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Stochastic Restricted Two-Parameter Estimator in Linear Mixed Measurement Error Models

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Abstract. In this study, the stochastic restricted and unrestricted two-parameter estimators of fixed and random effects are investigated in the linear mixed measurement error models. For this purpose, the asymptotic properties and then the comparisons under the criterion of mean squared error matrix (*MSEM*) are derived. Furthermore, the proposed methods are used for estimating the biasing parameters. Finally, a real data analysis and a simulation study are provided to evaluate the performance of the proposed estimators

Keywords. Linear Mixed Measurement Error Model, Two Parameter Estimation, Stochastic Restricted Two Parameter Estimation, Mean-square Error Matrix.

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1 Introduction

Nowadays, linear mixed models (LMMs) are widely used in data analysis that is obtained by repeating measurements, such as longitudinal data. Also, these data can be used in many fields of physical, biological, medical and social sciences. Linear mixed models are generalized simple linear models that provide the possibility of random and fixed effects with each other. Moreover, independent variables are often measured with unavoidable errors in statistical models. Using the maximum likelihood (ML) method to estimate the parameters of fixed and random effects in LMMs, without considering the effect of measurement errors, leads to inconsistent estimates. For solving this problem, Zhong et al. (2002) the method of corrected score, introduced by Nakamura (1990), and obtained estimates of fixed and random effects parameters. Also, Zare et al. (2012) introduced the corrected score estimates of variance components in these models.

Here, we consider the linear mixed measurement error model as follows:

$$\begin{aligned} y &= Z\beta + Ub + \varepsilon, \\ X &= Z + L, \end{aligned} \tag{1.1}$$

where y is an $n \times 1$ vector of observations, $U = [U_1, ..., U_l]$ with U_i is an $n \times q_i$ design matrix corresponding to the *i-th* random effect factor, such that $q = \sum_{i=1}^{l} q_i$, Z is an $n \times p$ matrix of regressor of the fixed effects. β is a $p \times 1$ parameter vector of unknown fixed effects, $b = [b_1', ..., b_l']'$ is a $q \times 1$ unobservable vector of random effects from $N(0, \sigma^2\Sigma)$, where Σ is a block diagonal matrix with the *i-th* block being $\gamma_i I_{q_i}$ for $\gamma_i = {\sigma_i}^2/\sigma^2$, $i = 1, ..., l$. Besides, ε is an $n \times 1$ unobservable vector of random errors from $N(0, \sigma^2 I_n)$ and X is the matrix of observed value of Z with the measurement error L, where L is an $n \times p$ random matrix from $MN(0, I_n \otimes \Lambda)$. Also Λ is a $p \times p$ matrix of known values. b, ε and L are mutually independent. Under model (1.1), y has a multivariate normal distribution with $E(y) = Z\beta$ and $Var(y) = \sigma^2 V$ where $V = I_n + U\Sigma U'$. The conditional distribution of b/y is $N(\Sigma U'V^{-1}(y - Z\beta), \sigma^2\Sigma T)$, where $T = (I_q + U'U\Sigma)^{-1}$. The corrected score estimators of β, b, σ^2 and σ_i^2 are given by

$$\begin{aligned} \tilde{\beta} &= (X'V^{-1}X - tr(V^{-1})\Lambda)^{-1}X'V^{-1}y, \\ \tilde{b} &= \Sigma U'V^{-1}(y - X\tilde{\beta}), \\ \tilde{\sigma}^2 &= \frac{1}{n}[(y - X\tilde{\beta})'V^{-1}(y - X\tilde{\beta}) - tr(V^{-1})\tilde{\beta}'\Lambda\tilde{\beta}], \end{aligned}$$

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$$\tilde{\sigma}_i^2 = \frac{[\tilde{b}_i'\tilde{b}_i - tr(\tilde{D}_i'\tilde{D}_i)\tilde{\beta}'\Lambda\tilde{\beta}]}{q_i - tr(T_{ii})}, i = 1, ..., l,$$

where $\tilde{b}_i = \tilde{D}_i(y - X\tilde{\beta})$ and $\tilde{D}_i = \tilde{\gamma}_i U_i' V^{-1}$ is the ij-th block of matrix T (see Zhong et al. 2002 and Zare et al. 2012).

In linear regression, we usually assume that the matrix of explanatory variables is linearly independent. However, in practice, there may be strong or near to strong linear relationships among the variables, which causes the problem of multicollinearity. To reduce the effects of multicollinearity, Hoerl and Kennard (1970) and Liu (1993) proposed the ridge estimator and the Liu estimator, respectively. Ozkale and Kaçıranlar (2007) introduced the restricted and unrestricted two parameter estimators. Also, Yang and Chang (2010) obtained another two parameter estimator, "Using the mixed estimation technique introduced by Theil and Goldberger (1961) and Theil (1963). They considered the prior information about β in the restricted form of $(d - k)\hat{\beta}(k) = \beta + \varepsilon_0$, where k, d and $\hat{\beta}(k)$ are respectively the ridge and Liu parameters and the ridge estimator". In LMMs, authors such as Gilmour et al. (2004), Jiang (2007) and Searl et al. (1992) considered a state where the matrix $Z'V^{-1}Z$ is singular. Eliot et al. (2011) and Liu and Hu (2013) inquired the ridge prediction. Kuran and Ozkale (2016) obtained the mixed and stochastic restricted ridge predictors by using Gilmour approach. They introduced stochastic linear restriction as

$$r = R\beta + e, \tag{1.2}$$

where r is an $m \times 1$ vector, R is an m × p known matrix of rank $m \leq p$ and e is an $m \times 1$ random vector with $E(e) = 0$ and $Var(e) = \sigma^2 W$. Also, e and ε are assumed to be independent. Ozkale and Can (2017) proposed ridge estimation of fixed and random effects in the context of Henderson's mixed model equations. In the linear mixed measurement error models, Ghapani (2019) assumed that the vector of parameters is subject to the stochastic linear restrictions $r = R\beta + e$ and obtained the stochastic restricted estimator and the stochastic restricted Liu estimator. Also, Yavarizadeh et al. (2019) obtained the ridge estimator and the stochastic restricted ridge estimator, respectively, as

$$\tilde{\beta}(k) = (X'V^{-1}X + kI_p - tr(V^{-1})\Lambda)^{-1}X'V^{-1}y,$$
$$\tilde{\beta}_r(k) = (X'V^{-1}X + kI_p + R'W^{-1}R - tr(V^{-1})\Lambda)^{-1}(X'V^{-1}y + R'W^{-1}r).$$

Our primary aim in this article is to obtain a new stochastic restricted and unrestricted two-parameter estimators in linear measurement error mixed models. In Section 2, we

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follow Nakamura's approach in LMMs with the measurement error to obtain the two parameter estimator. In Section 3, by setting stochastic linear restrictions on the vector of fixed effects parameters, we derive the stochastic restricted two parameter estimation. The asymptotic properties of the proposed estimators are obtained in Section 4. In Section 5, under the mean square error matrix (*MSEM*) sense, we offer comparisons of new two parameter estimators. In Section 6, the proposed methods are used for estimating of the biasing parameters. The simulation results are discussed in Section 7 and a real data analysis is given in Section 8. Finally, summary and some conclusions are mentioned in Section 9.

