

## Point Prediction for the Proportional Hazards Family under Progressive Type-II Censoring

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**Abstract.** In this paper, we discuss different predictors of times to failure of units censored in multiple stages in a progressively censored sample from proportional hazard rate models. The maximum likelihood predictors, best unbiased predictors and conditional median predictors are considered. We also consider Bayesian point predictors for the times to failure of units. A numerical example and a Monte Carlo simulation study are presented to illustrate all the prediction methods discussed in this paper.

### 1 Introduction

A progressive Type-II censoring scheme is an important scheme in life-testing experiments. The experimenter can remove units from a life-testing at various stages during the experiments, possibly resulting in a saving of costs and time.

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The two most common censoring schemes are termed conventional Type-I and Type-II censoring. Neither of the conventional censoring schemes allow for units to be lost or removed from the test at points other than the final termination point. This allowance will be desirable, as in the case of studies of wear, in which the study of the actual aging process requires items to be fully disassembled at various stages in the experiment. Intermediate removal may also be desirable when some of the surviving items in the experiment that are removed early on (particularly when the items under test are very difficult to obtain or very expensive) can be used for some other test. These reasons provide evidence for the usefulness and efficiency of progressive censoring (see Balakrishnan and Aggarwala (2000)).

The progressive Type-II censoring can be described as follows. Suppose  $n$  units are placed on a life test. Immediately following the first failure,  $R_1$  surviving units are removed from the test at random. Then, immediately following the second failure,  $R_2$  surviving units are removed from the test at random. This process continues until, at the time of the  $m$ -th failure, all the remaining  $R_m = n - R_1 - R_2 - \dots - R_{m-1} - m$  units are removed from the experiment. Here the  $R_i$ 's are fixed prior to study. If  $R_1 = R_2 = \dots = R_m = 0$ , then  $n = m$  which corresponds to the complete sample situation. If  $R_1 = R_2 = \dots = R_{m-1} = 0$ , we have  $R_m = n - m$  which corresponds to the conventional Type-II right censoring scheme. For further details on progressively censoring, inferences and their applications, one may refer to Balakrishnan and Aggarwala (2000).

Prediction of unobserved or censored observations is an interesting topic, especially in the viewpoint of actuarial, medical and engineering sciences. Viveros and Balakrishnan (1994) used the conditional method of inference to develop a conditional prediction interval for an observation from an independent future sample based on an observed progressively Type-II right censored sample. Balakrishnan and Lin (2002) discussed exact prediction intervals for last censored failure times in a progressively Type-II right censored sample from an exponential distribution, based on the best linear unbiased estimator (BLUE). Basak et al. (2006) presented a detailed discussion on the point prediction of censored failure times in a progressively Type-II right censored sample. Recently, Raqab et al. (2010) have discussed different predictors of times to failure of units censored in multiple stages in a progressively censored sample from the Pareto Type-II (Lomax) distribution.

Let  $F_0(\cdot)$  be a cumulative distribution function (cdf) with a corresponding hazard rate function  $r_0(\cdot)$ . The family of random variables with hazard rate function of the form  $\{\theta r_0(\cdot) : \theta > 0\}$  is called the proportional hazard rate (PHR) family and the cdf  $F_0(\cdot)$  is called the baseline cdf of that family. Therefore, if  $X$  is a member of proportional hazard family with the baseline cdf  $F_0(\cdot)$ , then the distribution function of  $X$  becomes

$$F(x; \theta) = 1 - [\bar{F}_0(x)]^\theta, \quad x \in B, \quad \theta > 0 \quad (1.1)$$

where  $\bar{F}_0(\cdot) = 1 - F_0(\cdot)$  is the baseline survival function and  $B$  is the support of the baseline cdf. Note also that the baseline cdf  $F_0(\cdot)$  corresponds to the case  $\theta = 1$ . This model was originally proposed by Cox (1972) and has been extensively discussed in the statistical literature. This family of distributions includes several well-known lifetime distributions such as exponential, Pareto (Types I and II), Beta, Burr type XII, and so on. The PHR model has been extensively discussed in the statistical literature, see Ahmadi et al. (2008, 2009), Asgharzadeh and Valiollahi (2009, 2010),

Generally, in PRH model, introduced by Cox (1972),  $\theta$  is usually regarded as a random variable which is a function  $\theta(z)$  of the covariates  $z = (z_1, z_2, \dots, z_k)$ . With the parameter  $\theta$  taken to be a function  $\theta = \theta(z)$  of covariates, the resulting model is

$$r(x|\theta(z)) = r(x)\theta(z).$$

Two most commonly used covariate functions are the linear

$$\theta(z) = \beta z,$$

and the log linear

$$\theta(z) = \exp(\beta z),$$

models, where  $\beta$  may be a vector parameter. When  $\theta = \theta(z)$  has the form log linear  $\theta(z) = \exp(\beta z)$ , the resulting model is often called the Cox model. Other functions of the covariates are sometimes used. For further details, see Lawless (2003) and Marshal and Olkin (2007).

From the model (1.1), the probability density function (pdf) is given, by

$$f(x; \theta) = \theta f_0(x) [\bar{F}_0(x)]^{\theta-1}, \quad -\infty \leq c < x < d \leq \infty, \quad \theta > 0, \quad (1.2)$$

where  $f_0(\cdot)$  is the pdf of  $F_0(\cdot)$ .

Let  $X_{1:m:n}, \dots, X_{m:m:n}$  denote the progressively Type-II order statistics from the PHR model given in (1.1) obtained from a sample of size  $n$  with the censoring scheme  $(R_1, \dots, R_m)$ . To simplify the notation, we will use  $X_i$  in place of  $X_{i:m:n}$ . The aim of this paper is to discuss the prediction of the life-lengths  $Y = X_{i,(s)}$  ( $s = 1, 2, \dots, R_i$ ,  $i = 1, 2, \dots, m$ ) of all censored units in all  $m$  stages of censoring based on observed data  $\mathbf{X} = (X_1, \dots, X_m)$ . Here  $Y = X_{i,(s)}$  denotes the  $s$ -th order statistic out of  $R_i$  removed units at stage  $i$  ( $i = 1, 2, \dots, m$ ). Various point predictors are considered in Section 2 under non-Bayesian and Bayesian approaches. In Section 3, we present a numerical example and a Monte Carlo simulation study to illustrate all the prediction methods discussed in this paper.

