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On Multivariate Likelihood Ratio Ordering among Generalized Order Statistics and their Spacings

Maryam Sharafi, Baha-Eldin Khaledi, Nasrin Hami

Department of Statistics, Razi University, Kermanshah, Iran.

Abstract. The most of the results obtained about stochastic properties of generalized order statistics and their spacings in the literature are based on equal model parameters. In this paper, with less restrictive conditions on the model parameters, we prove some new multivariate likelihood ratio ordering results between two sub-vectors of GOS's as well as two sub-vectors of p-spacings based on two continuous distribution functions. In particular, we apply the new results to obtain some computable bounds on the mean residual life of some unobserved progressive type II censored order statistics.

Keywords. Hazard rate order, log-convex density, MTP_2 and TP_2 functions, progressive Type-II censored order statistics, residual lifetime and univariate likelihood ratio order.

MSC: 60E15, 62G30, 62N01, 62N05.

Maryam Sharafi(mmaryamsharafi@gmail.com), Baha-Eldin Khaledi(✉)(bkhaledi @hotmail.com), Nasrin Hami(n_hami702@yahoo.com)
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1 Introduction

Let $X_{(1,n,\tilde{m},k)}, \cdots, X_{(n,n,\tilde{m},k)}$ be n generalized order statistics (GOS's) based on distribution function F with joint density function

$$f(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\prod_{j=1}^{n-1}(1-F(x_j)^{m_j}(1-F(x_n))^{k-1}\prod_{j=1}^{n}f(x_j), \tag{1.1}$$

where $n \in \mathbb{N}$, $k > 0$, $m_1, \cdots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \cdots, n-1\}$, and let $\widetilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$) (see Kamps, 1995a,b). It is known that the distribution of GOS's from F is the same as that of

$$(F^{-1}(U(1,n,\widetilde{m},k)), F^{-1}(U(2,n,\widetilde{m},k)), \ldots, F^{-1}(U(n,n,\widetilde{m},k))), \tag{1.2}$$

where $(U(1,n,\widetilde{m},k), U(2,n,\widetilde{m},k), \ldots, U(n,n,\widetilde{m},k))$, is the vector of GOS's from a uniform distribution over $(0,1)$ with density function

$$h(u_1,u_2,\ldots,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\prod_{j=1}^{n-1}(1-u_j)^{m_j}(1-u_n)^{k-1}, \ \ u_1 \leq \ldots \leq u_n. \tag{1.3}$$

From Eq. (1.3), upon integrating out appropriate variables, we obtain the joint density function of $(U(1,n,\widetilde{m},k), U(2,n,\widetilde{m},k), \ldots, U(i,n,\widetilde{m},k))$, for $1 \leq i \leq n$, as

$$h(u_1,u_2,\ldots,u_i) = c_{i-1}\left(\prod_{j=1}^{i-1}(1-u_j)^{m_j}\right)(1-u_i)^{\gamma_i-1}, \ \ 0 \leq u_1 \leq \ldots \leq u_i < 1.$$

where the constant c_{i-1} is defined by $c_{i-1} = \prod_{j=1}^{i}\gamma_j$, $i = 1, \cdots, n-1$, $c_0 = 0$, and $\gamma_n = k$.

Using specific set of parameters m_i's and k, various ordered statistical data like usual order statistics, record values, progressive censoring, sequential order statistics among others are special cases of GOS's. For more details the reader is referred to Kamps (1995a) and Khaledi (2005).

As defined in Xie and Hu (2009), for a given positive integer $p \leq n$, let denote the vector of p-spacings of $X_{(i,n,\widetilde{m},k)}$'s by

$$\mathbf{D_X}^{(p)} = (D_{X,1,n}^{(p)}, D_{X,2,n}^{(p)}, \ldots, D_{X,n-p,n}^{(p)}),$$

where $D^{(p)}_{X,r,n} = X_{(r+p,n,\widetilde{m},k)} - X_{(r,n,\widetilde{m},k)}$, $r = 0, 1, \cdots, n-p$ and $X_{(0,n,\widetilde{m},k)} = 0$.

Stochastic comparisons of GOS's as well as spacings of GOS's in one sample as well as two sample problems were discussed in Belzunce et al. (2005), Khaledi (2005), Khaledi and Kochar (2005), Hu and Zhuang (2005), Hu and Zhuang (2006), Fang et al. (2006), Hu et al. (2007), Zhuang and Hu (2007), Xie and Hu (2009) and Torrado et al. (2012). Recently, Balakrishnan et al. (2010) established some new stochastic ordering results among GOS's according to univariate as well as multivariate likelihood ratio orders, from which they obtained some interesting new results about stochastic comparisons of conditional GOS's which cover many results for usual order statistics obtained by Khaledi and Shaked (2007), Li and Zhao (2008), Zhao and Balakrishnan (2009) and many results for record values obtained by Khaledi and Shojaei (2007) and Khaledi et al. (2009).

The most of the results obtained about stochastic properties of GOS's and their spacings in the above mentioned references are based on the condition that the parameters m_i's in (1.1) are all equal. In this paper with less restrictive conditions on the m_i's, we prove some new multivariate likelihood ratio ordering results between two sub-vectors of GOS's as well as two sub-vectors of p-spacings based on two distribution functions F and G.

The notion of multivariate likelihood ratio order and univariate likelihood ratio order is defined next.

Definition 1.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate likelihood ratio order, denoted by $\mathbf{X} \leq_{\rm lr} \mathbf{Y}$, if

$$\begin{aligned} &f_{\mathbf{X}}(x_1, x_2, \cdots, x_n) f_{\mathbf{Y}}(y_1, y_2, \cdots, y_n) \\ &\leq f_{\mathbf{X}}(x_1 \wedge y_1, x_2 \wedge y_2, \cdots, x_n \wedge y_n) f_{\mathbf{Y}}(x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n) \end{aligned}$$

for all $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, where $\wedge$ and $\vee$ denote the minimum and maximum operations, respectively.

Given a random vector $\mathbf{X}$ with density function $f_{\mathbf{X}}$, we say that $\mathbf{X}$ or $f_{\mathbf{X}}$ is MTP$_2$ (multivariate totally positive of order 2) if $\mathbf{X} \leq_{\rm lr} \mathbf{X}$. It is known that the multivariate likelihood ratio order as well as MTP_2 property of a random vector is closed under the marginal operator.

In the univariate case, given two random variables X and Y with density functions f and g, respectively, we say that X is less than Y in the likelihood ratio order, denoted by $X \le_{\mathrm{lr}} Y$, if $f(t)g(s) \le f(s)g(t)$ for all $s < t \in \mathbb{R}$. For more details about these orderings the reader is referred to Shaked and Shanthikumar (2007, chapters 1 and 6).

Balakrishnan et al. (2010) considered the problem of stochastic comparisons of GOS's and proved the following result.

Theorem 1.2. *(Balakrishnan et al. (2010)) Let X and Y be two absolutely continuous random variables with distribution functions F and G, densities f and g, and hazard rates r_F and r_G, respectively. Let $\mathbf{X} = (X_{(1,n,m,k)}, \cdots, X_{(n,n,m,k)})$ and $\mathbf{Y} = (Y_{(1,n',m,k)}, \cdots, Y_{(n',n',m,k)})$ be random vectors of m-GOS's based on distributions F and G, respectively. For $r_1 \le r_2 < \cdots \le r_i \le n$, $r'_1 \le r'_2 \le \cdots \le r'_i \le n'$, $r'_1 - r_1 = r'_2 - r_2 = \cdots = r'_i - r_i \ge \max\{0, n' - n\}$, if either*

(i) $X \le_{\mathrm{lr}} Y$ *and* $m \ge 0$, *or*

(ii) $X \le_{\mathrm{hr}} Y$, $r_G(x)/r_F(x)$ *is increasing in* x, *and* $-1 \le m < 0$,

then

$$\begin{aligned}&(X_{(r_1,n,m,k)}, X_{(r_2,n,m,k)}, \ldots, X_{(r_i,n,m,k)}) \le_{\mathrm{lr}} \\ &\qquad (Y_{(r'_1,n',m,k)}, Y_{(r'_2,n',m,k)}, \ldots, Y_{(r'_i,n',m,k)}).\end{aligned}$$

Let F denote an absolutely distribution function of a non-negative random variable X and assume that the vector of parameters $m_X = (m_1, m_2, \ldots, m_{n-1})$ in (1.1) for $r_1 < r_2 < \cdots < r_i$, $\{r_1, r_2, \ldots, r_i\} \subset \{1, 2, 3, \ldots, n\}$, satisfy that

$m_1 = m_2 = \cdots = m_{r_1-1} = \mu_1$, $m_{r_1} = \mu^{(r_1)}$,
$m_{r_1+1} = m_{r_1+2} = \cdots = m_{r_2-1} = \mu_2$, $m_{r_2} = \mu^{(r_2)}$,
$m_{r_2+1} = m_{r_2+2} = \cdots = m_{r_3-1} = \mu_3$, $m_{r_3} = \mu^{(r_3)}$, $\ldots$,
$m_{r_{i-1}+1} = m_{r_{i-1}+2}, \cdots = m_{r_i-1} = \mu_i$, $m_{r_i} = \mu^{(r_i)}$,
and $m_{r_i+l} = \mu_{r_i+l}$, $l = 1, \cdots, n-1-r_i$. That is

$$\begin{aligned}\mathbf{m_X} \;=\; &(\mu_1, \ldots, \mu_1, \mu_1^{(r_1)}, \mu_2, \ldots, \mu_2, \mu_2^{(r_2)}, \ldots, \mu_i, \ldots, \mu_i, \mu^{(r_i)}, \\ &\mu_{r_{i+1}}, \mu_{r_{i+2}}, \ldots, \mu_{n-1}). \qquad (1.4)\end{aligned}$$