2 The Two Parameter Estimator

In this section, we obtain a two parameter estimator in the LMM with the measurement error. For this purpose, we move the restriction $(d-k)\tilde{\beta}(k) = \beta + \varepsilon_0$ used by Yang and Ghang (2010) in linear regression to the LMMs with the measurement error and produce the two parameter estimator using "penalized term" idea. In restriction $(d-k)\tilde{\beta}(k) = \beta + \varepsilon_0$, $0 < d < 1$, $k > 0$ and $\varepsilon_0 \sim N(0, \sigma^2 I_p)$. So, the new model is

$$\begin{array}{l} y = Z\beta + Ub + \varepsilon; X = Z + L \qquad subject \quad to \\ (d-k)\tilde{\beta}(k) = \beta + \varepsilon_0. \end{array} \tag{2.1}$$

The log-likelihood for model (1.1) is given by

$$l(\theta; Z, y) = \frac{-1}{2\sigma^2}\{(y - Z\beta)' V^{-1}(y - Z\beta)\} - \frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\ln|V|,$$

where $\theta = (\beta, \sigma^2, \gamma)$. Following Ozkale and Kuran (2018), The penalized log-likelihood function for model (2.1) is define as

$$l_P(\theta, k, d; Z, y) = l(\theta; Z, y) - \frac{1}{2\sigma^2}((d-k)\tilde{\beta}(k) - \beta)'((d-k)\tilde{\beta}(k) - \beta).$$

The estimators obtained from $l_P(\theta, k, d; Z, y)$ are not consistent (see Zare et al. 2012 for more details), to correct this using Nakamura's approach (1990), we define the corrected penalized log-likelihood function for $l_P(\theta, k, d; Z, y)$ as

$$\begin{aligned} l_P^*(\theta, k, d; X, y) = \frac{-1}{2\sigma^2}\{&(y - X\beta)' V^{-1}(y - X\beta) \\ &+((d-k)\tilde{\beta}(k) - \beta)'((d-k)\tilde{\beta}(k) - \beta) - tr(V^{-1})\beta'\Lambda\beta\} \end{aligned}$$

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$$-\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\ln \mid V \mid .$$

Let E^* denotes the conditional mean with respect to X given y . The $l^*_P(\theta, k, d; X, y)$ satisfies

$$E^*[\frac{\partial}{\partial\beta} l^*_P(\theta, k, d; X, y)] = \frac{\partial}{\partial\beta} l_P(\theta, k, d; Z, y),$$
$$E^*[\frac{\partial}{\partial\sigma^2} l^*_P(\theta, k, d; X, y)] = \frac{\partial}{\partial\sigma^2} l_P(\theta, k, d; Z, y),$$
$$E^*[\frac{\partial}{\partial\gamma_i} l^*_P(\theta, k, d; X, y)] = \frac{\partial}{\partial\gamma_i} l_P(\theta, k, d; Z, y), i = 1, ..., l.$$

From $l^*_P(\theta, k, d; X, y)$, we get the partial derivative with respect to β and σ^2, then set the equations to zero and by using $\tilde{\beta}(k, d)$ and $\tilde{\sigma}^2(k, d)$ to denote the solutions give

$$(X'V^{-1}X + I_p)\tilde{\beta}(k, d) - tr(V^{-1})\Lambda\tilde{\beta}(k, d) = X'V^{-1}y + (d - k)\tilde{\beta}(k),$$
$$\frac{n}{\tilde{\sigma}^2(k, d)} = \frac{1}{\tilde{\sigma}^4(k, d)}\{(y - X\tilde{\beta}(k, d))'V^{-1}(y - X\tilde{\beta}(k, d)) - tr(V^{-1})\tilde{\beta}'(k, d)\Lambda\tilde{\beta}(k, d)$$
$$+((d - k)\tilde{\beta}(k) - \tilde{\beta}(k, d))'((d - k)\tilde{\beta}(k) - \tilde{\beta}(k, d))\}.$$

By solving these equations, we obtain the two parameter estimator of β and σ^2 as

$$\tilde{\beta}(k, d) = (X'V^{-1}X + I_p - tr(V^{-1})\Lambda)^{-1}(X'V^{-1}y + (d - k)\tilde{\beta}(k)), \tag{2.2}$$

$$\tilde{\sigma}^2(k, d) = \frac{1}{n}\{(y - X\tilde{\beta}(k, d))'V^{-1}(y - X\tilde{\beta}(k, d)) - tr(V^{-1})\tilde{\beta}'(k, d)\Lambda\tilde{\beta}(k, d)$$
$$+((d - k)\tilde{\beta}(k) - \tilde{\beta}(k, d))'((d - k)\tilde{\beta}(k) - \tilde{\beta}(k, d))\}. \tag{2.3}$$

By putting $\tilde{\beta}(k) = (X'V^{-1}X + kI_p - tr(V^{-1})\Lambda)^{-1}X'V^{-1}y$ in equation (2.2), $\tilde{\beta}(k, d)$ is obtained as

$$\tilde{\beta}(k, d) = H(k, d)(X'V^{-1}y),$$

where
$H(k, d) = (X'V^{-1}X + I_p - tr(V^{-1})\Lambda)^{-1}(X'V^{-1}X+dI_p-tr(V^{-1})\Lambda)\times(X'V^{-1}X + kI_p - tr(V^{-1})\Lambda)^{-1}$.
In addition, the two parameter predictor of b is given by

$$\tilde{b}(k, d) = \Sigma U'V^{-1}(y - X\tilde{\beta}(k, d)). \tag{2.4}$$

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If the elements of γ are unknown, their two parameter estimator, given by $\tilde{\gamma}_i(k,d)$, will be substituted back into Σ to obtain $\tilde{\beta}(k,d)$, $\tilde{\sigma}^2(k,d)$ and $\tilde{b}(k,d)$. For the two parameter estimator of $\tilde{\gamma}_i(k,d)$, we consider the two parameter estimator of $\tilde{\sigma}_i^2(k,d)$ as

$$\tilde{\sigma}_i^2(k,d) = \frac{[\tilde{b}_i'(k,d)\tilde{b}_i(k,d) - tr(\tilde{D}_i'(k,d)\tilde{D}_i(k,d))\tilde{\beta}'(k,d)\Lambda\tilde{\beta}(k,d)]}{q_i - tr(T_{ii})}, i = 1, ..., l,$$

where $\tilde{b}_i(k,d) = \tilde{D}_i(k,d)(y - X\tilde{\beta}(k,d))$, $\tilde{D}_i(k,d) = \tilde{\gamma}_i(k,d)U_i'V^{-1}$ and T_{ij} is the *ij-th* block of matrix T.

3 The Stochastic Restricted Two Parameter Estimator

In this section, we obtain the stochastic restricted two parameter estimation. By unifying the model (1.1) with stochastic linear restrictions model (1.2), we have

$$\begin{bmatrix} y \\ r \end{bmatrix} = \begin{bmatrix} Z \\ R \end{bmatrix}\beta + \begin{bmatrix} U \\ 0 \end{bmatrix} b + \begin{bmatrix} \varepsilon \\ e \end{bmatrix},$$

or

$$y_r = Z_r\beta + U_r b + \varepsilon_r. \tag{3.1}$$

Also b and y_r are jointly distributed as

$$\begin{bmatrix} b \\ y_r \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ Z_r\beta \end{bmatrix}, \sigma^2 \begin{bmatrix} \Sigma & \Sigma U'_r \\ U_r\Sigma & V_r \end{bmatrix}\right),$$

where $V_r = \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix}$. Therefore, the conditional distribution of b/y_r is $N(\Sigma U_r' V_r^{-1}(y_r - Z_r\beta), \sigma^2\Sigma T_r)$, where $T_r = (I_q + U_r'U_r\Sigma)^{-1}$, and the log-likelihood of model (3.1) is given by

$$l(\theta; Z, y, r) = \frac{-1}{2\sigma^2}\{(y - Z\beta)'V^{-1}(y - Z\beta) + (r - R\beta)'W^{-1}(r - R\beta)\}$$
$$-\frac{n+m}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\ln|V| - \frac{1}{2}\ln|W|.$$

The penalized log-likelihood function for model (3.1) with $(d-k)\tilde{\beta}_r(k) = \beta + \varepsilon_0$ is define as follows:

$$l_P(\theta, k, d; Z, y, r) = l(\theta; Z, y, r) - \frac{1}{2\sigma^2}((d-k)\tilde{\beta}_r(k) - \beta)'((d-k)\tilde{\beta}_r(k) - \beta).$$

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Due to better performance in the results of the simulation study, we used the restricted ridge estimator instead of the unrestricted one. The corrected function for $l_P(\theta, k, d; Z, y, r)$ is

$$l_P^*(\theta, k, d; X, y, r) = \frac{-1}{2\sigma^2}\{(y - X\beta)'V^{-1}(y - X\beta)$$
$$+((d-k)\tilde{\beta}_r(k) - \beta)'((d-k)\tilde{\beta}_r(k) - \beta) + (r - R\beta)'W^{-1}(r - R\beta) - tr(V^{-1})\beta'\Lambda\beta\}$$
$$-\frac{n+m}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\ln|V| - \frac{1}{2}\ln|W|.$$

