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On Estimation Following Selection with Applications on k-Records and Censored Data

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Abstract. Let X_1 and X_2 be two independent random variables from gamma populations Π_1, Π_2 with means $\alpha\theta_1$ and $\alpha\theta_2$ respectively, where $\alpha(> 0)$ is the common known shape parameter and θ_1 and θ_2 are scale parameters. Let $X_{(1)} \leq X_{(2)}$ denote the order statistics of X_1 and X_2. Suppose that the population corresponding to the largest $X_{(2)}$ (or the smallest $X_{(1)}$) observation is selected. The problem of interest is to estimate the scale parameters θ_M (and θ_J) of the selected gamma population under an asymmetric scale invariant loss function. We characterize admissible estimators of θ_M (or θ_J) within the class of linear estimators of the form $cX_{(2)}$ (or $cX_{(1)}$). In estimating θ_M,

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we derive a minimax estimator and provide sufficient conditions for the inadmissibility of arbitrary invariant estimators of θ_M. We apply our results to k-Records and censored data. Finally, we extend our results to a subclass of exponential family of distributions.

1 Introduction

The problem of estimating parameter(s) of a selected population is an important estimation problem having wide practical applications in various agricultural, industrial or medical experiments and in some cases it is related to ranking and selection methodology. There are numerous such examples in the literature. As an example, an agricultural experimenter, who has selected the variety with the highest yield, would naturally be interested in estimating the average yield of the selected variety, see Kumar and Kar (2001), a commercial vehicle operator not only prefer to buy a vehicle with maximum fuel efficiency, but he also wants to estimate the average fuel efficiency of the selected vehicle, see Kumar and Gangopadhyay (2005), or a drug company selects the regimen with maximal efficacy or minimal toxicity from a set of regimens and estimates a treatment effect for the selected regimen, see Sill and Sampson (2007).

The problem of estimating after selection has been a subject of interest over the past three decades. Readers may refer to Gibbons et al. (1999) and Gupta and Panchapakesan (2002). Some other contributions in this area are: Sarkadi (1967), Dahiya (1974), Kumar and Kar (2001), Misra et al. (2006a,b) and Kumar et al. (2009).

Let X_1 and X_2 be two independent random variables from populations Π_1 and Π_2 having gamma distributions with means $\alpha\theta_1$ and $\alpha\theta_2$ respectively, where α is the common known shape parameter and θ_1, θ_2 are unknown scale parameters. Let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$ denote the order statistics of X_1, X_2. For selecting the best population, we use the natural selection rule and select the population corresponding to $X_{(2)}$ (or $X_{(1)}$). Optimum properties of the natural selection rule are studied in details by Eaton (1967). Our goal is to estimate the scale parameters associated with the larger and smaller selected population which are given by

$$\theta_M = \begin{cases} \theta_1 & X_1 \geq X_2 \\ \theta_2 & X_1 < X_2 \end{cases} \quad and \quad \theta_J = \begin{cases} \theta_2 & X_1 \geq X_2 \\ \theta_1 & X_1 < X_2. \end{cases}$$

Note that the parameters θ_M and θ_J are data-dependent and need not be the same as the maximum or minimum of the θ_i's, respectively.

The problem of estimating the scale parameter of selected gamma population has been receiving a lot of attention in the literature. Vellaisamy and Sharma (1988,1989) and Vellaisamy (1992,1993,1996) dealt with UMVU, admissible and minimax estimation of θ_M under the Squared Error Loss (SEL) function. Misra et al. (2006a,b) extended the admissibility and inadmissibility results of Vellaisamy and Sharma (1988) to the case of known and arbitrary shape parameter for estimation of θ_M and θ_J.

In this paper, we discuss the estimation of the scale parameter of a selected gamma population under the following asymmetric scale invariant loss function

$$L(\theta, \delta) = \left(\sqrt{\frac{\delta}{\theta}} - \sqrt{\frac{\theta}{\delta}}\right)^2 = \frac{\delta}{\theta} + \frac{\theta}{\delta} - 2. \tag{1.1}$$

The loss function (1.1) is strictly convex and asymmetric in δ and as a function of δ has a unique minimum at $\delta = \theta$. This loss is useful in situations where underestimation is more serious than overestimation. For example, in dam construction, an underestimation of the peak water level is usually much more serious than an overestimation, see Zellner (1986). Under the loss function (1.1), it is easy to show that the best scale invariant estimator of θ_i is $[\alpha(\alpha-1)]^{-\frac{1}{2}} X_i,\ \alpha > 1,\ i = 1, 2$.

We consider estimating the random parameters θ_M and θ_J of the selected gamma population under the loss function (1.1) with some applications on k-records and censored data. The paper is organized as follows. In section 2, we discuss the admissibility of invariant estimators in the form of $cX_{(2)}$ and $cX_{(1)}$ for estimating θ_M and θ_J, respectively. In section 3, we obtain minimax estimator of θ_M. In section 4, we employ the technique of Brewster and Zidek (1974) for providing sufficient conditions for the inadmissibility of arbitrary invariant estimators of θ_M. In section 5, we consider applications on k-records and censored data and an extension of the problem to some subclass of exponential family. Finally, we conclude the paper and discuss unsolved problems in section 6.