## 2 Point Prediction

Let  $X_1, X_2, \dots, X_m$  denote a progressively Type-II right censored sample from model (1.1), with  $(R_1, R_2, \dots, R_m)$  being the progressive censoring scheme. Our interest is to predict  $Y = X_{i,(s)}$  ( $s = 1, 2, \dots, R_i$ ;  $i = 1, 2, \dots, m$ ) in all  $m$  stages of censoring based on the observed progressively Type-II right censored sample  $\mathbf{x} = (x_1, \dots, x_m)$ .

### 2.1 Maximum Likelihood Predictor

Maximum likelihood prediction has been considered for predicting future observations by many authors. See, for examples, kaminsky and Rhodin (1985), Basak and Balakrishnan (2003) and Basak *et al.* (2006). For the case of progressive censoring, Basak *et al.* (2006) proved the existence of unique maximum likelihood predictor and discussed the condition for unbiasedness and consistency of maximum likelihood predictor.

In maximum likelihood prediction, the principle of maximum likelihood is applied to the joint prediction and estimation of a future random variable and an unknown parameter. We assume dependence between present and future, and the approach is non-Bayesian. Let  $\mathbf{X} = (X_1, X_2, \dots, X_m)$  and  $Y$  have the joint pdf  $f(\mathbf{x}, y; \theta)$  indexed by the parameter  $\theta$ . The problem here will be to predict the unobservable future value of  $Y$ , having observed  $\mathbf{X}$ . Thus, viewed as a function of  $y$  and  $\theta$ , we define

$$L(y, \theta; \mathbf{x}) = f(\mathbf{x}, y; \theta)$$

to be the predictive likelihood function (PLF) of  $y$  and  $\theta$ .

Suppose  $\hat{Y}_p = u(\mathbf{X})$  and  $\hat{\theta}_p = v(\mathbf{X})$  are statistics for which

$$L(u(\mathbf{x}), v(\mathbf{x}); \mathbf{x}) = \sup_{(y, \theta)} L(y, \theta; \mathbf{x}),$$

then  $u(\mathbf{X})$  is said to be the maximum likelihood predictor (MLP) of  $Y$  and  $v(\mathbf{X})$  the predictive maximum likelihood estimator (PMLE) of  $\theta$ . Note that the PLF can be rewritten as

$$L(y, \theta; \mathbf{x}) = f(y|\mathbf{x}; \theta)f(\mathbf{x}; \theta).$$

So, when  $\theta$  is known, an MLP for  $Y$  is also a mode of the conditional distribution of  $Y$  given  $\mathbf{X} = \mathbf{x}$ .

Now, let  $X_1, X_2, \dots, X_m$  be a progressively Type II censored sample from model (1.1), with progressive censoring scheme  $(R_1, R_2, \dots, R_m)$ . Then, the joint density function of  $\mathbf{X} = (X_1, X_2, \dots, X_m)$  is given (see Balakrishnan and Aggarwala (2000)) by

$$\begin{aligned} f(\mathbf{x}; \theta) &= A \prod_{i=1}^m [f(x_i; \theta)[1 - F(x_i; \theta)]^{R_i}] \\ &= A \left[ \prod_{i=1}^m \frac{f_0(x_i)}{F_0(x_i)} \right] \theta^m e^{-\theta T(\mathbf{x})}, \end{aligned} \tag{2.1}$$

where  $A = n(n-1-R_1)(n-2-R_1-R_2) \dots (n-m+1-R_1 \dots -R_{m-1})$  and  $T(\mathbf{x}) = -\sum_{i=1}^m (R_i + 1) \ln \bar{F}_0(x_i)$ .

From (2.1), the maximum likelihood estimator (MLE) of  $\theta$  is derived to be

$$\hat{\theta}_{ML} = \frac{m}{T(\mathbf{x})}.$$

It is well-known that the conditional distribution of  $X_{i,(s)}$  given  $\mathbf{X}$  is just the distribution of  $X_{i,(s)}$  given  $X_i = x_i$ , due to the Markovian property of progressively Type-II right censored-order statistics (see Balakrishnan and Aggarwala (2000)). This implies that the density of  $X_{i,(s)}$  given  $\mathbf{X} = \mathbf{x}$  is the same as the density of the  $s$ th order statistic out of  $R_i$  units from the population with density  $f(y)/(1-F(x_i)), y \geq x_i$  (left truncated density at  $x_i$ ). Therefore, the conditional density of  $Y = X_{i,(s)}$  given  $X_i = x_i$ , for  $y \geq x_i$ , is given by

$$\begin{aligned} f(y|x_i; \theta) &= s \binom{R_i}{s} f(y; \theta) [F(y; \theta) - F(x_i; \theta)]^{s-1} \\ &\times [1 - F(y; \theta)]^{R_i-s} [1 - F(x_i; \theta)]^{-R_i}. \end{aligned} \tag{2.2}$$

For model (1.1), (2.2) reduces to

$$\begin{aligned}
 f(y|x_i; \theta) &= s \binom{R_i}{s} \theta \frac{f_0(y)}{\bar{F}_0(y)} \left[ (\bar{F}_0(x_i))^\theta - (\bar{F}_0(y))^\theta \right]^{s-1} \\
 &\times \left[ (\bar{F}_0(y))^\theta \right]^{R_i-s+1} \\
 &\times \left[ (\bar{F}_0(x_i))^\theta \right]^{-R_i}, \quad y \geq x_i. \tag{2.3}
 \end{aligned}$$

Consequently, the predictive likelihood function (PLF) of  $Y$  and  $\theta$ , can be rewritten as

$$\begin{aligned}
 L(y, \theta; \mathbf{x}) &= f_{Y|X_i}(y|x_i, \theta) f(\mathbf{x}, \theta) \\
 &= cf(y; \theta) [F(y; \theta) - F(x_i; \theta)]^{s-1} \\
 &\times [1 - F(y; \theta)]^{R_i-s} \prod_{j=1}^m f(x_j; \theta) \\
 &\times \prod_{j=1, j \neq i}^m [1 - F(x_j; \theta)]^{R_j}, \quad y \geq x_i, \tag{2.4}
 \end{aligned}$$

where  $c$  denotes a constant factor. Apart from a constant term, the predictive log-likelihood function is

$$\begin{aligned}
 \ln L(y, \theta; \mathbf{x}) &= \ln f(y; \theta) + (s-1) \ln [F(y; \theta) - F(x_i; \theta)] \\
 &+ (R_i - s) \ln [1 - F(y; \theta)] + \sum_{j=1}^m \ln f(x_j; \theta) \\
 &+ \sum_{j=1, j \neq i}^m R_j \ln [1 - F(x_j; \theta)], \quad y \geq x_i. \tag{2.5}
 \end{aligned}$$