Similarly, we assume that G denote an absolutely distribution function of a non-negative random variable Y. Let, the vector of parameters $m_Y = (m_1, m_2, \ldots, m_{n-1})$ in (1.1) for $r_1 < r_2 < \cdots < r_i$, $\{r_1, r_2, \ldots, r_i\} \subset \{1, 2, 3, \ldots, n\}$, satisfy that

$m_1 = m_2 = \cdots = m_{r_1-1} = \gamma_1$, $m_{r_1} = \gamma^{(r_1)}$,
$m_{r_1+1} = m_{r_1+2} = \cdots = m_{r_2-1} = \gamma_2$, $m_{r_2} = \gamma^{(r_2)}$,
$m_{r_2+1} = m_{r_2+2} = \cdots = m_{r_3-1} = \gamma_3$, $m_{r_3} = \gamma^{(r_3)}$,$\ldots$,

$m_{r_{i-1}+1} = m_{r_{i-1}+2}, \cdots = m_{r_i-1} = \gamma_i$, $m_{r_i} = \gamma^{(r_i)}$,
and $m_{r_i+l} = \gamma_{r_i+l}$, $l = 1, \cdots, n-1-r_i$. That is

$$\begin{aligned} \mathbf{m_Y} &= (\gamma_1, \dots, \gamma_1, \gamma_1^{(r_1)}, \gamma_2, \dots, \gamma_2, \gamma_2^{(r_2)}, \dots, \gamma_i, \dots, \gamma_i, \gamma^{(r_i)}, \\ &\quad \gamma_{r_{i+1}}, \gamma_{r_{i+2}}, \dots, \gamma_{n-1}). \end{aligned} \tag{1.5}$$

In Section 2, we generalize Theorem 1.2 to the cases when the set of parameters of GOS's based on F is $\mathbf{m_X}$ and that of GOS's based on G is $\mathbf{m_Y}$ (Theorem 2.6). In particular, we applied this result to obtain some computable bounds on the mean residual of some unobserved progressive type II censored order statistics.

Hu and Zhuang (2006) considered the problem of stochastic comparisons of p-spacings of GOS's when $m_1 = m_2 = \dots m_{n-1} = m$ and proved that if $X \leq_{lr} Y$ and either X or Y has log-convex density, then $D^{(p)}_{X,r,n} \leq_{lr} D^{(p)}_{Y,r,n}$, $r = 0, \dots, n-p$. Xie and Hu (2009) further studied this problem and in one sample problem proved that if $m_1 \geq m_2 \geq \dots \geq m_{n-1} \geq 0$ and F has log-convex density, then $D^{(p)}_{X,r,n} \leq_{lr} D^{p}_{X,r+1,n}$ and $D^{(p)}_{X,r,n+1} \leq_{lr} D^{(p)}_{X,r,n}$. Fang et al. (2006) obtained a multivariate likelihood ratio order result between two vectors of 1-spacings of GOS's in one sample problem. They showed that if F has log-convex density and $m_j \geq 0$, $j = 1, \dots, n$, then

$$(D^{(1)}_{X,1,n+1}, D^{(1)}_{X,2,n+1}, \dots, D^{(1)}_{X,n,n+1}) \leq_{lr} (D^{(1)}_{Y,1,n}, D^{(1)}_{Y,2,n}, \dots, D^{(1)}_{Y,n,n})$$

In Section 3, we prove that if $X \leq_{lr} Y$ and either X or Y has log-convex density, then

$$(D^{(p)}_{X,r_1,n}, D^{(p)}_{X,r_2,n}, \dots, D^{(p)}_{X,r_n-p,n}) \leq_{lr} (D^{(p)}_{Y,r'_1,n}, D^{(p)}_{Y,r'_2,n}, \dots, D^{(p)}_{Y,r'_n-p,n}),$$

where $D^{(p)}_{X,r_j,n}$ is the jth p-spacing of GOS's based on distribution function F with parameter $\mathbf{m_X}$ given in (1.4) and $D^{(p)}_{Y,r'_j,n}$ is the jth p-spacing of GOS's based on distribution function G with parameter $\mathbf{m_Y}$ given in (1.5).

2 Multivariate lr Ordering among GOS's

We shall be using the notion of totally positive of order 2 in this paper. We say that a function $h(x, y)$ is totally positive of order 2 (TP_2) if $h(x, y) \geq 0$ and

$$\begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0, \tag{2.1}$$

whenever $x_1 < x_2$, $y_1 < y_2$. We use the following lemma of Karlin (1968 p. 99).

Lemma 2.3. *Let A, B and C be subsets of the real line and let $L(x,z)$ be TP_2 in (x,z) and $M(x,y,z)$ be TP_2 in each pairs of (z,x), (z,y) and (x,y) for $x \in A$, $z \in B$ and $y \in C$. Then $K(x,y) = \int L(x,z)M(x,y,z)\,d\mu(z)$ is TP_2 in (x,y) for $x \in A$, $y \in C$. Here μ is a sigma-finite measure.*

To prove the main results in this section we also need the following lemma which is a modified version of Lemma 2.5 in Balakrishnan et al. (2010).

Lemma 2.4. *Given random variables X and Y with distribution functions F and G, if $\mu \geq \gamma$ and*

(i) $X \leq_{lr} Y$, $\gamma \geq 0$ or

(ii) $X \leq_{hr} Y$, $r_G(x)/r_F(x)$ is increasing in x, $-1 \leq \gamma < 0$,

then

(a) $h_\gamma(G(x))/h_\mu(F(x))$ is increasing in $x \in \mathbb{R}$, and

(b) for $\mu \geq \gamma$, the function

$$l(x,y) = \frac{h_\gamma(G(y)) - h_\gamma(G(x))}{h_\mu(F(y)) - h_\mu(F(x))}$$

is increasing in $(x,y) \in \mathbb{R}$, where

$$h_m(x) = \begin{cases} \frac{1}{m+1}(1-(1-x)^{m+1}), & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases} \tag{2.2}$$

Proof. Let for $\mu \neq -1$, $\gamma \neq -1$, $h_1^*(x)$ and $h_2^*(x)$ be derivatives of $h_\mu(F(x))$ and $h_\gamma(G(x))$ with respect to x, respectively. That is $h_1^*(x) = f(x)(\bar{F}(x))^\mu$, $h_2^*(x) = g(x)(\bar{G}(x))^\gamma$.

Let (i) holds, we give the proofs of parts (a) and (b).

(a) For $x \geq 0$ and $i = 1,2$ denote

$$\varphi_i(x) = \int_0^\infty h_i^*(w).I_{(0<w\leq x)}(w)dw.$$

Since the likelihood ratio order implies the hazard rate order and $\mu \geq \gamma$, it follows that

$$\frac{h_2^*(x)}{h_1^*(x)} = \frac{g(x)}{f(x)}\left[\frac{\bar{G}(x)}{\bar{F}(x)}\right]^\gamma \left[\frac{1}{\bar{F}(x)}\right]^{\mu-\gamma}$$

is increasing in $x \in \Re_+$. That is, the function $h_i^*(w)$ is TP_2 in $(i, w) \in \{1, 2\} \times \Re_+$. On the other hand, the function $I_{(0<w\leq x)}(w)$ is TP_2 in $(w, x) \in (0, x) \times \Re_+$. Combining these observations, it follows from Lemma 2.3 that $\varphi_i(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times \Re_+$ which is the required result.

(b) For $y \geq x$ and $i = 1, 2$, let

$$\psi_i(x, y) = \int_0^{\infty} h_i^*(w) I_{(x<w\leq y)}(w) dw.$$

From part (a), the function $h_i^*(w)$ is TP_2 in $(i, w) \in \{1, 2\} \times \Re_+$. It is easy to show that the indicator function $I_{(x<w\leq y)}$ is TP_2 in $(w, x) \in \Re_+ \times (0, y)$ for each fixed $y \in \Re_+$, and is TP_2 in $(w, y) \in \Re_+ \times (x, \infty)$ for each fixed $x \in \Re_+$. Therefore, by Lemma 2.3, the function $\psi_i(x, y)$ is TP_2 in $(i, x) \in \{1, 2\} \times (0, y)$ for each fixed $y \in \Re_+$ and is TP_2 in $(i, y) \in \{1, 2\} \times (x, \infty)$ for each fixed $x \in \Re_+$ which are equivalent to that

$$l(x, y) = \frac{\psi_2(x, y)}{\psi_1(x, y)}$$

is increasing in $(x, y) \in \Re_+^2$, $y \geq x$.

