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Periodic Oscillations in the Analysis of Algorithms and Their Cancellations

Helmut Prodinger*

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits, 2050 Johannesburg, South Africa. (helmut@maths.wits.ac.za)

Abstract. A large number of results in analysis of algorithms contain fluctuations. A typical result might read "The expected number of ... for large n behaves like $\log_2 n + \text{constant} + \delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero." Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling. Often, there are huge cancellations to be noted, especially if one wants to compute variances. In order to see this, one needs identities for the Fourier coefficients of the periodic functions involved. There are several methods to derive such identities, which belong to the realm of modular functions. The most flexible method seems to be the calculus of residues. In some situations, Mellin transforms help. Often, known identities can be employed. This survey shows the various techniques by elaborating on the most important examples from the literature.

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1 Introduction

A surprisingly large number of results in *analysis of algorithms* contain *fluctuations*. A typical result might read "The expected number of . . . for large n behaves like $\log_2 n$ + constant + $\delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero." Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling; see the classic books by Flajolet, Knuth, Mahmoud, Sedgewick, Szpankowki [23, 16, 17, 18, 25] for background.

We use the name $\delta(x)$ in a generic sense; in concrete situations we call them $\delta_0(x)$, $\delta_1(x)$, etc. An important set of such functions is

$$\delta_j(x) := \frac{1}{L}\sum_{k\neq 0}\frac{\Gamma(j-\chi_k)}{j!}e^{2\pi ikx},$$

where we use the standard abbreviations $L = \log 2$ and $\chi_k = \frac{2\pi ik}{L}$.

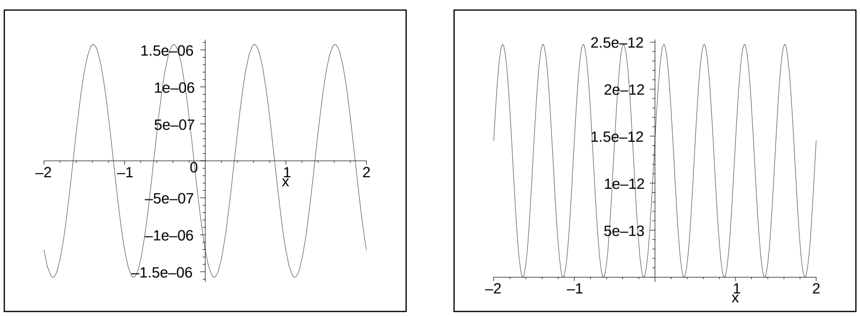

Figure 1: $\delta_0(x)$ and $\delta_0^2(x)$

As one can see from the picture, $\delta_0(x)$ has mean zero (the zeroth Fourier coefficient is not there). On the other hand, $\delta_0^2(x)$ is still periodic with period 1, but its mean is *not* zero. Why should we worry about a quantity apparently as small as $\approx 10^{-12}$?

The reason is the *variance* of such parameters, as it naturally contains the term "$-$expectation^2," and as such also $-\delta^2(x)$. That might not be a sufficient motivation for a casual reader if it were not the case that often *substantial cancellations* occur. In order to identify them, one has to know more about $\delta^2(x)$. If one ignores these terms, one gets wrong results, and the results are not wrong by $\approx 10^{-12}$, but *by an order of growth!* Path length in tries, Patricia

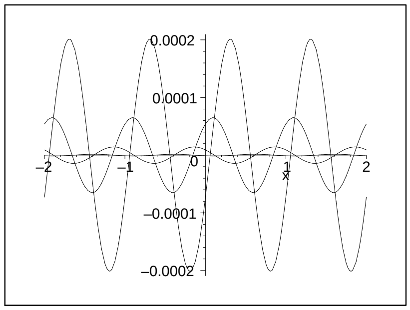

Figure 2: The functions $\delta_j(x) = \frac{1}{L}\sum_{k\neq 0}\frac{\Gamma(j-\chi_k)}{j!}e^{2\pi ikx}$ grow in amplitude

tries, and digital search trees [8, 15, 10] are such cases: the variance is in reality of order n only, but ignoring the fluctuations would lead to a (wrong) $\approx n^2$ result.