From Equations (1.1), (1.2) and (2.5), the log PLF of  $Y = y$  and  $\theta$ , for  $y \geq x_i$ , is given by

$$\begin{aligned}
 \ln L(y, \theta; \mathbf{x}) &= (m+1) \ln \theta + \ln \left[ \frac{f_0(y)}{\bar{F}_0(y)} \right] \\
 &+ (s-1) \ln \left[ 1 - \left( \frac{\bar{F}_0(y)}{\bar{F}_0(x_i)} \right)^\theta \right] \\
 &+ \theta (R_i - s + 1) [\ln \bar{F}_0(y) - \ln \bar{F}_0(x_i)] \\
 &+ \theta \sum_{j=1}^m (R_j + 1) \ln \bar{F}_0(x_j). \tag{2.6}
 \end{aligned}$$

By using (2.6), the predictive likelihood equations (PLEs) for  $y$  and  $\theta$  (for  $y \geq x_i$ ) are given by

$$\begin{aligned} \frac{\partial \ln L(y, \theta)}{\partial y} &= \frac{1}{\bar{F}_0(y)} \left[ \frac{f'_0(y)\bar{F}_0(y) + f_0^2(y)}{f_0(y)} - \theta(R_i - s + 1)f_0(y) \right. \\ &\quad \left. + \theta(s - 1)f_0(y) \frac{\left[\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right]^\theta}{1 - \left[\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right]^\theta} \right] = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ln L(y, \theta)}{\partial \theta} &= \frac{m + 1}{\theta} - (s - 1) \ln \left[ \frac{\bar{F}_0(y)}{\bar{F}_0(x_i)} \right] \frac{\left[\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right]^\theta}{1 - \left[\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right]^\theta} \\ &\quad + (R_i - s + 1) [\ln[\bar{F}_0(y)] - \ln[\bar{F}_0(x_i)]] \\ &\quad + \sum_{j=1}^m (R_j + 1) \ln[\bar{F}_0(x_j)] = 0. \end{aligned} \tag{2.7}$$

**Example 1.** (i) (Exponential distribution): For the case of exponential distribution, we have

$$\bar{F}_0(x) = e^{-x} \quad x > 0.$$

In this case, the PLEs reduce to:

$$\frac{\partial \ln L(y, \theta)}{\partial y} = -\theta(R_i - s + 1) + \theta(s - 1) \frac{e^{-\theta(y-x_i)}}{1 - e^{-\theta(y-x_i)}} = 0,$$

and

$$\begin{aligned} \frac{\partial \ln L(y, \theta)}{\partial \theta} &= \frac{m + 1}{\theta} - \left[ \sum_{j=1}^m (R_j + 1)x_j + (R_i - s + 1)(y - x_i) \right] \\ &\quad + (s - 1) \frac{(y - x_i)e^{-\theta(y-x_i)}}{1 - e^{-\theta(y-x_i)}} = 0. \end{aligned}$$

Now, the MLP of  $Y$  and PMLE of  $\theta$  can be obtained as

$$\hat{Y}_{MLP} = x_i + \frac{1}{\hat{\theta}_{PML}} \ln \left[ \frac{R_i}{R_i - s + 1} \right], \tag{2.8}$$

and

$$\hat{\theta}_{PML} = \frac{m + 1}{T_0(\mathbf{x})}, \tag{2.9}$$

where

$$T_0(\mathbf{x}) = \sum_{i=1}^m (R_i + 1)x_i. \quad (2.10)$$

(ii) (Pareto type I distribution): For the Pareto Type-I distribution with  $\bar{F}_0(x) = \frac{\eta}{x}$ ,  $x > \eta > 0$  (with known  $\eta$ ), the PLEs reduce to:

$$\begin{aligned} \frac{\partial \ln L(y, \theta)}{\partial y} &= -1 - \theta(R_i - s + 1) + \theta(s - 1) \frac{x_i^{2\theta}}{y^{2\theta} - x_i^{2\theta}} = 0, \\ \frac{\partial \ln L(y, \theta)}{\partial \theta} &= \frac{m + 1}{\theta} + (R_i - s + 1) \ln\left(\frac{x_i}{y}\right)^2 + \sum_{j=1}^m (R_j + 1) \ln\left(\frac{\eta}{x_j}\right) \\ &\quad - (s - 1) \ln\left(\frac{x_i}{y}\right)^2 \frac{x_i^{2\theta}}{y^{2\theta} - x_i^{2\theta}} = 0. \end{aligned}$$

Equation  $\partial \ln L(y, \theta)/\partial y = 0$ , yields the MLP of  $Y$  to be

$$\hat{Y}_{MLP} = x_i \left( \frac{\hat{\theta}_{PML} R_i + 1}{\hat{\theta}_{PML} (R_i - s + 1) + 1} \right)^{(1/2\hat{\theta}_{PML})}. \quad (2.11)$$

Equation  $\partial \ln L(y, \theta)/\partial \theta = 0$ , in conjunction with the MLP of  $Y$  in (2.11), reduces to

$$(m + 1) - \frac{1}{\hat{\theta}_{PML}} \ln \left( \frac{\hat{\theta}_{PML} (R_i - s + 1) + 1}{\hat{\theta}_{PML} R_i + 1} \right) - \hat{\theta}_{PML} T_1(\mathbf{x}) = 0, \quad (2.12)$$

where

$$T_1(\mathbf{x}) = - \sum_{i=1}^m (R_i + 1) \ln\left(\frac{\eta}{x_i}\right). \quad (2.13)$$

Since (2.12) can not be solved analytically for  $\hat{\theta}_{PML}$ , some numerical methods must be employed.

## 2.2 Best Unbiased Predictor

A statistic  $\hat{Y}$  which is used to predict  $Y = X_{i,(s)}$  is called a best unbiased predictor (BUP) of  $Y$ , if the predictor error  $\hat{Y} - Y$  has a mean zero and its prediction error variance  $\text{Var}(\hat{Y} - Y)$  is less than or equal to that of any other unbiased predictor of  $Y$ .

Since the conditional distribution of  $Y$  given  $\mathbf{X} = (X_1, \dots, X_m)$  is just the distribution of  $Y$  given  $X_i$ , therefore the BUP of  $Y$  is (see Nayak (2000))

$$\hat{Y}_{BUP} = E(Y|X_i = x_i).$$

By (2.3), we have

$$\begin{aligned} \hat{Y}_{BUP} &= \int_{x_i}^{\infty} yf(y|x_i, \theta)dy \\ &= \int_0^1 \bar{F}_0^{-1}\left(u^{\frac{1}{\theta}}\bar{F}_0(x_i)\right) \frac{u^{R_i-s}(1-u)^{s-1}}{\text{Beta}(R_i-s+1, s)}du. \end{aligned} \tag{2.14}$$

If the parameter  $\theta$  is unknown, we can replace  $\theta$  by its MLE and obtain an approximate BUP of  $Y$ .