2 Characterization of admissible estimators

Let X_1 and X_2 be two independent random variables from populations Π_1 and Π_2, respectively, where Π_i has probability density function (pdf)

$$f(x|\theta_i,\alpha) = \frac{1}{\theta_i^{\alpha}\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\frac{x}{\theta_i}},\ x>0,\ \alpha>0,\ \theta_i>0,\ i=1,2, \tag{2.1}$$

where the shape parameter α is known and $\theta_i,\ i=1,2$ are unknown. In estimating the unknown random parameters θ_M and θ_J under the loss function (1.1), the problem is invariant under the scale and permutation group of transformations $(X_1,X_2)\to(cX_2,cX_1),\ c>0$. Therefore, it is natural to consider only those estimators which are permutation and scale invariant, i.e. estimators satisfying $\delta(cX_1,cX_2) = c\delta(X_2,X_1),\ \forall c>0$. For this purpose, consider the following two subclasses of permutation and scale invariant estimators of θ_M and θ_J respectively

$$D_1 = \{\delta_{1c} : \delta_{1c}(X_1,X_2) = cX_{(2)},\ c>0\} \tag{2.2}$$

and

$$D_2 = \{\delta_{2c} : \delta_{2c}(X_1,X_2) = cX_{(1)},\ c>0\}. \tag{2.3}$$

In this section, we characterize the admissible estimators of θ_M and θ_J within the subclasses D_1 and D_2, respectively, under the loss function (1.1). The following lemma plays a key role in deriving the subsequent results of Sections 2 and 3.

Lemma 2.1. *Let X_1 and X_2 be two independent random variables such that $X_i,\ i=1,2$ has pdf (2.1) and $X_{(1)}\le X_{(2)}$ be the order statistics of X_1 and X_2. Let $S=\frac{X_{(2)}}{\theta_M}$, $U=\frac{X_{(1)}}{\theta_J}$, $\mu=\frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)}\ge 1$,*

$A(\mu) = \frac{2}{B(\alpha,\alpha)} \frac{\mu^\alpha}{(1+\mu)^{2\alpha}}$ *and for* $a, b > 0$, *define*

$$G_{a,b}(t) = \frac{1}{B(a,b)} \int_0^t x^{a-1}(1-x)^{b-1} dx \tag{2.4}$$

and

$$H_{a,b}(t) = G_{a,b}(t) + G_{a,b}(1-t) \tag{2.5}$$

where $B(.,.)$ *stands for the Beta function. Then*

(i) $E(S^k) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} H_{\alpha,\alpha+k}\big(\frac{1}{1+\mu}\big)$, *which is an increasing (a decreasing) function of* μ *for* $k < 0$ (> 0).

(ii) $E(U^k) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\big[2 - H_{\alpha,\alpha+k}\big(\frac{1}{1+\mu}\big)\big] = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} H_{\alpha+k,\alpha}\big(\frac{1}{1+\mu}\big)$, *which is an increasing (a decreasing) function of* μ *for* $k > 0$ (< 0).

(iii) $H_{\alpha,\alpha+1}(\frac{1}{1+\mu}) = 1 + \frac{2}{\alpha B(\alpha,\alpha)} \frac{\mu^\alpha}{(1+\mu)^{2\alpha}} = 1 + \frac{A(\mu)}{\alpha}$.

(iv) $H_{\alpha,\alpha-1}(\frac{1}{1+\mu}) = 1 - \frac{1}{(2\alpha-1)B(\alpha,\alpha)} \frac{\mu^{\alpha-1}}{(1+\mu)^{2(\alpha-1)}} = 1 - \frac{1}{2(2\alpha-1)} \frac{(1+\mu)^2}{\mu} A(\mu)$, $\alpha > 1$.

Proof. For a proof of (i), (iii) and (iv) see Lemma 2.1(i) of Motamed-Shariati and Nematollahi (2009) and Lemma 3.1(iii) and 3.1(iv) of Nematollahi and Motamed-Shariati (2009). For a proof of (ii), note that

$$E(S^k + U^k) = E\big[\big(\frac{X_2}{\theta_2}\big)^k + \big(\frac{X_1}{\theta_1}\big)^k\big] = 2\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$

Therefore, for all $\mu \geq 1$,

$$\frac{d}{d\mu} E(S^k) = \frac{\Gamma(2\alpha+k)}{\Gamma^2(\alpha)} \; \frac{\mu^{\alpha-1}}{(1+\mu)^{2\alpha+k}} \big(1 - \mu^k\big) \leq 0 \; (\geq 0), \quad k > 0 \; (< 0)$$

and

$$\frac{d}{d\mu} E(U^k) = \frac{\Gamma(2\alpha+k)}{\Gamma^2(\alpha)} \; \frac{\mu^{\alpha-1}}{(1+\mu)^{2\alpha+k}} \big(\mu^k - 1\big) \geq 0 \; (\leq 0), \quad k > 0 \; (< 0). \quad \blacksquare$$

In the following theorem, we characterize admissible estimators of θ_M within the subclass D_1 under the loss function (1.1).

Theorem 2.1. *Let* $u(\alpha) = \alpha B(\alpha, \alpha) 2^{2\alpha-1}$,
$c_1^\star = \left[\frac{1}{\alpha(\alpha-1)}\left\{1 - \frac{4\alpha-1}{(2\alpha-1)(u(\alpha)+1)}\right\}\right]^{\frac{1}{2}}$ *and* $c_2^\star = \left[\frac{1}{\alpha(\alpha-1)}\right]^{\frac{1}{2}}$, $\alpha > 1$. *Then, under the loss function (1.1), the estimators* $\delta_{1c}(X_1, X_2) = cX_{(2)}$ *are admissible within the subclass* D_1 *of invariant estimators of* θ_M*, if and only if* $c \in [c_1^\star, c_2^\star]$.