If W and V are known, by solving the equations $\partial l_p^*(\theta, k, d; X, y, r)/\partial\beta = 0$ and $\partial l_p^*(\theta, k, d; X, y, r)/\partial\sigma^2 = 0$, we have

$$(X'V^{-1}X + R'W^{-1}R + I_p - tr(V^{-1})\Lambda)\tilde{\beta}_r(k,d) = X'V^{-1}y + (d-k)\tilde{\beta}_r(k) + R'W^{-1}r,$$
$$\frac{(n+m)}{\tilde{\sigma}_r^2(k,d)} = \frac{1}{\tilde{\sigma}_r^4(k,d)}\{(y - X\tilde{\beta}_r(k,d))'V^{-1}(y - X\tilde{\beta}_r(k,d)) - tr(V^{-1})\tilde{\beta}_r'(k,d)\Lambda\tilde{\beta}_r(k,d)$$
$$+(r - R\tilde{\beta}_r(k,d))'W^{-1}(r - R\tilde{\beta}_r(k,d)) + ((d-k)\tilde{\beta}_r(k) - \tilde{\beta}_r(k,d))'((d-k)\tilde{\beta}_r(k) - \tilde{\beta}_r(k,d))\}.$$

So the stochastic restricted two parameter estimator of β and σ^2 will be

$$\tilde{\beta}_r(k,d) = (X'V^{-1}X + R'W^{-1}R + I_p - tr(V^{-1})\Lambda)^{-1}$$
$$\times(X'V^{-1}y + (d-k)\tilde{\beta}_r(k) + R'W^{-1}r), \tag{3.2}$$
$$\tilde{\sigma}_r^2(k,d) = \frac{1}{n+m}\{(y - X\tilde{\beta}_r(k,d))'V^{-1}(y - X\tilde{\beta}_r(k,d))$$
$$-tr(V^{-1})\tilde{\beta}_r'(k,d)\Lambda\tilde{\beta}_r(k,d) + (r - R\tilde{\beta}_r(k,d))'W^{-1}(r - R\tilde{\beta}_r(k,d))$$
$$+((d-k)\tilde{\beta}_r(k) - \tilde{\beta}_r(k,d))'((d-k)\tilde{\beta}_r(k) - \tilde{\beta}_r(k,d))\}.$$

If we put $\tilde{\beta}_r(k) = (X'V^{-1}X + kI_p + R'W^{-1}R - tr(V^{-1})\Lambda)^{-1}(X'V^{-1}y + R'W^{-1}r)$ in equation (3.2), the stochastic restricted two parameter estimator of β is given by

$$\tilde{\beta}_r(k,d) = H_R(k,d)(X'V^{-1}y + R'W^{-1}r), \tag{3.3}$$

where $H_R(k,d) = (X'V^{-1}X + R'W^{-1}R + I_p - tr(V^{-1})\Lambda)^{-1}(X'V^{-1}X + R'W^{-1}R + dI_p - tr(V^{-1})\Lambda)(X'V^{-1}X + R'W^{-1}R + kI_p - tr(V^{-1})\Lambda)^{-1}$.

Finally, the stochastic restricted two parameter predictor of b is given by

$$\tilde{b}_r(k,d) = \Sigma U'V^{-1}(y - X\tilde{\beta}_r(k,d)).$$

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If the elements of γ are unknown, their stochastic restricted two parameter estimator, denoted by $\tilde{\gamma}_{ri}(k,d)$, will be substituted back into Σ to obtain $\tilde{\beta}_r(k,d)$, $\tilde{\sigma}^2_r(k,d)$ and $\tilde{b}_r(k,d)$. For the stochastic restricted two parameter estimator of $\tilde{\gamma}_{ri}(k,d)$, we consider the stochastic restricted two parameter estimator of $\tilde{\sigma}^2_{ri}(k,d)$ as

$$\tilde{\sigma}^2_{ri}(k,d) = \frac{[\tilde{b}'_{ri}(k,d)\tilde{b}_{ri}(k,d) - tr(\tilde{D}'_{ri}(k,d)\tilde{D}_{ri}(k,d))\tilde{\beta}'_r(k,d)\Lambda\tilde{\beta}_r(k,d)]}{q_i - tr(T_{ii})}, i = 1, ..., l,$$

where $\tilde{b}_{ri}(k,d) = \tilde{D}_{ri}(k,d)(y - X\tilde{\beta}_r(k,d))$, $\tilde{D}_{ri}(k,d) = \tilde{\gamma}_{ri}(k,d)U'_i V^{-1}$ and T_{ij} is the *ij-th* block of matrix T.

4 Asymptotic Properties of Fixed Effect Estimators

In this section, using the large sample asymptotic approximation theory, we study the asymptotic properties of the proposed estimators. We assume the parameter β is identifiable. It is also assumed that as n tends to infinity, the limits of $n^{-1}tr(V^{-1})$, $n^{-1}(Z'V^{-1}Z + R'W^{-1}R + kI_p)$, $n^{-1}(Z'V^{-1}Z + R'W^{-1}R + dI_p)$ and $n^{-1}(Z'V^{-1}Z + R'W^{-1}R + I_p)$ exist.

Theorem 1. *$\tilde{\beta}_r(k,d)$ has an asymptotic normal distribution with mean vector $E(\tilde{\beta}_r(k,d)) \cong M_R(k,d)M_R^{-1}\beta$ and covariance matrix $AVar(\tilde{\beta}_r(k,d)) \cong M_R(k,d)(B + \sigma^2 M_R^{-1}) \times M_R(k,d)$, where $M_R = (Z'V^{-1}Z + R'W^{-1}R)^{-1}$, $M_R(k,d) = (Z'V^{-1}Z + R'W^{-1}R + I_p)^{-1}(Z'V^{-1}Z + R'W^{-1}R + dI_p)(Z'V^{-1}Z + R'W^{-1}R + kI_p)^{-1}$ and $B = (\beta' Z'V^{-2}Z\beta + \sigma^2 tr(V^{-1}))\Lambda$.*

Proof. Since $E(X'V^{-1}X) = Z'V^{-1}Z + tr(V^{-1})\Lambda$, by Fung et al. (2003), we have $X'V^{-1}X = Z'V^{-1}Z + tr(V^{-1})\Lambda + O_p(n^{1/2})$, so $\tilde{\beta}_r(k,d)$ can be written as follows

$$\begin{aligned}
\tilde{\beta}_r(k,d) &= (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + I_p) + O_p(n^{-1/2}))^{-1} \\
&\times (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + dI_p) + O_p(n^{-1/2})) \\
&\times (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + kI_p) + O_p(n^{-1/2}))^{-1} n^{-1}(X'V^{-1}y + R'W^{-1}r) \\
&= (I_p + O_p(n^{-1/2}))^{-1}(n^{-1}(Z'V^{-1}Z + R'W^{-1}R + I_p))^{-1}(I_p + O_p(n^{-1/2})) \\
&\times (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + dI_p))(I_p + O_p(n^{-1/2}))^{-1} \\
&\times (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + kI_p))^{-1} n^{-1}(X'V^{-1}y + R'W^{-1}r).
\end{aligned}$$

Using Taylor series expansion, we have $(I_p + O_p(n^{-1/2}))^{-1} = (I_p + O_p(n^{-1/2}))$, so

$$\begin{aligned}
\sqrt{n}\tilde{\beta}_r(k,d) &\cong (I_p + O_p(n^{-1/2}))F_d(n^{-1}(Z'V^{-1}Z + R'W^{-1}R + kI_p))^{-1} \\
&\times n^{-1/2}(X'V^{-1}y + R'W^{-1}r),
\end{aligned}$$