**Example 2.** (i) (Exponential distribution): For exponential distribution with  $\bar{F}_0(x) = e^{-x}$  and after replacing  $\theta$  by its MLE, we obtain the approximate BUP of  $Y$  as

$$\begin{aligned} \hat{Y}_{BUP} &= \int_0^1 -\ln\left(u^{\frac{T_0(\mathbf{x})}{m}}e^{-x_i}\right) \frac{u^{R_i-s}(1-u)^{s-1}}{\text{Beta}(R_i-s+1, s)}du \\ &= x_i + \frac{T_0(\mathbf{x})}{m} \int_0^1 (-\ln u) \frac{u^{R_i-s}(1-u)^{s-1}}{\text{Beta}(R_i-s+1, s)}du \\ &= x_i + \frac{T_0(\mathbf{x})}{m} E(-\ln U) \\ &= x_i + \frac{T_0(\mathbf{x})}{m} E(Z_{s:R_i}), \end{aligned} \tag{2.15}$$

where  $U$  has the  $\text{Beta}(R_i-s+1, s)$  distribution and  $Z_{s:R_i}$  denotes the  $s$ -th order statistics out of  $R_i$  units from a standard exponential distribution.

(ii) (Pareto Type-I distribution): For the Pareto Type-I distribution with  $\bar{F}_0(x) = \frac{\eta}{x}$ ,  $x > \eta > 0$ , with known  $\eta$ , and after replacing  $\theta$  by its MLE, we obtain the approximate BUP of  $Y$  as

$$\begin{aligned} \hat{Y}_{BUP} &= x_i \int_0^1 u^{-\frac{T_1(\mathbf{x})}{m}} \frac{u^{R_i-s}(1-u)^{s-1}}{\text{Beta}(R_i-s+1, s)} du \\ &= \frac{\text{Beta}(R_i-s-\frac{T_1(\mathbf{x})}{m}+1, s)}{\text{Beta}(R_i-s+1, s)} x_i. \end{aligned} \tag{2.16}$$

### 2.3 Conditional Median Predictor

Conditional median predictor (CMP) is another possible predictor which is suggested by Raqab and Nagaraja (1995). A predictor  $\hat{Y}$  is

called the CMP of  $Y$ , if it is the median of the conditional distribution of  $Y$  given  $X_i = x_i$ ; that is,

$$P_\theta(Y \leq \hat{Y}|X_i = x_i) = P_\theta(Y \geq \hat{Y}|X_i = x_i).$$

Using the relation

$$P_\theta(Y \leq \hat{Y}|X_i = x_i) = P_\theta \left[ \left( \frac{\bar{F}_0(Y)}{\bar{F}_0(X_i)} \right)^\theta \geq \left( \frac{\bar{F}_0(\hat{Y})}{\bar{F}_0(X_i)} \right)^\theta \mid X_i = x_i \right],$$

and using the fact that the distribution of  $\left( \frac{\bar{F}_0(Y)}{\bar{F}_0(X_i)} \right)^\theta$  given  $X_i = x_i$  is a  $Beta(R_i - s + 1, s)$  distribution, we obtain the CMP of  $Y$  as

$$\hat{Y}_{CMP} = \bar{F}_0^{-1} \left( \bar{F}_0(x_i) [Med(U)]^{\frac{1}{\theta}} \right),$$

where  $U$  has  $Beta(R_i - s + 1, s)$  distribution and  $Med(U)$  stands for median of  $U$ . By substituting  $\theta$  with its MLE, we obtain

$$\hat{Y}_{CMP} = \bar{F}_0^{-1} \left( \bar{F}_0(x_i) (Med(U))^{\frac{T(\mathbf{X})}{m}} \right),$$

as the CMP of  $Y$ .

**Example 3.** (i) Taking  $\bar{F}_0(x) = e^{-x}$ , for the case of exponential distribution, we obtain the CMP of  $Y$  as

$$\begin{aligned} \hat{Y}_{CMP} &= -\ln \left( e^{-x_i} [Med(U)]^{\frac{T_0(\mathbf{X})}{m}} \right) \\ &= x_i - \frac{T_0(\mathbf{X})}{m} \ln[Med(U)] \\ &= x_i + \frac{T_0(\mathbf{X})}{m} Med(Z_{s:R_i}) \end{aligned} \quad (2.17)$$

(ii) Taking  $\bar{F}_0(x) = \frac{\eta}{x}$ ,  $x > \eta > 0$ , with known  $\eta$ , for the case of Pareto Type-I distribution, we obtain

$$\hat{Y}_{CMP} = x_i (Med(U))^{-\frac{T_1(\mathbf{X})}{m}} \quad (2.18)$$

## 2.4 Bayesian Predictors

In this section, our interest is to predict  $Y = X_{i,(s)}$  ( $s = 1, 2, \dots, R_i$ ;  $i = 1, 2, \dots, m$ ) based on the observed progressively Type-II right censored sample  $\mathbf{x} = (x_1, \dots, x_m)$  from a Bayesian approach. Bayesian predictors are obtained from  $f^*(y|\mathbf{x})$ , the Bayes predictive density function of  $Y$  given  $\mathbf{X} = \mathbf{x}$ , and the given loss function. Let  $\hat{Y} = \delta(\mathbf{X})$  is an predictor  $Y$ . The loss function  $L(y, \hat{y})$  denote the loss for using  $\hat{y}$  as the predicted value of  $Y$  when the realized value is  $y$ .

In the Bayesian prediction problem, the most commonly used loss function is the squared error loss (SEL)

$$L(y, \hat{y}) = (\hat{y} - y)^2.$$

This loss is symmetric and its use is very popular, perhaps, because of its mathematical simplicity. The Bayes point predictor of  $y$ , under this loss function, ( $\hat{y}_{SEP}$ ) is  $E(y|\mathbf{x})$ . Another symmetric loss function is the absolute difference loss (ADL)

$$L(y, \hat{y}) = |y - \hat{y}|.$$

It is well known that the Bayes point predictor of  $y$ , under this loss function, ( $\hat{y}_{ADP}$ ) is the median of the Bayes predictive distribution.

In life testing and reliability problems, the nature of losses are not always symmetric and hence the uses of SEL and ADL are unacceptable in many situations.