Questions like that occurred in several writings of this author (together with various coauthors), as can be seen from the references. The techniques are extremely interesting, as one has to dig deep into classical analysis. So far, it seems that the *calculus of residues* is the most versatile approach in this context. Another approach is to use (modular) identities due to Dedekind, Ramanujan, Jacobi and others (which can often be proved by Mellin transform techniques); however, often they do not quite *fit*. The residue calculus approach directly addresses the formula that is ultimately needed.

In this survey paper, we discuss all these methods by looking at various examples. The paper has also a tutorial concern, as we want to encourage the interested reader to prove his/her own identities with the methods that are provided.

Oscillating functions are usually given as *Fourier* series $f = \sum_{k\neq 0} a_k e^{2\pi ikx}$, thus representing a periodic function of period 1, and since the term a_0 is missing, oscillating around zero. We often refer to the coefficient a_k by writing $[f]_k$.

Other cancellation phenomena concerning oscillations related to Patricia tries and compositions (resp. words) were only discovered recently and presented in [22], at ANALCO*04* (dedicated to Hosam Mahmoud).

Here are some examples from the literature.

Approximate counting [5, 11, 20, 21]

After n successive increments the average content $\overline{C}_n$ of the counter satisfies:

$$\overline{C}_n \sim \log_2 n + \frac{\gamma}{L} - \alpha + \frac{1}{2} - \delta_0(\log_2 n),$$

with

$$\alpha = \sum_{k\geq 1} \frac{1}{2^k - 1} \quad \text{and} \quad \delta_0(x) = \frac{1}{L} \sum_{k\neq 0} \Gamma(-\chi_k) e^{2\pi i k x},$$

with $L = \log 2$ and $\chi_k = \frac{2\pi i k}{L}$. The identity that one needs is

$$[\delta_0^2]_0 = \frac{1}{L^2} \sum_{k\neq 0} \Gamma(\chi_k)\Gamma(-\chi_k) = \frac{\pi^2}{6L^2} - \frac{11}{12} - \frac{2}{L} \sum_{h\geq 1} \frac{(-1)^{h-1}}{h\,(2^h - 1)}. \qquad (1)$$

We will present various proofs of this identity, which will be our running example, in the next sections. The methods are residue calculus (Section 2), Mellin transform (Section 3), and identities of Ramanujan (Section 4).

Maximum of a sample of n geometric random variables [26, 13]

Assume that X is a geometric random variable such that $\mathbb{P}\{X = k\} = 2^{-k}$ (for simplicity, we only discuss this case, not the slightly more general $\mathbb{P}\{X = k\} = (1-q)q^{k-1}$). We consider n independent trials and look for their maximum. This is a natural parameter which is also useful in the analysis of various algorithms (e. g., skiplists [14]).

The expected value is given by

$$E_n \sim \log_2 n + \frac{\gamma}{L} + \frac{1}{2} - \delta_0\,(\log_2 n)$$

with the same periodic function as before.

Tries [12, 8, 7]

The expected number of internal nodes in a trie built from n random data is

$$l_n = \frac{n}{L} + n\sigma(\log_2 n) + O(1),$$

with

$$\sigma(x) = \frac{1}{L} \sum_{k\neq 0} \chi_k \Gamma(1 - \chi_k) e^{2\pi i k x}.$$

The formula that one needs is

$$[\sigma^2]_0 = 3 - \frac{1}{L} - \frac{1}{L^2} + \frac{2}{L}\sum_{j\ge 2}\frac{(-1)^j j}{(j+1)(j-1)(2^j-1)}. \tag{2}$$

Partial match queries in tries [9]

The average cost (defined in the paper [9]), for random tries constructed from n random data, is

$$l_n = \sqrt{n}\Big(\sqrt{\pi}\frac{1+\sqrt{2}}{2L} + \tau\big(\log_2\sqrt{n}\,\big)\Big) + O(1),$$

where the fluctuating function $\tau(x) = \sum_{k\ne 0}\tau_k e^{2k\pi i x}$ has the Fourier coefficients

$$\tau_k = \frac{1}{2L}\Big(1+\sqrt{2}(-1)^k\Big)\Gamma\Big(\frac{-1-\chi_k}{2}\Big)\Big(\frac{-1+\chi_k}{2}\Big).$$

