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On Burr III-Inverse Weibull Distribution with COVID-19 Applications

Fiaz Ahmad Bhatti [1], **Sedigheh Mirzaei Salehabadi** [2], **and Gholamhossein G. Hamedani**[3]

[1] National College of Business Administration and Economics, Lahore, PAKISTAN.
[2] St. Jude Children's Research Hospital, Memphis, TN, USA.
[3] Marquette University, Milwaukee, WI 53201-1881, USA.

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Abstract. We introduce a flexible lifetime distribution called Burr III-Inverse Weibull (BIII-IW). The new proposed distribution has well-known sub-models. The BIII-IW density function includes exponential, left-skewed, right-skewed and symmetrical shapes. The BIII-IW model's failure rate can be monotone and non-monotone depending on the parameter values. To show the importance of the BIII-IW distribution, we establish various mathematical properties such as random number generator, ordinary moments, conditional moments, residual life functions, reliability measures and characterizations. We address the maximum likelihood estimates (MLE) for the BIII-IW parameters and estimate the precision of the maximum likelihood estimators via a simulation study. We consider applications to two COVID-19 data sets to illustrate the potential of the BIII-IW model.

Keywords. Moment, Reliability, Characterizations, Maximum Likelihood Estimation.

MSC: 62E10,62E15, 62F10, 62P10.

Corresponding Author: Fiaz Ahmad Bhatti (fiazahmad72@gmail.com)
Sedigheh Mirzaei Salehabadi (Sedigheh.Mirzaei@stjude.org)
Gholamhossein G. Hamedani (g.hamedani@mu.edu)

1 Introduction

The selection of a suitable model for data analysis is generally quite challenging. If a wrong model is applied to analyze the dataset it leads to loss of information and invalid inferences. It is obligatory to identify the most suitable model for the given dataset. In the recent decade, many continuous distributions have been introduced in statistical literature. Some of these distributions, however, are not flexible enough for analysis of lifetime data. Hence, more versatile models are needed. The generalization techniques such as either inserting one or more shape parameters or transforming of the parent distribution are useful to (i) increase the applicability of a parent distribution; (ii) explore skewness and tail properties and (iii) improve the goodness-of-fit of the generalized distributions.

The inverse Weibull distribution (Keller and Kanath, 1982), developed to study the decay of mechanical components in survival and reliability analysis, is a highly flexible yet simple model. Variations and generalizations of the inverse Weibull distribution have been of great interest in literature: beta inverse Weibull (B-IW) (Khan , 2010), Kumaraswamy-Inverse Weibull (Kw-IW) (Shahbaz, et al. , 2012), reflected generalized beta inverse Weibull (Elbatal, et al. , 2016)), Topp-Leone inverse Weibull (Abbas, et al. , 2017), Odd Frechet inverse Weibull (OF-IW) (Fayomi , 2019) and gamma inverse Weibull(G-IW) (Abbas, et al. , 2020).

The main idea of the present paper is to incorporate the inverse Weibull distribution into a larger family through an application of the Burr III distribution. In fact, based on the T-X transform defined by Alzaatreh, et al. (2013), we construct the BIII-IW distribution. The new model has flexible shapes to model various lifetime data sets. Additionally, its special sub-models produce better fits than other well-known models. Here we study the BIII-IW distribution with COVID-19 applications.

Our study of the BIII-IW distribution is based on the following motivations: (i) to generate distributions with symmetrical, right-skewed, left-skewed, J, reverse-J, exponential shaped as well as high kurtosis; (ii) to have monotone and non-monotone failure rate function; (iii) to derive mathematical properties such as random number generator, ordinary moments, conditional moments, residual life functions, reliability measures and characterizations; (iv) to estimate the precision of the maximum likelihood estimators via a simulation study; (v) to reveal the potentiality and utility of the BIII-IW model; (vi) to work as the preeminent substitute model to other existing models; (vii) to deliver better fits than other models and (viii) to infer empirically from goodness of fit statistics (GOFs) and graphical tools.

This article is structured as follows. Section 2 derives the BIII-IW model and studies its basic structural properties, random number generator and sub-models. Section 3 presents certain mathematical properties such as ordinary moments, conditional moments, residual life functions, reliability measures and characterizations. In Section 4, we address the maximum likelihood estimation for the BIII-IW parameters. We evaluate the precision of the maximum likelihood estimators via a simulation study. In Section 5, we consider applications to two COVID-19 data sets to elucidate the potentiality of the BIII-IW model. In Section 6, we offer some concluding remarks.

2 The BIII-IW Distribution

We derive the BIII-IW distribution from the T-X family technique. We highlight the shapes of the density and failure rate functions.

2.1 T-X Family Technique

The cumulative distribution function (cdf) and probability density function (pdf) of the inverse Weibull distribution are given, respectively, by

$$G(x;\lambda,\eta) = \exp(-\lambda x^{-\eta}), \qquad x \geq 0,$$

and

$$g(x;\lambda,\eta) = \lambda\eta x^{-\eta-1}\exp(-\lambda x^{-\eta}), \qquad x > 0, \lambda > 0, \eta > 0,$$

The odds ratio for the inverse Weibull random variable X is

$$W\Big(G(x;\lambda,\eta)\Big) = \frac{G(x;\lambda,\eta)}{\overline{G}(x;\lambda,\eta)} = \frac{e^{-\lambda x^{-\eta}}}{1-e^{-\lambda x^{-\eta}}} = [\exp(\lambda x^{-\eta})-1]^{-1}.$$

The cdf of the T-X family (Alzaatreh, et al. , 2013) of distributions has the form

$$F(x) = \int_{a}^{W[G(x;\xi)]} r(t)\,dt, \qquad x \in \mathbb{R}, \tag{2.1}$$

where $r(t)$ is the pdf of the random variable (rv) T, where $T \in [a,b]$ for $-\infty < a < b < \infty$ and $W[G(x;\xi)]$ is a function of the baseline cdf of a rv X with the parameter vector ξ, which satisfies the conditions:

- $W[G(x;\xi)] \in [a,b]$,
- $W[G(x;\xi)]$ is differentiable and monotonically non-decreasing and
- $\lim_{x\to-\infty} W[G(x;\xi)] \to a$ and $\lim_{x\to\infty} W[G(x;\xi)] \to b$.

