022, 4.215)
350	θ	(0.484, 1.326)	(0.352, 1.458)
325	λ	(0, 2.496)	(0, 2.943)
400	θ	(1.290,1.430)	(1.267, 1.452)
375	λ	(1.382, 3.433)	(1.060, 3.755)
450	θ	(1.311,1.426)	(1.293,1.445)
425	λ	(1.402, 2.748)	(1.191, 2.959)
500	θ	(1.243,1.331)	(1.229,1.345)
475	λ	(1.329, 2.378)	(1.164, 2.545)

9 Conclusions

In this paper, we considered an extended generalized half logistic distribution and derived some properties of this distribution. We also obtained maximum likelihood equations for estimating the distribution parameters based on complete data, Type-I and Type-II censored data. In addition, asymptotic variance and covariance of the estimators were given. We also evaluated the properties of maximum likelihood estimation through the mean squared error, relative absolute bias and relative error. Furthermore, asymptotic confidence intervals of the estimators derived. Finally, some simulation results are presented.

From Tables 3 - 11, we conclude that if the sample size n is increased, then

1. the MSE, RAB and RE of $\hat{\theta}$ and $\hat{\lambda}$ are decreased. This indicates that the maximum likelihood estimates provide asymptotically

normally distributed and consistent estimators for the parameters, (see Tables 3 - 5);

2. the asymptotic variances of the estimators are decreased, (see Tables 6 - 8);

3. The confidence interval lengths of the parameters are decreased. Also, the confidence interval length at $1-\alpha = 0.95$ is smaller than the confidence interval length at $1-\alpha = 0.99$, (see Tables 9 - 11).

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