A useful asymmetric loss function is the general entropy loss (GEL) function introduced by Zellner (1986), Dey (1999) and Soliman (2005):

$$L(y, \hat{y}) \propto \left(\frac{\hat{y}}{y}\right)^q - q \ln \left(\frac{\hat{y}}{y}\right) - 1, \quad q \neq 0. \quad (2.19)$$

The Bayes point predictor  $\hat{y}_{GEP}$  of  $y$  under the general entropy loss (2.19) can be shown to be

$$\hat{y}_{GEP} = (E_y(y^{-q}))^{-\frac{1}{q}}, \quad (2.20)$$

where  $E_y(\cdot)$  denotes the posterior expectation with respect to the Bayes predictive density function of  $y$ .

Under the assumption that the parameter  $\theta$  is unknown, we can use the conjugate gamma prior  $\Gamma(\alpha, \beta)$ , with pdf

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0, \quad (\beta > 0, \alpha > 0). \quad (2.21)$$

The posterior density function of  $\theta$  given the data, denoted by  $\pi(\theta | \underline{x})$ , can be obtained using (2.1) and (2.21) as

$$\pi(\theta | \underline{x}) = \frac{[\beta + T(\underline{x})]^{m+\alpha}}{\Gamma(m + \alpha)} \theta^{m+\alpha-1} e^{-\theta[\beta+T(\underline{x})]}. \tag{2.22}$$

The Bayes predictive density function of  $Y$  given  $X_i = x_i$  is given by

$$f^*(y|x_i) = \int_{\Theta} f(y|x_i, \theta)\pi(\theta|\underline{x})d\theta. \tag{2.23}$$

By substituting (2.3) and (2.22) into (2.23), we get

$$\begin{aligned} f^*(y|x_i) &= \int_0^\infty s \binom{R_i}{s} \theta \frac{f_0(y)}{\bar{F}_0(y)} [(\bar{F}_0(y))^\theta]^{R_i-s+1} \\ &\times [(\bar{F}_0(x_i))^\theta - (\bar{F}_0(y))^\theta]^{s-1} [(\bar{F}_0(x_i))^\theta]^{-R_i} \\ &\times \frac{[\beta + T(\underline{x})]^{m+\alpha}}{\Gamma(m + \alpha)} \theta^{m+\alpha-1} e^{-\theta[T(\underline{x})+\beta]} d\theta, \quad y \geq x_i \end{aligned} \tag{2.24}$$

Using bivariate expansion, we have

$$[(\bar{F}_0(x_i))^\theta - (\bar{F}_0(y))^\theta]^{s-1} = \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j [\bar{F}_0(y)]^{\theta j} [\bar{F}_0(x_i)]^{\theta(s-j-1)}. \tag{2.25}$$

From (2.25), the equation (2.24) can be rewritten as

$$\begin{aligned} f^*(y|x_i) &= s \binom{R_i}{s} \frac{f_0(y)}{\bar{F}_0(y)} \frac{m + \alpha}{\beta + T(\underline{x})} \\ &\times \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \\ &\times \left[ 1 - \frac{(R_i - s + j + 1) \ln(\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)})}{\beta + T(\underline{x})} \right]^{-(m+\alpha+1)}. \end{aligned} \tag{2.26}$$

From (2.26), under ADL, the Bayes point predictor of  $Y$ , say  $\hat{Y}_{ADP}$ , can be obtained from the following equation:

$$\begin{aligned} \frac{1}{2} &= \int_{x_i}^{\hat{Y}_{ADP}} f^*(y|x_i)dy \\ &= s \binom{R_i}{s} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \frac{1}{R_i - s + j + 1} \end{aligned}$$

$$\times \left[ 1 - \left( 1 - \frac{(R_i - s + j + 1) \ln\left(\frac{\bar{F}_0(\hat{Y}_{ADP})}{\bar{F}_0(x_i)}\right)}{\beta + T(\underline{x})} \right)^{-(m+\alpha)} \right]. \quad (2.27)$$

Using (2.20), the Bayes point predictor of  $Y$  under GE loss function is given by

$$\begin{aligned} \hat{Y}_{GEP} &= \left( \int_{x_i}^{\infty} y^{-q} f^*(y|x_i) dy \right)^{-\frac{1}{q}} \\ &= \left( s \binom{R_i}{s} \frac{m + \alpha}{\beta + T(\underline{x})} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j I(j) \right)^{-\frac{1}{q}} \end{aligned} \quad (2.28)$$

where

$$I(j) = \int_{x_i}^{\infty} y^{-q} \frac{f_0(y)}{\bar{F}_0(y)} \left[ 1 - \frac{(R_i - s + j + 1) \ln\left(\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right)}{\beta + T(\underline{x})} \right]^{-(m+\alpha+1)} dy.$$

Note that the Bayes point predictor  $\hat{Y}_{SEP}$  under SEL can be obtained by setting  $q = -1$  in (2.28).

For a special case, when  $s = 1$ , the equations (2.27) and (2.28) are reduced to:

$$\frac{1}{2} = \left[ 1 - \frac{R_i}{\beta + T(\underline{x})} \ln\left(\frac{\bar{F}_0(\hat{Y}_{ADP})}{\bar{F}_0(x_i)}\right) \right]^{-(m+\alpha)}$$

and

$$\hat{Y}_{GEP} = \left( \frac{m + \alpha}{\beta + T(\underline{x})} R_i I(0) \right)^{-\frac{1}{q}},$$

where

$$I(0) = \int_{x_i}^{\infty} y^{-q} \frac{f_0(y)}{\bar{F}_0(y)} \left[ 1 - \frac{R_i \ln\left(\frac{\bar{F}_0(y)}{\bar{F}_0(x_i)}\right)}{\beta + T(\underline{x})} \right]^{-(m+\alpha+1)} dy.$$

In this case, we obtain the Bayes point predictor  $\hat{Y}_{ADP}$  as

$$\hat{Y}_{ADP} = \bar{F}_0^{-1} \left( \bar{F}_0(x_i) \exp \left( \frac{\beta + T(\underline{x})}{m + \alpha} \left[ 1 - \left(\frac{1}{2}\right)^{m+\alpha} \right] \right) \right).$$