The pdf of the $T-X$ family can be expressed as

$$f(x) = \left\{\frac{\partial}{\partial x} W\Big[G(x;\xi)\Big]\right\} r\Big\{W[G(x;\xi)]\Big\}, \qquad x \in \mathbb{R}. \tag{2.2}$$

We derive the cdf of the BIII-IW distribution from the T-X family technique by setting

$$r(t) = \alpha\beta t^{-\beta-1}(1+t^{-\beta})^{-\alpha-1}, \quad t>0, \alpha>0, \beta>0,$$

and

$$W\Big(G(x;\lambda,\eta)\Big) = \frac{G(x;\lambda,\eta)}{\overline{G}(x;\lambda,\eta)} = \frac{e^{-\lambda x^{-\eta}}}{1-e^{-\lambda x^{-\eta}}} = [\exp(\lambda x^{-\eta})-1]^{-1}.$$

The cdf of the BIII-IW distribution takes the form

$$F(x) = \Big[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\Big]^{-\alpha} \qquad x \geq 0, \tag{2.3}$$

where $\alpha, \beta, \lambda, \eta$ are the parameters. The BIII-IW density can be expressed as

$$f(x) = \alpha\beta\lambda\eta x^{-\eta-1} e^{\lambda x^{-\eta}}\Big(e^{\lambda x^{-\eta}}-1\Big)^{\beta-1}\Big[1+\Big(e^{\lambda x^{-\eta}}-1\Big)^{\beta}\Big]^{-\alpha-1} \qquad x>0. \tag{2.4}$$

Hereafter, a random variable with pdf (2.4) is denoted by $X \sim BIII-IW(\alpha,\beta,\lambda,\eta)$. If $X \sim BIII-IW(\alpha,\beta,\lambda,\eta)$, the survival function, failure rate, reverse failure rate and cumulative failure rate of X are given, respectively, by (for $x>0$)

$$S(x) = 1-\Big[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\Big]^{-\alpha},$$

$$\lambda(x) = \frac{\alpha\beta\lambda\eta x^{-\eta-1} e^{\lambda x^{-\eta}}\Big(e^{\lambda x^{-\eta}}-1\Big)^{\beta-1}\Big[1+\Big(e^{\lambda x^{-\eta}}-1\Big)^{\beta}\Big]^{-\alpha-1}}{1-\Big[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\Big]^{-\alpha}},$$

$$r(x) = \frac{d}{dx}\ln\Big[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\Big]^{-\alpha},$$

and

$$\Lambda(x) = -\ln\Big\{1-\Big[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\Big]^{-\alpha}\Big\}.$$

The quantile function of X (for $0 < q < 1$) follows from

$$x_q = \left\{ \ln\left[(q^{-\frac{1}{\alpha}} - 1)^{\frac{1}{\beta}} + 1 \right]^{\frac{1}{\lambda}} \right\}^{-\frac{1}{\eta}},$$

and its random number generator with $Z \sim$ Uniform (0,1) is the solution of the nonlinear equation

$$X = \left\{ \ln\left[(Z^{-\frac{1}{\alpha}} - 1)^{\frac{1}{\beta}} + 1 \right]^{\frac{1}{\lambda}} \right\}^{-\frac{1}{\eta}}.$$

The sub-models of BIII-IW distribution are (i) For $\eta = 2$, the BIII-IW distribution reduces to Burr III-inverse Rayleigh (BIII-IR); (ii) For $\eta = 1$, the BIII-IW distribution reduces to Burr III-inverse exponential (BIII-IE); (iii) For $\beta = 1$, the BIII-IW distribution reduces to inverse Lomax-inverse Weibull (IL-IW); (iv) For $\beta = 1$ and $\eta = 2$,the BIII-IW distribution reduces to inverse Lomax- inverse Rayleigh (IL-IR); (v) For $\beta = \eta = 1$, the BIII-IW distribution reduces to inverse Lomax- inverse exponential (IL-IE); (vi) For $\alpha = 1$, the BIII-IW distribution reduces to the log-logistic -inverse Weibull (LL-IW); (vii) For $\alpha = 1$ and $\eta = 2$, the BIII-IW distribution reduces to the log-logistic-inverse Rayleigh (LL-IR) and (viii) For $\alpha = \eta = 1$, the BIII-IW distribution reduces to the log-logistic-inverse exponential (LL-IE) distribution.

2.2 Shapes of the BIII-IW Density and Hazard Rate Functions

We plot the density and failure rate functions of the BIII-IW distribution for selected parameter values. The BIII-IW density can display numerous shapes such as bimodal, symmetrical, right-skewed, left-skewed, J, reverse-J and exponential (Figure 1). The failure rate function can highlight shapes as modified bathtub, inverted bathtub, decreasing, increasing (Figure 2). Therefore, the BIII-IW distribution is quite flexible and can be applied to numerous data sets.

2.3 Useful Expansions

Consider the two binomial series

$$(1+z)^{-\alpha} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+i-1}{i} z^i \qquad |z| < 1, \alpha > 0, \tag{2.5}$$

and

$$(1-z)^{\alpha} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} z^m. \tag{2.6}$$

The pdf in (2.4) can be expressed as

$$f(x) = \alpha\beta\lambda\eta x^{-\eta-1}\exp(\lambda x^{-\eta})\Big[\frac{\exp(-\lambda x^{-\eta})}{1-\exp(-\lambda x^{-\eta})}\Big]^{-\beta+1}\underbrace{\Big\{1+\Big[\frac{1-\exp(-\lambda x^{-\eta})}{\exp(-\lambda x^{-\eta})}\Big]^{\beta}\Big\}^{-\alpha-1}}_{\text{A(x)}}, \quad (2.7)$$

applying (2.5) to $A(x)$ in (2.7), we obtain

$$f(x) = \alpha\beta\lambda\eta x^{-\eta-1}\exp(\lambda x^{-\eta})\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i}{i}\Big[\frac{1-\exp(-\lambda x^{-\eta})}{\exp(-\lambda x^{-\eta})}\Big]^{\beta(i+1)-1},$$

$$f(x) = \alpha\beta\lambda\eta x^{-\eta-1}\exp(\lambda x^{-\eta})\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i}{i}\Big[\exp(-\lambda x^{-\eta})\Big]^{\beta(i+1)-1}\underbrace{\Big[1-\exp(-\lambda x^{-\eta})\Big]^{\beta(i+1)-1}}_{\text{B(x)}}. \quad (2.8)$$

Now applying (2.6) to $B(x)$ in (2.8), we obtain

$$f(x) = \lambda\eta x^{-\eta-1}\sum_{i,m=0}^{\infty}c_{i,m}\exp\Big\{-\lambda[m-\beta(i+1)]x^{-\eta}\Big\}, \quad (2.9)$$

which is density of exponentiated inverse Weibull distribution with scale parameter $\lambda[m-\beta(i+1)]$ and

$$c_{i,m} = c_{i,m}(\alpha,\beta) = \alpha\beta(-1)^{i+m}\binom{\alpha+i}{i}\binom{\beta(\alpha+i)-1}{m}.$$

Equation (2.9) is the main result of this section. It reveals that the BIII-IW density is a linear combination of IW density. So, some of its mathematical properties can be easily determined from those of IW model.