Now, let (ii) holds. It is seen that

▲ For $-1 < \gamma < 0$,

$$\begin{aligned} \frac{h_2^*(x)}{h_1^*(x)} &= \frac{g(x)}{f(x)} \left[\frac{\bar{G}(x)/g(x)}{\bar{F}(x)/f(x)}\right]^{\gamma} \left[\frac{g(x)}{f(x)}\right]^{\gamma} \\ &= \left[\frac{g(x)}{f(x)}\right]^{\gamma+1} \times \left[\frac{r_F(x)}{r_G(x)}\right]^{\gamma} \times \left[\frac{1}{\bar{F}(x)}\right]^{\mu-\gamma} \end{aligned} \tag{2.3}$$

is increasing in $x \in \Re_+$, since if $r_G(x)/r_F(x)$ is increasing in x and $X \leq_{hr} Y$ then $X \leq_{lr} Y$ (cf. Belzunce et al., (2001), Lemma 3.5).

▲ For $\gamma = -1$ and $\mu = -1$, if $r_G(x)/r_F(x)$ is increasing in x, then

$$\begin{aligned} \frac{h_{-1}(G(x))}{h_{-1}(F(x))} &= \frac{-\log \bar{G}(x)}{-\log \bar{F}(x)} \\ &= \frac{\int_0^u r_G(u)du}{\int_0^x r_F(u)du} \end{aligned} \tag{2.4}$$

is increasing in $x \in \Re_+$.

Now, the required results follows by the same arguments used to prove the results under the assumptions given in (i).

▲ For $\gamma = -1$ and $\mu > -1$, define derivative of $h_{-1}(G(x))$ by $h_3^*(x) = \frac{g(x)}{\bar{G}(x)}$, and

$$\varsigma_i(x) = \int_0^\infty h_i^*(w).I_{(0<w\leq x)}(w)dw,$$

where $x \geq 0$ and $i = 1, 3$. Since the hazard rate order satisfies and $\mu + 1 > 0$, thus

$$\frac{h_3^*(x)}{h_1^*(x)} = \frac{r_G(x)}{r_F(x)} \left[\frac{1}{\bar{F}(x)}\right]^{\mu+1}$$

is increasing in $x \in \Re_+$. That is, the function $h_i^*(w)$ is TP_2 in $(i, w) \in \{1, 3\} \times \Re_+$. On the other hand, the function $I_{(0<w\leq x)}(w)$ is TP_2 in $(w, x) \in (0, x) \times \Re_+$. With using these observations and again applying Lemma 2.3, we conclude that $\varsigma_i(x)$ is TP_2 in $(i, x) \in \{1, 3\} \times \Re_+$. This completes the proof of the Lemma. □

Let $\mathbf{X}^{(\mathbf{i})}_{\mathbf{m_X}} = (X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$ be a subvector of GOS's based on absolutely continuous distribution function F, where $\mathbf{m_X}$ is the same as that in (1.4). To prove the main results we need to derive an expression for the density function of $\mathbf{X}^{(\mathbf{i})}_{\mathbf{m_X}}$, denoted by $f_{r_1,r_2,\ldots,r_i}$, which is of independent interest, includes the m-GOS's as an special case, covers the results of Lemma 2.6 in Balakrishnan et al. (2010) and generalizes the result of Lemma 3.1.7. in Kamps (1995a).

Lemma 2.5. *Given a random vector $(X_{(1,n,\mathbf{m_X},k)}, \cdots, X_{(n,n,\mathbf{m_X},k)})$ of GOS's from an absolutely continuous distribution function F and density function f, the joint density of $\mathbf{X}^{(\mathbf{i})}_{\mathbf{m_X}}$ for $r_1 < r_2 < \cdots < r_i$, $i \leq n$ and $x_{r_1} < x_{r_2} < \cdots < x_{r_i}$ is given by*

$$\begin{aligned} & f_{(r_1,r_2,\ldots,r_i)}(x_{r_1}, x_{r_2}, \ldots, x_{r_i}) \\ &= \frac{c_{r_i-1}}{(r_1-1)!\prod_{j=1}^{i-1}(r_{j+1}-r_j-1)!}(\overline{F}(x_{r_i}))^{k+n-r_i+M_{r_i}-1} f(x_{r_i}) h_{\mu_1}^{r_1-1}(F(x_{r_1})) \\ &\times \prod_{j=1}^{i-1}(\overline{F}(x_{r_j}))^{\mu^{(r_j)}} [h_{\mu_{j+1}}(F(x_{r_{j+1}})) - h_{\mu_{j+1}}(F(x_{r_j}))]^{r_{j+1}-r_j-1} f(x_{r_j}), \end{aligned} \tag{2.5}$$

where $c_{r_i-1} = \prod_{i=1}^{r_i} \gamma_i$ and h_m and $\mathbf{m_X}$ are as given in (2.2) and (1.4), respectively.

Proof. It follows from (1.2) that the distribution of

$$(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$$

is the same as that of

$$(F^{-1}(U(r_1,n,\mathbf{m_X},k)),F^{-1}(U(r_2,n,\mathbf{m_X},k)),\ldots,F^{-1}(U(r_i,n,\mathbf{m_X},k))). \tag{2.6}$$

The joint density function of

$$(U(r_1,n,\mathbf{m_X},k),U(r_2,n,\mathbf{m_X},k),\ldots,U(r_i,n,\mathbf{m_X},k))$$

can be derived as follows:

$$\begin{aligned}
&h_{(r_1,r_2,\ldots,r_i)}(u_{r_1},u_{r_2},\ldots,u_{r_i})\\
&= \int_0^{u_{r_1}}\int_{u_1}^{u_{r_1}}\cdots\int_{u_{r_1-2}}^{u_{r_1}}\int_{u_{r_1}}^{u_{r_2}}\int_{u_{r_1+1}}^{u_{r_2}}\cdots\int_{u_{r_2-2}}^{u_{r_2}}\cdots\\
&\quad \int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}}\int_{u_{r_{i-1}}}^{u_{r_i}}\int_{u_{r_{i-1}+1}}^{u_{r_i}}\cdots\int_{u_{r_i-3}}^{u_{r_i}}\int_{u_{r_i-2}}^{u_{r_i}} h(u_1,u_2\cdots,u_{r_i})\\
&\quad du_{r_i-1}du_{r_i-2}\cdots du_{r_{i-1}+2}du_{r_{i-1}+1}du_{r_{i-1}-1}\cdots\\
&\quad du_{r_2-1}\cdots du_{r_1+2}du_{r_1+1}du_{r_1-1}\cdots du_2du_1\\
&= \int_0^{u_{r_1}}\int_{u_1}^{u_{r_1}}\cdots\int_{u_{r_1-2}}^{u_{r_1}}\int_{u_{r_1}}^{u_{r_2}}\int_{u_{r_1+1}}^{u_{r_2}}\cdots\\
&\quad \int_{u_{r_2-2}}^{u_{r_2}}\cdots\int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}}\int_{u_{r_{i-1}}}^{u_{r_i}}\int_{u_{r_{i-1}+1}}^{u_{r_i}}\cdots\\
&\quad \int_{u_{r_i-3}}^{u_{r_i}}\int_{u_{r_i-2}}^{u_{r_i}} c_{r_i-1}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1}\Pi_{j=1}^{r_1-1}h'_{\mu_1}(u_j)(1-u_{r_1})^{m_{r_1}}\\
&\quad \times\Pi_{j=r_1+1}^{r_2-1}h'_{\mu_2}(u_j)(1-u_{r_2})^{m_{r_2}}\Pi_{j=r_2+1}^{r_3-1}h'_{\mu_3}(u_j)\cdots\\
&\quad \times(1-u_{r_{i-1}})^{m_{r_{i-1}}}\Pi_{j=r_{i-1}+1}^{r_i-1}h'_{\mu_i}(u_j)\\
&\quad du_{r_i-1}du_{r_i-2}\cdots du_{r_{i-1}+2}du_{r_{i-1}+1}du_{r_{i-1}-1}\ldots du_{r_2-1}\cdots\\
&\quad du_{r_1+2}du_{r_1+1}du_{r_1-1}\cdots du_2du_1\\
&= c_{r_i-1}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1}\\
&\qquad \times\int_0^{u_{r_1}}\int_{u_1}^{u_{r_1}}\cdots\int_{u_{r_1-2}}^{u_{r_1}}\Pi_{j=1}^{r_1-1}h'_{\mu_1}(u_j)(1-u_{r_1})^{m_{r_1}}\\
&\quad \int_{u_{r_1}}^{u_{r_2}}\int_{u_{r_1+1}}^{u_{r_2}}\cdots\int_{u_{r_2-2}}^{u_{r_2}}\Pi_{j=r_1+1}^{r_2-1}h'_{\mu_2}(u_j)(1-u_{r_2})^{m_{r_2}}\cdots\\
&\quad \int_{u_{r_{i-1}-2}}^{u_{r_{i-1}}}\int_{u_{r_{i-1}}}^{u_{r_i}}\int_{u_{r_{i-1}+1}}^{u_{r_i}}\cdots\\
&\quad \int_{u_{r_i-3}}^{u_{r_i}}\int_{u_{r_i-2}}^{u_{r_i}}\Pi_{j=r_{i-1}+1}^{r_i-1}h'_{\mu_i}(u_j)(1-u_{r_{i-1}})^{m_{r_{i-1}}}
\end{aligned}$$