The formula one needs is

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2}\big(3+2\sqrt{2}\,\big) + \frac{3-2\sqrt{2}}{L}F(L) + \frac{2\sqrt{2}}{L}F\Big(\frac{L}{2}\Big) \tag{3}$$

with

$$F(x) = \sum_{k\ge 1}\frac{e^{-kx}}{1+e^{-2kx}}.$$

2 Proofs by residue calculus

In this section we will show how to use residue calculus in order to prove the relevant identities. As examples of the technique, we concentrate on the identities (1), (2), and (4). However, after going through these representative examples, the reader will surely be able to prove his/her own identities, following the technique.

The following approach ("residue calculus") to evaluate $[\delta^2]_0$ seems to be the easiest and most flexible. We start with the following example:

$$\delta_0(x) = \frac{1}{L}\sum_{k\ne 0}\Gamma(-\chi_k)e^{2\pi i k x}.$$

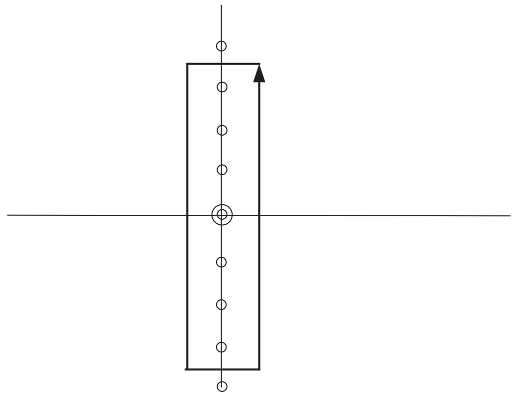

Figure 3: Path of integration; poles at χ_k are indicated, the double pole at 0 by a double circle

Find a function $F(z)$ so that $[\delta_0^2]_0$ is (apart from a few extra terms) the sum of the residues along the imaginary axis. Here, take

$$F(z) = \frac{L}{e^{Lz}-1}\Gamma(-z)\Gamma(z).$$

If we set

$$I_1 = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(z)dz,$$

then by shifting and collecting residues,

$$I_1 = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} F(z)dz + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

What happens here is often called *closing the box*, compare Figure 3, see e. g. [25, 18]. One integrates along a rectangle with corners $\pm\frac{1}{2} \pm iM$. One can evaluate it by collecting the residues inside the rectangle. And one can let the parameter M go to infinity. In this type of problems, the integrals along the horizontal lines disappear, and we can express one integral along a vertical line by an integral along another vertical line, plus a few residues. The justification that these integrals along the horizontal lines disappear comes from the fact that the Gamma function (which is always present in our examples) becames small extremely fast for large imaginary parts, see [27]. Since all our examples are of that nature, we will perform the relevant operations without further comments.

The emphasis of this survey is to prove identities, and this is to some extent a more algebraic than analytic endeavour.—

Now one writes

$$\frac{1}{e^z-1} = -1 - \frac{1}{e^{-z}-1}$$

and gets, by a simple change of variable $z := -z$,

$$I_1 = -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz - I_1 + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

The integral

$$I_2 = -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz$$

can be computed by collecting the negative residues right to the line $\Re z = -\frac{1}{2}$, viz.

$$I_2 = -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz = \sum_{l\geq 1}\frac{(-1)^l}{l!}(l-1)! = -L.$$

Altogether we have

$$2I_1 = -L + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

On the other hand, integral I_1 is also the sum of the negative residues right of the line $\Re z = \frac{1}{2}$, i. e.,

$$I_1 = -L\sum_{l\geq 1}\frac{(-1)^l}{l!(2^l-1)}(l-1)! = -L\sum_{l\geq 1}\frac{(-1)^l}{l(2^l-1)}.$$

Combining these results, we get

$$-2L\sum_{l\geq 1}\frac{(-1)^l}{l(2^l-1)} = -L + \sum_{k\neq 0}\Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

This is the identity we wanted.