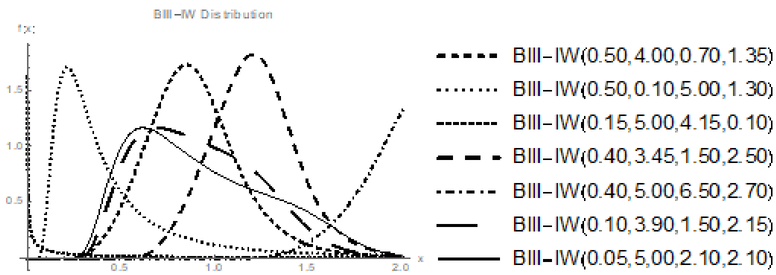


Figure 1: Plots of the BIII-IW density.

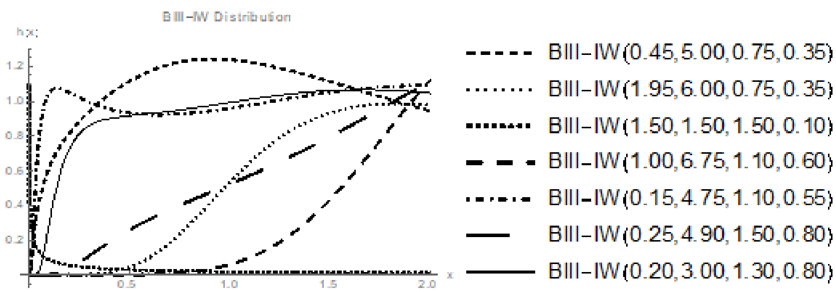


Figure 2: Plots of the BIII-IW hazard rate.

3 Mathematical Properties

We derive some of its mathematical properties including the ordinary moments, conditional moments, residual life functions, reliability measures and characterizations.

3.1 Moments

The moments are significant tools for statistical analysis in pragmatic sciences. The rth ordinary moment of X with the BIII-IW distribution is,

$$E(X^r) = \int_0^\infty x^r \lambda\eta x^{-\eta-1} \sum_{i,m=0}^{\infty} c_{i,m} \exp\Big\{-\lambda\big[m-\beta(i+1)\big]x^{-\eta}\Big\}dx,$$

$$E(X^r) = \lambda\eta \sum_{i,m=0}^{\infty} c_{i,m} \int_0^\infty x^r x^{-\eta-1} \exp\Big\{-\lambda\big[m-\beta(i+1)\big]x^{-\eta}\Big\}dx,$$

Letting $\lambda\Big[m-\beta(i+1)\Big]x^{-\eta}=w,\ -\lambda\eta x^{-\eta-1}dx=\frac{dw}{\Big[m-\beta(i+1)\Big]},\ x^{r}=\lambda^{\frac{r}{\eta}}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}}w^{\frac{r}{\eta}}$, we will have

$$E(X^r)=\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}[m-\beta(i+1)]^{\frac{r}{\eta}-1}\int_0^{\infty}w^{1-\frac{r}{\eta}-1}e^{-w}dw$$
$$=\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}[m-\beta(i+1)]^{\frac{r}{\eta}-1}\Gamma(1-\frac{r}{\eta}), \tag{3.1}$$

where $r<\eta$ and $\Gamma(\cdot)$ is the gamma function.

The rth cenaral moment (μ_r), skewness (γ_1) and kurtosis (γ_2) for the BIII-IW model are obtained from

$$\mu_r=\sum_{l=1}^{r}(-1)^l\binom{r}{l}\mu'_l\mu'_{r-l}\,, \qquad \gamma_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \quad \text{and} \quad \beta_2=\frac{\mu_4}{(\mu_2)^2}.$$

3.2 Conditional Moments

Life expectancy, mean waiting time and inequality measures can be obtained from the incomplete moments.

The rth lower incomplete moment $E_{X\le z}(X^r)$ is

$$M'_r(z)=E_{X\le z}(X^r)=\int_0^{z}x^r\lambda\eta x^{-\eta-1}\sum_{i,m=0}^{\infty}c_{i,m}\exp\Big\{-\lambda\Big[m-\beta(i+1)\Big]x^{-\eta}\Big\}dx,$$
$$E_{X\le z}(X^r)=\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\int_0^{\infty}w^{1-\frac{r}{\eta}-1}e^{-w}dw,$$
$$M'_r(z)=E_{X\le z}(X^r)=\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\gamma\Big[\Big(1-\frac{r}{\eta}\Big),z\Big], \tag{3.2}$$

where $\int_0^{w}w^{1-\frac{r}{\eta}-1}e^{-w}dw=\gamma\Big[\Big(1-\frac{r}{\eta}\Big),w\Big]$ is the lower incomplete gamma function.

The rth conditional moment $E\Big(X^r|X>z\Big)$ is

$$E\Big(X^r|X>z\Big)=\frac{1}{S(z)}\Big\{\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\Gamma\Big[\Big(1-\frac{r}{\eta}\Big),z\Big]\Big\},$$

where $\int_w^\infty w^{1-\frac{r}{\eta}-1}e^{-w}dw = \Gamma\Big[\Big(1-\frac{r}{\eta}\Big), w\Big]$ is the upper incomplete gamma function.

The rth conditional moment $E\Big(X^r|X \leq z\Big)$ is

$$E\Big(X^r|X \leq z\Big) = \frac{1}{F(z)}\Big\{\lambda^{\frac{r}{\eta}} \sum_{i,m=0}^{\infty} c_{i,m}[m-\beta(i+1)]^{\frac{r}{\eta}-1}\gamma\Big[\Big(1-\frac{r}{\eta}\Big), z\Big]\Big\}.$$

The mean deviation about the mean ($\delta_1 = E|X-\mu|$) and about the median($\delta_2 = E|X-\tilde{\mu}|$)) can be written as $\delta_1 = 2\mu\Big(F(\mu) - M_1'(\mu)\Big)$ and $\delta_2 = \mu - 2M_1'(\tilde{\mu})$, respectively, where $\mu = E(X)$ and $\tilde{\mu} = x_{0.5}$. The quantities $M_1'(\mu)$ and $M_1'(\tilde{\mu})$ can be obtained from (3.2). For specific probability p, Lorenz and Bonferroni curves are computed as $L(p) = \frac{M_1'(q)}{\mu}$ and $B(p) = L(p)|p$, where $q = Q(p)$.