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where $F_d = (Z'V^{-1}Z + R'W^{-1}R + I_p)^{-1}(Z'V^{-1}Z + R'W^{-1}R + dI_p)$ .The limit of $C^{-1} = (n^{-1}(Z'V^{-1}Z + R'W^{-1}R + kI_p))^{-1}$ and F_dC^{-1} exist, therefore $\sqrt{n}\tilde{\beta}_r(k,d)$ can be written as

$$\sqrt{n}\tilde{\beta}_r(k,d) \cong F_dC^{-1}h + O_p(n^{-1/2}),$$

where $h = n^{-1/2}(X'V^{-1}y + R'W^{-1}r)$. Since $E(h) = n^{-1/2}M_R^{-1}\beta$, so we have $\sqrt{n}(\tilde{\beta}_r(k,d) - M_R(k,d)M_R^{-1}\beta) \cong F_dC^{-1}(h - E(h)) + O_p(n^{-1/2})$, which indicates that $\sqrt{n}(\tilde{\beta}_r(k,d) - \beta)$ is asymptotically normal with $E(\sqrt{n}(\tilde{\beta}_r(k,d) - \beta)) \cong 0$. Therefore, $AVar(\sqrt{n}\tilde{\beta}_r(k,d)) \cong F_dC^{-1}Var(h)C^{-1}F_d$ and the variance of h can be obtained by

$$\begin{aligned} Var(h) &= E_{y_r}[Var(h/y_r)] + Var_{y_r}[E(h/y_r)] \\ &= n^{-1}E_{y_r}(y'V^{-2}y)\Lambda + n^{-1}Var_{y_r}(Z'V^{-1}y + R'W^{-1}r), \end{aligned}$$

where E_{y_r} and Var_{y_r} denote the expectation and variance with respect to the random vector $y_r' = (y', r')$, respectively. Since $E_{y_r}(y'V^{-2}y) = \beta'Z'V^{-2}Z\beta + \sigma^2 tr(V^{-1})$ and $Var_{y_r}(Z'V^{-1}y + R'W^{-1}r) = \sigma^2 M_R^{-1}$, therefore $Var(h) = n^{-1}(B + \sigma^2 M_R^{-1})$, whose limit exists as n tends to infinity. Thus, $AVar(\tilde{\beta}_r(k,d)) \cong M_R(k,d) \times (B + \sigma^2 M_R^{-1})M_R(k,d)$ and this theorem is proved.

Corollary 1. *$\tilde{\beta}(k,d)$ has an asymptotic normal distribution with mean vector $E(\tilde{\beta}(k,d)) \cong M(k,d)M^{-1}\beta$ and covariance matrix $AVar(\tilde{\beta}(k,d)) = M(k,d) \times (B + \sigma^2 M^{-1})M(k,d)$, where $M = (Z'V^{-1}Z)^{-1}$ and $M(k,d) = (Z'V^{-1}Z + I_p)^{-1} \times (Z'V^{-1}Z + dI_p)(Z'V^{-1}Z + kI_p)^{-1}$.*

Corollary 2. *$\tilde{\beta}$ has an asymptotic normal distribution with mean vector β and covariance matrix $AVar(\tilde{\beta}) = M(B + \sigma^2 M^{-1})M$, where $M = (Z'V^{-1}Z)^{-1}$.*

Corollary 3. *$\tilde{\beta}_r$ has an asymptotic normal distribution with mean vector β and covariance matrix $AVar(\tilde{\beta}_r) = M_R(B + \sigma^2 M_R^{-1})M_R$, where $M_R = (Z'V^{-1}Z + R'W^{-1}R)^{-1}$.*

5 Comparison of Estimators

In this section, we compare the estimator $\tilde{\beta}(k,d)$ with $\tilde{\beta}$, the estimator $\tilde{\beta}_r(k,d)$ with $\tilde{\beta}_r$, and the estimator $\tilde{\beta}_r(k,d)$ with $\tilde{\beta}(k,d)$ using the asymptotic mean squares error matrix ($AMSEM$). The mean-square error matrix for the estimators $\tilde{\beta}(k,d)$ is given by

$$AMSEM(\tilde{\beta}(k,d), \beta) = AVar(\tilde{\beta}(k,d)) + bias(\tilde{\beta}(k,d))bias(\tilde{\beta}(k,d))',$$

where $bias(\tilde{\beta}(k,d)) = E(\tilde{\beta}(k,d) - \beta) = (M(k,d)M^{-1} - I_p)\beta$.

Then, $AMSEM(\tilde{\beta}(k,d), \beta) = M(k,d)(B + \sigma^2 M^{-1})M(k,d) + (M(k,d)M^{-1} - I_p)\beta\beta'(M(k,d)M^{-1} - I_p)'$ (Ghapani and Babadi 2020). Also, we obtain the asymptotic mean square error

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matrices of $\tilde{\beta}_r(k,d)$, $\tilde{\beta}_r$ and $\tilde{\beta}$ as follows

$$AMSEM(\tilde{\beta}_r(k,d),\beta) = M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d) \\ +(M_R(k,d)M_R^{-1} - I_p)\beta\beta'(M_R(k,d)M_R^{-1} - I_p)',$$

$$AMSEM(\tilde{\beta}_r,\beta) = M_R(B+\sigma^2 M_R^{-1})M_R,$$

$$AMSEM(\tilde{\beta},\beta) = M(B+\sigma^2 M^{-1})M.$$

According to Rao and Toutenburg (1995), if two estimators $\tilde{\beta}_1$ and $\tilde{\beta}_2$ of β are given, the estimator $\tilde{\beta}_2$ is superior to $\tilde{\beta}_1$ with respect to the *MSEM* sense, if and only if $\Delta(\tilde{\beta}_1,\tilde{\beta}_2) = MSEM(\tilde{\beta}_1) - MSEM(\tilde{\beta}_2) > 0$, that is, $\Delta(\tilde{\beta}_1,\tilde{\beta}_2)$ is a positive definite (pd) matrix.

5.1 Comparison of $\tilde{\beta}(k,d)$ with $\tilde{\beta}$

In order to compare $\tilde{\beta}(k,d)$ with $\tilde{\beta}$ in the *MSEM* sense, we consider the asymptotic *MSEM* difference as

$$\Delta_1 = AMSEM(\tilde{\beta},\beta) - AMSEM(\tilde{\beta}(k,d),\beta) = D_1 - b_1 b_1',$$

where $D_1 = M(B+\sigma^2 M^{-1})M - M(k,d)(B+\sigma^2 M^{-1})M(k,d)$ and $b_1 = (M(k,d)M^{-1} - I_p)\beta$. Thus, we have the following theorem.

Theorem 2. *For $0 < d < 1$ and $k > 0$, the estimator $\tilde{\beta}(k,d)$ is superior to the estimator $\tilde{\beta}$ under the MSEM sense, if and only if $b_1' {D_1}^{-1} b_1 \leq 1$.*

***Proof*.** According to Farebrother (1976), if $D_1 > 0$, then the necessary and sufficient condition for $\tilde{\beta}(k,d)$ to be superior to $\tilde{\beta}$ is $b_1' {D_1}^{-1} b_1 \leq 1$. So we must prove D_1 is a pd matrix. We can write D_1 as

$$D_1 = MBM - M(k,d)BM(k,d) + \sigma^2[M - M(k,d)M^{-1}M(k,d)].$$

Since $Z'V^{-1}Z$ is a pd matrix, then M, $M(k,d)$ and B are pd matrices. By Theorem A.52 of Rao et al.(2008) and Theorem 2 of Ghapani (2019), $MBM - M(k,d)BM(k,d)$ is always a pd matrix. Besides, we can write $\sigma^2[M - M(k,d)M^{-1}M(k,d)]$ as

$$\sigma^2[M - M(k,d)M^{-1}M(k,d)] = \sigma^2 S\{2(k+1-d)M^{-1}M^{-1} + \\ +(4k+k^2+1-d^2)M^{-1} + k^2 M + 2kI_p + 2k^2 I_p\}S,$$

where $S = (M^{-1}+I_p)^{-1}(M^{-1}+kI_p)^{-1}$. Since $M^{-1} = Z'V^{-1}Z$ is a pd matrix and $(Z'V^{-1}Z)'(Z'V^{-1}Z) = (Z'V^{-1}Z)(Z'V^{-1}Z)$ then, $M^{-1}M^{-1}$ and S are pd matrices. For $0 < d < 1$ and $k > 0$, we can write $(k-d+1) > 0$ and $(4k+k^2+1-d^2) > 0$ so, $\sigma^2[M - M(k,d)M^{-1}M(k,d)]$ is a pd matrix. Consequently, D_1 is a pd and the proof is completed.

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5.2 Comparison of $\tilde{\beta}_r(k,d)$ with $\tilde{\beta}_r$

The asymptotic *MSEM* difference between $\tilde{\beta}_r(k,d)$ and $\tilde{\beta}_r$ is

$$\Delta_2 = AMSEM(\tilde{\beta}_r,\beta) - AMSEM(\tilde{\beta}_r(k,d),\beta) = D_2 - b_2 b_2',$$

where $D_2 = M_R(B+\sigma^2 M_R^{-1})M_R - M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d)$ and $b_2 = (M_R(k,d)M_R^{-1} - I_p)\beta$. The comparison between $\tilde{\beta}_r(k,d)$ and $\tilde{\beta}_r$ is stated in the following theorem .