With not much more effort one can also compute the coefficients $[\delta_0^2]_k$, for $k \neq 0$. For this, one works with the function

$$F(z) = \frac{L}{e^{Lz}-1}\Gamma(-z-\chi_k)\Gamma(z).$$

One obtains

$$[\delta_0^2]_k = \frac{1}{L^2}\sum_{j\ne 0,\,\ne k}\Gamma(-\chi_j)\Gamma(-\chi_k+\chi_j)$$
$$= \frac{2}{L}\sum_{l\ge 1}\frac{(-1)^l\Gamma(-\chi_k+l)}{l!(2^l-1)} + \frac{2}{L^2}\Gamma(-\chi_k)\big(\psi(-\chi_k)+\gamma\big).$$

We omit the details.

Guy Louchard, who is interested in higher moments, asked to compute the coefficients $[\delta_0^3]_k$. Here is the instance $k=0$, the general case is very involved and not too attractive:

$$[\delta_0^3]_0 = -1 - \frac{2\zeta(3)}{L^3} - \frac{1}{L}\sum_{l\ge 1}\frac{(-1)^l}{l(2^l-1)} + \frac{6}{L^2}\sum_{l\ge 1}\frac{(-1)^l H_{l-1}}{l(2^l-1)} + \frac{2\log 3}{L}$$
$$+ \frac{2}{L}\sum_{l,j\ge 1}\frac{(-1)^{l+j}}{(l+j)(2^l-1)}\left[\frac{1}{2^j-1}+\frac{1}{2^{j+l}-1}\right]\binom{l+j}{j}.$$

(In this formula, the harmonic numbers $H_n := \sum_{1\le k\le n}\frac{1}{k}$ appear.)

This has been tested numerically as well and gives
$9.42817763095796606421903 \times 10^{-25}$.

Let us straight ahead do another example (identity (2)), which also occurs often:

$$\sigma(x) = \frac{1}{L}\sum_{k\ne 0}\chi_k\Gamma(-1-\chi_k)e^{2\pi ikx}.$$

Here, we take

$$F(z) = -\frac{L}{e^{Lz}-1}z^2\Gamma(-1-z)\Gamma(-1+z).$$

Then

$$I_1 = \frac{1}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}F(z)dz + \sum_{k\ne 0}\chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)+1$$

and

$$2I_1 = LI_2 + \sum_{k\ne 0}\chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)+1$$

with

$$I_2 = \frac{1}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}z^2\Gamma(-1-z)\Gamma(-1+z)dz$$

$$= \sum_{l\ge 2} l^2 \frac{(-1)^{l+1}}{(l+1)!}(l-2)! + \frac{L}{4}$$
$$= \sum_{l\ge 2} \frac{(-1)^{l+1} l}{(l+1)(l-1)} = -L + \frac{1}{4} + \frac{1}{4} = -L + \frac{1}{2}.$$

Therefore

$$2I_1 = -L^2 + \frac{L}{2} + \sum_{k\ne 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k) + 1.$$

But I_1 is also

$$I_1 = -\frac{L}{4} + L^2 + L\sum_{l\ge 2} \frac{l^2}{2^l-1}\frac{(-1)^{l+1}}{(l+1)!}(l-2)!$$
$$= -\frac{L}{4} + L^2 + L\sum_{l\ge 2} \frac{(-1)^{l+1} l}{(2^l-1)(l+1)(l-1)}.$$

Putting things together, we find

$$2I_1 = -L^2 + \frac{L}{2} + \sum_{k\ne 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)$$
$$= -\frac{L}{2} + 2L^2 + 2L\sum_{l\ge 2} \frac{(-1)^{l+1} l}{(2^l-1)(l+1)(l-1)} + 1,$$

or

$$\sum_{k\ne 0} \chi_k(-\chi_k)\Gamma(-1-\chi_k)\Gamma(-1+\chi_k)$$
$$= -1 - L + 3L^2 + 2L\sum_{l\ge 2} \frac{(-1)^{l+1} l}{(2^l-1)(l+1)(l-1)},$$

which is the identity in question, as it expresses the quantity $L^2[\sigma^2]_0$ in two different ways.