Proof. For fixed $\mu (\geq 1)$, the risk function

$$R(\theta_M, cX_{(2)}) = E\left(c\frac{X_{(2)}}{\theta_M} + \frac{\theta_M}{cX_{(2)}} - 2\right) = cE(S) + \frac{1}{c}E(S^{-1}) - 2$$

is a strictly convex function of c and minimizes at $c = c_1(\mu)$, where

$$c_1(\mu) = \left(\frac{E(S^{-1})}{E(S)}\right)^{\frac{1}{2}}. \tag{2.6}$$

Using Lemma 2.1(i), $[E(S)]^{-1}$ and $E(S^{-1})$ are continuous, increasing and positive functions of $\mu(\geq 1)$. Therefore the function $c_1(\mu)$ given by (2.6) is a continuous and increasing function of $\mu(\geq 1)$, and hence

$$\sup_{\mu \geq 1} c_1(\mu) = \lim_{\mu \to \infty} c_1(\mu) = \left[\frac{1}{\alpha(\alpha-1)} \frac{H_{\alpha,\alpha-1}(0)}{H_{\alpha,\alpha+1}(0)}\right]^{\frac{1}{2}} = c_2^\star.$$

and

$$\inf_{\mu \geq 1} c_1(\mu) = c_1(1) = \left[\frac{1}{\alpha(\alpha-1)}\left\{1 - \frac{4\alpha-1}{(2\alpha-1)(u(\alpha)+1)}\right\}\right]^{\frac{1}{2}} = c_1^\star$$

Thus, any value of $c \in [c_1^\star, c_2^\star)$ minimizes the risk function $R(\theta_M, \delta_{1c})$ for some values of $\mu \geq 1$ and hence such a c corresponds to an admissible estimator. The admissibility of the estimator $\delta_{c_2^\star}$, follows from the continuity of the risk function.

Also, for each fixed $\mu \geq 1$, the risk function $R(\theta_M, cX_{(2)})$ is an increasing function of c if $c > c_1(\mu)$ and it is a decreasing function of c if $c < c_1(\mu)$. Since $c_1^\star \leq c_1(\mu) \leq c_2^\star$, $\forall \mu \geq 1$, we conclude that the estimators $\delta_{1c} = cX_{(2)}$ for $c \in (0, c_1^*) \cup (c_2^*, \infty)$ are inadmissible in estimating θ_M, which completes the proof. ■

In the following theorem, we characterize admissible estimators of θ_J within the subclass D_2 under the loss function (1.1).

Theorem 2.2. *Let* $c_2^\star = [\frac{1}{\alpha(\alpha-1)}]^{\frac{1}{2}}$ *and* $c_3^\star = \left[\frac{1}{\alpha(\alpha-1)}\left\{1 + \frac{4\alpha-1}{(2\alpha-1)(u(\alpha)-1)}\right\}\right]^{\frac{1}{2}}$, $\alpha > 1$. *Then, under the loss function (1.1), the estimators* $\delta_{2c}(X_1, X_2) = cX_{(1)}$ *are admissible within the subclass* D_2 *of invariant estimators of* θ_J*, if and only if* $c \in [c_2^\star, c_3^\star]$.

Proof. As in the proof of Theorem 2.1, for fixed $\mu(\geq 1)$, the risk function

$$R(\theta_J, cX_{(1)}) = E\left(c\frac{X_{(1)}}{\theta_J} + \frac{\theta_J}{cX_{(1)}} - 2\right) = cE(U) + \frac{1}{c}E(U^{-1}) - 2$$

is a strictly convex function of c and minimizes at $c = c_2(\mu)$, where

$$c_2(\mu) = \left(\frac{E(U^{-1})}{E(U)}\right)^{\frac{1}{2}}. \tag{2.7}$$

Using Lemma 2.1(ii), $[E(U)]^{-1}$ and $E(U^{-1})$ are continuous, positive and decreasing functions of $\mu(\geq 1)$, so $c_2(\mu)$ is a decreasing function of $\mu(\geq 1)$, and hence

$$\inf_{\mu\geq 1} c_2(\mu) = \lim_{\mu\to\infty} c_2(\mu) = \left[\frac{1}{\alpha(\alpha-1)}\frac{H_{\alpha-1,\alpha}(0)}{H_{\alpha+1,\alpha}(0)}\right]^{\frac{1}{2}} = c_2^\star$$

and

$$\sup_{\mu\geq 1} c_2(\mu) = c_2(1) = \left[\frac{1}{\alpha(\alpha-1)}\frac{H_{\alpha-1,\alpha}(\frac{1}{2})}{H_{\alpha+1,\alpha}(\frac{1}{2})}\right]^{\frac{1}{2}} = c_3^\star.$$

Now, an argument analogous to the one in the proof of Theorem 2.1 completes the proof. ■

Figure 1 shows the graphs of $R_2(c) = R(\theta_M, cX_{(2)})$ and $R_1(c) = R(\theta_J, cX_{(1)})$ for $\alpha = 2, 3$ and some values of c. Note that from Theorems 2.1 and 2.2, $R_2(c_1^\star) < R_2(c)$ for $c < c_1^\star$, $R_2(c_2^\star) < R_2(c)$ for $c > c_2^\star$, $R_1(c_2^\star) < R_1(c)$ for $c < c_2^\star$ and $R_1(c_3^\star) < R_1(c)$ for $c > c_3^\star$, which are clear from Figure 1.

