Tracking Interval for Type II Hybrid Censoring Scheme

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Abstract. The purpose of this paper is to obtain the tracking interval for difference of expected Kullback-Leibler risks of two models under Type II hybrid censoring scheme. This interval helps us to evaluate proposed models in comparison with each other. We drive a statistic which tracks the difference of expected Kullback–Leibler risks between maximum likelihood estimators of the distribution in two different models and obtain an estimator of the variance of this statistic under Type II hybrid censoring scheme. Monte Carlo simulations are performed to verify the theoretical results. A real data set representing micro-droplet splashing reported in 90° spray angle is used to illustrate the results for the tracking interval. Furthermore, because of the great importance of prediction in coating industries, pivotal method is considered to obtain the prediction interval of future observation of the droplet splashing based on Type II hybrid censored sample.

Keywords. Burr distribution, Kullback-Leibler risk, model selection, prediction interval, tracking interval, type II hybrid censoring, Vuong's test.

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1   Introduction

An important problem in statistics is to test a sample of n independent and identically distributed observations coming from a specified distribution. Model selection has followed two approaches in the literature: hypothesis testing and model selection criteria. Since then, several articles have been published on model selection based on complete data, for example, Cox (1961; 1962) modified the classical hypothesis testing to compare the non-nested hypothesis, Vuong (1989) tested the equivalence of two models. The null hypothesis of Vuong test is the expectation under the true model of the log-likelihood ratio of the two rival models are equal to zero, which means that, two proposed models are equivalent. This expectation however is unknown. But Vuong test works, because the decision making procedure by Vuong test does not depend on this unknown quantity. The results in Vuong have been extended and applied in a number of ways, including, Vuong and Wang (1993), Lien and Vuong (1987), Commenges et. al. (2008), Sayyareh et. al. (2011), Sayyareh (2012a). Akaike (1973) introduced a criterion to select the best model under parsimony. One problem with Akaike information criterion (AIC) is that its value has no intrinsic meaning; in particular, AIC is not invariant to a one-to-one transformation of the random variables and values of AIC depend on the number of observations. So, Commenges et. al. (2008) and Sayyareh (2012b) considered the normalized difference of AIC as an estimate of a difference of Kullback-Leibler (KL) risks between two models and then constructed the tracking interval to verify the equivalence of two rival models.

However, in many experimental and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. Type I (time) censoring, where the life testing experiment will be terminated at a prescribed time T, and Type II (failure) censoring, where the life testing experiment will be terminated upon the rth failure are the two most popular censoring schemes used in the reliability and experimental studies. The mixture of Type I and Type II censoring schemes is known as hybrid censoring scheme. Several authors considered different aspects of these censoring schemes; (see for example, Balakrishnan and Kundu (2013), Kundu and Gupta (2007), Kundu and Raqab (2012), Park et. al. (2008), Kim and Yum (2011), Joarder et. al. (2011), Panahi and Sayyareh (2013).

Although several articles have been done on the hybrid censoring scheme but we have not come across any article on the tracking interval

under this censoring. Thus, the main aim of this paper is two folds. First we focus on the behavior of the two rival models under Type II hybrid censoring scheme. We want to decide whether or not the two rival models are two equivalent models. For this purpose, we use the tracking interval which should contain the difference of risks with a given probability. When rival models are non-nested, we propose a test statistic that convergence in distribution to the normal distribution and use it to test the null hypothesis that the rival models are equally close to the data generating model against the alternative hypothesis that one model is closer.

The second aim of this paper is to analyze the micro-droplet splashing data and then compare the rival models using the tracking interval. Thermal sprayed coatings are widely used to protect components exposed to corrosion, wear or heat. The mechanical properties of coatings are known to depend strongly on the shape of splats formed by individual droplets as they impact and freeze. As a good-coated surface is extremely important in the industry, one of the most important phenomena, which cause deterioration of the coated surface, is droplet splashing.

Splashing occurs when a single droplet breaks up on impact, producing secondary, or satellite, droplets. Figure 1 illustrates splashing via a sequence of photographs of the impact of a molten tin droplet onto a hot surface.
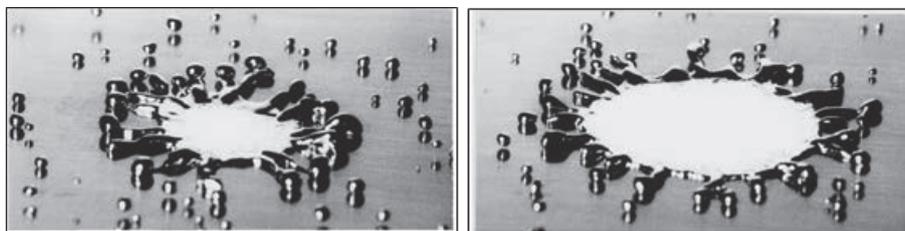

Figure 1: Splashing of a molten droplet during impact on a stainless steel surface (Ref., Aziz and Chandra; 2000)

Splashing degrade coating quality since they leave voids in the deposit, increasing its porosity and reducing its strength. The physical mechanisms of splashing are still not completely understood and the splash study is an extremely interesting and attractive phenomenon. Moreover, prediction of micro- droplets splashing can potentially reduce the cost of the development of new micro splashing considerably. While much research has been done on the study of droplet splashing, lack

attention has been paid to the distribution modeling and statistical prediction of this phenomenon. We know that the degree of splashing is always positive and therefore, it is reasonable to analyze the splashing data using the probability distribution, which has support only on the positive real axis. Thus, we have considered different two-parameter distributions namely, Burr XII, Weibull, generalized Rayleigh and Burr III distributions. Then we construct the tracking intervals to verify the equivalence of two rival models under different censoring schemes. Moreover, we consider the prediction interval of the future observation based on the Type II hybrid censored sample.