Here is a third example, dealing with the function

$$\frac{1}{L}\sum_{k\ne 0} \Gamma(j-\chi_k)e^{2\pi i k x},$$

for $j \ge 1$, and the computation of the constant term of its square. The technique should be familiar by now. Consider the function

$$L\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}.$$

Therefore we have

$$\sum_{k\ne0}\Gamma(j+\chi_k)\Gamma(j-\chi_k)=\frac{L}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz \\ -\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz-\Gamma(j)^2.$$

($\Gamma(j)^2$ is the residue at $z=0$.)

Now we use again the decomposition

$$\frac{1}{e^{Lz}-1}=-1-\frac{1}{e^{-Lz}-1}$$

for the second integral and get

$$-\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz \\ =\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\Gamma(j+z)\Gamma(j-z)dz \\ +\frac{L}{2\pi i}\int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{-Lz}-1}dz \\ =\frac{L}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(j+z)\Gamma(j-z)dz \\ +\frac{L}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(j-z)\Gamma(j+z)}{e^{Lz}-1}dz.$$

Therefore

$$\sum_{k\ne0}\bigl|\Gamma(j+\chi_k)\bigr|^2 \\ =\frac{2L}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}dz \\ +\frac{L}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(j+z)\Gamma(j-z)dz-\Gamma(j)^2$$

$$= I_1 + I_2 - \Gamma(j)^2.$$

Integral I_1 is evaluated by shifting the contour to the *right* and collecting the *negative* residues, which gives

$$I_1 = -2L \sum_{m \geq j} \frac{\Gamma(j+m)}{e^{Lm}-1} \frac{(-1)^{j-m+1}}{(m-j)!}$$

and with $m = h + j$

$$= 2L \sum_{h \geq 0} \frac{(h+2j-1)!(-1)^h}{h!} \frac{1}{2^{h+j}-1}$$

$$= 2L(2j-1)! \sum_{h \geq 0} \binom{-2j}{h} \frac{1}{2^{h+j}-1}.$$

Integral I_2 is of interest for itself and appears already in early references to the *Mellin transform technique* as by Nielsen [19, p. 224]. (It could, however, by computed as in the previous examples.)

We start with the function

$$f(x) = \frac{x^j}{(1+x)^{2j}}$$

and perform its Mellin transform (see, e.g., [6] for definitions)

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx = B(j+s, j-s) = \frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)}$$

with the Beta function $B(z,w)$ (compare [1]). The *fundamental strip* is $\langle -j, j \rangle$. Therefore the inversion formula for the Mellin transform gives

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)} x^{-s} ds.$$

Now we may evaluate at $x = 1$ and get the formula

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(j+s)\Gamma(j-s) ds = \Gamma(2j) 2^{-2j}.$$

This produces the formula

$$\sum_{k \neq 0} \Gamma(j+\chi_k)\Gamma(j-\chi_k)$$

$$= 2L(2j-1)!\sum_{h\ge0}\binom{-2j}{h}\frac{1}{2^{h+j}-1} + L(2j-1)!2^{-2j} - (j-1)!^2. \tag{4}$$

This formula was essential in the paper [13].

Remark. The computation of the integral I_2 (as in the examples above) sometimes leads to series like

$$\sum_{l\ge1}(-1)^l l.$$

There is nothing wrong here. The correct interpretation is as an *Abel limit*

$$\lim_{t\to1-}\sum_{l\ge1}(-1)^l l t^l = \lim_{t\to1-}\frac{-t}{(1+t)^2} = -\frac{1}{4}.$$

3 Using the Mellin transform to prove identities

Let us start with our running example (1) and show how this can be proved using the Mellin transform. The Mellin transform is very prominent in the analysis of algorithms, and we refer to [6] for a nice survey.