3.3 Residual Life Functions

The nth moments about origin of residual life, say $M_n(z) = E\Big[(X-z)^n|X > z\Big]$, of X is

$$\begin{aligned} m_n(z) &= \frac{1}{S(z)} \int_z^\infty (x-z)^r f(x)dx \\ &= \frac{1}{S(z)} \sum_{r=0}^{n} \binom{n}{r} (-z)^{n-r} E_{X>z}(X^r) \\ &= \frac{1}{1-F(z)} \sum_{r=0}^{n} \binom{n}{r} (-z)^{n-r} \Big\{\lambda^{\frac{r}{\eta}} \sum_{i,m=0}^{\infty} c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\Gamma\Big[\Big(1-\frac{r}{\eta}\Big), z\Big]\Big\}. \end{aligned}$$

The average remaining lifetime of a component at time z, say $m_1(z)$, or life expectancy called mean residual life (MRL) function is

$$m_1(z) = \frac{1}{1-F(z)} \sum_{r=0}^{1} \binom{1}{r} (-z)^{1-r} \Big\{\lambda^{\frac{r}{\eta}} \sum_{i,m=0}^{\infty} c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\Gamma\Big[\Big(1-\frac{r}{\eta}\Big), z\Big]\Big\}.$$

The nth moments about origin reverse residual life, say $M_n(z) = E\Big[(z-X)^n|X \leq z\Big]$, of X

with BIII-IW distribution is

$$\begin{aligned}M_n(z) &= \frac{1}{F(z)}\int_0^z (z-x)^r f(x)dx\\ &= \frac{1}{F(z)}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(z)^{n-r}E_{X\le z}(X^r)\\ &= \frac{1}{F(z)}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(z)^{n-r}\Big\{\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}\Big[m-\beta(i+1)\Big]^{\frac{r}{\eta}-1}\gamma\Big[\Big(1-\frac{r}{\eta}\Big),z\Big]\Big\}.\end{aligned}$$

The waiting time z for failure of a component has passed with condition that this failure had happened in the interval $[0,z]$ is called mean waiting time (MWT) or mean inactivity time. The waiting time z for failure of a component for the BIII-IW distribution is defined by

$$M_1(z) = \frac{1}{F(z)}\sum_{r=0}^{1}(-1)^r\binom{1}{r}(z)^{1-r}\Big\{\lambda^{\frac{r}{\eta}}\sum_{i,m=0}^{\infty}c_{i,m}[m-\beta(i+1)]^{\frac{r}{\eta}-1}\gamma\Big[\Big(1-\frac{r}{\eta}\Big),z\Big]\Big\}.$$

3.4 Reliability Estimation in Multi-Component Stress-Strength Model

The multi-component stress-strength model provides about the awareness of life of a component or a system in reliability. Consider a system with κ identical elements, out of which s elements are operative. Let $X_i, i = 1,2...\kappa$, represent strengths of κ elements with the cdf F while, the stress Y enforced on the elements has the cdf G. The strengths X_i and stress Y are independently and identically distributed (i.i.d.). The probability that system operates properly, is the reliability of the system, i.e.

$$\begin{aligned}R_{s,\kappa} &= P\Big[\text{strengths}(X_i, i = 1,2,....,\kappa) > \text{stress}(Y)\Big]\\ &= P\Big[\text{at the minimum}`s\text{'of } (X_i, i = 1,2,....,\kappa)\ \text{ exceed } (Y)\Big].\end{aligned}$$

Then, we can write this probability (Bhattacharyya and Johnson , 1974) as follows:

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_{-\infty}^{\infty}[1-F(y)]^{\ell}[F(y)]^{\kappa-\ell}dG(y). \tag{3.3}$$

Let $X \sim$ BIII-IW $(\alpha_1,\beta,\lambda,\eta)$, $Y \sim$ BIII-IW $(\alpha_2,\beta,\lambda,\eta)$ with unknown α_1 and α_2, common β, λ, η where X and Y are independent. The reliability in multi-component stress-strength for the BIII-ME distribution is given by

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^{\infty}\left\{1-\left[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\right]^{-\alpha_1}\right\}^{\ell}\left[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\right]^{-\alpha_1(\kappa-\ell)}\alpha_2\beta\lambda\eta x^{-\eta-1}\left(e^{\lambda x^{-\eta}}-1\right)^{\beta-1}\left[1+\left(e^{\lambda x^{-\eta}}-1\right)^{\beta}\right]^{-\alpha_2-1}dx.$$

Letting $u = \left[1+(e^{\lambda x^{-\eta}}-1)^{\beta}\right]^{-\alpha_2}$, we obtain

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1 (1-u^{\nu})^{\ell}u^{\nu(\kappa-\ell)}du,$$

where $\nu = \frac{\alpha_1}{\alpha_2}$.

Let $w = u^{\nu},\ u = w^{\frac{1}{\nu}},\ du = \frac{1}{\nu}w^{\frac{1}{\nu}-1}dw$, then we have

$$\begin{aligned} R_{s,\kappa} &= \sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1 (1-w)^{\ell}w^{(\kappa-\ell)}\frac{1}{\nu}w^{\frac{1}{\nu}-1}dw, \\ &= \frac{1}{\nu}\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}B(\ell+1,\kappa-\ell+\frac{1}{\nu}), \end{aligned} \tag{3.4}$$

where $B(\cdot)$ is the Beta function and $\nu = \frac{\alpha_1}{\alpha_2}$. The probability in (3.4) is known as the reliability in multi-component stress-strength model. For $s = k = 1$, the multi-component stress-strength model reduces to the stress-strength model (Kotz, et al. , 2003) $R_{1,1} = \frac{\alpha_1}{(\alpha_1+\alpha_2)}$, where $\alpha_1 + \alpha_2 > 0$.

3.5 Characterization via Truncated Moment

We characterize the BIII-IW distribution via truncated moment (Glänzel , 1987).