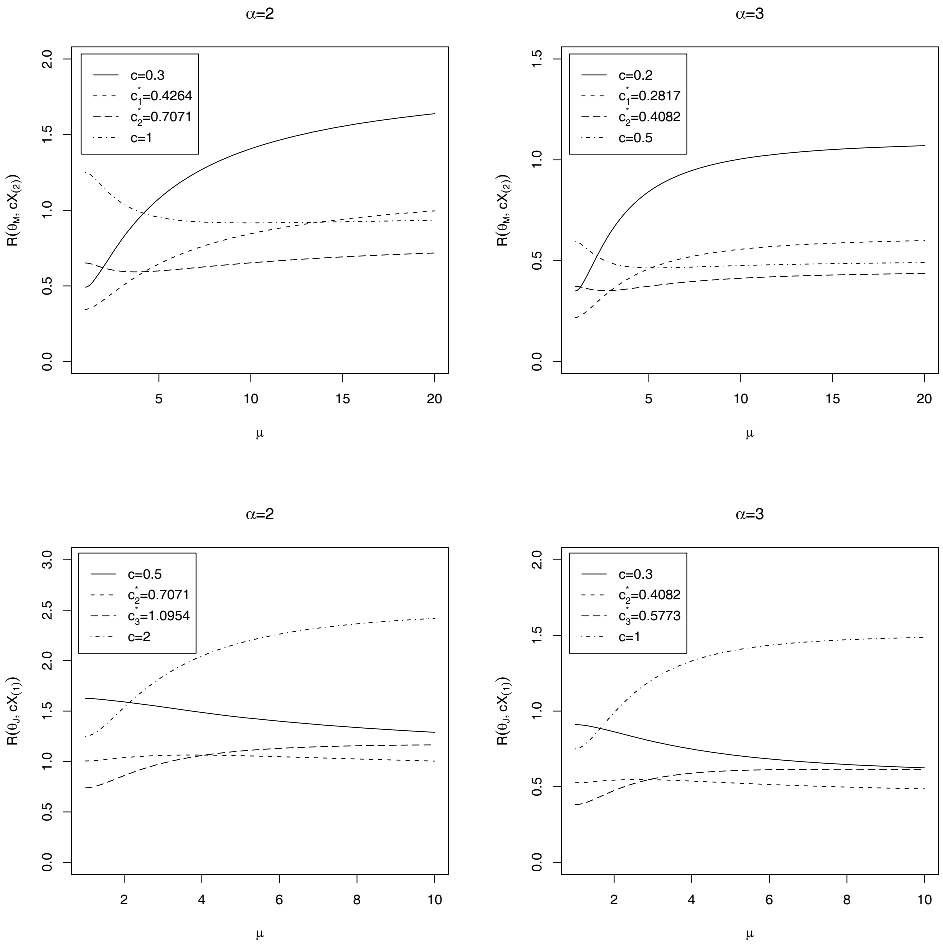

Figure 1: Graphs of $R(\theta_M, cX_{(2)})$ and $R(\theta_J, cX_{(1)})$ for $\alpha = 2, 3$

3 Minimax estimation of θ_M

In this section, we deal with minimax estimation of θ_M. For finding minimax estimator of θ_M, we use the results of Sackrowitz and Samuel-Cahn (1987). So, we first deal with the minimax estimation in the component problem for $\theta_i,\ i=1,2$. Assuming IGamma(υ,β)-prior for $\theta_i,\ i=1,2$, with pdf

$$\pi_i^{\upsilon,\beta}(\theta_i)=\frac{\beta^{\upsilon}}{\Gamma(\upsilon)\theta_i^{\upsilon+1}}\ e^{-\frac{\beta}{\theta_i}},\quad \theta_i>0,\quad \upsilon>0,\ \beta>0,\ i=1,2,\tag{3.1}$$

the posterior pdf of θ_i given $X_i=x_i$ is $IGamma(\upsilon+\alpha,\beta+x_i)$. It is easy to show that the Bayes estimator of θ_i with respect to (w.r.t.) the prior (3.1) and under the loss function (1.1) is given by

$$\delta^i_{\upsilon,\beta}(X_i)=\left(\frac{E(\theta_i|X_i)}{E(\theta_i^{-1}|X_i)}\right)^{\frac{1}{2}}=\frac{X_i+\beta}{\sqrt{(\alpha+\upsilon)(\alpha+\upsilon-1)}},\quad i=1,2.\tag{3.2}$$

Also the posterior risk of $\delta^i_{\upsilon,\beta}(X_i)$ under the loss function (1.1) is

$$r(x_i,\delta^i_{\upsilon,\beta}(x_i))=2\left[\sqrt{\frac{\alpha+\upsilon}{\alpha+\upsilon-1}}-1\right],\tag{3.3}$$

which does not depend on x_i. Therefore the Bayes risk of $\delta^i_{\upsilon,\beta}(X_i)$ is also

$$r^{\star}(\pi_i^{\upsilon,\beta},\delta^i_{\upsilon,\beta})=2\left[\sqrt{\frac{\alpha+\upsilon}{\alpha+\upsilon-1}}-1\right],\quad i=1,2.\tag{3.4}$$

Now, we consider Bayes estimation of θ_M under the loss function (1.1). Suppose θ_1 and θ_2 are two independent and identically distributed (i.i.d.) random variables with inverted gamma priors whose densities is given in (3.1). Then using (3.2) and Lemma 3.2 of Sackrowitz and Samuel-Cahn (1987), the unique Bayes estimator of θ_M under the loss function (1.1) w.r.t. the prior $\pi^{\upsilon,\beta}=(\pi_1^{\upsilon,\beta},\pi_2^{\upsilon,\beta})$ is given by

$$\delta^I_{\upsilon,\beta}(X_1,X_2)=\frac{X_{(2)}+\beta}{\sqrt{(\alpha+\upsilon)(\alpha+\upsilon-1)}}.$$

Notice that, the limiting Bayes estimator of θ_M, i.e., $\delta^I_{0,0}(X_1,X_2)=\frac{X_{(2)}}{\sqrt{\alpha(\alpha-1)}}$, $\alpha>1$, is the generalized Bayes estimator of θ_M w.r.t. non-informative prior $\pi(\theta_1,\theta_2)=(\theta_1\theta_2)^{-1}$, $\theta_1,\theta_2\in(0,\infty)=\Re_+$. Since the posterior risk (3.3) for the component problem is independent of $x = (x_1, x_2)$, therefore by Theorem 3.1 of Sackrowitz and Samuel-Cahn (1987), the Bayes risk of $\delta^I_{\upsilon,\beta}(X_1, X_2)$, is the same as the one given in (3.4), i.e.,

$$r^\star(\pi^{\upsilon,\beta}, \delta^I_{\upsilon,\beta}) = r^\star(\pi_i^{\upsilon,\beta}, \delta^i_{\upsilon,\beta}) = 2\left[\sqrt{\frac{\alpha+\upsilon}{\alpha+\upsilon-1}} - 1\right],\ i = 1, 2.$$

Hence,

$$\lim_{\upsilon\to 0} r^\star(\pi^{\upsilon,\beta}, \delta^I_{\upsilon,\beta}) = 2\left[\sqrt{\frac{\alpha}{\alpha-1}} - 1\right], \quad \alpha > 1.$$

Now using the Theorem 3.2 of Sackrowitz and Samuel-Cahn (1987), the estimator $\delta_M(X_1, X_2)$ is minimax for θ_M if

$$R(\theta_M, \delta_M) \le \lim_{\upsilon\to 0} r^\star(\pi^{\upsilon,\beta}, \delta^I_{\upsilon,\beta}) = 2\left[\sqrt{\frac{\alpha}{\alpha-1}} - 1\right], \qquad \alpha > 1,\ \forall\theta = (\theta_1, \theta_2) \tag{3.5}$$

where $R(\theta_M, \delta_M)$ is the risk function of δ_M under the loss function (1.1).