We will treat again our identity (1) and might for instance start with the series

$$\sum_{h\ge1}\frac{(-1)^{h-1}}{h\,(2^h-1)}$$

and interpret it as $g(\log 2)$ with

$$g(x) := \sum_{h\ge1}\frac{(-1)^{h-1}}{h\,(e^{hx}-1)} = \sum_{h,k\ge1}\frac{(-1)^{h-1}}{h}e^{-hkx}.$$

Now one computes the Mellin transform $g^*(s)$:

$$\begin{aligned} g^*(s) &= \sum_{h,k\ge1}\frac{(-1)^{h-1}}{h}e^{-hkx} = \sum_{h,k\ge1}\frac{(-1)^{h-1}}{h}h^{-s}k^{-s}\Gamma(s) \\ &= (1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s). \end{aligned}$$

The Mellin transform exists in the fundamental strip $\langle 1, \infty\rangle$; whence we can invoke the inversion formula for the Mellin transform. We may choose e. g. the line $\Re z = \frac{3}{2}$ since $\frac{3}{2}$ lies in the fundamental strip. So we get

$$\begin{aligned}
g(x) &= \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}(1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s}ds \\
&= \frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}+\frac{1}{2\pi i}\int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty}(1-2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s}ds \\
&= \frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24} \\
&\quad+\frac{1}{2\pi i}\int\limits_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty}(2^s-1)\zeta(s+1)\zeta(s)\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\Big(\frac{s+1}{2}\Big)x^{-s}ds.
\end{aligned}$$

This form was obtained by taking 3 residues out and invoking the duplication formula of the Γ-function. (Observe that the exponential smallness of the Γ-function along vertical lines justifies the shifting of the line integral.) We now use the functional equation for $\zeta(s)$, namely

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s)=\pi^{s-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s), \tag{5}$$

and continue:

$$\begin{aligned}
g(x) &= \frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24} \\
&\quad+\frac{1}{2\pi i}\int\limits_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty}(2^s-1)\frac{1}{2}\pi^{2s-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)\Gamma\Big(\frac{-s}{2}\Big)\zeta(-s)x^{-s}ds \\
&= \frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24} \\
&\quad+\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}(2^{-s}-1)\frac{1}{2}\pi^{-2s-\frac{1}{2}}\Gamma\Big(\frac{1+s}{2}\Big)\zeta(1+s)\Gamma\Big(\frac{s}{2}\Big)\zeta(s)x^{s}ds \\
&= \frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24} \\
&\qquad-\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}(1-2^{-s})\pi^{-2s}\zeta(1+s)\zeta(s)\Gamma(s)x^{s}2^{-s}ds,
\end{aligned}$$

and so

$$g(x)=\frac{\pi^2}{12x}-\frac{L}{2}+\frac{x}{24}-g\Big(\frac{2\pi^2}{x}\Big). \tag{6}$$

This is the formula we need, since we can also rewrite the left side of (1) in terms of this $g(x)$ function:

$$\begin{aligned}[\delta_0^2]_0&=\frac{1}{L^2}\sum_{k\neq 0}\Gamma(\chi_k)\Gamma(-\chi_k)\\&=\frac{1}{L}\sum_{k\ge 1}\frac{1}{k\sinh(2k\pi^2/L)}=\frac{2}{L}\sum_{k\ge 1}\frac{e^{kz}}{k(e^{2kz}-1)},\end{aligned}$$

with $z=2\pi^2/L$. But

$$\begin{aligned}\sum_{k\ge 1}\frac{e^{kz}}{k(e^{2kz}-1)}&=\sum_{k\ge 1,\,j\ge 0}\frac{1}{k}e^{-k(2j+1)z}\\&=\sum_{k\ge 1,\,j\ge 1}\frac{1}{k}e^{-kjz}-2\sum_{k\ge 1,\,j\ge 1}\frac{1}{2k}e^{-2kjz}\\&=\sum_{k\ge 1,\,j\ge 1}\frac{(-1)^{k-1}}{k}e^{-kjz}=\sum_{k\ge 1}\frac{(-1)^{k-1}}{k(e^{kz}-1)}=g(z),\end{aligned}$$

and so

$$[\delta_0^2]_0=\frac{2}{L}g\Big(\frac{2\pi^2}{L}\Big).$$

Let us do a more complicated example in the same style: We want to rewrite $[\tau^2]_0$, to get identity (3). Note that

$$[\tau^2]_0=2\sum_{k\ge 1}\tau_k\tau_{-k}=\frac{2}{4L^2}\sum_{k\ge 1}\Big(3+2\sqrt{2}(-1)^k\Big)\,\Gamma\Big(\frac{1-\chi_k}{2}\Big)\Gamma\Big(\frac{1+\chi_k}{2}\Big).$$