Proposition 3.1. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h_1(x) = \frac{1}{\alpha}[1+(e^{\lambda x^{-\eta}}-1)^{\beta}]^{\alpha+1},\quad x > 0$ and $h_2(x) = \frac{2}{\alpha}(e^{\lambda x^{-\eta}}-1)^{\beta}[1+(e^{\lambda x^{-\eta}}-1)^{\beta}]^{\alpha+1},\quad x > 0$. The random variable X has the pdf* (2.4) *iff, the function $v(x) = \frac{E[h_1(X)|X\geq x]}{E[h_2(X)|X\geq x]}$ has the form $v(x) = (e^{\lambda x^{-\eta}}-1)^{\beta}, x > 0$.*

Proof. if X has the pdf (2.4), then

$$\begin{aligned} \left(1-F(x)\right)E\left(h_1(x)|X\geq x\right) &= (e^{\lambda x^{-\eta}}-1)^{\beta}, x>0, \\ \left(1-F(x)\right)E\left(h_2(x)|X\geq x\right) &= (e^{\lambda x^{-\eta}}-1)^{2\beta}, x>0, \\ \frac{E[h_1(X)|X\geq x]}{E[h_2(X)|X\geq x]} &= \nu(x) = (e^{\lambda x^{-\eta}}-1)^{-\beta}, x>0. \end{aligned}$$

Conversely, if $\nu(x)$ has the given form, then $\nu'(x) = \beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}(e^{\lambda x^{-\eta}}-1)^{-\beta-1}, x > 0.$ The differential equation

$$s'(x) = \frac{\nu'(x)h_2(x)}{\nu(x)h_2(x) - h_1(x)} = \frac{2\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}}{(e^{\lambda x^{-\eta}} - 1)},$$

has soltion $s(x) = \ln(e^{\lambda x^{-\eta}} - 1)^{-2\beta}, x > 0$. Therefore, in light of Theorem G (Glänzel , 1987), X has the pdf (2.4). □

Corollary 3.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h_2(x) = \frac{2}{\alpha}(e^{\lambda x^{-\eta}} - 1)^{\beta}[1 + (e^{\lambda x^{-\eta}} - 1)^{\beta}]^{\alpha+1}, \quad x > 0$. The pdf of X is (2.4) iff there exist function $\nu(x)$ and $h_1(x)$ satisfying the differential equation*

$$\frac{\nu'(x)h_2(x)}{\nu(x)h_2(x) - h_1(x)} = \alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}(e^{\lambda x^{-\eta}} - 1)^{-\beta-1}\Big[1 + (e^{\lambda x^{-\eta}} - 1)^{\beta}\Big]^{-\alpha-1}. \tag{3.5}$$

Remark 1. *The general solution of (3.5) is given by*

$$\nu(x) = (e^{\lambda x^{-\eta}} - 1)^{-2\beta}\Big\{\int\Big[-\alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}(e^{\lambda x^{-\eta}} - 1)^{\beta-1}\Big[1 + (e^{\lambda x^{-\eta}} - 1)^{\beta}\Big]^{-\alpha-1}h_1(x)\Big]dx + D\Big\}$$

where D is a constant.

4 Statistical Inference

First, we adopt the maximum likelihood estimation technique for the BIII-IW parameters. We evaluate the behavior of the maximum likelihood estimators of the BIII-IW parameters via a simulation study. We explain the utility of the BIII-IW model among its family and class using real data set.

4.1 Parameter Estimation

Let $\xi = (\alpha, \beta, \lambda, \eta)^T$ be the unknown parameter vector. The log-likelihood function $\ell(\xi)$ for the BIII-IW distribution is

$$\begin{aligned}\ell = \ln L(\xi) =& n\ln(\alpha) + n\ln(\beta) + n\ln(\lambda) + n\ln(\eta) - (\eta + 1)\sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n}\ln x_i^{-\eta} \\ &+ (\beta - 1)\sum_{i=1}^{n}\ln\Big(e^{\lambda x_i^{-\eta}} - 1\Big) - (\alpha + 1)\sum_{i=1}^{n}\ln\Big[1 + \Big(e^{\lambda x_i^{-\eta}} - 1\Big)^{\beta}\Big].\end{aligned} \tag{4.1}$$

The first derivatives of $\ell(\xi)$ with respect to α, β, λ and η are as given below.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln\left[1 + \left(e^{\lambda x_i^{-\eta}} - 1\right)^{\beta}\right],$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(e^{\lambda x_i^{-\eta}} - 1\right) + (\alpha + 1) \sum_{i=1}^{n} \left[1 + \left(e^{\lambda x_i^{-\eta}} - 1\right)^{-\beta}\right]^{-1} \ln\left(e^{\lambda x_i^{-\eta}} - 1\right),$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{-\eta} - (\beta+1) \sum_{i=1}^{n} \frac{x_i^{-\eta}}{1 - e^{-\lambda x_i^{-\eta}}} + (\alpha+1)\beta \sum_{i=1}^{n} \left[1 + \left(e^{\lambda x_i^{-\eta}} - 1\right)^{\beta}\right]^{-1} \left(e^{\lambda x_i^{-\eta}} - 1\right)^{\beta-1} x_i^{-\eta} e^{\lambda x_i^{-\eta}},$$

$$\frac{\partial \ell}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^{-\eta} \ln x_i + (\beta+1)\lambda \sum_{i=1}^{n} \frac{x_i^{-\eta} \ln x_i}{\left(1 - e^{-\lambda x_i^{-\eta}}\right)} + (\alpha+1)\beta \sum_{i=1}^{n} \frac{\left(e^{\lambda x_i^{-\eta}} - 1\right)^{\beta-1} x_i^{-\eta} e^{\lambda x_i^{-\eta}} \ln x_i}{\left[1 + \left(e^{\lambda x_i^{-\eta}} - 1\right)^{\beta}\right]}.$$

One can compute the maximum likelihood estimators (MLEs) of $\xi = (\alpha, \beta, \lambda, \eta)^T$ by solving the following equations $\frac{\partial \ell}{\partial \alpha} = 0$, $\frac{\partial \ell}{\partial \beta} = 0$, $\frac{\partial \ell}{\partial \lambda} = 0$ and $\frac{\partial \ell}{\partial \eta} = 0$ simultaneously, either directly or using quasi-Newton procedure, computer packages/softwares such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

5 Simulation Study

We evaluate the behavior of the MLEs of the BIII-IW parameters regarding the sample size n. We generate 10000 samples of sizes n =30, 50, 100, 200, 300, 500 from the inverse cdf of the BIII-IW distribution with true parameter settings $(\alpha, \beta, \lambda, \eta)$ =(0.1, 1.25, 0.75, 0.1), (0.25, 1.5, 1.1, 0.25), (0.5, 1.75, 1.50, 0.5) and (0.75, 2.0, 1.75, 0.75). We estimate the MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\eta})$ for 10000 samples from the non-linear optimization techniques. We also compute the means, biases and mean squared errors (MSE) of the MLEs. We infer from the simulation results (Table 1) that as the sample size n increases, the means approach to the true parameter value, the estimated MSE decrease, and estimated biases drop to zero. We observe that as the shape parameter increases, MSE of estimated parameters increases. Finally, we infer that the MLEs for the BIII-IW distribution are consistent.