In the following theorem we show that the generalized Bayes estimator $\delta^I_{0,0}(X_1, X_2) = \frac{X_{(2)}}{\sqrt{\alpha(\alpha-1)}}$ is a minimax estimator of θ_M.

Theorem 3.1. *Let X_1 and X_2 be two independent gamma random variables with pdf (2.1). If $X_{(2)} = \max(X_1, X_2)$, then under the loss function (1.1), the generalized Bayes estimator $\delta^I_{0,0}(X_1, X_2) = \frac{X_{(2)}}{\sqrt{\alpha(\alpha-1)}}$, $\alpha > 1$ is a minimax estimator of θ_M.*

Proof. Using Lemma 2.1, the risk function of $\delta^I_{0,0}(X_1, X_2) = \frac{X_{(2)}}{\sqrt{\alpha(\alpha-1)}}$ for $\alpha > 1$ is given by

$$\begin{aligned} R(\theta_M, \delta^I_{0,0}) &= \frac{1}{\sqrt{\alpha(\alpha-1)}}\, E(S) + \sqrt{\alpha(\alpha-1)}\, E(S^{-1}) - 2 \\ &\le 2\left[\sqrt{\frac{\alpha}{\alpha-1}} - 1\right] - \frac{1}{\sqrt{\alpha(\alpha-1)}}\left[\frac{2\alpha}{2\alpha-1} - 1\right] A(\mu) \\ &< 2\left[\sqrt{\frac{\alpha}{\alpha-1}} - 1\right] \end{aligned}$$

where $A(\mu) > 0$ is given in Lemma 2.1. Now, the result follows from (3.5). ■

Remark 3.1. *From Theorem 2.1, the minimax and natural estimator* $\delta^{I}_{0,0}(X_1,\ X_2) = \frac{X_{(2)}}{\sqrt{\alpha(\alpha-1)}}$, $\alpha > 1$, *of* θ_M, *which is the analog of the best scale invariant estimators of* θ_2, *is admissible within the subclass* D_1 *of invariant estimators of* θ_M.

Remark 3.2. *Using similar argument that leads to (3.5), we can show that an estimator* $\delta_J(X_1, X_2)$ *is minimax for* θ_J *if*

$$R(\theta_J, \delta_J) \leq 2\left[\sqrt{\frac{\alpha}{\alpha-1}} - 1\right], \quad \alpha > 1, \quad \forall\theta = (\theta_1, \theta_2). \tag{3.6}$$

We cannot find an estimator δ_J *that satisfies (3.6), so the problem of finding minimax estimator of* θ_J *remains unsolved.*

4 Sufficient Conditions for Inadmissibility

Consider the following class of invariant estimators

$$D_3 = \{\delta_\psi : \delta_\psi(X_1, X_2) = X_{(2)}\psi(Y)\}, \tag{4.1}$$

for θ_M, where $Y = \frac{X_{(1)}}{X_{(2)}}$ and ψ is some real valued function defined on $(0, 1]$. In this section we give sufficient conditions for inadmissibility of some permutation and scale invariant estimators for θ_M in the class D_3 under the loss function (1.1) by deriving dominating estimators. For deriving dominating estimators, we use the technique of Brewster and Zidek (1974). The following lemma is useful in deriving the improved estimators for estimating θ_M.

Lemma 4.1. *Let* $Y = \frac{X_{(1)}}{X_{(2)}}$, $\mu = \frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)}$ *and* ψ *be a real valued function defined on* $(0, 1]$. *For* $\alpha > \frac{1}{2}$, $x > 0$ *and* $\mu \geq 1$ *define the function* $\eta_x(\mu)$ *as*

$$\eta_x(\mu) = 2\alpha(2\alpha - 1)\ \frac{\mu(x + \mu)^{-(2\alpha+1)} + (1 + x\mu)^{-(2\alpha+1)}}{\mu^{-1}(x + \mu)^{-(2\alpha-1)} + (1 + x\mu)^{-(2\alpha-1)}}.$$

(i) *For* $y \in (0,1]$, *the conditional pdf of* $S = \frac{X_{(2)}}{\theta_M}$ *given* $Y = y$ *is*

$$f_{S|Y=y}(s) = \frac{y^{\alpha-1}s^{2\alpha-1}}{\Gamma^2(\alpha)f_Y(y)}\left[\mu^{-\alpha}e^{-(\frac{y}{\mu}+1)s} + \mu^{\alpha}e^{-(1+\mu y)s}\right], \quad s > 0.$$

where $f_Y(y)$ *denotes the pdf of* Y.

(ii) *For* $\alpha > \frac{1}{2}$ *and* $y \in (0,1]$

$$\sup_{\mu \geq 1} \eta_y(\mu) = \frac{2\alpha(2\alpha-1)}{(1+y)^2} = \left(\frac{1}{\psi^{\star}(y)}\right)^2. \tag{4.2}$$

Proof. **(i)** For a proof, see Lemma 16(i) of Misra et al. (2006a).