Now we use the formula (equivalent to the reflection formula for the Gamma function, cf. [1]) $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ and obtain

$$\begin{aligned}\Gamma\Big(\frac{1-\chi_k}{2}\Big)\Gamma\Big(\frac{1+\chi_k}{2}\Big)&=\frac{\pi}{\sin(\pi/2+ik\pi^2/L)}=\frac{\pi}{\cos(ik\pi^2/L)}\\&=\frac{\pi}{\cosh(k\pi^2/L)}=2\pi\frac{e^{-k\pi^2/L}}{1+e^{-2k\pi^2/L}},\end{aligned}$$

so that

$$[\tau^2]_0=\frac{\pi}{L^2}\sum_{k\ge 1}\Big(3+2\sqrt{2}(-1)^k\Big)\,\frac{e^{-k\pi^2/L}}{1+e^{-2k\pi^2/L}}. \tag{7}$$

Let us define two new functions

$$F(x) = \sum_{k\geq 1} \frac{e^{-kx}}{1+e^{-2kx}} \qquad \text{and} \qquad G(x) = \sum_{k\geq 1} \frac{(-1)^{k-1}e^{-kx}}{1+e^{-2kx}}.$$

Then, (7) in terms of $F(x)$ and $G(x)$ becomes

$$[\tau^2]_0 = \frac{3\pi}{L^2} F\Big(\frac{\pi^2}{L}\Big) - \frac{2\sqrt{2}\pi}{L^2} G\Big(\frac{\pi^2}{L}\Big). \tag{8}$$

We use a series transformation for $F(x)$ and $G(x)$. We start with

$$F(x) = \sum_{j\geq 0} (-1)^j \sum_{k\geq 1} e^{-k(2j+1)x} = \sum_{j\geq 0} \chi(j) \frac{1}{e^{jx}-1}$$

where

$$\chi(j) = \begin{cases} 0, & \text{for } j \text{ even;} \\ 1, & \text{for } j \equiv 1 \bmod 4; \\ -1, & \text{for } j \equiv 3 \bmod 4. \end{cases}$$

Once we know that

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x} F\Big(\frac{\pi^2}{x}\Big), \tag{9}$$

for $x > 0$, as we shall show soon, then $G(x) = F(x) - 2F(2x)$, hence

$$G(x) = \frac{1}{4} + \frac{\pi}{x} F\Big(\frac{\pi^2}{x}\Big) - \frac{\pi}{x} F\Big(\frac{\pi^2}{2x}\Big).$$

Applying the above to (8) we finally obtain

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2}\big(3+2\sqrt{2}\,\big) + \frac{3-2\sqrt{2}}{L} F(L) + \frac{2\sqrt{2}}{L} F\Big(\frac{L}{2}\Big).$$

To prove (9) we proceed as follows. Let

$$\beta(s) = \sum_{j\geq 0} (-1)^j \frac{1}{(2j+1)^s}.$$

We have

$$F(x) = \sum_{k\geq 1} \frac{e^{-kx}}{1+e^{-2kx}} = \sum_{j\geq 0} (-1)^j \sum_{k\geq 1} e^{-k(2j+1)x},$$

so that the Mellin transform $F^*(s) = \int_0^\infty F(x)x^{s-1}dx$ of $F(x)$ becomes $F^*(s) = \Gamma(s)\zeta(s)\beta(s)$. By the Mellin inversion formula this yields

$$F(x) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(s)\zeta(s)\beta(s)x^{-s}ds.$$

Now we take the two residues $s = 1$ and $s = 0$ out from the above integral (observe that $\beta(0) = 1/2$ and $\beta(1) = \pi/4$, cf. [1]) and apply the duplication formula for $\Gamma(s)$ to obtain