The rest of the paper is organized as follows. In Section 2, as preliminary, we briefly mention about the Type II hybrid censoring scheme, the theory about models and KL divergence. In Section 3, we bring the main results which we need to construct the tracking interval for the difference of the expected KL divergence of two non-nested rival models under Type II hybrid censoring scheme. Simulation results and the analysis of a real data are presented in Section 4, and the article concludes in Section 5.

2   Preliminary

2.1   Type II Hybrid Censoring Scheme

A hybrid censoring scheme is a mixture of Type I and Type II censoring schemes. In Type I hybrid censoring scheme, the experiment terminates as soon as either the rth failure or the pre-specified censoring time T occurs. However, under the Type I hybrid censoring model, the experiment may be terminated too early resulting in very few failures. For this reason, Childs et al. (2003) has focused on Type II hybrid censoring, under which the experiment terminates when the latter of the rth failure and the censoring time T occurs. The main aim of this paper is to focus on the non-nested model selection under Type II hybrid censoring scheme. Type II hybrid censoring scheme has the advantage of guaranteeing at least r failures to be observed by the end of the experiment and it can be described as follows:

Put n identical items on test, and then stop the experiment at the random time $T^* = \max(T, Y_r)$. Therefore, under this censoring scheme we can observe the following three Cases of observations when $Y_1, ..., Y_n$ denote the ordered sample of $X_1, ..., X_n$.

Case I: $\{y_1 < ... < y_r\}$ if $y_r > T$
Case II: $\{y_1 < ... < y_r < y_{r+1} < ... < y_m < T < y_{m+1}\}$ if $r \le m < n$
and $y_m < T < y_{m+1}$

$CaseIII :$ $\{y_1 < ... < y_n < T\}$ \hfill (1)

Note that, in case II, we do not observe y_{m+1} but $y_m < T < y_{m+1}$ means that the mth failure took place before T and no failure took place between y_m and T.

2.2   Statistical Models and Kullback–Leibler (KL) Divergence

Consider a sample of independently identically distributed (*i.i.d.*) random variables, $X_1, ..., X_n$ having probability density function $h(.)$. Let us consider two rival models:

$$F^\alpha = \{f^\alpha(.), \ \alpha \in \mathrm{M} \subset R^p\} \quad \text{and} \quad G^\beta = \left\{g^\beta(.), \ \beta \in \mathrm{B} \subset R^q\right\}.$$

Definition 2.1.   *(i) (f)* and *(g)* are non overlapping if $(f)\bigcap(g) = \phi$; *(ii) (f)* is nested in *(g)* if $(f) \subset (g)$; *(iii) (f)* is well- specified if there is a value $\alpha_0 \in \mathrm{M}$ such that $f^{\alpha_0}(.) = h$; otherwise it is mis-specified. If the model is well-specified then $\alpha_0 = \alpha_*$, where $\alpha_* = arg\ max_{\alpha \in M} E_h(L_n^f(\alpha))$, and refer to as the pseudo-true value of the α. We consider the $f^\alpha(.)$ as a proposed model, then quasi log-likelihood function is given by $L_n^f(\alpha) = \sum_{i=1}^n \log f^\alpha(x_i)$. Under the following condition, $\hat\alpha_n$ is a quasi maximum likelihood estimator (QMLE):

$$L_n^f(\hat\alpha_n) = sup_{\alpha \in M} L_n^f(\alpha)$$

The KL information in favor of $h(x)$ against $f^\alpha(.)$ is defined in Kullback–Leibler (1951) to be

$$KL(h, f^\alpha) = E_h\left(\log \frac{h(X)}{f^\alpha(X)}\right) = \int_{-\infty}^\infty h(x)\log \frac{h(x)}{f^\alpha(x)}\ dx$$

We have $KL(h, f^\alpha) \ge 0$ and $KL(h, f^\alpha) = 0$, implies that $h = f^\alpha$, that is $\alpha = \alpha_0$. The KL divergence is often intuitively interpreted as a distance between the two probability measures, but this is not mathematically a distance; in particular, the KL divergence is not symmetric. It may be felt that this is a drawback. But, this feature may also have a deep meaning in some model selection problem when there is no symmetry between the true and the proposed models. We assume that, there is a

value $\alpha_* \in \mathrm{M}$ which minimizes $KL(h, f^\alpha)$. If the model is well-specified $\alpha_0 = \alpha_*$. Also, $\hat{\alpha}_n$ is a consistent estimator of α_*. We can say that (f) is closer to h than (g) if $KL(h,\ f^{\alpha_*}) < KL(h,\ g^{\beta_*})$. We cannot estimate $KL(h, f^{\alpha_*})$ because the entropy of h, $E_h(\ \log h(X)\ )$, cannot be correctly estimated. However, we can estimate the difference of risks $\Delta_{hybrid}(f^{\alpha_*}, g^{\beta_*}) = EKL(h, f^{\alpha_*}) - EKL(h, g^{\beta_*})$, a quantitative measure of the difference of misspecification by $[-n^{-1}(L_n^f(\hat{\alpha}_n) - L_n^g(\hat{\beta}_n))]$. This result may not be completely satisfactory in practice if n is not very large because the distribution we will use is $f^{\hat{\alpha}_n}$ rather than f^{α_*}. Thus it is reasonable to consider the risk $E_h\{\ \log(h(X)/f^{\hat{\alpha}_n}(X))\}$ that we call the expected KL risk and that we denote by $EKL(h, f^{\hat{\alpha}_n})$.

3   Main Results

In this section, we consider the difference quasi log-likelihood functions of the Type II hybrid censoring scheme. From (1), the quasi log-likelihood functions for three different cases follow.