Theorem 3. *For $0 < d < 1$ and $k > 0$, the estimator $\tilde{\beta}_r(k,d)$ is superior to the estimator $\tilde{\beta}_r$ under the MSEM sense, if and only if $b_2' D_2{}^{-1} b_2 \leq 1$.*

***Proof*.** According to Farebrother (1976), if $D_2 > 0$, then the necessary and sufficient condition for Δ_2 to be pd is $b_2' D_2{}^{-1} b_2 \leq 1$. So, it must be proved that D_2 is pd matrix. We note that, since $Z'V^{-1}Z$ and $R'W^{-1}R$ are pd matrices, then $M_R^{-1} = Z'V^{-1}Z + R'W^{-1}R$, M_R and $M_R(k,d)$ are pd matrices, too. Also, $M_R B M_R - M_R(k,d) B M_R(k,d)$ is always a pd matrix. We can write

$$\sigma^2[M_R - M_R(k,d)M_R^{-1}M_R(k,d)] = \sigma^2 S_R\{2(k+1-d)M_R^{-1}M_R^{-1}$$
$$+(4k+k^2+1-d^2)M_R^{-1} + k^2 M_R + 2kI_p + 2k^2 I_p\}S_R,$$

where $S_R = (M_R^{-1} + I_p)^{-1}(M_R^{-1} + kI_p)^{-1}$. Because M_R^{-1} is a pd and $(M_R^{-1})' M_R^{-1} = M_R^{-1} M_R^{-1}$ then, $M_R^{-1} M_R^{-1}$ and S_R are pd matrices. Therefore, $\sigma^2[M_R - M_R(k,d)M_R^{-1} \times M_R(k,d)]$ and D_2 are pd.

5.3 Comparison of $\tilde{\beta}_r(k,d)$ with $\tilde{\beta}(k,d)$

The asymptotic *MSEM* difference between $\tilde{\beta}(k,d)$ and $\tilde{\beta}_r(k,d)$ is equal to

$$\Delta_3 = AMSEM(\tilde{\beta}(k,d),\beta) - AMSEM(\tilde{\beta}_r(k,d),\beta) =$$
$$M(k,d)(B+\sigma^2 M^{-1})M(k,d) - M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d) + b_1 b'_1 - b_2 b'_2.$$

Theorem 4. *If the maximum eigenvalue of $M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d)$ $\times[M(k,d)(B+\sigma^2 M^{-1})M(k,d)]^{-1}$ is less than 1, the estimator $\tilde{\beta}_r(k,d)$ is superior to the estimator $\tilde{\beta}(k,d)$ in the MSEM sense, if and only if*

$$b_2'[M(k,d)(B+\sigma^2 M^{-1})M(k,d) - M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d) + b_1 b_1']^{-1} b_2 \leq 1.$$

***Proof*.** To show that $\Delta_3 \geq 0$, a requirement is that $M(k,d)(B+\sigma^2 M^{-1})M(k,d) - M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d)$ be a pd matrix. Because $M(k,d)(B+\sigma^2 M^{-1}) \times M(k,d) > 0$ and $M_R(k,d)(B+\sigma^2 M_R^{-1})M_R(k,d) > 0$, based on Lemma 2.1 of Güler and Kaçıranlar (2009), $M(k,d)(B+$

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$\sigma^2 M^{-1})M(k,d) > M_R(k,d)(B + \sigma^2 M_R^{-1})M_R(k,d)$ if and only if $\lambda < 1$, where λ is the maximum eigenvalue of $M_R(k,d)(B + \sigma^2 M_R^{-1})M_R(k,d)[M(k,d)(B + \sigma^2 M^{-1})M(k,d)]^{-1}$. So, $M(k,d) \times (B + \sigma^2 M^{-1})M(k,d) - M_R(k,d)(B + \sigma^2 M_R^{-1})M_R(k,d)$ is pd if and only if $\lambda < 1$, and, using Theorem 2 of Trenkler and Toutenburg (1990), Δ_3 is a pd matrix if and only if

$$b_2'[M(k,d)(B + \sigma^2 M^{-1})M(k,d) - M_R(k,d)(B + \sigma^2 M_R^{-1})M_R(k,d) + b_1 b_1']^{-1} b_2 \leq 1.$$

6 Selection of Parameters *k* and *d*

In this section, based on the *MSEM* sense, we propose methods for estimating the biasing parameters k and d. In the linear mixed measurement error models, Yavarizadeh et al. (2019) and Ghapani (2019) used the *MSEM* difference to estimate k and d in the Ridge and Liu estimators, respectively. In addition, since $MSE(\tilde{\beta}(k,d)) = tr[MSEM(\tilde{\beta}(k,d))]$, non-diameter elements of the matrix are ignored and for this reason we did not use the *MSE* minimization method, which is a common method. We obtain the parameters k and d so that Δ_1 is the maximum possible of a pd matrix. Consider Δ_1 as follows

$$\begin{aligned}\Delta_1 &= MBM - M(k,d)BM(k,d) + \sigma^2 M \\ &-[\sigma^2 M(k,d)M^{-1}M(k,d) + (M(k,d)M^{-1} - I_p)\beta\beta'(M(k,d)M^{-1} - I_p)].\end{aligned}$$

In Δ_1, the $MBM - M(k,d)BM(k,d) + \sigma^2 M$ is always a pd matrix, therefore if we get k and d so that $\sigma^2 M(k,d)M^{-1}M(k,d) + (M(k,d)M^{-1} - I_p)\beta\beta'(M(k,d)M^{-1} - I_p)$ takes the minimum values then Δ_1 is most likely to be a pd matrix. Let $\alpha = P'\beta$, so we can write $\sigma^2 M(k,d)M^{-1}M(k,d) + (M(k,d)M^{-1} - I_p)\beta\beta'(M(k,d)M^{-1} - I_p) = PTP' = Pdiag(\tau_1, ..., \tau_p)P'$, where $\tau_i = \frac{\sigma^2\lambda_i(\lambda_i+d)^2}{(\lambda_i+1)^2(\lambda_i+k)^2} + \frac{((k+1-d)\lambda_i+k)^2\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+k)^2}$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p > 0$ are the ordered eigenvalues of $Z'V^{-1}Z$ (Ghapani 2019). Let d be fixed, and we get the estimate k by minimizing τ_i. Therefore, by settin $\frac{\partial}{\partial k}\tau_i = 0$, we have

$$k = \frac{\sigma^2(\lambda_i + d) - \lambda_i\alpha_i^2(1 - d)}{\tilde{\alpha}_i^2(\lambda_i + 1)}, \tag{6.1}$$

since in equation (6.1), k depends on unknown α and σ^2, we can get the estimate of k by substituting $\tilde{\alpha}$ and $\tilde{\sigma}^2$ as follows:

$$\tilde{k} = \frac{\tilde{\sigma}^2(\lambda_i + d) - \lambda_i\tilde{\alpha}_i^2(1 - d)}{\tilde{\alpha}_i^2(\lambda_i + 1)}. \tag{6.2}$$

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Based on the estimator of k, proposed by Kibria (2003) and Hoerl and Kennard (1970), the harmonic mean value of k in (6.2) is

$$\tilde{k}_{HM} = \frac{p}{\sum_{i=1}^{p} \frac{\tilde{\alpha}_i^2(\lambda_i+1)}{\tilde{\sigma}^2(\lambda_i+d)-\lambda_i\tilde{\alpha}_i^2(1-d)}}. \tag{6.3}$$

Following Ozkale and Kaçıranlar (2007), we now get the estimate of d according to equation (6.1) so that the estimate of k is always positive. In equation (6.1), if $\frac{\sigma^2(\lambda_i+d)-\lambda_i\alpha_i^2(1-d)}{\alpha_i^2(\lambda_i+1)} > 0$ then the values of k are positive. Since $\alpha_i^2(\lambda_i + 1) > 0$, $\sigma^2(\lambda_i + d) - \lambda_i\alpha_i^2(1 - d)$ must be positive for all i. Then we get $d > \frac{1-\sigma^2/\alpha_i^2}{1+\sigma^2/(\lambda_i\alpha_i^2)}$ and, because this lower bound depends on the unknown parameters α_i^2 and σ^2, so $\tilde{\alpha}_i$ and $\tilde{\sigma}^2$ are replaced. Therefore, $\tilde{k}_{HM}$ is always positive if $\tilde{d}$ is selected as $\tilde{d} > \max\left\{\frac{1-\tilde{\sigma}^2/\tilde{\alpha}_i^2}{1+\tilde{\sigma}^2/(\lambda_i\tilde{\alpha}_i^2)}\right\}$. Note that $\frac{1-\tilde{\sigma}^2/\tilde{\alpha}_i^2}{1+\tilde{\sigma}^2/(\lambda_i\tilde{\alpha}_i^2)}$ is always less than one and since d must be between zero and one, we consider

$$\max\left\{\frac{1-\tilde{\sigma}^2/\tilde{\alpha}_i^2}{1+\tilde{\sigma}^2/(\lambda_i\tilde{\alpha}_i^2)}, o\right\} < \tilde{d} < 1. \tag{6.4}$$

In practice we obtain $\tilde{d}$ first and then $\tilde{k}_{HM}$. Also, unknown parameters are replaced with appropriate estimates.