$$
\begin{aligned}
& du_{r_i-1}du_{r_i-2}\cdots du_{r_{i-1}+2}du_{r_{i-1}+1}du_{r_{i-1}-1}\dots du_{r_2-1}\cdots \\
& du_{r_1+2}du_{r_1+1}du_{r_1-1}\cdots du_2 du_1 \\
= & \frac{c_{r_i-1}}{(r_i-r_{i-1}-1)!(r_{i-1}-r_{i-2}-1)!\dots(r_2-r_1-1)!(r_1-1)!} \\
& \times(1-u_{r_i})^{k+n-r_i+M_{r_i}-1}(1-u_{r_{i-1}})^{m_{r_{i-1}}}\dots(1-u_{r_1})^{m_{r_1}} \\
& \times[h_{\mu_i}(u_{r_i})-h_{\mu_i}(u_{r_{i-1}})]^{r_i-r_{i-1}-1} \\
& \times[h_{\mu_{i-1}}(u_{r_{i-1}})-h_{\mu_{i-1}}(u_{r_{i-2}})]^{r_{i-1}-r_{i-2}-1}\dots \\
& \times[h_{\mu_2}(u_{r_2})-h_{\mu_2}(u_{r_1})]^{r_2-r_1-1}h_{\mu_1}^{r_1-1}(u_{r_1}) \\
= & \frac{c_{r_i-1}}{(r_1-1)!\prod_{j=1}^{i-1}(r_{j+1}-r_j-1)!}(1-u_{r_i})^{k+n-r_i+M_{r_i}-1}h_{\mu_1}^{r_1-1}(u_{r_1}) \\
& \times\prod_{j=1}^{i-1}(1-u_{r_j})^{m_{r_j}}[h_{\mu_{r_{j+1}}}(u_{r_{j+1}})-h_{\mu_{j+1}}(u_{r_j})]^{r_{j+1}-r_j-1}. \qquad (2.7)
\end{aligned}
$$

The result in (2.5) follows by using (2.6) in (2.7). □

It is well known that for specific sets of parameters, n, k and m_i, $i=1,\dots,n-1$, the GOS's are reduced to the well known ordered random variables. Below we discuss the ordinary order statistics, k record values and Pfeifer's record values.

(A) Order Statistics corresponding to a random sample from an absolutely continuous distribution function F denoted by $X_{1:n}\leq\dots\leq X_{n:n}$ are special cases of GOS's with $m_1=\dots=m_{n-1}=0$, $k=1$. In this case $\gamma_r=n-r+1$, $r=1,\dots,n-1$. With these settings the joint density function given in (2.5) is reduced to the joint density function of $X_{r_1:n},\dots,X_{r_i:n}$ and it is given by

$$
\begin{aligned}
& f_{X_{r_1:n},\dots,X_{r_i:n}}(x_{r_1},\dots,x_{r_i}) \\
= & \frac{n(n-1)\dots(n-r_i+1)}{(r_1-1)!\prod_{j=1}^{i-1}(r_{j+1}-r_j-1)!}f(x_{r_1})F(x)^{r_1-1} \\
& \times\prod_{j=1}^{i-1}(F(x_{r_{j+1}})-F(x_{r_j}))^{r_{j+1}-r_j-1} \\
& \times f(x_{r_j})f(x_{r_i})\overline{F}^{n-r_i}(x_{r_i}). \qquad (2.8)
\end{aligned}
$$

It was used in Yao et al. (2008) to investigate dependence properties of generalized spacings of ordinary order statistics.

(B) **k-Records.** Let $\{X_i, i\geq 1\}$ be a sequence of i.i.d random variables from an absolutely continuous distribution F and let k be a positive integer. The random variables $L^{(k)}(n)$ given by $L^{(k)}(1) = 1$,

$L^{(k)}(n+1) = min\{j \in N; X_{j:j+k-1} > X_{L^{(k)}(n):L^{(k)}((n)+k-1)}\},\ n \geq 1,$

are called the nth k-th record times and the quantities $X_{L^{(k)}(n):L^{(k)}((n)+k-1)}$ which we denote by R_n^k are termed the nth k-records (cf. Kamps, 1995a, p.34).

In the case $m_1 = \ldots = m_{n-1} = -1$ and $k \in N$, the first n GOS's based on the distribution F are reduced to the first n k-records corresponding to a sequence of independent random variables from F. In this case $\gamma_r = k$, $r = 1, \ldots, n-1$. With these settings the joint density function given in (2.5) is reduced to the joint density function of $R_{r_1}^k, \ldots, R_{r_i}^k$ and is given by

$$
\begin{aligned}
&f_{R_{r_1}^k, \ldots, R_{r_i}^k}(x_{r_1}, \ldots, x_{r_i}) \\
&\quad = \frac{k^{r_i}}{(r_1-1)! \prod_{j=1}^{i-1}(r_{j+1}-r_j-1)!} \\
&\qquad \times (\overline{F}(x_{r_i})^{k-1} f(x_{r_i}))(-\log(\overline{F}(x_{r_1})))^{r_1-1} \\
&\qquad \times \left(\prod_{j=1}^{i-1} \left((-\log(\overline{F}(x_{r_{j+1}}))\right) - (-\log(\overline{F}(x_{r_j})))) \frac{f(x_{r_j})}{\overline{F}(x_{r_j})} \right). \quad (2.9)
\end{aligned}
$$

According to the best of our knowledge the above result is only available in the literature for the case when $i = 2$ (cf. Kampsa, 1995, p. 68).

(C) Pfeifer Model For $k = 1$ the k-records model reduces to the well know classic record model and for this model it is known that successive record values follows the conditional distribution given by

$$
P(R_n > x | R_{n-1} = x) = \frac{1 - F(y)}{1 - F(x)}, \quad \text{for } y > x. \qquad (2.10)
$$

Pfeifer (1982a) generalized the above model and consider a model in which the successive (upper) records values constitute a Markov chain with nonstationary transition distribution given by

$$
P(R_r^* > x | R_{r-1}^* = x) = \frac{1 - F_r(y)}{1 - F_r(x)}, \quad \text{for } y > x.
$$

Such a dependence structure for the record value sequence can be produced as follows. Suppose we have a double array of independent random variables $\{X_{l,j}; l, j \geq 1\}$ such that $X_{l,j}$ distribution function F_l, $l \geq 1$. Now take $R_1^* = X_{1,1}$ and define

$\delta_{l+1} = min\{j : X_{l+1,j} > R_l^*\}$ and $R_l^* = X_{l,\delta_l}$ for $l \geq 1$, such that $\delta_1 = 1$. This setting is called *Pfeifer record model* (cf. Arnold, Balakrishnan and Nagaraja, 1998, p.198 and Kampsa, 1995, p. 36). Pfeifer (1982b) showed that the sequence of jump-time generated by a pure birth process is identically distributed with records from Pfeifer models. Therefore the new results obtained here can be applied to this kind of Process.

For given positive real numbers $\beta_1, \ldots \beta_n$, the model of GOS's based on distribution F with parameters $m_i = \beta_i - \beta_{i+1} - 1$, $i = 1, \ldots, n-1$, $k = \beta_n$ and therefore $\gamma_i = \beta_i$, $i = 1, \ldots, n-1$, is reduced to the Pfeifer's record model based on distribution

$$F_r(t) = 1 - (1 - F(t))^{\beta_r}, \tag{2.11}$$

where F_r is the underlying distribution function until the rth record occurs.

Now, let for $r_1 < r_2 < r_3 < n$, $\beta_1 = \ldots = \beta_{r_1}$, $\beta_{r_1+1} = \ldots = \beta_{r_2}$, $\beta_{r_2+1} = \ldots = \beta_{r_3}$ and $\beta_l > 0$, $l = r_3 + 1, \ldots, n$. That is $m_{\mathbf{X}}$ in (1.4) is

$$\begin{aligned}(&-1, \ldots, -1, \beta_{r_1} - \beta_{r_1+1} - 1, -1, \ldots, -1, \beta_{r_2} - \beta_{r_2+1} - 1, \qquad (2.12)\\ &-1, \ldots, -1, \beta_{r_3} - \beta_{r_3+1} - 1, \beta_l - \beta_{l+1} - 1, l = r_3 + 1, \ldots n - 1).\end{aligned}$$

Therefore, $\mu_1 = \mu_2 = \mu_3 = -1$, $\mu_1^{(r_1)} = \beta_{r_1} - \beta_{r_1+1} - 1$, $\mu_2^{(r_2)} = \beta_{r_2} - \beta_{r_2+1} - 1$ and $\mu_3^{(r_3)} = \beta_{r_3} - \beta_{r_3+1} - 1$. Using these observations, then the joint density function of $R^*_{r_1}$, $R^*_{r_2}$ and $R^*_{r_3}$ is

$$\begin{aligned} &f_{R^*_{r_1}, R^*_{r_2}, R^*_{r_3}}(x_{r_1}, x_{r_2}, x_{r_3}) \\ &= \frac{\beta_r^{r_1} \beta_{r+1}^{r_2 - r_1} \beta_{r_3}^{r_3 - r_2}}{(r_1 - 1)!(r_2 - r_1 - 1)!(r_3 - r_2 - 1)!} \qquad (2.13)\\ &\times \ (\overline{F}(x_{r_3})^{\beta_n - 1} f(x_{r_3}))(-\log(\overline{F}(x_{r_1})))^{r_1 - 1} \\ &\times \ \overline{F}(x_{r_1})^{\beta_{r_1} - \beta_{r_1+1} - 1} \left(-\log(\overline{F}(x_{r_2})) - (-\log(\overline{F}(x_{r_1})))\right)^{r_2 - r_1 - 1} f(x_{r_1}) \\ &\times \ \overline{F}(x_{r_2})^{\beta_{r_2} - \beta_{r_2+1} - 1} \left(-\log(\overline{F}(x_{r_3})) - (-\log(\overline{F}(x_{r_2})))\right)^{r_3 - r_2 - 1} f(x_{r_2}). \end{aligned}$$

In this section we use this new expression to compare the above vector of Pfeifer records with that of classic records values according to the multivariate likelihood ratio ordering.