Table 1: Mean, Bias and MSE of BIII-IW distribution

Sample	Statistics	$\alpha = 0.1$	$\beta = 1.25$	$\lambda = 0.75$	$\eta = 0.1$	$\alpha = 0.25$	$\beta = 1.5$	$\lambda = 1.1$	$\eta = 0.25$
30	Mean	0.1616	3.2743	1.1538	0.0778	0.2702	4.9168	1.8524	0.2966
	Bias	0.0616	2.0243	0.4038	-0.0222	0.0202	3.4168	0.7524	0.0466
	MSE	0.0549	22.3957	1.0981	8 E-04	0.1218	72.8266	3.6263	0.0135
50	Mean	0.1649	2.2468	1.1830	0.0758	0.2846	2.9445	1.7614	0.2799
	Bias	0.0649	0.9968	0.4330	-0.0242	0.0346	1.4445	0.6614	0.0299
	MSE	0.0505	6.7098	1.1651	8 E-04	0.1134	14.880	3.3008	0.0063
200	Mean	1.4220	1.0979	0.0745	0.2893	0.2893	1.7197	1.3438	0.2716
	Bias	0.0398	0.1720	0.3479	-0.0255	0.0393	0.2197	0.2438	0.0216
	MSE	0.0197	0.2796	0.6033	7 E-04	0.0643	0.3111	1.0288	0.0017
300	Mean	0.1319	1.3633	1.0789	0.0746	0.2821	1.6322	1.2172	0.2723
	Bias	0.0319	0.1133	0.3289	-0.0254	0.0321	0.1322	0.1172	0.0223
	MSE	0.0126	0.1452	0.5064	7 E-04	0.0442	0.1184	0.5211	0.0013
500	Mean	0.1273	1.3146	1.0364	0.0746	0.2695	1.5732	1.1383	0.2733
	Bias	0.0273	0.0646	0.2864	-0.0254	0.0195	0.0732	0.0383	0.0233
	MSE	0.0087	0.0611	0.3623	7 E-04	0.0245	0.0434	0.2682	0.0010
Sample	Statistics	$\alpha = 0.5$	$\beta = 1.75$	$\lambda = 1.5$	$\eta = 0.5$	$\alpha = 0.75$	$\beta = 2.0$	$\lambda = 1.75$	$\eta = 0.75$
30	Mean	0.4792	4.1390	2.2859	0.9614	1.0345	3.4565	2.0767	2.341
	Bias	-0.0208	2.3890	0.7859	0.4614	0.2845	1.4565	0.3267	1.591
	MSE	0.3300	58.812	7.0945	0.6397	4.8049	33.970	5.5384	8.557
50	Mean	0.4777	2.8230	2.0593	0.8383	0.9115	2.6572	1.9040	1.8318
	Bias	-0.0223	1.0730	0.5593	0.3383	0.1615	0.6572	0.1540	1.0818
	MSE	0.2705	11.248	4.7741	0.2486	2.6098	7.0496	3.5698	3.3994
100	Mean	0.5114	2.2001	1.6104	0.7673	0.7628	2.3556	1.6278	1.5034
	Bias	0.0114	0.4501	0.1104	0.2673	0.0128	0.3556	-0.1222	0.7534
	MSE	0.1799	1.4944	1.9890	0.1212	0.7491	1.7569	1.7727	1.0839
200	Mean	0.5250	1.9752	1.3225	0.7492	0.7366	2.2318	1.3774	1.3909
	Bias	0.0250	0.2252	-0.1775	0.2492	-0.0134	0.2318	-0.3726	0.6409
	MSE	0.1212	0.4107	0.8442	0.0909	0.2205	0.8524	0.8580	0.608
300	Mean	0.5204	1.8994	1.2061	0.7468	0.7460	2.1856	1.2524	1.3557
	Bias	0.0204	0.1494	-0.2939	0.2468	-0.0040	0.1856	-0.4976	0.6057
	MSE	0.0867	0.2259	0.4776	0.0822	0.1486	0.5962	0.6426	0.5145
500	Mean	0.5170	1.8369	1.1116	0.7468	0.7637	2.1559	1.1407	1.3246
	Bias	0.0170	0.0869	-0.3884	0.2468	0.0137	0.1559	-0.6093	0.5746
	MSE	0.0537	0.1071	0.3046	0.0744	0.1061	0.4183	0.5442	0.4419

6 COVID-19 Applications

We consider two applications with real datasets to assess the interest in the BIII-IW model. The considered data, called the COVID-19 datasets, are presented below. Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2) is the strain of Coronavirus that causes respiratory illness known as coronavirus disease 2019 (COVID-19). COVID-19 is recalled as the 'the flu of 2020'. SARS-CoV-2 is a positive-sense single stranded RNA virus (contagious in humans). Here, in the first data set, the daily new COVID-19 confirmed cases in Pakistan from 21 March to 29 May 2020 (inclusive) are analyzed (Bantan, et al. , 2020). The second data set presents the analysis of the daily new deaths due to COVID-19 in China from 23 January to 28 March 2020 (Eliwa, et al. , 2020). Table 2 reports some descriptive measures for two data sets.

Table 2: Descriptive Statistics of two datasets

Data Sets	n	Min	Max	Mean	Median	Standard deviation	Skewness	Kurtosis
Daily New Confirmed cases	70	89	2636	941.7571	767	743.526	0.6172	2.182
Daily New Deaths	66	3	150	49.7424	33	43.8730	0.8365	2.450

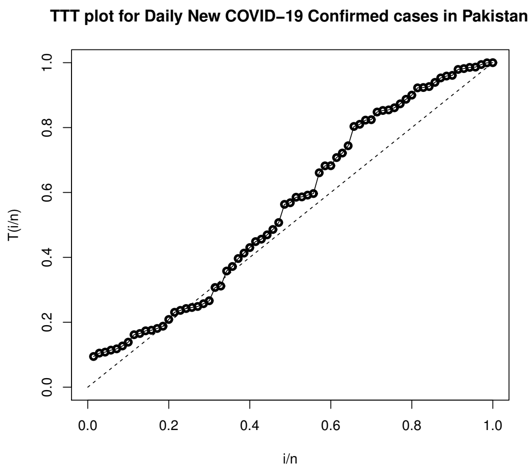
We compare the BIII-IW distribution with models such as BIII-IR, Burr XII inverse Weibull (BXII-IW), Burr XII inverse Rayleigh (BXII-IR) (Hafida and Haitham , 2019), odd Frechet inverse Weibull (OF-IW) (Fayomi , 2019), Kumaraswamy-Inverse Weibull (Kw-IW) (Shahbaz, et al. , 2012), W-IW, Weibull and inverse Weibull. For selection of the optimum distribution, we compute the estimate of goodness of statistics such as Cramer-von Mises W^*), Anderson Darling (A^*) and Kolmogorov- Smirnov (K-S) statistic with p-values and various model selection criteria such as the estimate of likelihood ratio statistic ($-2\hat{\ell}$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC) and Bayesian information criterion (BIC) for all competing and sub distributions. We compute the MLEs for the parameters and their standard errors (SEs) (in parentheses).