Case I: $L_n^f(\alpha) = \sum_{i=1}^r \log f^\alpha(y_i) + (n-r)\log \overline{F}^\alpha(y_r)$
Case II: $L_n^f(\alpha) = \sum_{i=1}^m \log f^\alpha(y_i) + (n-m)\log \overline{F}^\alpha(T)$
Case III: $L_n^f(\alpha) = \sum_{i=1}^n \log f^\alpha(y_i)$

where, $\overline{F}^\alpha(.) = 1 - F^\alpha(.)$. Therefore, the quasi log-likelihood function of combined Cases I, II, and III, can be written as:

$$L_n^f(\alpha) = \sum_{i=1}^R \log f^\alpha(y_i) + (n-R)\log \overline{F}^\alpha(U)$$

Here, R denotes the number of the total failures in experiment and $U = y_r$ if R = r and $U = T$ if R = m > r.
In other words,

$$R = \begin{cases} r & \text{for caseI} \\ m > r & \text{for caseII} \\ n & \text{for caseIII} \end{cases}$$

and

$$U = \begin{cases} y_r & \text{for caseI} \\ T & \text{for caseII and caseIII} \end{cases}$$

Therefore, the differences of the quasi log-likelihood functions of the two rival models can be obtained as:

$$L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) = L_n^f(\hat{\alpha}_n) - L_n^g(\hat{\beta}_n) = \sum_{i=1}^R \log \frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)} + (n-R) \log \frac{\overline{F}^{\hat{\alpha}_n}(U)}{\overline{G}^{\hat{\beta}_n}(U)}$$
(2)

where, $\hat{\alpha}_n$ is the quasi maximum likelihood estimator for the parameter α. Suppose that $Y_1, ..., Y_R$ are distributed as the order statistic of a random sample of size R from truncated distribution at U by probability density function (pdf) h^*. Now, if $\frac{r}{n} \to p$ as $n \to \infty$ such that $Y_r \xrightarrow{P} \zeta_p$, the pth percentile of true distribution, then from Voung (1989) and the property of Continuous Mapping, we have

$$\frac{1}{n} \sum_{i=1}^R \log \frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)} \xrightarrow{P} \breve{p} E_{h^*} \left[\log \frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)} \right];$$

and

$$\frac{1}{n}(n-R) \log \frac{\overline{F}^{\hat{\alpha}_n}(U)}{\overline{G}^{\hat{\beta}_n}(U)} \xrightarrow{P} (1-\breve{p}) \log \frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)}$$

where,

$$\lim_{n\to\infty} \frac{R}{n} = \breve{p} = \begin{cases} p = \lim_{n\to\infty} \frac{r}{n} & if \ R=r \\ p^* = \lim_{n\to\infty} \frac{m}{n} & if \ R = m > r \end{cases}$$

and

$$\breve{\zeta}_p = \begin{cases} \zeta_p; & if \ F(T) < p \\ \zeta_{F(T)}; & o.w. \end{cases}$$

Then the difference quasi log-likelihood function of two rival models is converges in probability as below:

$$\frac{1}{n} L_n^{f/g}\left(\hat{\alpha}_n, \hat{\beta}_n\right) \xrightarrow{P} \left\{ \breve{p} E_{h^*} \left[\log \frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)} \right] + (1-\breve{p}) \log \frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)} \right\}$$

where

$$\alpha_* = \arg\max_{\alpha \in \mathrm{M}} \left\{ \breve{p} E_{h^*} \left[\log f^\alpha(Y) \right] + (1-\breve{p}) \log \overline{F}^\alpha(\breve{\zeta}_p) \right\}$$

$$\beta_* = \arg\max_{\beta \in B} \left\{ \breve{p} E_{h^*} \left[\log g^\beta(Y) \right] + (1 - \breve{p}) \log \overline{G}^\beta(\breve{\zeta}_p) \right\}$$

α_* and β_* are pseudo-true values of α and β, respectively. Also quasi maximum likelihood estimator of α say $\hat{\alpha}_n$, can be obtain as a solution of $\frac{\partial}{\partial \alpha} L_n^f(\alpha) = 0$. The minimum assumptions,$\Re$, for compact neighborhood M of α_* are:

$\Re_1$:The parameter space M is an open interval in R.

$\Re_2$:All rival probability density functions have the same support.

$\Re_3$:For almost all x, the derivatives $(\frac{\partial}{\partial \alpha}) \log f^\alpha(x)$, $(\frac{\partial^2}{\partial \alpha^2}) \log f^\alpha(x)$ and $(\frac{\partial^3}{\partial \alpha^3}) \log f^\alpha(x)$ all exist for every α.

$\Re_4$:For every α, $\left| \frac{\partial \log f^\alpha(x)}{\partial \alpha} . \frac{\partial \log f^\alpha(x)}{\partial \alpha'} \right|$ and $\left| \frac{\partial^2 \log f^\alpha(x)}{\partial \alpha \partial \alpha'} \right|$ are dominated by integrable functions independent on α.

$\Re_5$:$\int (\frac{\partial}{\partial \alpha}) L^f(\alpha) d(H^*) = 0$ has a unique solution α_* on M, where H^* is the true distribution function related to the true density h^*.

Theorem 3.1. **(Asymptotic Distribution of the $L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n)$ Statistic):** Given Assumptions $\Re_1 - \Re_5$, suppose that the proposed model is misspecified and $f^{\alpha*} \neq g^{\beta*}$, then,

$$\sqrt{n} \left(\frac{1}{n} L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) - \breve{p} E_{h^*} \left[\log \frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)} \right] - (1 - \breve{p}) \log \frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)} \right) \xrightarrow{D} N \left(0, \omega_{*hybrid}^2 \right)$$

$$(3)$$

where,

$$\omega_{*hybrid}^2 = Var_h \left(\log \frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)} \right) + (1 - \breve{p}) Var_{h_1^*} \left(\log \frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)} \right) \quad (4)$$

and $w = (w_1, ..., w_n)$ =the complete data, $z = (z_1, ..., z_{n-R})$ =the complete data of size $n - R$, from the left truncated population with density function,