As discussed in Kamps (1995a,b), there are many other models like sequential order statistics, order statistics with non-integral sample size etc which can also be expressed as special cases of GOS's.

Next result generalizes the results of Theorem 3.19 in Balakrishnan et al. (2010).

Theorem 2.6. *Let X and Y be two absolutely continuous random variables with distribution F and G, densities f and g, and hazard rates r_F and r_g, respectively. Let $(X_{(1,n,\mathbf{m_X},k)}, \cdots, X_{(n,n,\mathbf{m_X},k)})$ and $(Y_{(1,n,\mathbf{m_Y},k')}, \cdots, Y_{(n,n,\mathbf{m_Y},k')})$ be random vectors of GOS's based on distributions F and G, respectively, where $\mathbf{m_X}$ and $\mathbf{m_Y}$ are as given in* (1.4) *and* (1.5). *Let also $r_1 \le r_2 < \cdots \le r_i \le n$, $r'_1 \le r'_2 < \cdots \le r'_i \le n'$ and $r'_1 - r_1 = r'_2 - r_2 = \cdots = r'_i - r_i \ge \max\{0, n' - n\}$. For $j = 1, \ldots, i$, $\gamma_j \le \mu_j$, $\gamma^{(r'_j)} \le \mu^{(r_j)}$, $k \ge k'$ and $(n - n') + (r'_i - r_i) + (M_{r_i} - M_{r'_i}) \ge 0$ if*

(i) $X \le_{\rm lr} Y$ and $\gamma_j \ge 0, \gamma^{(r'_j)} \ge 0$ or

(ii) $X \le_{\rm hr} Y$, $r_G(x)/r_F(x)$ is increasing in x, and $-1 \le \gamma_j \le 0$, $-1 \le \gamma^{(r_j)} \le 0$,

then

$$(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)}) \\ \le_{\rm lr} (Y_{(r'_1,n',\mathbf{m_Y},k')}, Y_{(r'_2,n',\mathbf{m_Y},k')}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k')}). \tag{2.14}$$

Proof. (i). It is known that the vector of GOS's is MTP_2 (cf. Belzunce et al. (2005)). It is also known that any sub-vector of a MTP_2 vector is MTP_2. That is MTP_2 property is preserved under marginalization. Combining these observations, it follows that both vectors $(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$ and $(Y_{(r'_1,n',\mathbf{m_Y},k')}, Y_{(r'_2,n',\mathbf{m_Y},k')}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k')})$ are MTP_2. Using this observation, to prove the required result, we only need to show that

$$\frac{g_{r'_1,r'_2,\cdots,r'_i}(x_1, x_2, \ldots, x_i)}{f_{r_1,r_2,\cdots,r_i}(x_1, x_2, \ldots, x_i)} \text{ is increasing in } (x_1 < x_2 < \ldots < x_i) \in \mathbb{R}^i, \tag{2.15}$$

where $f_{r_1,r_2,\cdots,r_i}$ and $g_{r'_1,r'_2,\cdots,r'_i}$ denote the joint densities of $(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$ and $(Y_{(r'_1,n',\mathbf{m_Y},k')}, Y_{(r'_2,n',\mathbf{m_Y},k')}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k')})$, respectively (cf. Hu, Khaledi and Shaked (2003), Remark 3.1).

Let $\delta_1 = \mu^{(r_j)} - \gamma^{(r'_j)}$ and $\delta_2 = (k - k') + (n - n') + (r'_i - r_i) + (M_{r_i} - M_{r'_i})$.

Now, using (2.5), for $x_1 < x_2 < \ldots < x_i$,

$$\frac{g_{r'_1,r'_2,\cdots,r'_i}(x_1,x_2,\ldots,x_i)}{f_{r_1,r_2,\cdots,r_i}(x_1,x_2,\ldots,x_i)} \propto \frac{g(x_i)}{f(x_i)}\left(\frac{\overline{G}(x_i)}{\overline{F}(x_i)}\right)^{k'+n'-r'_i+M_{r'_i}}$$
$$\times \quad \left(\frac{1}{\overline{F}(x_i)}\right)^{\delta_2}\prod_{j=1}^{i-1}\left(\frac{\overline{G}(x_j)}{\overline{F}(x_j)}\right)^{\gamma^{(r'_j)}}\left(\frac{1}{\overline{F}(x_j)}\right)^{\delta_1}\frac{g(x_j)}{f(x_j)}$$
$$\times \quad \left(\frac{(h_{\gamma_1}(G(x_1)))^{r'_1-1}}{(h_{\mu_1}(F(x_1)))^{r_1-1}}\right)\prod_{j=1}^{i-1}\left(\frac{(h_{\gamma_{j+1}}(G(x_{j+1}))-h_{\gamma_{j+1}}(G(x_j)))^{r'_{j+1}-r'_j-1}}{(h_{\mu_{j+1}}(F(x_{j+1}))-h_{\mu_{j+1}}(F(x_j)))^{r_{j+1}-r_j-1}}\right).$$

Under the assumptions given in (i),

$$\frac{g(x_i)}{f(x_i)},\quad \frac{g(x_j)}{f(x_j)},\ \frac{\overline{G}(x_i)}{\overline{F}(x_i)}^{k'+n'-r'_i+M_{r'_i}},$$
$$\left(\frac{1}{\overline{F}(x_i)}\right)^{\delta_2},\ \left(\frac{1}{\overline{F}(x_j)}\right)^{\delta_1}\ \textit{and}\left(\frac{\overline{G}(x_j)}{\overline{F}(x_j)}\right)^{\gamma^{(r'_j)}} \tag{2.16}$$

are increasing in $x_i \in \mathbb{R}$. On the other hand, for $\gamma_1 \le \mu_1$, it follows from Lemma 2.4 (a) that $\frac{\left(h_{\gamma_1}(G(x_1))\right)^{r'_1-1}}{\left(h_{\mu_1}(F(x_1))\right)^{r_1-1}}$ is increasing in $x_1 \in \mathbb{R}$ and for $\gamma_j \le \mu_j$, it follows from Lemma 2.4 (b) that

$$\frac{\left(h_{\gamma_{j+1}}(G(x_{j+1}))-h_{\gamma_{j+1}}(G(x_j))\right)^{r'_{j+1}-r'_j-1}}{\left(h_{\mu_{j+1}}(F(x_{j+1}))-h_{\mu_{j+1}}(F(x_j))\right)^{r_{j+1}-r_j-1}}$$

is increasing function in x_j as well as x_{j+1}.

Combining these observations, (2.15) is proved.

The proof of part (ii) is similar to that of part (i) and is omitted. □

Example 2.7. Let $\{R^k_n, n \ge 1\}$ be the sequence of k record values introduced in (B) based on an absolutely continuous distribution function G and $\{R^*_n, n \ge 1\}$ be the sequence of Pfeifer's record values introduced in (C). If $\beta_i \ge \beta_{i+1},\ i = 1,\ldots,n-1,\ k \ge \beta_n$. That is $\mathbf{m_X}$ and $\mathbf{m_Y}$ in the statement of Theorem 2.6 are respectively given by (2.12) and $(-1,-1,\ldots,-1,k)$. With these setting if $F \le_{hr} G$ and $r_G(x)/r_F(x)$ is increasing in x, then it follows from Theorem 2.6 (ii) that for $r_1 < r_2 < r_3$,

$$(R^*_{r_1}, R^*_{r_2}, R^*_{r_3}) \le_{lr} (R^k_{r_1}, R^k_{r_2}, R^k_{r_3})$$

Theorem 2.8. *Under the assumptions of Theorem* 2.6

$$\begin{aligned}&[X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)}|\\&X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)} \in L] \leq_{\mathrm{lr}}\\&[Y_{(r'_1,n',\mathbf{m_Y},k')}, Y_{(r'_2,n',\mathbf{m_Y},k')}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k')}|\\&Y_{(r'_1,n',\mathbf{m_Y},k')}, Y_{(r'_2,n',\mathbf{m_Y},k')}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k')} \in L]\end{aligned}$$

for all sublattices $L \subseteq \mathbb{R}^n$.

Proof. Using (2.14), the required result follows from Theorem 6.E.2 of Shaked and Shanthikumar (2007). □

The reader is referred to Balakrishnan et al. (2010) for various examples of this result, given the different interesting sublattices. An example is given next.