7 A Simulation Study

We perform a simulation study in order to inquire the performance of $\tilde{\beta}$, $\tilde{\beta}(k, d)$, $\tilde{\beta}_r$ and $\tilde{\beta}_r(k, d)$. For this purpose, we calculate the estimated mean square error (*EMSE*) with various values of sample size, variance and degree of collinearity. Following McDonald and Galarneau (1975), we generate the explanatory variables as

$$z_{ijt} = (1-\rho^2)^{1/2}w_{ijt} + \rho w_{ij,p+1}, i = 1, ..., l, j = 1, ..., n_i, t = 1, ..., p, \tag{7.1}$$

where w_{ijt} are independent standard normal pseudo-random numbers and ρ^2 is the correlation between any two fixed effects. The value of ρ^2 is set to be 0.75, 0.85 and 0.95. We generate the *j-th* set of simulated data as

$$y_j = Z\beta + Ub_j + \varepsilon_j, X_j = Z + L_j,$$
$$r_j = R\beta + e_j, j = 1, ..., 1000,$$

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where $y_j = (y_{11j}, ..., y_{1n_1j}, y_{21j}, ..., y_{2n_2j}, ..., y_{l1j}, ..., y_{ln_lj})$, $b_j = (b_{1j}, b_{2j}, ..., b_{lj})'$, $Z = (z^{(1)}, ..., z^{(p)})$ and $z^{(t)} = (z^{(t)}_{11}, ..., z^{(t)}_{1n_1}, z^{(t)}_{21}, ..., z^{(t)}_{2n_2}, ..., z^{(t)}_{l1}, ..., z^{(t)}_{ln_l})'$, $t = 1, ..., p$, $U = 1_{n_i} \oplus 1_{n_i} \oplus ... \oplus 1_{n_i}$ is a $n \times q$ matrix and 1_{n_i} is a $n_i \times 1$ vector with all elements 1. In this dataset, $n = \sum_{i=1}^{l} n_i$ is the total size and l and n_i are the number of independent groups and the group size, respectively. The random error ε_j and the random effects b_j are generated from the normal distribution $N(0, \sigma^2 I_{n_i})$ and $N(0, \sigma_1{}^2 I_q)$, respectively. In addition, $r_j = (r_{1j}, r_{2j}, ..., r_{mj})'$, $R = (R^{(1)}, ..., R^{(p)})$, $R^{(t)} = (R_{1j}, ..., R_{mj})'$, $R^{(t)}_{ij} \sim N(0, 1)$ and $e_j \sim N(0, \sigma^2 I_m)$ where $W = I_m$ (see Ghapani 2019). We consider two designs that in the first design $n_i = 3$ and in the second design $n_i = 7$. Also, the same value $l = 9$, $p = 3$, $m = 2$ and $q = 9$ are taken in both designs. Following Ozkale and Can (2017), "The β vector was chosen as the eigenvector corresponding to the largest eigenvalue of the $X'V^{-1}X$ matrix". We consider $\sigma^2 = 0.5, 1$, $\sigma_1^2 = 0.5, 1$ and use two case of the matrix Λ as $\Lambda_1 = diag(0.01, 0.01, 0.01)$ and $\Lambda_2 = diag(0.05, 0.05, 0.05)$. The trial was replicated 1000 times by generating new error terms. For each simulated dataset, we derived $\tilde{\beta}$, $\tilde{\sigma}^2$, $\tilde{k}_{HM}$, $\tilde{d}$ and $\tilde{Z}$. An estimate of Z can be obtained as $\tilde{Z} = X + \tilde{\sigma}_v^{-2} \tilde{v} \tilde{\beta}' \Lambda$, where $\tilde{v}_i = y_i - x_i' \tilde{\beta} - u_i' \tilde{b}$ and $\tilde{\sigma}_v^2 = \tilde{\sigma}^2 + \tilde{\beta}' \Lambda \tilde{\beta}$ (see , Zare et al. 2012). We calculate the estimated mean squared error ($EMSE$) of all estimators as follows:

$$EMSE(\tilde{\beta}) = \frac{1}{1000} \sum_{i=1}^{1000} (\tilde{\beta}_{(i)} - \beta)'(\tilde{\beta}_{(i)} - \beta),$$

where $\tilde{\beta}_{(i)}$ is the estimates of β in the *i-th* replication of the experiment. For relative comparison of $\tilde{\beta}$ with $\tilde{\beta}^*$, we calculated the relative mean square ($RMSE$) as

$$RMSE(\tilde{\beta} : \tilde{\beta}^*) = \frac{EMSE(\tilde{\beta})}{EMSE(\tilde{\beta}^*)}.$$

When $RMSE$ is greater than one, it indicates that the estimator $\tilde{\beta}^*$ is superior to the estimator $\tilde{\beta}$. In Tables 1 to 3, we replace unknown parameters by suitable estimates and obtain the values of $EMSE$ and $RMSE$ for $\tilde{\beta}$, $\tilde{\beta}(k, d)$, $\tilde{\beta}_r$ and $\tilde{\beta}_r(k, d)$. The following results are based on the values in Tables 1-3:

- In all Tables, the $EMSE$ values of $\tilde{\beta}(k, d)$ is less than $\tilde{\beta}$. Also, the $EMSE$ values of $\tilde{\beta}_r(k, d)$ is less than $\tilde{\beta}_r$. In general, the $EMSE$ values of $\tilde{\beta}_r(k, d)$ is less than all estimators.

- As ρ^2, Λ, σ_1^2 and σ^2 increase, the $EMSE$ values of the estimators increase.

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Table 1: Estimated *MSE* and *RMSE* values with $\tilde{k}_{HM}$, $\tilde{d}$ and $l = 9$ at $\rho = 0.75$

$(\Lambda, \sigma^2, \sigma_1{}^2)$	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,1,1)	$(\Lambda_1$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0001315	7.2129e-05	0.0002654	0.0001400
$EMSE(\tilde{\beta}_r)$	0.0001047	5.9267e-05	0.0002124	0.0001156
$EMSE(\tilde{\beta}(k,d))$	0.0001072	6.1887e-05	0.0001697	0.0001064
$EMSE(\tilde{\beta}_r(k,d))$	8.9682e-05	5.2487e-05	0.0001510	9.2901e-05
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.2269	1.1654	1.5635	1.3151
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.1685	1.1654	1.4065	1.2448
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.1954	1.1790	1.1240	1.1459
$(\Lambda, \sigma^2, \sigma_1{}^2)$	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,1,1)	$(\Lambda_2$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0001327	8.9629e-05	0.0002679	0.0001621
$EMSE(\tilde{\beta}_r)$	0.0001010	7.0613e-05	0.0002076	0.0001297
$EMSE(\tilde{\beta}(k,d))$	0.0001128	7.6994e-05	0.0001876	0.0001235
$EMSE(\tilde{\beta}_r(k,d))$	8.9401e-05	6.2641e-05	0.0001586	0.0001044
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.1760	1.1640	1.4277	1.3122
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.1303	1.1272	1.3087	1.2415
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.2624	1.2291	1.1829	1.1825

- In all Tables, the values of *RMSE* are greater than one. Also, with increasing values of ρ^2, σ^2 and σ_1^2, the values of *RMSE* increased.