$$h_1^* = \frac{f^\alpha(z)}{\overline{F}^\alpha(U)}; \qquad z > U.$$

Proof. From the Taylor expansion of $L_n^f(\alpha_*)$ around the $\hat{\alpha}_n$, we can write

$$L_n^f(\alpha_*) = L_n^f(\hat{\alpha}_n) + \frac{n}{2}(\hat{\alpha}_n - \alpha_*)' \widehat{J}_{f_{hybrid}} (\hat{\alpha}_n - \alpha_*) + o_p(1)$$

and

$$L_n^g(\beta_*) = L_n^f(\hat{\beta}_n) + \frac{n}{2}(\hat{\beta}_n - \beta_*)' \widehat{J}_{g_{hybrid}} (\hat{\beta}_n - \beta_*) + o_p(1)$$

where,

$$\widehat{J}_{f_{hybrid}} = E\left(\frac{\partial^2 \log (f^\alpha(Y))}{\partial\alpha\,\partial\alpha'}\right) \quad \text{and} \quad \widehat{J}_{g_{hybrid}} = E\left(\frac{\partial^2 \log (g^\beta(Y))}{\partial\beta\,\partial\beta'}\right)$$

Thus,

$$L_n^{f/g}(\hat\alpha_n,\hat\beta_n) = L_n^{f/g}(\alpha_*,\beta_*) - \frac{n}{2}(\hat\alpha_n - \alpha_*)'J_{f_{hybrid}}(\hat\alpha_n - \alpha_*) \\ + \frac{n}{2}(\hat\beta_n - \beta_*)'J_{g_{hybrid}}(\hat\beta_n - \beta_*) + o_p(1)$$

It is known that $\sqrt{n}(\hat\alpha_n - \alpha_*)$ and $\sqrt{n}(\hat\beta_n - \beta_*)$ are $O_p(1)$. So, we have

$$\sqrt{n}\left(\frac{1}{n}L_n^{f/g}(\hat\alpha_n,\hat\beta_n) - \breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] - (1-\breve{p})\log\frac{\overline{F}^{\alpha_*}(\breve\zeta_p)}{\overline{G}^{\beta_*}(\breve\zeta_p)}\right) =$$

$$\sqrt{n}\left\{\frac{1}{n}L_n^{f/g}(\alpha_*,\beta_*) - \breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] - (1-\breve{p})\log\frac{\overline{F}^{\alpha_*}(\breve\zeta_p)}{\overline{G}^{\beta_*}(\breve\zeta_p)}\right\} + o_p(1)$$

But from the multivariate central Theorem, the first term in the right hand side converges in distribution to $N(0,\omega_{*hybrid}^2)$. It now suffices to show that $\omega_{*hybrid}^2 = Var_h\left(\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right) + (1-\breve{p})Var_{h_1^*}\left(\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right)$. From missing information principle developed in Louis (1982), we can write

$$\sum_{i=1}^{R}\log f^\alpha(y_i) = \sum_{i=1}^{n}\log f^\alpha(w_i) - \sum_{i=1}^{n-R}\log f^\alpha(z_i\,|Y=y)$$

Where, $w = (w_1,...,w_n)$ and $z = (z_1,...,z_{n-R})$ are defined as before in (4). Note that, the sequences of random variables W's and Z's are independent. For simplicity, we use $f^\alpha(z_i)$ instead of $f^\alpha(z_i\,|\,y)$ in what follows. Therefore, from (2) we can write

$$\begin{aligned}\omega_{*hybrid}^2 &= \frac{1}{n}Var\left(\sum_{i=1}^{R}\log\frac{f^{\alpha_*}(Y_i)}{g^{\beta_*}(Y_i)} + (n-R)\log\frac{\overline{F}^{\alpha_*}(U)}{\overline{G}^{\beta_*}(U)}\right) \\ &= \frac{1}{n}Var\left[\left(\sum_{i=1}^{n}\log\frac{f^{\alpha_*}(W_i)}{g^{\beta_*}(W_i)} - \sum_{i=1}^{n-R}\log\frac{f^{\alpha_*}(Z_i)}{g^{\beta_*}(Z_i)}\right.\right. \\ &\quad\left.\left. +(n-R)\log\frac{\overline{F}^{\alpha_*}(U)}{\overline{G}^{\beta_*}(U)}\right)\right]\end{aligned}$$

Now, If, $\frac{n-R}{n} \to 1-\breve{p}$ as $n \to \infty$ such that $U \to \breve\xi_p$ in probability, then using Continuous Mapping Theorem

$$\omega_{*hybrid}^2 = Var_h\left(\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right) + (1-\breve{p})Var_{h_1^*}\left(\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right).$$

Hence, we propose the following statistics:

$$
\begin{aligned}
\hat{\omega}_{hybrid}^2 \quad = \quad & \frac{1}{n}\sum_{i=1}^{n}\left(\log\frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\log\frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right)\right)^2 \\
& + (1-\frac{R}{n})\left[\frac{1}{n-R}\sum_{i=1}^{n-R}\left(\log\frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)^2\right. \\
& \left. - \left(\frac{1}{n-R}\sum_{i=1}^{n-R}\left(\log\frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)\right)^2\right] \quad\quad (5)
\end{aligned}
$$

Thus, based on Vuong (1989), we consider the hypotheses as

$$
\begin{aligned}
H_0 &: E_{h*}\left[\log\frac{f^{\alpha*}(Y)}{g^{\beta*}(Y)}\right] = 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha*}(\breve{\zeta}_p)}{\overline{G}^{\beta*}(\breve{\zeta}_p)} = 0\right] \\
H_f &: E_{h*}\left[\log\frac{f^{\alpha*}(Y)}{g^{\beta*}(Y)}\right] > 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha*}(\breve{\zeta}_p)}{\overline{G}^{\beta*}(\breve{\zeta}_p)} > 0\right] \\
H_g &: E_{h*}\left[\log\frac{f^{\alpha*}(Y)}{g^{\beta*}(Y)}\right] < 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha*}(\breve{\zeta}_p)}{\overline{G}^{\beta*}(\breve{\zeta}_p)} < 0\right]
\end{aligned}
$$