Example 2.9. Let $(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$ and $(Y_{(r'_1,n',\mathbf{m_Y},k)}, Y_{(r'_2,n',\mathbf{m_Y},k)}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k)})$ be two sub-vectors of GOS's defined under the setup of Theorem 2.6. Let also that $(x_{(1)} \leq \cdots \leq x_{(n)})$ denote a non-decreasing arrangement of the components of a vector $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Now, consider the sublattice $L_1 = [(x_{(1)}, \cdots, x_{(n)}) \in \mathbb{R}^n | x_{(1)} > t_1, x_{(3)} < t_2]$. Then, from Theorem 2.8 for $t_1, t_2 \in \mathbb{R}$ and $t_1 \leq t_2$, we have that

$$\begin{aligned}&[X_{(r_2,n,\mathbf{m_X},k)} - t_1 | X_{(r_1,n,\mathbf{m_X},k)} > t_1, X_{(r_3,n,\mathbf{m_X},k)} < t_2]\\&\quad \leq_{\mathrm{lr}} [Y_{(r'_2,n',\mathbf{m_Y},k)} - t_1 | Y_{(r'_1,n',\mathbf{m_Y},k)} > t_1, Y_{(r'_3,n',\mathbf{m_Y},k)} < t_2].\end{aligned}$$

In particular, let $r'_1 = 4, r'_2 = 6, r'_3 = 8, r_1 = 2, r_2 = 4, r_3 = 6$ then

$$\begin{aligned}&[X_{(4,n,\mathbf{m_X},k)} - t_1 | X_{(2,n,\mathbf{m_X},k)} > t_1, X_{(6,n,\mathbf{m_X},k)} < t_2]\\&\quad \leq_{\mathrm{lr}} [Y_{(6,n',\mathbf{m_Y},k)} - t_1 | Y_{(4,n',\mathbf{m_Y},k)} > t_1, Y_{(8,n',\mathbf{m_Y},k)} < t_2].\end{aligned}$$

Note that this comparison does not follow from the results obtained in Balakrishnan et al. (2010).

3 Multivariate *lr* Ordering among Spacings of GOS's

A random variable X with density function f is decreasing likelihood ratio (DLR) if f is log-convex and is increasing likelihood ratio (ILR) if f is log-concave. It is known that if f is DLR, then $\overline{F}$ is log-convex and

$$\frac{f(x+\delta)}{f(x)} \quad \textit{is increasing in } x \geq 0;$$

and if f is ILR, then $\overline{F}$ is log-concave and

$$\frac{f(x+\delta)}{f(x)} \quad \textit{is decreasing in } x \geq 0.$$

For more details of DLR and ILR, the reader is referred to Barlow and Proschan(1981), Shaked and Shanthikumar (1987) and Righter and Shanthikumar (1992).
We need the following lemma to prove the main result in this section.

Lemma 3.10. *Let X be a DLR (ILR) nonnegative random variable with distribution function F and $\lambda \geq 1$. Then,*
(a) for, $x > 0$ and $u_2 > u_1 > 0$,

$$\phi(x,u) = \frac{\bar{F}^{\lambda}(x+u_2) - \bar{F}^{\lambda}(u_2)}{\bar{F}^{\lambda}(x+u_1) - \bar{F}^{\lambda}(u_1)} \tag{3.1}$$

is increasing [decreasing] in x.
(b) The function

$$\psi_{\delta}(x,u) = \frac{\bar{F}^{\lambda}(x+u+\delta) - \bar{F}^{\lambda}(u+\delta)}{\bar{F}^{\lambda}(x+u) - \bar{F}^{\lambda}(u)} \tag{3.2}$$

is increasing [decreasing] in $(x,u) \in \mathbb{R}^2_+$ for each $\delta > 0$.
(c) For $u > 0$ and $x_2 > x_1 > 0$, the function

$$\psi_{\delta}(x,u) = \frac{\bar{F}^{\lambda}(x_2+u+\delta) - \bar{F}^{\lambda}(u+\delta)}{\bar{F}^{\lambda}(x_1+u) - \bar{F}^{\lambda}(u)} \tag{3.3}$$

is increasing [decreasing] in u.

Proof. Let Y be a random variable with survival function $\bar{G}(x) = [\bar{F}(x)]^{\lambda}$ $(\lambda \geq 1)$ and density function $g(x) = (\lambda) f(x) \bar{F}^{\lambda-1}(x)$. Now, for any $\delta > 0$,

$$\frac{g(x+\delta)}{g(x)} = \lambda \frac{f(x+\delta)}{f(x)} \left(\frac{\bar{F}(x+\delta)}{\bar{F}(x)} \right)^{\lambda-1}$$

is increasing in x. That is the random variable Y is DLR. Now, part (a) follows from Lemma 2.2 in Xu and Li (2006) and parts (b) and (c) follow from Lemma 2.3 in Yao et al. (2008). □

Let $(X_{(r_1,n,\mathbf{m_X},k)}, X_{(r_2,n,\mathbf{m_X},k)}, \ldots, X_{(r_i,n,\mathbf{m_X},k)})$ be an i-dimensional sub-vectors of GOS's as given in Lemma 2.5. With $r_j - r_{j-1} = p$,

$j = 2, \ldots, i$ and $p \geq 1$, the multivariate density function of the vector of p-spacings

$$\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})} = (D_{X,r_1,n}^{(p)}, D_{X,r_2,n}^{(p)}, \ldots, D_{X,r_i-p,n}^{(p)}) \tag{3.4}$$

is given by

$$f_{\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}}(y_1, y_2, \cdots, y_i)$$

$$= C_{\mathbf{r}} \left[\bar{F}\left(\sum_{l=1}^{i} y_l \right) \right]^{k+n-r_i+M_{r_i}-1} \prod_{j=1}^{i-1} \left[\bar{F}\left(\sum_{l=1}^{j} y_l \right) \right]^{\mu^{(r_j)}} \tag{3.5}$$

$$\prod_{j=1}^{i} \left[h_{\mu_j}\left(F\left(\sum_{l=1}^{j} y_l \right) \right) - h_{\mu_j}\left(F\left(\sum_{l=1}^{j-1} y_l \right) \right) \right]^{p-1} f\left(\sum_{l=1}^{j} y_l \right), \tag{3.6}$$

where $C_{\mathbf{r}} = \frac{c_{r_i-1}}{(r_1-1)!(p-1)^{i-1}}$, $D_{X,r_j,n}^{(p)} = X_{(r_j+p,n,\mathbf{m_X},k)} - X_{(r_j,n,\mathbf{m_X},k)}$, $j = 1, \cdots, i-1$, $X_{(0,n,\mathbf{m_X},k)} = 0$ and $h_{\mu_j}(\bar{F}(\sum_{l=1}^{0} y_l)) = 0$.

Next we prove the MTP_2 property of $\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}$ given in (3.4).

Theorem 3.11. *Let* $(X_{(1,n,\mathbf{m_X},k)}, \cdots, X_{(n,n,\mathbf{m_X},k)})$ *be random vector of GOS's based on distribution* F *with* $\mathbf{m_X}$ *given in* (1.4) *for which* $\mu_j \geq 0$ *and* $\mu^{(r_j)} \geq 0$*,* $j = 1, \cdots, i$*. If* F *is DLR, then* $\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}$ *is* MTP_2*.*

Proof. The assumption that f is log-convex implies that $\overline{F}$ is log-convex which in turn is equivalent to that $\overline{F}(u+v))$ is TP_2 in u and v. Using this observation, the function $\left[\bar{F}\left(\sum_{l=1}^{i} y_l \right) \right]^{k+n-r_i+M_{r_i}-1}$ is TP_2 in every pair of variables when the remaining variables are held fixed. On the other hand, the support of the spacings is lattice. Now, it follows from Karlin and Rinott (1980, p.469) that $\left[\bar{F}\left(\sum_{l=1}^{i} y_l \right) \right]^{k+n-r_i+M_{r_i}-1}$ is MTP_2. Using the same kind of arguments, we find that the functions $\prod_{j=1}^{i-1} \left[\bar{F}\left(\sum_{l=1}^{j} y_l \right) \right]^{\mu^{(r_j)}}$ in (3.5) and $f\left(\sum_{l=1}^{j} y_l \right)$ in (3.6) are MTP_2.