**Example 4.** (i) (Exponential distribution): For exponential distribution with  $\bar{F}_0(x) = e^{-x}$ , the Bayes point predictors  $\hat{Y}_{ADP}$  and  $\hat{Y}_{GEP}$

of  $Y$  can be obtained from the following equations:

$$\begin{aligned} \frac{1}{2} &= s \binom{R_i}{s} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \frac{1}{R_i - s + j + 1} \\ &\times \left[ 1 - \left( 1 + \frac{(R_i - s + j + 1)(\hat{Y}_{ADP} - x_i)}{\beta + T_0(\underline{x})} \right)^{-(m+\alpha)} \right]. \end{aligned} \quad (2.29)$$

and

$$\hat{Y}_{GEP} = \left( s \binom{R_i}{s} \frac{m + \alpha}{\beta + T_0(\underline{x})} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j I_0(j) \right)^{-\frac{1}{q}}, \quad (2.30)$$

where

$$I_0(j) = \int_{x_i}^{\infty} y^{-q} \left[ 1 + \frac{(R_i - s + j + 1)(y - x_i)}{\beta + T_0(\underline{x})} \right]^{-(m+\alpha+1)} dy.$$

(ii) (Pareto Type I distribution): For Pareto Type-I distribution with  $\bar{F}_0(x) = \frac{\eta}{x}$ , the Bayes point predictors  $\hat{Y}_{ADP}$  and  $\hat{Y}_{GEP}$  of  $Y$  can be obtained from the following equations:

$$\begin{aligned} \frac{1}{2} &= s \binom{R_i}{s} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \frac{1}{R_i - s + j + 1} \\ &\times \left[ 1 - \left( 1 + \frac{(R_i - s + j + 1) \ln(\frac{x_i}{\hat{Y}_{ADP}})}{\beta + T_1(\underline{x})} \right)^{-(m+\alpha)} \right]. \end{aligned} \quad (2.31)$$

and

$$\hat{Y}_{GEP} = \left( s \binom{R_i}{s} \frac{m + \alpha}{\beta + T_1(\underline{x})} \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j I_1(j) \right)^{-\frac{1}{q}}, \quad (2.32)$$

where

$$I_1(j) = \int_{x_i}^{\infty} y^{-q-1} \left[ 1 + \frac{(R_i - s + j + 1) \ln(\frac{x_i}{y})}{\beta + T_1(\underline{x})} \right]^{-(m+\alpha+1)} dy.$$

### 3 Numerical Computations

In this section, we present the results of numerical experiments of the different point predictors investigated in the previous sections. We present the analysis of a data set and a Monte Carlo simulation to compare the performance of the point predictors with respect to their biases and mean square prediction errors (MSPE's). We consider the exponential distribution  $E(1/\theta)$  with cdf

$$F(x, \theta) = 1 - e^{-\theta x}, \quad x > 0, \quad \theta > 0,$$

as a special case from the model (1.1). Here, we have

$$\bar{F}_0(x) = e^{-x} \quad \text{and} \quad T_0(\underline{x}) = \sum_{j=1}^m (R_j + 1) x_j.$$

#### 3.1 Numerical Example

According to the following steps, the different point predictors are obtained as described in Section 2.

- (i) For given values of  $\alpha = 2$  and  $\beta = 1$ , we sample  $\theta = 1.952$  from the prior pdf (2.21).
- (ii) Using the value  $\theta = 1.952$  from step (i), we generate a progressively Type-II censored sample of size  $m = 8$  with the censoring scheme

$$R_1 = (0, 0, 3, 0, 3, 0, 0, 5)$$

from the exponential distribution according to the algorithm presented in Balakrishnan and Aggarwala (2000). The sample generated is

0.0322 0.1201 0.1249 0.1435 0.1633 0.1805 0.2101 0.2158

- (iii) Using this sample, we obtain the non-Bayesian point predictors MLP, BUP and CMP and Bayesian point predictors ADP, SEP and GEP for  $X_{i,(s)}$  ( $s = 1, 2, \dots, R_i$ ;  $i = 3, 5, 8$ ). The results are displayed in Table 1.

**Table 1.** Different point predictors

	MLP	BUP	CMP	ADP	SEP	GEP				
						$q = -0.8$	$q = -0.5$	$q = -0.3$	$q = 0.3$	$q = 0.5$
$X_{3,(1)}$	0.125	0.199	0.176	0.216	0.436	0.555	0.816	1.069	2.539	3.442
$X_{3,(2)}$	0.209	0.311	0.280	0.400	0.785	0.885	1.088	1.268	2.156	2.632
$X_{3,(3)}$	0.354	0.535	0.478	0.753	1.483	1.469	1.487	1.527	1.799	1.953
$X_{5,(1)}$	0.163	0.238	0.215	0.255	0.499	0.621	0.879	1.120	2.413	3.155
$X_{5,(2)}$	0.248	0.349	0.315	0.438	0.847	0.945	1.137	1.304	2.088	2.489
$X_{5,(3)}$	0.393	0.573	0.516	0.791	1.546	1.521	1.524	1.551	1.765	1.889
$X_{8,(1)}$	0.215	0.260	0.246	0.270	0.491	0.620	0.886	1.129	2.372	3.052
$X_{8,(2)}$	0.262	0.316	0.300	0.362	0.666	0.788	1.026	1.231	2.185	2.668
$X_{8,(3)}$	0.322	0.391	0.371	0.490	0.898	0.992	1.187	1.342	2.010	2.326
$X_{8,(4)}$	0.407	0.503	0.475	0.676	1.247	1.296	1.392	1.475	1.834	1.999
$X_{8,(5)}$	0.552	0.727	0.673	1.031	1.945	1.840	1.727	1.676	1.622	1.634

### 3.2 Simulation Study and Discussion

The expressions of the predictors show that an analytic comparison of these predictors is not possible. Therefore, a Monte Carlo simulation study is used to evaluate the biases and MSPEs for the predictors MLP, BUP, CMP, ADP, SEP and BGP. We randomly generated 1000 progressively censored sample from exponential distribution with  $\theta = 0.75, 1, 2$ . We also used two censoring schemes  $R_1 = (0, 0, 3, 0, 3, 0, 0, 5)$  and  $R_2 = (0, 0, 0, 0, 5)$ . To compute Bayesian predictors, since we do not have any prior information, we assume that  $\alpha = \beta = 0$ . Although it implies an improper prior on  $\theta$ , but the corresponding posterior is proper. Tables 2 and 3 display the biases and MSPEs of different predictors obtained from this simulation study. All the computations are performed using Visual Maple (V12) package.

From Tables 2 and 3, as anticipated, we observe that the BUPs produce the best results in terms of biases and MSPEs for two censoring schemes and for all different values of  $\theta$  considered. The SEPs are the second best predictors. We also note that the MLP does not work well.

From Tables 2 and 3, we observe that most of predictors usually underpredict the life-lengths  $Y = X_{i,(s)}$  ( $s = 1, 2, \dots, R_i$ ;  $i = 1, 2, \dots, m$ ), except the SEP and BGP, which overpredict all the times.