Thus, from (Vuong; Theorem 5.1),

$$
\begin{aligned}
\text{under} \quad\quad H_0 &: \Im = \frac{L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n)}{\sqrt{n}\hat{\omega}_{hybrid}} \xrightarrow{D} N(0,1) \\
\text{under} \quad\quad H_f &: \Im = \frac{L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n)}{\sqrt{n}\hat{\omega}_{hybrid}} \xrightarrow{a.s} +\infty \\
\text{under} \quad\quad H_g &: \Im = \frac{L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n)}{\sqrt{n}\hat{\omega}_{hybrid}} \xrightarrow{a.s} -\infty
\end{aligned}
$$

If the value of the statistic $\Im$ is higher than $Z_{1-\alpha}$ then one rejects the null hypothesis that the model are equivalent in favor of F^{α} being better than G^{β}. If $\Im$ is smaller than $-Z_{1-\alpha}$ then one rejects the null hypothesis in favor of G^{β} being better than F^{α}, finally if $|\Im| < Z_{1-\alpha}$ then one cannot discriminate between the two rival models based on the given data. Also, $Z_{1-\alpha}$ is $(1-\alpha)^{th}$ quantile of standard normal distribution.

3.1 Tracking Interval for a Difference of KL Divergences

In model selection context, selection the null hypothesis is not easy and on the other hand we faced with many alternatives. Generally in hypothesis testing when we decide about null hypothesis we do not add more and more alternative hypothesis, in fact in hypothesis testing we select the one best alternative to compare against. Confidence

intervals are equivalent to encapsulating the results of many hypothesis tests. Thus, we propose the new interval say tracking interval for a difference of expected Kullback-Leibler risks, $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = EKL(h, f^{\hat{\alpha}_n}) - EKL(h, g^{\hat{\beta}_n})$, for two rival models. The proposed confidence interval contains the difference of Kullbak-Leibler risks with a fixed probability. This interval has another interpretation for the use of AICs. In fact we are not in a situation to detect the best model but we are in search for a model which has the relatively less risk compared to other models. This interval is not a usual confidence interval because $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ changes with n. Although it converges toward $\Delta_{hybrid}(f^{\alpha_*}, g^{\beta_*})$, we wish to approach $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ for values of n for which the Akaike correction is not negligible. Akaikes approach was revisited by Linhart and Zucchini (1986) who showed that:

$$EKL\left(h, f^{\hat{\alpha}_n}\right) = KL\left(h, f^{\alpha_*}\right) + \frac{1}{2n} \, Tr\left(I_{f_{hybrid}} J^{-1}_{f_{hybrid}}\right) + o(n^{-1})$$

Where,

$$J_{f_{hybrid}} = -E_h \left(\frac{\partial^2 \ln f^\alpha(Y)}{\partial \alpha \, \partial \alpha'}\right)\bigg|_{\alpha_*}$$

and

$$I_{f_{hybrid}} = E_h \left(\frac{\partial \ln f^\alpha(Y)}{\partial \alpha} . \frac{\partial \ln f^\alpha(Y)}{\partial \alpha'}\right)\bigg|_{\alpha_*}.$$

This can be nicely interpreted by saying that the risk $EKL(h, f^{\hat{\alpha}_n})$, is the sum of the misspecification risk $KL(h, f^{\alpha_*})$ plus the statistical risk $\frac{1}{2n} \, Tr(I_{f_{hybrid}} J^{-1}_{f_{hybrid}})$. Note that if (f) is well-specified, we have

$$KL(h, f^{\alpha_*}) = 0 \quad and \quad EKL\left(h, f^{\hat{\alpha}_n}\right) = \frac{p}{2n} + o(n^{-1}).$$

Also based on Commenges et. al. (2008), we have

$$EKL(h, f^{\hat{\alpha}_n}) = -E_h(n^{-1}L_n^f(\hat{\alpha}_n)) + F(h) + \frac{1}{n}Tr(I_{f_{hybrid}} J^{-1}_{f_{hybrid}}) + o_p(n^{-1}) \tag{6}$$

Here we have essentially estimated $E_h(\log f^{\alpha_*}(X))$ by $E_h(\frac{1}{n}L_n^f(\hat{\alpha}_n))$, but because of the overestimation bias, the factor $1/2$ in the last term disappears. Akaike criterion $(AIC(f^{\hat{\alpha}_n}) = -2L_n^f(\hat{\alpha}_n) + 2p)$ follows from (6) by multiplying by $2n$, deleting the constant term, $F(h)$ which we cannot estimate that, and replacing the expected value of the normalized version of maximized likelihood function by its empirical version. Thus,

we can estimate the difference of risks $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ as:

$$\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) =$$
$$E_h \left\{ -\frac{1}{n} \left[L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) - Tr(I_{f_{hybrid}} J_{f_{hybrid}}^{-1}) + Tr(I_{g_{hybrid}} J_{g_{hybrid}}^{-1}) \right] \right\}$$

Now, using the Akaike approximation, $Tr(I_{f_{hybrid}} J_{f_{hybrid}}^{-1}) \approx p$, the simple estimator of $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ is

$$
\begin{aligned}
D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) &= \frac{1}{2n}[AIC(f^{\hat{\alpha}_n}) - AIC(g^{\hat{\beta}_n})] \\
&= -\frac{1}{n}\left[L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) - (p - q) \right]
\end{aligned}
$$