To prove the required result we show that

$$\frac{\left[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l \vee y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l \vee y_l)) \right]}{\left[h_{\mu_j}(F(\sum_{l=1}^{j} x_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l)) \right]}$$

$$\geq \frac{\left[h_{\mu_j}(F(\sum_{l=1}^{j} y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} y_l)) \right]}{\left[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l \wedge y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l \wedge y_l)) \right]}.$$

Now, for $(x_1,\dots,x_j)$ and $(y_1,\dots,y_j)$, let $\delta=\sum_{l=1}^{j-1}(x_l\vee y_l)-\sum_{l=1}^{j-1}x_l$. Since

$$\sum_{l=1}^{j-1}(x_l\vee y_l)+\sum_{l=1}^{j-1}(x_l\wedge y_l)=\sum_{l=1}^{j-1}x_l+\sum_{l=1}^{j-1}y_l,$$

then

$$\delta=\sum_{l=1}^{j-1}y_l-\sum_{l=1}^{j-1}(x_l\wedge y_l). \tag{3.7}$$

First, assume that $x_j\geq y_j$, then

$$\begin{aligned}
&\frac{\left[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l\vee y_l))-h_{\mu_j}(F(\sum_{l=1}^{j-1}x_l\vee y_l))\right]}{\left[h_{\mu_j}(F(\sum_{l=1}^{j}x_l))-h_{\mu_j}(F(\sum_{l=1}^{j-1}x_l))\right]}\\
&=\frac{\left[h_{\mu_j}(F(x_j+\sum_{l=1}^{j-1}x_l+\delta))-h_{\mu_j}(F(\sum_{l=1}^{j-1}x_l+\delta))\right]}{\left[h_{\mu_j}(F(x_j+\sum_{l=1}^{j-1}x_l))-h_{\mu_j}(F(\sum_{l=1}^{j-1}x_l))\right]}\\
&=\frac{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}x_l+\delta)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l+\delta))}{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}x_l))-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l))}\\
&\geq\frac{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}x_l\wedge y_l+\delta)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l\wedge y_l+\delta)}{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}x_l\wedge y_l)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l\wedge y_l)}\\
&=\frac{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}y_l)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}y_l)}{\bar F^{\mu_{j+1}}(x_j+\sum_{l=1}^{j-1}x_l\wedge y_l)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l\wedge y_l)} &(3.8)\\
&\geq\frac{\bar F^{\mu_{j+1}}(\sum_{l=1}^{j}y_l)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}y_l)}{\bar F^{\mu_{j+1}}(\sum_{l=1}^{j}x_l\wedge y_l)-\bar F^{\mu_{j+1}}(\sum_{l=1}^{j-1}x_l\wedge y_l)}\\
&=\frac{\left[h_{\mu_j}(F(\sum_{l=1}^{j}y_l))-h_{\mu_j}(F(\sum_{l=1}^{j-1}y_l))\right]}{\left[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l\wedge y_l))-h_{\mu_j}(F(\sum_{l=1}^{j-1}x_l\wedge y_l))\right]} &(3.9)
\end{aligned}$$

The first inequality follows from Lemma 3.10 (b), since $\sum_{l=1}^{j-1}x_l\geq\sum_{l=1}^{j-1}x_l\wedge y_l$. The equality in (3.8) follows from (3.7). The second inequality follows from Lemma 3.10 (a), since $x_j\geq y_j$.

Now, let $x_j \le y_j$,

$$\begin{aligned}
&\frac{\Big[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l \vee y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l \vee y_l))\Big]}{\Big[h_{\mu_j}(F(\sum_{l=1}^{j} x_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l))\Big]} \\
&= \frac{\Big[h_{\mu_j}(F(y_j + \sum_{l=1}^{j-1} x_l + \delta)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l + \delta))\Big]}{\Big[h_{\mu_j}(F(x_j + \sum_{l=1}^{j-1} x_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l))\Big]} \\
&= \frac{\bar{F}^{\mu_{j+1}}(y_j + \sum_{l=1}^{j-1} x_l + \delta) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l + \delta))}{\bar{F}^{\mu_{j+1}}(x_j + \sum_{l=1}^{j-1} x_l)) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l))} \\
&\ge \frac{\bar{F}^{\mu_{j+1}}(y_j + \sum_{l=1}^{j-1} x_l \wedge y_l + \delta) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l \wedge y_l + \delta)}{\bar{F}^{\mu_{j+1}}(x_j + \sum_{l=1}^{j-1} x_l \wedge y_l) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l \wedge y_l)} \quad (3.10) \\
&= \frac{\bar{F}^{\mu_{j+1}}(y_j + \sum_{l=1}^{j-1} y_l) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} y_l)}{\bar{F}^{\mu_{j+1}}(x_j + \sum_{l=1}^{j-1} x_l \wedge y_l) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l \wedge y_l)} \quad (3.11) \\
&= \frac{\bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j} y_l) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} y_l)}{\bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j} x_l \wedge y_l) - \bar{F}^{\mu_{j+1}}(\sum_{l=1}^{j-1} x_l \wedge y_l)} \\
&= \frac{\Big[h_{\mu_j}(F(\sum_{l=1}^{j} y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} y_l))\Big]}{\Big[h_{\mu_j}(F(\sum_{l=1}^{j}(x_l \wedge y_l)) - h_{\mu_j}(F(\sum_{l=1}^{j-1} x_l \wedge y_l))\Big]} \quad (3.12)
\end{aligned}$$

The inequality (3.10) follows from part (c) of Lemma 3.10, since $x_j < y_j$ and $\sum_{l=1}^{j-1} x_l \ge \sum_{l=1}^{j-1}(x_l \wedge y_l)$. The equality (3.11) follows from (3.7).

Therefore, the density function of $\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}$ given in (3.4) is a product of four non-negative MTP_2 functions which in turn is MTP_2. This completes the proof of the required result. □

Remark 3.12. Theorem 3.11 generalized the result of Theorem 3.1 in Yao et al. (2008) from the particular case of usual order statistics to GOS's.

Let $(Y_{(r'_1,n',\mathbf{m_Y},k)}, Y_{(r'_2,n',\mathbf{m_Y},k)}, \ldots, Y_{(r'_i,n',\mathbf{m_Y},k)})$ be another i-dimensional vector of GOS's based on distribution G as given in the statement of Theorem 2.6. With $r'_j - r'_{j-1} = p$, $j = 2, \ldots, i$, $p \ge 1$, the corresponding vector of p-spacings is denoted by

$$\mathbf{D}_{\mathbf{Y}}^{(\mathbf{p})} = (D_{Y,r'_1,n}^{(p)}, D_{Y,r'_2,n}^{(p)}, \ldots, D_{Y,r'_i-p,n}^{(p)}), \quad (3.13)$$

where $D_{Y,r'_j,n}^{(p)} = Y_{(r'_j+p,n,\mathbf{m_Y},k)} - Y_{(r'_j,n,\mathbf{m_Y},k)}$, $j = 1, \cdots, i-1$ and $Y_{(0,n,\mathbf{m_Y},k)} = 0$.

Theorem 3.13. *Let* $\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}$ *and* $\mathbf{D}_{\mathbf{Y}}^{(\mathbf{p})}$ *be as defined by* (3.4) *and* (3.13). *If either* X *or* Y *is DLR then under the assumptions of Theorem 2.6,*

$$\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})} \leq_{\mathrm{lr}} \mathbf{D}_{\mathbf{Y}}^{(\mathbf{p})}. \tag{3.14}$$

Proof. If we show that the ratio

$$\frac{f_{\mathbf{D}_{\mathbf{Y}}^{(\mathbf{p})}}(y_1, y_2, \cdots, y_i)}{f_{\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}}(y_1, y_2, \cdots, y_i)} \tag{3.15}$$

is increasing in y_j, $j = 1, \cdots, i$, then the required result follows from Remark 3.1 in Hu et al. (2003), since by Theorem 3.11 the vector $\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}$ is MTP_2. That is we have to show that the ratio

$$\frac{f_{\mathbf{D}_{\mathbf{Y}}^{(\mathbf{p})}}(y_1, y_2, \cdots, y_i)}{f_{\mathbf{D}_{\mathbf{X}}^{(\mathbf{p})}}(y_1, y_2, \cdots, y_i)}$$

$$\propto \left[\frac{\bar{G}(\sum_{l=1}^{i} y_l)}{\bar{F}(\sum_{l=1}^{i} y_l)}\right]^{n'-r'_i+M_{r'_i}} \left(\frac{1}{\bar{F}(\sum_{l=1}^{i} y_l)}\right)^{\delta_2} \prod_{j=1}^{i-1} \left(\frac{\bar{G}(\sum_{l=1}^{j} y_l)}{\bar{F}(\sum_{l=1}^{j} y_l)}\right)^{\gamma^{(r'_j)}} \tag{3.16}$$

$$\times \left(\frac{1}{\bar{F}(\sum_{l=1}^{j} y_l)}\right)^{\delta_1} \frac{g(\sum_{l=1}^{j} y_l)}{f(\sum_{l=1}^{j} y_l)} \tag{3.17}$$

$$\times \prod_{j=1}^{i} \left[\frac{\left[h_{\mu_j}\left(G\left(\sum_{l=1}^{j} y_l\right)\right) - h_{\mu_j}\left(G\left(\sum_{l=1}^{j-1} y_l\right)\right)\right]}{\left[h_{\gamma_j}\left(F\left(\sum_{l=1}^{j} y_l\right)\right) - h_{\gamma_j}\left(F\left(\sum_{l=1}^{j-1} y_l\right)\right)\right]}\right]^{p-1} \tag{3.18}$$

is increasing y_j, $j = 1, \ldots, i$. Using the assumption that $X \leq_{lr} Y$ and $X \leq_{hr} Y$, we get that the functions $\frac{g(\sum_{l=1}^{j} y_l)}{f(\sum_{l=1}^{j} y_l)}$ and $\frac{\bar{G}(\sum_{l=1}^{j} y_l)}{\bar{F}(\sum_{l=1}^{j} y_l)}$ are increasing in y_j, $j = 1, \ldots, i$. On the other hand, using the facts that the function $I_{(x+y_k, x+y_k+y_j)}(w)$ is TP_2 in x and w, $X \leq_{lr} Y$ and the similar arguments used to prove Lemma 2.4, we observe that for $j = 1, \ldots, i$ and $x = \sum_{l \neq k}^{j-1} y_l$, the function

$$\frac{\left[h_{\gamma_j}\left(G\left(\sum_{l=1}^{j} y_l\right)\right) - h_{\gamma_j}\left(G\left(\sum_{l=1}^{j-1} y_l\right)\right)\right]}{\left[h_{\mu_j}\left(F\left(\sum_{l=1}^{j} y_l\right)\right) - h_{\mu_j}\left(F\left(\sum_{l=1}^{j-1} y_l\right)\right)\right]}$$
$$= \frac{h_{\gamma_j}\left(G(x + y_k + y_j)\right) - h_{\gamma_j}\left(G(x + y_k)\right)}{h_{\mu_j}\left(F(x + y_k + y_j)\right) - h_{\mu_j}\left(F(x + y_k)\right)},$$

is increasing in y_k, $k = 1, \ldots, j-1$ and y_j. Combining these observations, it follows that the ratio given in (3.15) is increasing in each y_k, $k =$

$1, \ldots, i$, when the remaining arguments are held fixed. This complete the proof of the required result. $\square$

4 An Application

In this section we introduce an important particular case of GOS's called progressive type II censored order statistics and applied some of the new results to obtain some computable bounds on the mean residual life of unobserved progressive type II censored order statistics.