- As ρ^2 increases, the difference between the *EMSE* values of $\tilde{\beta}(k,d)$ and $\tilde{\beta}$ and the difference between the *EMSE* values of $\tilde{\beta}_r(k,d)$ and $\tilde{\beta}_r$ increase. This indicates an increase in the performance improvement of the two-parameter estimator in reducing the EMSE compared to other estimators.

- As $n = \sum_{i=1}^{t} n_i$ increases from 27 to 63 for fixed ρ^2, Λ, σ_1^2 and σ^2, the *EMSE* values of the estimators decrease.

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Table 2: Estimated *MSE* and *RMSE* values with $\tilde{k}_{HM}$, $\tilde{d}$ and $l = 9$ at $\rho = 0.85$

$(\Lambda, \sigma^2, {\sigma_1}^2)$	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,1,1)	$(\Lambda_1$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0002144	0.0001262	0.0004259	0.0002395
$EMSE(\tilde{\beta}_r)$	0.0001507	9.1531e-05	0.0003025	0.0001758
$EMSE(\tilde{\beta}(k,d))$	0.0001530	9.7718e-05	0.0002064	0.0001535
$EMSE(\tilde{\beta}_r(k,d))$	0.0001191	7.6053e-05	0.0001821	0.0001269
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.4014	1.2922	2.0631	1.5606
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.2649	1.2035	1.6605	1.3855
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.2845	1.2848	1.1333	1.2093
$(\Lambda, \sigma^2, {\sigma_1}^2)$	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,1,1)	$(\Lambda_2$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0002329	0.0001675	0.0004479	0.0002882
$EMSE(\tilde{\beta}_r)$	0.0001525	0.0001149	0.0003029	0.0002031
$EMSE(\tilde{\beta}(k,d))$	0.0001819	0.0001320	0.0002579	0.0001901
$EMSE(\tilde{\beta}_r(k,d))$	0.0001283	9.6536e-05	0.0002063	0.0001490
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.2801	1.2690	1.7365	1.5156
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.1883	1.1907	1.4679	1.3624
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.4172	1.3674	1.2499	1.2755

8 Real Data Analysis

We consider the Boston Housing data collection to illustrate the behavior of proposed estimators. For this dataset, Harrison and Rubinfeld (1978) considered ways for "using housing mart information to assessment the tendency to payment for clean air". Zhong et al. (2002) collected the data of 132 census tract in the 15 areas of the Boston city (as a part of 506 observations on census tracts in the Boston Standard Metropolitan Statistical Area (SMSA) in 1970). They considered the dependent variable as the (logarithm) median value of the houses occupied by the owner in the census tract, and the fixed effects variables considered as Average number of rooms per dwelling (RM), proportions of owner-occupied units built prior to 1940 (AGE), the variable 1000(b-0.63)2, where b is the proportion of Blacks (B), a percentage of lower status population (LSTAT), crime rate per capita (CRIM), a dummy variable with two levels,

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Table 3: Estimated *MSE* and *RMSE* values with $\tilde{k}_{HM}$, $\tilde{d}$ and $l = 9$ at $\rho = 0.95$

$(\Lambda, \sigma^2, \sigma_1{}^2)$	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,0.5,0.5)	$(\Lambda_1$,1,1)	$(\Lambda_1$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0007086	0.0004244	0.0013256	0.0007602
$EMSE(\tilde{\beta}_r)$	0.0003199	0.0002177	0.0006237	0.0004051
$EMSE(\tilde{\beta}(k,d))$	0.0003637	0.0002076	0.0003680	0.0002443
$EMSE(\tilde{\beta}_r(k,d))$	0.0002193	0.0001416	0.0003095	0.0002050
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.9483	2.0441	3.6023	3.1113
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.4586	1.5369	2.0149	1.9762
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.6581	1.4657	1.1888	1.1918
$(\Lambda, \sigma^2, \sigma_1{}^2)$	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,0.5,0.5)	$(\Lambda_2$,1,1)	$(\Lambda_2$,1,1)
n_i	3	7	3	7
$EMSE(\tilde{\beta})$	0.0007998	0.0004484	0.0013922	0.0007246
$EMSE(\tilde{\beta}_r)$	0.0003473	0.0002517	0.0006444	0.0004219
$EMSE(\tilde{\beta}(k,d))$	0.0005442	0.0003251	0.0006323	0.0004342
$EMSE(\tilde{\beta}_r(k,d))$	0.0002744	0.0002033	0.0004083	0.0003060
$RMSE(\tilde{\beta} : \tilde{\beta}(k,d))$	1.4696	1.3794	2.2017	1.6685
$RMSE(\tilde{\beta}_r : \tilde{\beta}_r(k,d))$	1.2655	1.2381	1.5782	1.3787
$RMSE(\tilde{\beta}(k,d) : \tilde{\beta}_r(k,d))$	1.9828	1.5988	1.5486	1.4191

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1 if tract border to Charles River and 0 otherwise (CHAS) and levels of nitrogen oxides concentration (parts per 10 million) per town (NOX). All independent variables can be measured precisely except the pollution variable NOXSQ which is taken to have measurement errors. Therefore, we fit the dataset by model (1.1), which is a linear mixed measurement error model. In this model, y is the 132×1 vector of response variables and X and U are regression matrix with dimensions 132×8 and 132×15, respectively. Note that X is the matrix of observed value of Z. First, we estimated the variance components by considering $\sigma_1{}^2 = 0.5$ and $\sigma^2 = 0.5$. Then, by calculating the eigenvalues of $X'V^{-1}X$, the condition number 354.25 is obtained, which indicate severe multicollinearity. Following Ghapani (2019), we considered the stochastic linear restrictions as $r = R\beta + e$, $e \sim N(0, \sigma^2 I_5)$ where $W = I_m$ and selected 5 available data from nearby Boston. Using the method introduced in section 6, estimates k and d are calculated as $\tilde{k}_{HM} = 7.99881\text{e}{-08}$ and $\tilde{d} = 0.9149$, respectively. In Table 4, the estimated *MSE* values of the estimators are obtained by replacing in the corresponding theoretical

MSE equations. We can see the estimated *MSE* values of $\tilde{\beta}(k,d)$ is less than $\tilde{\beta}$. Also, the estimated *MSE* values of $\tilde{\beta}_r(k,d)$ is less than $\tilde{\beta}_r$. In general, the estimated *MSE* values of $\tilde{\beta}_r(k,d)$ is less than all estimators. So, we conclude that the stochastic restricted two-parameter estimators performs better than the other estimators. In section 6, we obtained the parameters k and d such that the Δ_1 is a pd matrix. Δ_1 eigenvalues are 3.338933e-03, 1.674830e-03, 2.927536e-04, 1.982075e-05, 2.653215e-09, 1.167239e-10, 5.571947e-11 and 1.587284e-11, all of which are positive, indicating that Δ_1 is pd matrix. In Figure 1, a plot of the estimated *MSE* values of the estimators against k in the interval [0,0.5] with fixed $\tilde{d} = 0.9149$ is drawn. In this Figure, we can see that the estimated *MSE* values of $\tilde{\beta}_r$ is always less than $\tilde{\beta}$. Also in the interval [0,0.18) the estimated *MSE* values of $\tilde{\beta}_r(k,d)$ is less than the other estimators. In addition, Figure 2 shows another plot of the estimated *MSE* values of the estimators against d in the interval (0,1) with fixed $\tilde{k}_{HM} = 7.99881e{-}08$. Figure 2, demonstrates that estimated *MSE* values of the $\tilde{\beta}_r(k,d)$ is less than the other estimators when d is selected in interval (0.6, 1). Altogether, it is obvious that the two parameter estimators can perform better than the $\tilde{\beta}$ in *MSEM* criterion under conditions.

Table 4: Parameter estimates and *MSE* values of the proposed estimators (the t test statistics are in parentheses).