Where, p and q are the number of parameters in two rival models. Also $E_h \left[D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - \Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) \right]$ is an $o(n^{-1})$. Thus, in contrast with AIC, $D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ has an interpretation since its expectation tracks the quantity of main interest $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ with pretty good accuracy. Now, we emphasis on the case where $f^{\alpha_*} \neq g^{\beta_*}$. Thus using theorem 3.1, we have

$$n^{1/2} \left(D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - \Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) \right) \xrightarrow{D} N(0, \omega_{*hybrid}^2)$$

From this, the tracking interval for $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ is given by

$$\left[D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - n^{-1/2} z_{\alpha/2} \, \hat{\omega}_{hybrid} , D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) + n^{-1/2} z_{\alpha/2} \, \hat{\omega}_{hybrid} \right]$$
(7)

This interval has the property as

$$P_h \left[\, A_n < \Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) < B_n \, \right] \to 1 - \alpha$$

where,

$$A_n = D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - n^{-1/2} z_{\alpha/2} \, \hat{\omega}_{hybrid}$$

and

$$B_n = D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) + n^{-1/2} z_{\alpha/2} \, \hat{\omega}_{hybrid}$$

and P_h represents the probability with density h. Tracking interval helps us to evaluate proposed models in comparison with each other. In other words, if the calculated distance includes zero, it can be concluded that

based on the predetermined confidence, both proposed models are equivalent. This interval could be useful in a wide variety of applications. For example, in the dynamics models, we may use an information complexity criterion and AIC to select between models. But, we could not quantitatively assess the difference between models. So, Commenges et. al. (2008) considered the tracking interval between two models in two applications. The first is a study of the relationship between body-mass index and depression in elderly people. The second is the choice between models of HIV dynamics, where one model makes the distinction between activated CD4+T lymphocytes and the other does not.

4   Simulations and Data Analysis

4.1   Simulations

In this subsection we present some numerical experiments, mainly to observe how the two rival models behave for different sample sizes, different parameter values of generated data and for different censoring schemes. because of the application of the Burr distribution (Burr; 1942) in the study of biological, engineering, industrial, reliability and life testing, and several industrial and economic experiments (see for example, Panahi and Sayyareh (2012), Rastogi and Tripathi (2012), Abd-Elfattah and Alharbey (2012), Raqab and Kundu (2012)),we generate 10^4 Monte-Carlo data sets of sample size n from $Burr$ XII ($h_{BXII} = \alpha\beta\ x^{\beta-1}(1 + x^\beta)^{-\alpha-1}$) distribution (which plays the role of the true model). The two rival models (misspecified models) are considered as $f_W^{(p,b)} = pbx^{b-1}e^{-px^b}$ and $g_{BIII}^{(p,b)} = pbx^{-b-1}(1 + x^{-b})^{-p-1}$. We choose different sample sizes, namely n=40 and 50, whereas (α, β) for different sample sizes are taken as $(\alpha, \beta) = (1.5, 1)$, $(2, 1)$ and $(2.5, 1)$. For each case, we estimate the unknown parameters of different rival models using the maximum likelihood method. Then we compute the $D_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$, $\hat{\omega}_{hybrid}$ and construct a 0.95 tracking interval from (7). The results are reported in Table 1. Some of the points are quite clear from Table 1. It is observed that for r=n (complete sample), both limits of tracking intervals are negative, which indicates that the Burr III is better than the Weibull distribution to estimate the true model. Note that, we say one model is better than the other one when the tracking interval does not contain zero. In other words, both limits of the tracking intervals are negative or positive. For the censored observation, the tracking intervals contain zero, which indicates that the Burr III and

Weibull are equal or observationally equal to estimate the true model. Moreover, (i) for fixed r and T as n increases from 40 to 50 the length of the tracking interval decrease, (ii) for fixed n and T as r increase, the length of the tracking interval decrease, (iii) for fixed n, r and T the length of the tracking interval decrease as parameter α increases. Also, it is important to examine how well our proposed interval works for comparing the two rival models under different censoring schemes. So, we compared the tracking intervals in terms of coverage probabilities in Table 1. It is observed that, the coverage probabilities of the tracking intervals are all close to the desired level of 0.95.

Table 1: Choice between Burr III and Weibull models using tracking interval.

$n = 40$		$T = 1$	
Parameters	$r = 30$	$r = 35$	$r = 40$
$(\alpha = 1.5, \beta = 1)$	(-4.02649,3.32243)	(-1.960619,1.53603)	(-0.05856,-0.03034)
	(0.941)	(0.945)	(0.953)
$(\alpha = 2 , \beta = 1)$	(-1.26938,0.97791)	(-0.66210,0.43669)	(-0.02305,-0.01082)
	(0.944)	(0.946)	(0.952)
$(\alpha = 2.5, \beta = 1)$	(-0.42545,0.39794)	(-0.41227,0.40209)	(-0.01864,-0.01007)
	(0.944)	(0.948)	(0.952)
$n = 40$ (Continued)		$T = 2$	
$(\alpha = 1.5, \beta = 1)$	(-1.45283, 1.00837)	(-1.23375, 1.12345)	(-0.06436, -0.03537)
	(0.942)	(0.953)	(0.952)
$(\alpha = 2 , \beta = 1)$	(-0.32106, 0.30367)	(-0.31232, 0.25679)	(-0.02104, -0.00995)
	(0.944)	(0.946)	(0.949)
$(\alpha = 2.5, \beta = 1)$	(-0.12034, 0.04098)	(-0.10622, 0.051583)	(-0.01805, -0.00854)
	(0.947)	(0.952)	(0.951)
$n = 50$		$T = 1$	
Parameters	$r = 40$	$r = 45$	$r = 50$
$(\alpha = 1.5, \beta = 1)$	(-2.12008, 1.61159)	(-0.87032, 0.52300)	(-0.05481, -0.03276)
	(0.943)	(0.945)	(0.952)
$(\alpha = 2 , \beta = 1)$	(-1.22309, 0.93760)	(-0.36080, 0.21071)	(-0.02484, -0.01418)
	(0.946)	(0.948)	(0.949)
$(\alpha = 2.5, \beta = 1)$	(-0.41308, 0.31154)	(-0.19061, 0.13823)	(-0.01399, -0.00641)
	(0.947)	(0.951)	(0.949)
$n = 50$ (Continued)		$T = 2$	
$(\alpha = 1.5, \beta = 1)$	(-1.36443, 1.01692)	(-0.25908, 1.04317)	(-0.05591, -0.03461)
	(0.944)	(0.947)	(0.952)
$(\alpha = 2 , \beta = 1)$	(-0.26457, 0.08886)	(-0.23157, 0.15087)	(-0.01512, -0.02558)
	(0.953)	(0.948)	(0.950)
$(\alpha = 1.5, \beta = 1)$	(-0.10775, 0.03190)	(-0.08056, 0.01870)	(-0.00945, -0.001409)
	(0.948)	(0.951)	(0.949)