We note that for  $q$  close to -1, the GEPs are close to the SEPs. These tables show that the Bayes predictors relative to asymmetric loss function (GE) are sensitive to the value of the shape parameter  $q$ .

One of the referees raised a valid point that why the likelihood prediction is often not efficient when compared to other prediction methods. It may be mentioned that in the likelihood prediction when we condition on the given data, the likelihood function used only the last observed failure time and the rest of the data are not playing a big role in the prediction except when estimating the parameter. Moreover, the MLPs are not usually easiest to compute. The MLPs usually involve solving non-linear equations and they need be calculated by some iterative processes such as the Newton-Raphson method which may converge to wrong root.

It would be interesting also to investigate how the predictors compare to each other, when  $\theta$  is decomposed as  $\theta = \beta z$  or  $\theta = \exp(\beta z)$ , where  $z = (z_1, \dots, z_k)$  is a vector of covariates and  $\beta = (\beta_1, \dots, \beta_k)$  may be a vector of parameters. Because of dealing with the vector parameter  $\beta$  as well as the variable  $z$  to be predicted, the analysis may not be straightforward. More work is needed in this direction.

Finally, it should be mentioned here that all of the results obtained in this study can be specialized to: (a) usually Type-II censored case (for  $R_m = n - m$ ,  $R_i = 0$ ,  $i = 1, \dots, m - 1$ ) (b) complete sample case (for  $n = m$  and  $R_i = 0$ ,  $i = 1, 2, \dots, m$ ).

**Table 2.** Biases and MSPE's of point predictors for the censoring scheme  $R_1$

			MLP	BUP	CMP	ADP	SEP	BGP		
								$q = -0.8$	$q = -0.5$	$q = -0.3$
$\theta = 0.75$	$X_{3,(1)}$	Bias	-0.506	-0.035	-0.159	-0.143	0.042	0.048	0.063	0.102
		MSPE	0.406	0.117	0.213	0.186	0.148	0.166	0.196	0.262
	$X_{3,(2)}$	Bias	-0.707	-0.022	-0.225	-0.180	0.051	0.067	0.292	0.392
		MSPE	1.155	0.338	0.825	0.638	0.420	0.551	0.711	0.736
	$X_{3,(3)}$	Bias	-1.459	-0.148	-0.479	-0.660	0.239	0.693	1.255	1.510
		MSPE	2.316	0.652	1.220	1.830	1.579	1.782	2.733	3.408
	$X_{5,(1)}$	Bias	-0.456	-0.014	-0.150	-0.136	0.019	0.021	0.063	0.131
		MSPE	0.502	0.179	0.468	0.384	0.199	0.214	0.330	0.389
	$X_{5,(2)}$	Bias	-0.768	-0.088	-0.290	-0.248	0.102	0.194	0.489	0.628
		MSPE	1.600	0.635	0.964	0.883	0.781	0.892	1.236	1.390
	$X_{5,(3)}$	Bias	-1.264	-0.325	-0.393	-0.385	0.353	0.529	0.885	1.157
		MSPE	2.616	1.254	1.433	1.994	1.399	1.525	1.882	2.133
	$X_{8,(1)}$	Bias	-0.255	-0.051	-0.143	-0.133	0.094	0.149	0.223	0.317
		MSPE	0.506	0.349	0.644	0.538	0.258	0.745	0.844	0.948
	$X_{8,(2)}$	Bias	-0.397	-0.097	-0.185	-0.165	0.121	0.164	0.336	0.427
MSPE		0.698	0.168	0.485	0.401	0.266	0.377	0.785	0.987	
$X_{8,(3)}$	Bias	-0.559	-0.129	-0.217	-0.166	0.134	0.355	0.839	1.078	
	MSPE	0.628	0.276	0.393	0.301	0.308	0.430	1.071	1.616	
$X_{8,(4)}$	Bias	-0.837	-0.214	-0.398	-0.359	0.329	0.601	1.267	1.582	
	MSPE	1.818	0.526	0.874	0.639	0.613	1.376	2.516	3.139	
$X_{8,(5)}$	Bias	-1.183	-0.409	-1.210	-1.080	0.498	0.653	0.967	1.148	
	MSPE	2.214	0.770	1.982	1.638	0.903	1.100	1.605	1.914	
$\theta = 1$	$X_{3,(1)}$	Bias	-0.356	-0.073	-0.159	-0.151	0.132	0.220	0.238	0.255
		MSPE	0.426	0.110	0.228	0.203	0.188	0.307	0.324	0.367
	$X_{3,(2)}$	Bias	-0.437	-0.084	-0.191	-0.159	0.137	0.193	0.219	0.301
		MSPE	0.540	0.134	0.289	0.219	0.204	0.306	0.349	0.434
	$X_{3,(3)}$	Bias	-0.593	-0.180	-0.220	-0.207	0.197	0.229	0.273	0.449
		MSPE	0.619	0.173	0.330	0.197	0.185	0.364	0.360	0.458
	$X_{5,(1)}$	Bias	-0.339	-0.027	-0.123	-0.113	0.087	0.155	0.190	0.219
		MSPE	0.428	0.087	0.128	0.102	0.101	0.119	0.134	0.163
	$X_{5,(2)}$	Bias	-0.464	-0.058	-0.178	-0.152	0.144	0.172	0.205	0.283
		MSPE	0.697	0.102	0.188	0.145	0.139	0.249	0.282	0.360
	$X_{5,(3)}$	Bias	-0.515	-0.264	-0.387	0.321	0.294	0.497	0.549	0.596
		MSPE	0.888	0.369	0.449	0.417	0.397	0.634	0.628	0.749
	$X_{8,(1)}$	Bias	-0.136	-0.036	-0.067	-0.064	0.052	0.059	0.083	0.095
		MSPE	0.132	0.045	0.078	0.073	0.064	0.075	0.098	0.110
	$X_{8,(2)}$	Bias	-0.187	-0.085	-0.158	-0.142	0.121	0.159	0.192	0.235
MSPE		0.212	0.122	0.153	0.131	0.119	0.137	0.178	0.241	
$X_{8,(3)}$	Bias	-0.302	-0.123	-0.165	-0.155	0.134	0.156	0.177	0.209	
	MSPE	0.293	0.180	0.231	0.201	0.188	0.137	0.170	0.234	
$X_{8,(4)}$	Bias	-0.387	-0.205	-0.274	-0.230	0.229	0.298	0.313	0.382	
	MSPE	0.351	0.189	0.262	0.285	0.275	0.332	0.368	0.419	
$X_{8,(5)}$	Bias	-0.469	-0.250	-0.321	-0.307	0.277	0.302	0.351	0.402	
	MSPE	0.574	0.294	0.380	0.346	0.322	0.373	0.401	0.499	