6.1 Daily New COVID-19 Confirmed Cases in Pakistan

The Total Time on Test (TTT) plot for daily new confirmed cases is first convex and then concave (Figure 3 [left]) which infers shaped modified tub shaped hazard rate. The boxplot for daily new confirmed cases is positively skewed (Figure 3 [right]). So, the BIII-IW distribution is suitable to model daily new confirmed cases data.

Table 3 reports the MLEs (SE) and measures W^*, A^*, K-S (p-values). Table 4 displays $-2\hat{\ell}$, AIC, CAIC and BIC.

From the Tables 3 and 4, it is clear that our proposed model is best fitted, with the smallest values for all statistics and maximum p-value.



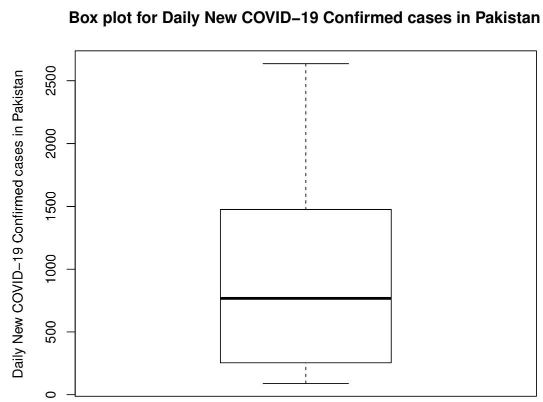


Figure 3: The Total Time on Test (TTT) plot (left), Boxplot for daily new COVID-19 confirmed cases in Pakistan (right).

Table 3: MLE (SE) and W^*, A^*, K-S (p-values) for daily new COVID-19 confirmed cases in Pakistan

Model	α	β	λ	η	W^*	A^*	K-S p-value
BIII-IW	0.0091 (0.0061)	45.5689 (33.4263)	221.63 (103.29)	0.7314 (0.0602)	0.0496	0.3390	0.0738 (0.8139)
BIII-IR	92.528 (517.01)	0.5135 (0.0473)	20.926 (234.19)	—	0.3859	2.3401	0.1376 (0.1286)
BXII-IW	32.155 (51.379)	2.5230 (2.6114)	7.7965 (12.2749)	0.2313 (0.1453)	0.1726	1.1537	0.1151 (0.2895)
BXII-IR	0.0030 (0.0001)	78.493 (18.875)	5416.41 (128.3)	—	0.5300	3.1422	0.2239 (0.0015)
KIW	89.1834 (54.506)	0.2853 (0.1560)	183.66 (2.5263)	3.5573 (1.7404)	0.3892	2.3574	0.1455 (0.0931)
WIW	0.0034 (0.0009)	0.6860 (0.2698)	5.6749 (3.7754)	1.4694 (0.4970)	0.1604	1.0863	0.1153 (0.2871)
OF-IW	—	0.0680 (0.0377)	10.840 (6.0189)	1.0360 (0.2307)	0.4003	2.4195	0.1404 (0.1147)
BIII	426.272 (218.155)	1.0224 (0.0914)	—	—	0.3875	2.3487	0.138 (0.1265)
IW	423.106 (218.146)	1.0215 (0.0919)	—	—	0.3880	2.351	0.1382 (0.1252)

Table 4: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for daily new COVID-19 confirmed cases in Pakistan

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BIII-IW	**1077.652**	**1085.652**	**1086.267**	**1094.646**	**1089.224**
BIII-IR	1111.445	1117.445	1117.809	1124.191	1120.125
BXIII-IW	1095.391	1103.391	1104.006	1112.385	1106.963
BXII-IR	1140.84	1146.84	1147.204	1153.585	1149.519
KIW	1114.757	1122.757	1123.372	1131.751	1126.329
WIW	1098.409	1106.409	1107.025	1115.403	1109.982
OF-IW	1112.548	1118.548	1118.912	1125.293	1121.227
BIII	1111.525	1115.525	1115.704	1120.022	1117.311
IW	1111.547	1115.547	1115.726	1120.044	1117.334

Figure 4 infers that the proposed model is closely fitted to daily new COVID-19 confirmed cases.

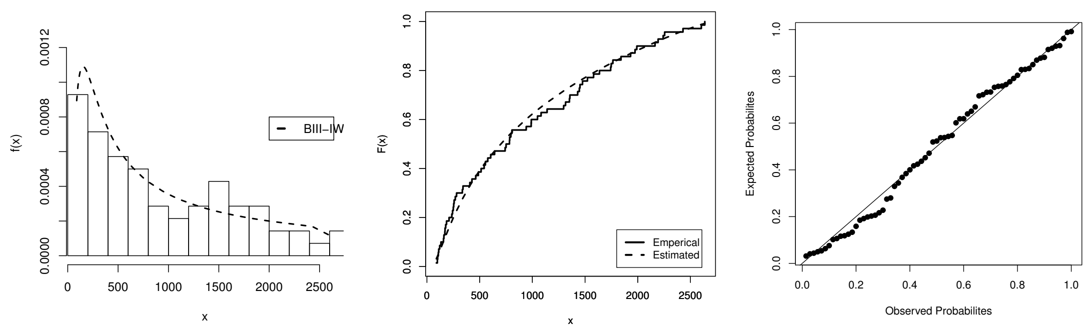


Figure 4: Fitted pdf (left), cdf (center) and P-P(right) plots of the BIII-IW distribution for daily new COVID-19 confirmed cases in Pakistan

6.2 Daily New COVID-19 Deaths in China

The TTT plot for daily new deaths data is first convex and then concave (Figure 5 [left]) which infers modified tub shaped hazard rate.

The boxplot for daily new deaths due to COVID-19 data is positively skewed

(Figure 5 [right]). So, the BIII-IW distribution is suitable to model these data.