(ii) Note that

$$\eta_y(1) = \frac{2\alpha(2\alpha-1)}{(1+y)^2}.$$

Thus, it suffices to show that

$$\eta_y(\mu) \leq \frac{2\alpha(2\alpha-1)}{(1+y)^2} \quad \forall \mu \geq 1 \tag{4.3}$$

Now, with some awkward algebraic calculations, it can be shown that the inequality (4.3) holds if and only if

$$\left(\frac{y+\mu}{1+y\mu}\right)^{2\alpha+1} \geq 1,$$

which is satisfied for $y \in (0,1]$ and $\mu \geq 1$. Hence the result follows. ∎

The next theorem gives a sufficient condition for inadmissibility of arbitrary invariant estimators $\delta_{\psi}(X_1, X_2) \in D_3$.

Theorem 4.1. *Let* $\delta_{\psi}(X_1, X_2) \in D_3$ *be an invariant estimator of* θ_M, $\psi_{11}(y)$ *a function defined on* $(0,1]$ *such that* $\psi_{11}(y) \leq \psi^{\star}(y)$, $\forall y \in (0,1]$ *and* $P_{\theta}\big(\psi(Y) < \psi_{11}(Y)\big) > 0$, $\forall \theta = (\theta_1, \theta_2) \in \Re_+ \times \Re_+ = \Re_+^2$. *Then under the loss function (1.1), the invariant estimator* δ_{ψ} *is inadmissible for estimating* θ_M, *and is dominated by* $\delta_{\psi_1}(X_1, X_2) =$

$X_{(2)}\psi_1(Y)$, *where for* $0 < y \le 1$,

$$\psi_1(Y) = \begin{cases} \psi_{11}(Y) & \psi(Y) < \psi_{11}(Y) \\ \\ \psi(Y) & o.w. \end{cases}$$

Proof. For $\mu \ge 1$, the risk difference of δ_ψ and δ_{ψ_1} is

$$\begin{aligned} \Delta(\mu) &= R(\theta_M, \delta_\psi) - R(\theta_M, \delta_{\psi_1}) \\ &= E_\theta\big\{D_\theta(y)\big\}, \end{aligned}$$

where for $y \in (0, 1]$,

$$D_\theta(y) = \\ [\psi_1(y) - \psi(y)]E_\theta\big(S^{-1}|Y = y\big)\left\{\frac{1}{\psi_1(y)\psi(y)} - \frac{E_\theta\big(S|Y = y\big)}{E_\theta\big(S^{-1}|Y = y\big)}\right\}. \quad (4.4)$$

Now from Lemma 4.1(i), we have

$$\begin{aligned} K_y(\mu) &= E_\theta\big(S^{-1}|Y = y\big) \\ &= \frac{\Gamma(2\alpha-1)y^{\alpha-1}\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \left[\mu^{-1}(y+\mu)^{-(2\alpha-1)} + (1+y\mu)^{-(2\alpha-1)}\right], \end{aligned}$$

and

$$E_\theta\big(S|Y = y\big) = \frac{\Gamma(2\alpha+1)y^{\alpha-1}\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \left[\mu(y+\mu)^{-(2\alpha+1)} + (1+y\mu)^{-(2\alpha+1)}\right].$$

So, by substituting the above formula in (4.4), we have

$$D_\theta(y) = [\psi_1(y) - \psi(y)]K_y(\mu)\left\{\frac{1}{\psi_1(y)\psi(y)} - \eta_y(\mu)\right\},$$

where $\eta_y(\mu)$ is defined in Lemma 4.1. Clearly, if $\psi(y) \ge \psi_{11}(y)$, then $D_\theta(y) = 0,\ \forall\theta \in \Re^2_+$ and $\forall y \in (0, 1]$. For $\psi(y) < \psi_{11}(y)$, using (4.2) we have

$$D_\theta(y) \ge [\psi_{11}(y) - \psi(y)]K_y(\mu)\left[\frac{1}{\psi_{11}(y)\psi(y)} - \left(\frac{1}{\psi^\star(y)}\right)^2\right] > 0, \forall\theta \in \Re^2_+.$$

Since $P_\theta(\psi(Y) < \psi_{11}(Y)) > 0,\ \forall\theta \in \Re_+^2$, it follows that $\Delta(\mu) > 0,\ \forall\theta \in \Re_+^2$. ■

The following corollary is an immediate consequence of the Theorem 4.1.

Corollary 4.1. *Let $\delta_\psi(X_1, X_2) \in D_3$ be an invariant estimator of θ_M. If $P_\theta\big(\psi(Y) < \psi^\star(Y)\big) > 0,\ \forall\theta = (\theta_1, \theta_2) \in \Re_+^2$, then under the loss function (1.1), the invariant estimator δ_ψ is inadmissible for estimating θ_M, and is dominated by $\delta_{\psi_1}(X_1, X_2) = X_{(2)}\psi_1(Y)$, where for $0 < y \leq 1$,*

$$\psi_1(Y) = \begin{cases} \psi^\star(Y) & \psi(Y) < \psi^\star(Y) \\ \\ \psi(Y) & o.w. \end{cases}$$

Remark 4.1. *Consider the following class of convex combination estimators of θ_M*

$$\begin{aligned} \delta_{p,\psi}(X_1, X_2) &= pX_{(2)} + (1-p)X_{(1)} \\ &= X_{(2)}[p + (1-p)Y] = X_{(2)}\psi(Y), \end{aligned}$$

where $p \in [0, 1]$, and let $A = \frac{1}{\sqrt{2\alpha(2\alpha-1)}}$. If $p < A \leq \frac{1}{2} \leq 1 - A$, which is true for $\alpha \geq \frac{5}{4}$, then $P_\theta\big(\psi(Y) < \psi^\star(Y)\big) = P_\theta\big(p + (1-p)Y \leq A(Y+1)\big) = P_\theta\big(Y \leq \frac{A-p}{1-p-A}\big) > 0$. So, by Corollary 4.1, the estimator $\delta_{p,\psi}(X_1, X_2)$ is inadmissible and is dominated by

$$\delta^\star_{p,\psi}(X_1, X_2) = \begin{cases} \frac{X_{(1)}+X_{(2)}}{\sqrt{2\alpha(2\alpha-1)}} & p + (1-p)Y < \frac{1+Y}{\sqrt{2\alpha(2\alpha-1)}} \\ \\ \delta_{p,\psi}(X_1, X_2) & o.w. \end{cases}$$

when $0 \leq p \leq \frac{1}{\sqrt{2\alpha(2\alpha-1)}}$ and $\alpha \geq \frac{5}{4}$. Figure 2 shows the graph of risk functions of the estimators $\delta_{p,\psi}$ and $\delta^\star_{p,\psi}$ for some values of α and p. It is evident from these graphs that the estimator $\delta^\star_{p,\psi}$ dominates the estimator $\delta_{p,\psi}$.