Table 2. Continued

			MLP	BUP	CMP	ADP	SEP	BGP		
								$q = -0.8$	$q = -0.5$	$q = -0.3$
$\theta = 2$	$X_{3,(1)}$	Bias	-0.164	-0.025	-0.061	-0.056	0.046	0.072	0.123	0.149
		MSPE	0.239	0.015	0.107	0.095	0.082	0.120	0.164	0.207
	$X_{3,(2)}$	Bias	-0.384	-0.105	-0.223	-0.209	0.198	0.254	0.296	0.345
		MSPE	0.275	0.174	0.212	0.200	0.187	0.232	0.248	0.267
	$X_{3,(3)}$	Bias	-0.571	-0.155	-0.279	-0.243	0.228	0.292	0.344	0.351
		MSPE	0.948	0.224	0.252	0.227	0.209	0.285	0.334	0.421
	$X_{5,(1)}$	Bias	-0.112	-0.043	-0.073	-0.061	0.056	0.075	0.084	0.105
		MSPE	0.137	0.031	0.107	0.094	0.084	0.108	0.118	0.127
	$X_{5,(2)}$	Bias	-0.150	-0.083	-0.125	-0.102	0.097	0.115	0.128	0.142
		MSPE	0.178	0.102	0.158	0.121	0.114	0.139	0.146	0.165
	$X_{5,(3)}$	Bias	-0.566	-0.168	-0.487	-0.352	0.287	0.323	0.387	0.432
		MSPE	0.514	0.217	0.477	0.369	0.332	0.354	0.409	0.448
	$X_{8,(1)}$	Bias	-0.103	-0.036	-0.087	-0.064	0.052	0.049	0.073	0.095
		MSPE	0.132	0.035	0.079	0.064	0.046	0.037	0.086	0.107
	$X_{8,(2)}$	Bias	-0.123	-0.048	-0.111	-0.091	0.072	0.091	0.102	0.113
		MSPE	0.222	0.054	0.172	0.154	0.097	0.123	0.152	0.189
	$X_{8,(3)}$	Bias	-0.234	-0.081	-0.208	-0.174	0.151	0.172	0.187	0.215
		MSPE	0.351	0.144	0.301	0.230	0.203	0.246	0.252	0.323
	$X_{8,(4)}$	Bias	-0.314	-0.152	-0.282	-0.252	0.216	0.233	0.261	0.297
		MSPE	0.483	0.219	0.414	0.372	0.306	0.339	0.367	0.435
$X_{8,(5)}$	Bias	-0.554	-0.208	-0.505	-0.463	0.425	0.472	0.500	0.538	
	MSPE	0.838	0.284	0.729	0.686	0.596	0.665	0.721	0.787	

**Table 3.** Biases and MSPE's of point predictors for the censoring scheme  $R_2$

			MLP	BUP	CMP	ADP	SEP	BGP		
								$q = -0.8$	$q = -0.5$	$q = -0.3$
$\theta = 0.75$	$X_{5,(1)}$	Bias	-0.225	-0.042	-0.199	-0.171	0.080	0.122	0.154	0.185
		MSPE	0.177	0.037	0.163	0.144	0.064	0.111	0.131	0.154
	$X_{5,(2)}$	Bias	-0.251	-0.078	-0.217	-0.204	0.123	0.161	0.179	0.205
		MSPE	0.250	0.102	0.228	0.210	0.140	0.172	0.198	0.219
	$X_{5,(3)}$	Bias	-0.543	-0.213	-0.501	-0.491	0.367	0.416	0.481	0.528
		MSPE	0.870	0.350	0.798	0.725	0.429	0.627	0.686	0.751
	$X_{5,(4)}$	Bias	-0.768	-0.365	-0.725	-0.687	0.470	0.605	0.620	0.679
		MSPE	1.151	0.570	1.113	0.986	0.811	0.842	0.905	0.963
	$X_{5,(5)}$	Bias	-1.109	-0.517	-1.075	-0.995	0.738	0.916	0.977	1.044
		MSPE	1.932	0.846	1.687	1.376	1.029	1.305	1.398	1.502
$\theta = 1$	$X_{5,(1)}$	Bias	-0.208	-0.038	-0.188	-0.172	0.068	0.120	0.159	0.175
		MSPE	0.157	0.031	0.141	0.128	0.044	0.106	0.113	0.129
	$X_{5,(2)}$	Bias	-0.231	-0.060	-0.207	-0.192	0.099	0.158	0.188	0.202
		MSPE	0.164	0.056	0.149	0.131	0.081	0.107	0.117	0.146
	$X_{5,(3)}$	Bias	-0.391	-0.103	-0.375	-0.332	0.217	0.311	0.342	0.368
		MSPE	0.318	0.109	0.298	0.269	0.192	0.253	0.278	0.294
	$X_{5,(4)}$	Bias	-0.483	-0.216	-0.440	-0.417	0.296	0.398	0.414	0.423
		MSPE	0.766	0.308	0.694	0.650	0.420	0.652	0.678	0.687
	$X_{5,(5)}$	Bias	-0.933	-0.483	-0.841	-0.818	0.657	0.798	0.814	0.823
		MSPE	1.399	0.505	1.146	0.995	0.752	0.973	0.992	1.118
$\theta = 2$	$X_{5,(1)}$	Bias	-0.156	-0.031	-0.141	-0.128	0.071	0.105	0.123	0.130
		MSPE	0.123	0.024	0.115	0.103	0.065	0.087	0.095	0.102
	$X_{5,(2)}$	Bias	-0.210	-0.048	-0.196	-0.179	0.066	0.163	0.179	0.192
		MSPE	0.141	0.044	0.123	0.109	0.068	0.099	0.106	0.117
	$X_{5,(3)}$	Bias	-0.309	-0.078	-0.280	-0.257	0.168	0.239	0.251	0.271
		MSPE	0.235	0.092	0.206	0.189	0.142	0.184	0.195	0.201
	$X_{5,(4)}$	Bias	-0.419	-0.248	-0.403	-0.390	0.304	0.371	0.384	0.393
		MSPE	0.634	0.294	0.604	0.569	0.398	0.520	0.558	0.581
	$X_{5,(5)}$	Bias	-0.714	-0.328	-0.684	-0.657	0.498	0.608	0.632	0.667
		MSPE	0.980	0.387	0.882	0.814	0.593	0.768	0.814	0.852

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