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{\sqrt{\pi}} 2^{s-1}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)x^{-s}\zeta(s)\beta(s)ds.$$

We now use the functional equations for $\zeta(s)$ and $\beta(s)$, namely

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)$$

and

$$\beta(1-s)\Gamma\Big(1-\frac{s}{2}\Big) = 2^{2s-1}\pi^{-s+\frac{1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\beta(s).$$

The first identity is Riemann's functional equation for $\zeta(s)$, and the second is an immediate consequence of the functional equation for Hurwitz's ζ-function $\zeta(s,a)$ (cf. [2]), and the fact that

$$\beta(s) = 4^{-s}\Big[\zeta\big(s, \tfrac{1}{4}\big) - \zeta\big(s, \tfrac{3}{4}\big)\Big].$$

Substituting $1 - s = u$, we get

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \pi^{1-2u}\Gamma(u)x^{u-1}\zeta(u)\beta(u)du,$$

which proves (9).

Using the above scheme, several other identities which one needs in the analysis of algorithms can be proved. We refer to Szpankowski's book [25].

4 Modular identities

Formulæ like (6) belong to the realm of modular functions. Many of them can be found in the literature, and are due to Jacobi, Dedekind, Ramanujan and others. Berndt's book [4] contains a wealth of information about the subject, compare also [3].

Here is a little bit of background: Let H be the upper complex halfplane $\{z \in \mathbb{C} \mid \Im z > 0\}$. Then the Dedekind η function is defined by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n\geq 1} \left(1 - e^{2\pi in\tau}\right), \quad \tau \in H;$$

there is a transformation formula:

$$\eta\Big(-\frac{1}{\tau}\Big) = (-i\tau)^{1/2}\eta(\tau).$$

C. L. Siegel [24] gave an elegant proof of this transformation formula using residue calculus.

Ramanujan considered series

$$f(z) := \sum_{k\geq 1} \frac{k^m}{e^{2kz}-1}, \quad m \text{ an odd integer},$$

and could relate them to $f(\pi^2/z)$. For the reader's convenience, we give these formulæ here:

Set $m = 2N+1$ and $N \in \mathbb{N}$, $\alpha, \beta > 0$, and $\alpha\beta = \pi^2$, then

$$\begin{aligned}
\alpha^{-N}\Big\{\frac{1}{2}\zeta(2N+1) + \sum_{k\geq 1} \frac{k^{-2N-1}}{e^{2\alpha k}-1}\Big\} \\
&= (-\beta)^{-N}\Big\{\frac{1}{2}\zeta(2N+1) + \sum_{k\geq 1} \frac{k^{-2N-1}}{e^{2\beta k}-1}\Big\} \\
&\qquad - 2^{2N}\sum_{k=0}^{N+1}(-1)^k \frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\alpha^{N+1-k}\beta^k
\end{aligned}$$

(this covers the exponents $-3, -5, \dots$); the B_k's are the Bernoulli numbers. Then

$$\sum_{k\geq 1} \frac{1}{k(e^{2\alpha k}-1)} - \frac{1}{4}\log\alpha + \frac{\alpha}{12} = \sum_{k\geq 1} \frac{1}{k(e^{2\beta k}-1)} - \frac{1}{4}\log\beta + \frac{\beta}{12},$$

which covers the exponent -1. Furthermore,

$$\alpha\sum_{k\geq 1} \frac{k}{e^{2\alpha k}-1} + \beta\sum_{k\geq 1} \frac{k}{e^{2\beta k}-1} = \frac{\alpha+\beta}{24} - \frac{1}{4}$$

which covers the exponent 1, and finally for $N \geq 2$,

$$\alpha^N \sum_{k\geq 1} \frac{k^{2N-1}}{e^{2\alpha k}-1} - (-\beta)^N \sum_{k\geq 1} \frac{k^{2N-1}}{e^{2\beta k}-1} = \left(\alpha^N - (-\beta)^N\right) \frac{B_{2N}}{4N},$$

which covers the exponents $3, 5, \ldots$.

The instance $m = -1$ is equivalent to the functional equation for Dedekind's eta function.

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