Remark 4.2. *Let* $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2,$ *be two independent random samples from* $\Pi_i,\ i = 1, 2,$ *where for each* i, Π_i *has pdf* (2.1). *Then* $T_i(\boldsymbol{X}_i) = \sum_{i=1}^{n} X_{ij},\ i = 1, 2,$ *is a complete sufficient statistic for* θ_i *and has a gamma distribution with parameters* $(n\alpha, \theta_i)$, *respectively, where* $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{in})$. *Therefore, the results of Sections 2-4 hold for this case if we replace* α *by* $n\alpha$ *and* X_i *by* $T_i(\boldsymbol{X}_i),\ i = 1, 2.$

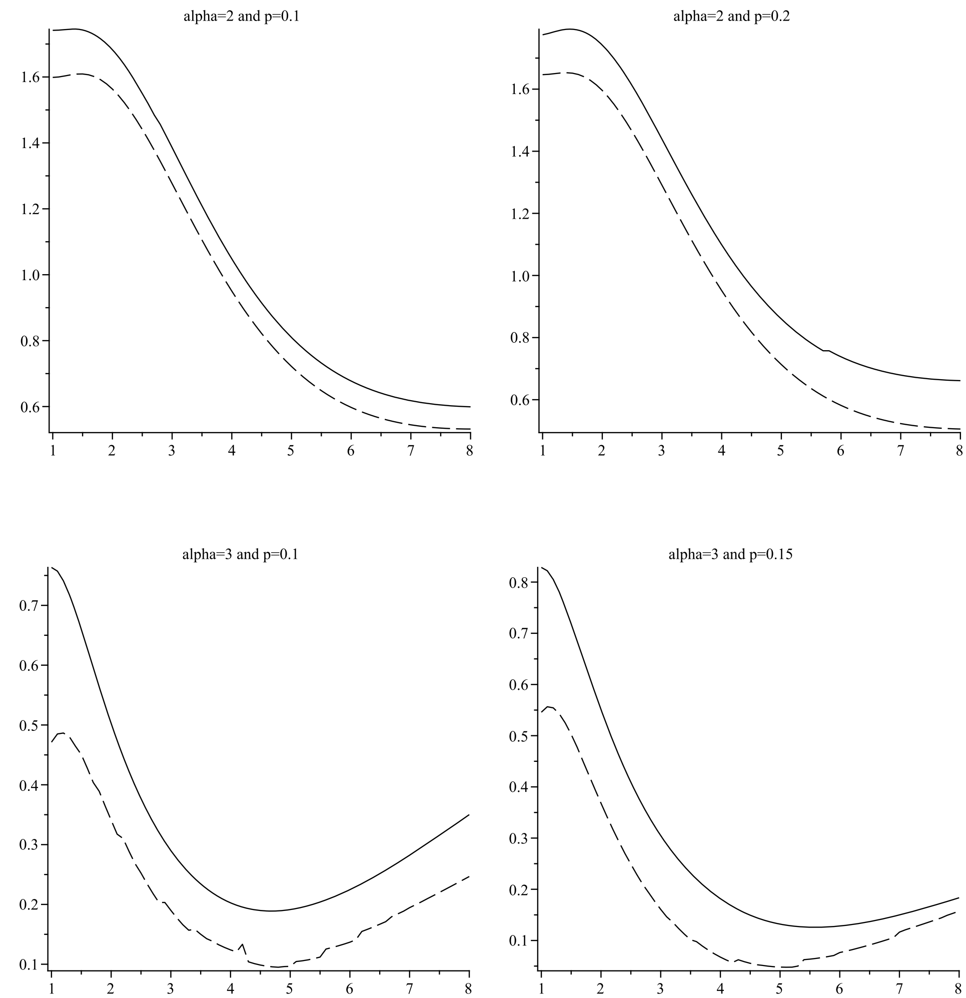

Figure 2: Graph of risk functions of the estimators $\delta_{p,\psi}$,— and $\delta^{\star}_{p,\psi}$,$---$

5 Applications and extensions

In this section, an application of estimation after selection in k-records and Type-II censored data and extension of the results of Sections 2-4 to a subclass of exponential family are considered.

5.1 Estimation After Selection Based on k-Record data

Research in the area of records has progressed steadily since 1952's, where, Chandler began studying the distributions of lower records, record times and inter-record times for i.i.d. sequences of random variables. Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2$, be a pair of independent random samples from negative exponential populations with pdf

$$f(x|\theta_i) = \frac{1}{\theta_i}\, e^{-\frac{x}{\theta_i}}, \quad x > 0, \quad \theta_i > 0, \quad i = 1, 2, \tag{5.1}$$

where θ_1, θ_2 are unknown scale parameters. Let $R^i_{m(k)}$ be the upper k-records of i-th sample, $i = 1, 2$ and $R^{(1)}_{m(k)} \le R^{(2)}_{m(k)}$ denote the order statistics of $R^1_{m(k)}$ and $R^2_{m(k)}$. Suppose the population corresponding to the largest $R^{(2)}_{m(k)}$ (or the smallest $R^{(1)}_{m(k)}$) observation is selected. Our aim is to estimate the following random parameters:

$$\theta^m_M = \begin{cases} \theta_1 & R^1_{m(k)} \ge R^2_{m(k)} \\ \theta_2 & R^1_{m(k)} < R^2_{m(k)} \end{cases} \quad and \quad \theta^m_J = \begin{cases} \theta_2 & R^1_{m(k)} \ge R^2_{m(k)} \\ \theta_1 & R^1_{m(k)} < R^2_{m(k)}. \end{cases}$$

It is easy to verify that $kR^i_{m(k)}$ has a Gamma$\big(m, \theta_i\big)$-distribution, see Arnold et al. (1998), Nevzorov (2001). Therefore, the results of Sections 2-4 hold for this case if we replace α by m and X_i by $kR^i_{m(k)},\ i = 1, 2$.