The first and second rows represent the average tracking intervals and the corresponding coverage probabilities.

4.2   Real Data Analysis

In this subsection we consider one real data set to construct the tracking interval proposed in the section 3. It is degree of micro-droplet splashing data originally reported by Montavon et. al. (1997). The authors are thankful to Dr. Asadi (2008; 2012), for providing the data. The degree of splashing, *D.S*, is defined as

$$D.S = \frac{1}{4\pi}.\frac{P^2}{A}$$

where, A is the area of the selected feature and P is the perimeter. The degrees of splashing droplet data are reported in different spray angles. We use the data of 90° spray angle (See, Montavon et. al.; 1997). First we want to check whether the Burr XII distribution fits the data set or not, and that we have use the complete data set. For this purpose, we present the q-q plot of this data in Figure 2. This plot shows a strong relationship supporting the appropriateness of the Burr XII distribution.
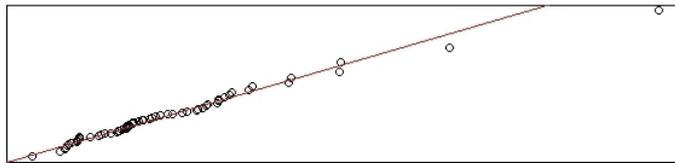

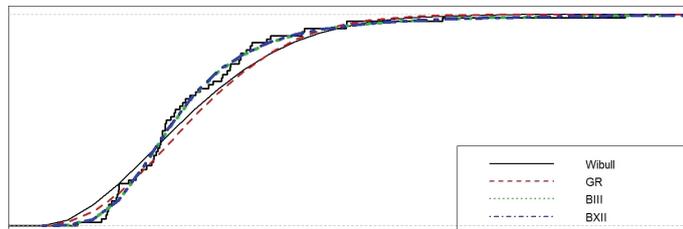Figure 2: The q-q plot of droplet splashing data

Figure 3: Empirical survival function and the fitted survival functions for droplet splashing data.

For comparison purposes, we fit Burr XII (BXII), Weibull (W), generalized Rayleigh (GR) and Burr III (BIII) distributions to the complete observation. The plot of the empirical and the fitted cumulative distribution functions for different distributions is presented in Figure

3. The estimated parameter values, AIC values, Kolmogorov-Smirnov (K-S) distances and the corresponding p-values are presented in Table 2. From the K-S distances, AIC values and p-values of Table 2, it is quite clear that the Burr XII model with estimated parameters $p = 0.89478$ and $b = 3.6601$ provides much better fit than the other distributions.

Table 2: Estimated parameters, K-S distances and AIC values for different distribution functions of droplet splashing data.

Distribution	Estimated parameters	K-S (p-value)	AIC
W	$p = 0.56168, b = 1.85330$	0.1134 (0.4237)	113.6794
BXII	$p = 0.89478, b = 3.66013$	0.0744 (0.8944)	98.4622
BIII	$p = 1.12448, b = 3.38375$	0.0756 (0.8825)	98.5607
GR	$p = 1.09763, b = 0.73915$	0.1281 (0.2786)	167.5384

$$f_W^{(p,b)} = pbx^{b-1}e^{-px^b}, f_{BXII}^{(p,b)} = pbx^{b-1}(1+x^b)^{-p-1}, f_{BIII}^{(p,b)} = pbx^{-b-1}(1+x^{-b})^{-p-1}, f_{GR}^{(p,b)} = 2pb^2xe^{-(bx)^2}(1-e^{-(bx)^2})^{p-1}$$

Furthermore, we present the tracking intervals of two rival models using (7). It is assumed the following four different cases of rival models and censoring schemes (n=60):

Case 1: Burr III (misspecified model; (f)) and Weibull (misspecified model;(g)) & $r = 55$, $T = 3$ (R=58).

Case 2: Burr III (misspecified model; (f)) and Weibull (misspecified model; (g)) & $r = 38$, $T = 1$ (R=38).

Case 3: Burr XII (well-specified model;(f)) and Weibull (misspecified; (g)) & $r = 55$, $T = 3$ (R=58).

Case 4: Burr XII (well-specified model; (f)) and Weibull (misspecified; (g)) & $r = 38$, $T = 1$ (R=38).