Figure 6 infers that the proposed model is closely fitted to daily new deaths. Table 5 reports the MLEs (SE) and measures W^{*}, A^{*}, K-S (p-values).Table 6 displays $-2\hat{\ell}$, AIC, CAIC and BIC.

From the Tables 5 and 6, it is clear that our proposed model is best fitted, with the smallest values for all statistics and maximum p-value.

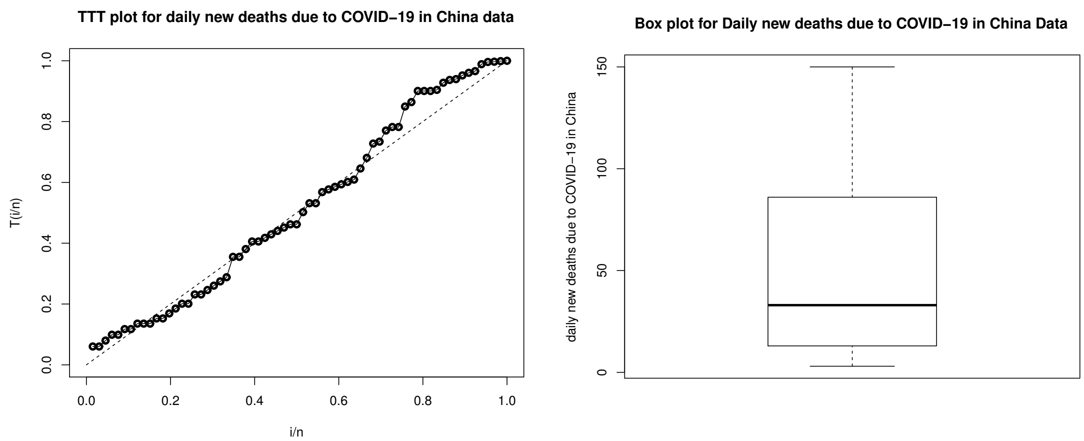


Figure 5: The Total Time on Test (TTT) plot (left), Boxplot for daily new deaths due to COVID-19 in China (right).

Table 5: MLE (SE) and W^{*}, A^{*}, K-S (p-values) for daily new COVID-19 confirmed cases in Pakistan

Model	α	β	λ	η	W^{*}	A^{*}	K-S p-value
BIII-IW	0.0057 (0.0023)	67.6042 (33.9034)	21.8514 (9.4536)	0.6870 (0.0858)	0.0239	0.1749	0.0683 (0.9176)
BIII-IR	5.0489 (2.8062)	0.4807 (0.0403)	12.6634 (14.2521)	—	0.2592	1.6224	0.1276 (0.2324)
BXII-IW	84.976 (250.66)	2.9006 (2.9932)	3.3697 (3.7244)	0.1722 (0.1074)	0.0905	0.7054	0.0914 (0.6402)
BXII-IR	0.4935 (0.1376)	0.6869 (0.1438)	69.5780 (23.8442)	—	0.3968	2.4200	0.196 (0.0126)
KIW	84.9999 (47.6766)	0.5911 (0.8534)	0.0406 (0.1846)	1.5365 (2.2344)	0.2859	1.7669	0.1348 (0.1814)
WIW	0.1212 (0.0675)	0.3714 (0.1894)	139.90 (286.04)	2.7208 (1.4777)	0.0547	0.4410	0.0876 (0.6914)
OF-IW	—	25.4543 (22.2713)	0.7462 (0.0480)	0.0260 (0.0226)	0.2900	1.7893	0.1298 (0.2159)
BIII	15.6815 (3.4122)	0.9489 (0.0810)	—	—	0.2695	1.6797	0.1258 (0.247)
IW	13.5316 (3.1093)	0.9159 (0.0837)	—	—	0.2843	1.7583	0.1284 (0.2263)

Table 6: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for daily new COVID-19 confirmed cases

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BIII-IW	**629.9200**	**637.9201**	**638.5758**	**646.6787**	**641.381**
BIII-IR	660.2998	666.2997	666.6868	672.8687	668.8954
BXIII-IW	646.6522	654.6523	655.308	663.4109	658.1132
BXII-IR	673.3036	679.3036	679.6906	685.8725	681.8993
KIW	665.4548	673.4548	674.1106	682.2134	676.9158
WIW	640.0214	648.0214	648.6771	656.7800	651.4823
OF-IW	662.657	668.6569	669.044	675.2259	671.2526
BIII	661.3306	665.3306	665.5211	669.7099	667.0611
IW	662.2032	666.2032	666.3936	670.5825	667.9336

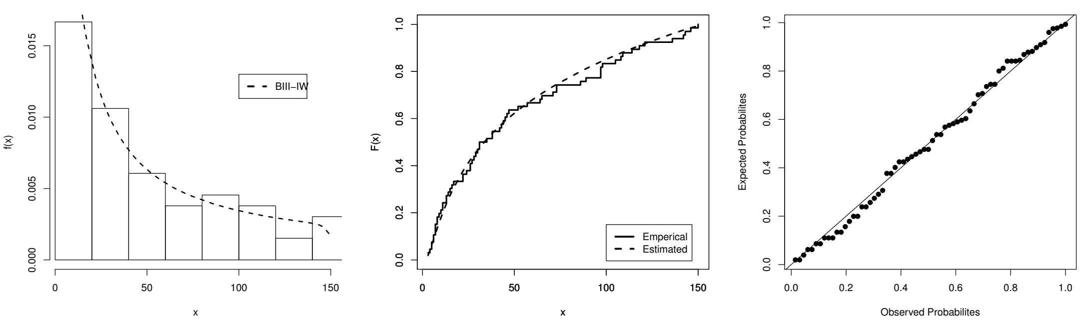


Figure 6: Fitted pdf (left), cdf (center) and P-P(right) plots of the BIII-IW distribution for daily new COVID-19 confirmed cases in China

7 Conclusion

We proposed a new probability distribution, called BIII-IW distribution, based on inverse Weibull and Burr III distribution via T-X family method. Its pdf and hrf shapes have very flexible forms. We studied certain mathematical properties such as random number generator, sub-models, ordinary moments, conditional moments, residual life functions, reliability measures and characterizations. We addressed the maximum likelihood estimation for the BIII-IW parameters and evaluated the precision of the maximum likelihood estimators via a simulation study. We considered applications to daily new COVID-19 confirmed cases in Pakistan and daily new COVID-19 deaths in China. We computed various model selection criteria for the BIII-IW distribution. The BIII-IW distribution is flexible, competitive and parsimonious with potential for wide ranging applications.

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