5.2 Estimation after selection using Type-II censored data

The most common censoring scheme is so called Type-II censoring. This is the situation that occurs when, for example, n items are put on test and the test is terminated after a predetermined number of items have failed. Complete observations on the first r (r fixed) order statistics $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$ are available, and the remaining $n-r$ unobserved lifetimes are known to be greater than $X_{(r)}$. For the case of negative exponential in (5.1), it is easy to show that in this scheme $T_i = \sum_{j=1}^{r} X_{i(j)} + (n-r)X_{i(r)},\ i = 1, 2,$ has a Gamma(r, θ_i)-distribution, see Lehmann and Romano (2005). Let $T_{(1)} = \min(T_1, T_2)$ and $T_{(2)} = \max(T_1, T_2)$ and suppose that the population corresponding to the largest $T_{(2)}$ (or the smallest $T_{(1)}$) is selected. Our goal is to estimate the random parameters

$$\theta_M = \begin{cases} \theta_1 & T_1 \geq T_2 \\ \theta_2 & T_1 < T_2 \end{cases} \quad and \quad \theta_J = \begin{cases} \theta_2 & T_1 \geq T_2 \\ \theta_1 & T_1 < T_2. \end{cases}$$

Since $T_i,\ i = 1, 2,$ has a Gamma$\big(r, \theta_i\big)$-distribution, therefore, the results of Sections 2-4 hold if we replace α by r and X_i by $T_i,\ i = 1, 2,$ in this case.

5.3 Extension to a subclass of exponential family

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \cdots, X_{in}),\ i = 1, 2,$ be a random sample of size n from the ith population $\Pi_i,\ i = 1, 2,$ with the joint scale probability density function

$$f(\mathbf{x}_i, \tau_i) = \frac{1}{\tau_i^n} f\big(\frac{\mathbf{x}_i}{\tau_i}\big), \quad i = 1, 2,$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{in})$. In some cases the above model reduces to

$$f(\mathbf{x}_i, \theta_i) = C(\mathbf{x}_i, n)\theta_i^{-\gamma} e^{-T_i(\mathbf{x}_i)/\theta_i}, \qquad i = 1, 2, \tag{5.2}$$

where $C(\mathbf{x}_i, n)$ is a function of $\mathbf{x}_i$ and n, $\theta_i = \tau_i^r$ for some $r > 0, \gamma$ is a function of n and $T_i(\mathbf{X}_i)$ is a complete sufficient statistic for θ_i with *Gamma*(γ, θ_i)- distribution, see Parsian and Nematollahi (1996).

Since $T_i = T_i(\mathbf{X}_i), i = 1, 2,$ has a *Gamma*(γ, θ_i)-distribution, therefore we can extend the results of Sections 2-4 to a subclass of the exponential family (5.2) with replacing α and X_i by γ and $T_i(\mathbf{X}_i)$, respectively.

The results of Section 2-4 can also be extended to the family of transformed chi-square distributions introduced by Rahman and Gupta (1993) and which includes Pareto and beta distributions. For details see Jafari Jozani et al. (2002).

6 Further investigation

In the previous sections, we discuss estimation after selection under the loss function (1.1). Now, consider a generalization of the loss function (1.1) with the following structure

$$L(\theta,\delta)=\left[\left(\frac{\delta}{\theta}\right)^{\frac{p}{2}}-\left(\frac{\theta}{\delta}\right)^{\frac{p}{2}}\right]^2=\left(\frac{\delta}{\theta}\right)^p+\left(\frac{\theta}{\delta}\right)^p-2,\quad p>0. \quad (6.1)$$

and use this loss function for the problem of estimating the scale parameter of selected gamma population. Using the argument as in Section 2, it is easy to verify that under the loss function (6.1), the estimators $\delta_{1c}(X_1,X_2)=cX_{(2)}$ are admissible within the subclass D_1 of invariant estimators of θ_M, if and only if $c\in[c_1^\star,c_2^\star]$, where

$$c_1^\star=\left[\frac{\Gamma(\alpha-p)}{\Gamma(\alpha+p)}\frac{H_{\alpha,\alpha-p}(\frac{1}{2})}{H_{\alpha,\alpha+p}(\frac{1}{2})}\right]^{\frac{1}{2p}} \quad \textit{and} \quad c_2^\star=\left[\frac{\Gamma(\alpha-p)}{\Gamma(\alpha+p)}\right]^{\frac{1}{2p}}$$

provided $\alpha>p$. Also, under the loss function (6.1), the estimators $\delta_{2c}(X_1,\ X_2)=cX_{(1)}$ are admissible within the subclass D_2 of invariant estimators of θ_J, if and only if $c\in[c_2^\star,c_3^\star]$, where

$$c_3^\star=\left[\frac{\Gamma(\alpha-p)}{\Gamma(\alpha+p)}\frac{H_{\alpha-p,\alpha}(\frac{1}{2})}{H_{\alpha+p,\alpha}(\frac{1}{2})}\right]^{\frac{1}{2p}}$$

We cannot find minimax estimator for θ_M and sufficient conditions for inadmissibility of some permutation and scale invariant estimators for θ_M in the class D_3 under the loss function (6.1). So, these problems remained unsolved.

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