	$\tilde{\beta}$	$\tilde{\beta}(k,d)$	$\tilde{\beta}_r$	$\tilde{\beta}_r(k,d)$
RM	-0.00119(-0.46)	-0.00106(-0.41)	-0.00989 (-1.61)	-0.00912 (-1.48)
AGE	0.00127(0.68)	0.00119 (0.64)	0.02579 (6.69)	0.02595 (6.71)
DIS	0.56280 (2.80)	0.53389(2.77)	3.05560 (8.69)	2.93664 (8.33)
B	0.54819(3.43)	0.52794 (3.42)	0.14965 (0.38)	0.14280 (0.38)
LSTAT	-0.47691(-7.14)	-0.47488 (-7.22)	-1.17260 (-8.34)	-1.18214 (-8.53)
CRIM	-0.00626 (-4.56)	-0.00633 (-4.61)	-0.00374 (-1.11)	-0.00389 (-1.15)
CHAS	-0.08809(-0.98)	-0.08757 (-0.99)	-0.01633 (-0.07)	-0.02704 (-0.12)
NOX	-0.00848 (-1.66)	-0.00858 (-1.69)	0.06969 (7.41)	0.07057 (7.50)
σ^2	0.02695	0.0260	0.1704	0.1645
σ_1^2	75.6890	76.318	1.2691	1.2293
EMSE	0.07827427	0.07294793	0.06110634	0.05750057

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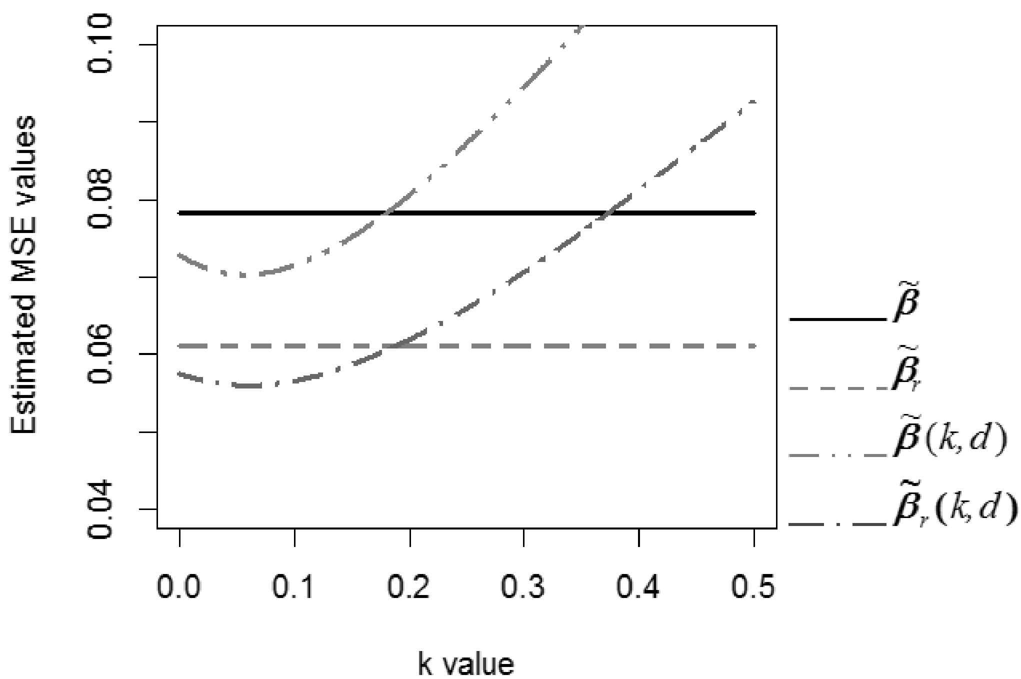

Figure 1: The estimated mean square error values of the estimators versus k with $\tilde{d}$.

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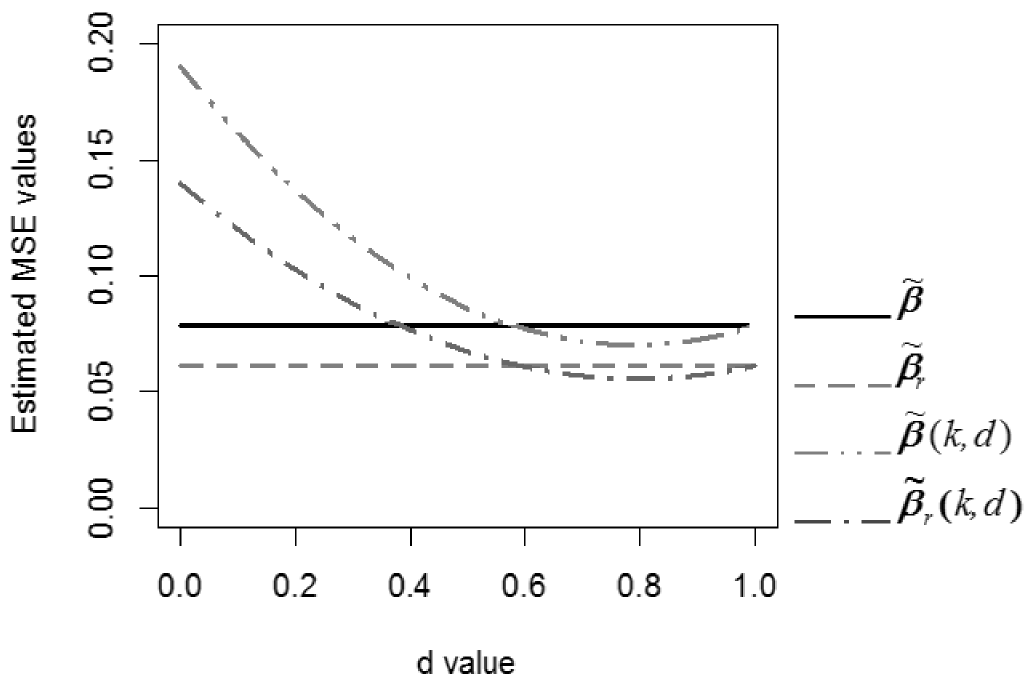

Figure 2: The estimated mean square error values of the estimators versus d with $\tilde{k}_{HM}$.

9 Conclusion

In this study, the Nakamura's approach is used to obtain the two parameter estimator and the stochastic restricted two parameter estimator in linear mixed measurement error models. Also, the asymptotic properties of the fixed effect estimators are obtained and then comparisons between proposed estimators and other estimators are made using MSEM sense. Therefore, the methods for estimating biasing parameters were proposed. Furthermore, a simulation study is provided and it is found that the stochastic restricted two parameter estimator has the smallest *MSE* value compared to other estimators. Finally, a data example has been given to illustrate the performance of the new estimators. Both the numerical example results and simulation study indicate that the use of stochastic linear restrictions is useful when there is collinearity in the dataset. Also, the performance of the stochastic restricted two parameter estimator is better than the other estimators under certain conditions in the *MSEM* sense.

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Appendix

In the following, for the convenience of the reader, we repeat some theorems and lemmas from other researches without proofs, thus making our exposition self-contained.

Lemma 1 (Farebrother, 1976). Let A be a positive definite matrix, namely $A > 0$ and α be some vector, then $A - \alpha\alpha' \geq 0$ if and only if $\alpha' A \alpha \leq 1$.

Lemma 2 (Rao et al. 2008, Theorem A. 52). Let $A > 0$ and $B > 0$, Then $B - A > 0$ if and only if $A^{-1} - B^{-1} > 0$.

Lemma 3 (Ghapani, 2019, Theorem 2). When $B > 0$ and $G_d B G_d > 0$, where $G_d = (Z'V^{-1}Z + I_p)^{-1}(Z'V^{-1}Z + dI_p)$; that is B and $G_d B G_d$ are positive definite (p.d.) matrix, we have $B - G_d B G_d > 0$.

Lemma 4 (Güler and Kaçıranlar. 2009, Lemma 2.1). Let $A > 0$ and $B > 0$, Then $B - A > 0$ if and only if $\lambda_i^B(A) < 1$.

Lemma 5 (Trenkler and Toutenburg. 1990, Theorem 2). Let, $\hat{\beta}_j = A_j y$, $j = 1, 2$ be two competing estimators of β. Suppose that $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) > 0$, where $Cov(\hat{\beta}_j)$, $j = 1, 2$ denote the covariance matrix of $\hat{\beta}_j$. Then $\Delta(\hat{\beta}_1, \hat{\beta}_2) = MSEM(\hat{\beta}_1) - MSEM(\hat{\beta}_2) \geq 0$

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if and only if $b_2'(D + b_1 b_1')^{-1} b_2 \leq 1$, where $MSEM(\hat{\beta}_j)$ and b_j denote the mean squared error matrix and bias vector of $\hat{\beta}_j$, respectively.

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