Note that, the true model (h) is Burr XII distribution. In all the cases we have estimated the unknown parameters using the MLEs and then constructed the tracking intervals. For cases 1, 2, 3 and 4, the tracking intervals are (-0.165910, -0.014091) and (-3.809721, 3.784794), (-0.166438, -0.012277) and (-3.815360, 3.791280) respectively. For case 1, it is observed that both limits of the tracking interval are negative, which indicates that the Burr III is better than the Weibull density to estimate the true model for splashing data. We have plotted the different estimated density functions and the relative histogram of this case in Figure 4. For case 2, zero is well inside this interval, so there is no
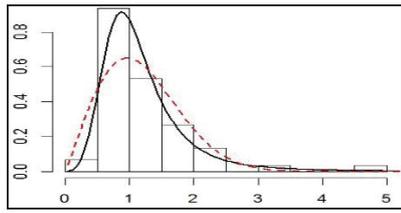
Figure 4: The two fitted rival models (Case 1), Weibull (dashed line) and Burr III (solid line) for droplet splashing data.
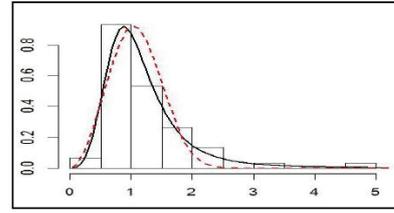

Figure 5: The two fitted rival models (Case 2), Weibull (dashed line) and Burr III (solid line) for droplet splashing data.

confidence that we incur a lower risk using Burr III rather than Weibull distribution (see also Figure 5). As we expected, the Burr XII is better than the Weibull density to estimate the true model for case 3. But with decreasing the number of failures (R), the Burr XII and Weibull distributions are equivalent to consider as an estimate for the true model.

For more comparison, we consider the Voung test (Voung; 1989) to confirm the results of tracking interval. For cases 1-4, the Voung statistics $\Im$ are (2.27215), (0.00621), (2.32380) and (0.00643) respectively. It is observed that for cases 2 and 4, $|\Im| < Z_{1-\alpha}$. Thus, we cannot discriminate between the two rival models. Also, for cases 1 and 3, $\Im > Z_{1-\alpha}$ and we can conclude that the Burr III and Burr XII are better than Weibull density to estimate the true model for a given data respectively.

Another important problem in engineering experiments namely the prediction interval of the future observation, based on the current available observation. So, we obtain the prediction interval of the $Z = Y_{d+R}$; $d = 1, 2, ..., n - R$ using observed data $\underline{y} = (y_1, ..., y_R)$. From Asgharzadeh et. al. (2013), the conditional distribution of $(d + R)^{th}$ order statistic under Type II hybrid censoring scheme is given by

$$f(z \,\big|\, \underline{y}, p, b) = \frac{(n - R)!}{(d - 1)!(n - R - d)!} \frac{(F(z) - F(U))^{\,d-1}(1 - F(z))^{\,n-R-d} f(z)}{(1 - F(U))^{\,n-R}}$$

(8)

For Burr XII distribution, $f(z \,\big|\, \underline{y}, p, b)$ can be written as

$$f(z \,\big|\, \underline{y}, p, b) \propto pbz^{b-1} \times \left[(1 + U^b)^{-p} - (1 + z^b)^{-p}\right]^{\,d-1} \\ \times (1 + U^b)^{p\,(n-R)} \times (1 + z^b)^{-p\,(n-R-d+1)-1}$$

Now, we consider the pivotal quantity $W = 1 - \frac{(1+Z^b)^{-p}}{(1+U^b)^{-p}}$ to obtain the prediction interval of Z under Type II hybrid censored sample. Obvi-

ously, the distribution of W given $\underline{Y} = \underline{y}$ is a $Beta\,(d\,,\,n - R - d + 1)$ distribution with pdf

$$f(w) = \frac{w^{d-1}(1-w)^{n-R-d}}{Beta(d, n-R-d+1)}; \quad 0 < w < 1.$$

Therefore, the $100(1 - \wp)\%$ prediction interval for the $(d + R)^{th}$ order statistic Z is given by

$$\{[(1 - B_{\wp/2})^{-1/p}(1 + U^b) - 1]^{1/b}, [(1 - B_{1-\wp/2})^{-1/p}(1 + U^b) - 1]^{1/b}\}$$

where, $B_\wp$ is the $\wp^{th}$ percentile of the Beta distribution with parameters d and $n - R - d + 1$ respectively. Now, we replace the unknown parameters with their MLEs and then obtain the 95% prediction interval for Z. The results for different censoring schemes are presented in Table 3.

Table 3: The %95 prediction intervals (PIs) for $Z = Y_{d+R}$ and their real values ($n = 60$).

d	Real Values	PIs	d	Real Values	PIs
1	1.35531	(1.31942 , 1.36752)	1	2.41025	(2.40494 , 2.52847)
9	1.57509	(1.47290 , 1.64380)	2	2.41758	(2.45509 , 2.78697)
12	1.66300	(1.58434 , 1.83109)	3	3.17948	(2.57347 , 3.21704)
8	2.41758	(2.04480 , 2.72187)	4	4.63004	(2.82505 , 4.55312)

5 Conclusion

In the present work, we examined how the two rival models behave under Type II hybrid censoring scheme. We considered an interval, say, tracking interval for differences of the expected KL of two rival models. Our approach enlightens the variability of any criterion based on log-likelihood function, like AIC and their variants. To introduce the tracking interval, we proposed a statistic which tracks the difference of the expected KL risks between maximum likelihood estimators in two non-nested rival models. When the models are nested minus two times of the log-likelihood function is comparable with our idea. We compared the behavior of two rival models using Monte Carlo simulation and using the tracking interval for different sample sizes and different censoring schemes. The results of our simulation study were in agreement with the theoretical results. For an application, we considered several statistical

distribution functions to analyze the micro-droplet splashing data. Using several statistical criteria, like minimum Kolmogorov distance and minimum AIC value, the Burr XII distribution with estimated parameters (0.89478, 3.66013) appears to be more appropriate statistical distribution for this data set. Also, we have obtained the tracking intervals for comparing the two rival models based on different censoring schemes and found that as R decreases, the tracking intervals for splashing data contain zero which indicate that the two rival models are equivalent to consider as an estimate for the Burr XII distribution. These results have been observed using other criteria. One important problem in engineering sciences is that the prediction of the future observations. So we considered the prediction intervals of future droplet splashing observations based on Type II hybrid censored sample. It is observed that these intervals work well. It may be mentioned although we have mainly considered Type II hybrid censoring case, but this interval can be extended for other censoring schemes also. More work is needed in these directions.

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