Let $X_1, \ldots, X_N$ be independent lifetimes of N identical units, with X_i having absolutely continuous distribution function F. These units are placed on test at time $t = 0$. At the time of the ith failure, R_i, $1 \leq i \leq n$, number of surviving units are randomly withdrawn from the experiment . Thus, if n failures are observed then $R_1 + \ldots + R_n$ number of units are progressively censored; hence $N = n + R_1 + \ldots + R_n$. The censoring scheme is denoted by the vector $\widetilde{R} = (R_1, \ldots, R_n)$ and $X_{i:n:N}^{\widetilde{R}}$, $i = 1, \ldots, n$, the ith failure time, is called the ith progressive type II censored order statistic. The progressive type II censored order statistic are special cases of GOS's with $m_i = R_i$, $i = 1, \ldots, n-1$ and $k = R_n + 1$. For a detailed description of other special cases the readers is referred to Kamps (1995a, b), Balakrishnan and Aggarwala (2000), Belzunce et al. (2005), Balakrishnan (2007) and Balakrishnan et al. (2010).

Let $n \geq 2$ (and hence $N \geq 2$),

$$c_{r_i-1,n} = \prod_{j=1}^{r_i} \gamma_{j,n}, \quad 1 \leq r_i \leq n \quad \text{and}$$

$$a_{l,r_i,n} = \prod_{j=1, j\neq l}^{r_i} \frac{1}{\gamma_{j,n} - \gamma_{l,n}}, \qquad 1 \leq l \leq r_i \leq n, \tag{4.1}$$

where $\gamma_{j,n} = n - j + 1 + \sum_{k=j}^{n} R_k$, for $j = 1, \ldots, n$ and the empty product $\prod_{\emptyset}$ is defined to be 1. From the definition of the γ's it can be easily derived that $N = \gamma_{1,n} > \ldots > \gamma_{n,n} \geq 1$ such that $\gamma_{j,n} \neq \gamma_{l,n}$ for $l \neq j$.

The marginal survival function of $X_{r_i:n:N}^{\widetilde{R}}$ based on F is given by

$$F_{X_{r_i:n:N}}^{\widetilde{R}}(x) = c_{r_i-1,n} \sum_{l=1}^{r_i} \frac{a_{l,r_i,n}}{\gamma_{l,n}} \left[\bar{F}(x)\right]^{\gamma_{l,n}}, \qquad r_i = 1, \ldots, n \tag{4.2}$$

(cf. Kamps and Cramer, 2001, Lemma 1). If F is absolutely continuous with density function f, then we obtain density function of $X^{\widetilde{R}}_{r_i:n:N}$ as

$$f^{\widetilde{R}}_{X_{r_i:n:N}}(x) = c_{r_i-1,n} \sum_{l=1}^{r_i} a_{l,r_i,n} \left[\bar{F}(x)\right]^{\gamma_{l,n}-1} f(x), \qquad r_i = 1, \ldots, n. \quad (4.3)$$

Under some restrictions on the hazard rate function $r_F(x)$ we obtain some computable bounds on the expected conditional residual life of the unobserved censored order statistics.

First, we state a proposition that is used later in this section.

Proposition 4.14. *Let $\mathbf{E}^{\lambda,\widetilde{R}} = (E^{\lambda,\widetilde{R}}_{1:n:N}, \ldots, E^{\lambda,\widetilde{R}}_{n:n:N})$ be n progressively Type II censored order statistics corresponding to an exponential distribution with hazard rate λ. Then, for $1 \leq r_1 < r_2 < r_3 \leq n$, we have*

$$\begin{aligned}
&E\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right] = \\
&\sum_{k=1}^{r_3-r_2} \frac{1}{\lambda\gamma_{k,n-r_2}} + \left[\sum_{k=1}^{r_2-r_1} \frac{1}{\lambda\gamma_{k,n-r_1}} + t_1 + \frac{\sum_{i=1}^{r_1} a_{i,r_1,n}\gamma_{i,n}^{-2}\lambda^{-1}e^{-\lambda\gamma_{i,n}t_1}}{\sum_{i=1}^{r_1} a_{i,r_1,n}\gamma_{i,n}^{-1}e^{-\lambda\gamma_{i,n}t_1}}\right] \\
&\quad\times \frac{e^{\lambda t_2}\sum_{i=1}^{r_1}\frac{a_{i,r_1,n}}{\gamma_{i,n}}e^{-(\lambda\gamma_{i,n})t_1}}{c_{r_2-r_1,n-r_1}\sum_{i=1}^{r_1}\sum_{j=1}^{r_2-r_1} a_{i,r_1,n}a_{j,r_2-r_1,n-r_1}\dfrac{e^{-\lambda\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)t_1}}{\gamma_{j,n-r_1}\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)}}.
\end{aligned} \quad (4.4)$$

Proof. The left hand side of (4.4), after some manipulations, can be simplified as

$$\begin{aligned}
&E\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right] \\
&= \int_0^\infty \mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 > x | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right] dx \\
&= \int_0^\infty \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 > x, E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} dx \\
&= \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 > x | E^{\lambda,\widetilde{R}}_{r_1:n:N} = u, E^{\lambda,\widetilde{R}}_{r_2:n:N} = v\right] \\
&\quad \times f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v)\, dv\, du\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 > x | E^{\lambda,\widetilde{R}}_{r_2:n:N} = v\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\quad \times f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v) dv du dx \\
&= \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - v + v - t_2 > x | E^{\lambda,\widetilde{R}}_{r_2:n:N} = v\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\quad \times f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v) dv du dx \\
&= \int_0^\infty \int_{t_1}^\infty \int_{t_2}^\infty \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_3-r_2:n-r_2:N-\sum_{l=1}^{r_2} R_l - r_2} + v - t_2 > x\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\quad \times f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v) dv du dx \qquad (4.5) \\
&= \int_{t_1}^\infty \int_{t_2}^\infty \frac{\{E\left[E^{\lambda,\widetilde{R}}_{r_3-r_2:n-r_2:N-\sum_{l=1}^{r_2} R_l - r_2}\right] + v - t_2\}}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\quad \times f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v) dv du \\
&= E\left[E^{\lambda,\widetilde{R}}_{r_3-r_2:n-r_2:N-\sum_{l=1}^{r_2} R_l - r_2}\right] \\
&\quad + E\left[E^{\lambda,\widetilde{R}}_{r_2:n:N} - t_2 | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right] \qquad (4.6)
\end{aligned}
$$

The second expression of the bracket in (4.6) can be simplified as follows:

$$
\begin{aligned}
&= \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\times \int_0^\infty \int_{t_1}^\infty \mathbb{P}\left[E^{\lambda,\widetilde{Q}}_{r_2:n:N} > t_2 + x | E^{\lambda,\widetilde{R}}_{r_1:n:N} = u\right] \frac{f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(u)}{\bar{F}_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(t_1)} du dx \\
&= \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]} \\
&\times \int_0^\infty \int_{t_1}^\infty \mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_2-r_1:n-r_1:N-\sum_{l=1}^{r_1} R_l - r_1} + u - t_2 > x\right] \frac{f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(u)}{\bar{F}_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(t_1)} du dx
\end{aligned}
$$

$$= \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right]}$$
$$\times \left[E\left[E^{\lambda}_{r_2-r_1:n-r_1:N-\sum_{l=1}^{r_1} R_l - r_1} - t_2\right] + \int_{t_1}^{\infty} \frac{u f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(u)}{\bar{F}_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(t_1)} du\right]. \quad (4.7)$$

$E^{\lambda,\widetilde{R}}_{j-i:n-i:N-\sum_{l=1}^{i} R_l - i} \overset{\text{st}}{=} \sum_{k=1}^{j-i} Z'_k$, where Z'_ks are the corresponding normalized spacings $Z'_1 = N E^{\lambda,\widetilde{R}}_{1:n:N}$ and $Z'_k = (n-k+1-R_1-\ldots-R_{k-1})(E^{\lambda,\widetilde{R}}_{k:n:N} - E^{\lambda,\widetilde{R}}_{k-1:n:N})$, for $k = 2, \ldots, n$.

It is known that Z'_k are independent exponential random variables with hazard rate $\lambda\gamma_{k,n-i}$ (cf. Cramer and Kamps, 2003, Theorem 3.1; Hu and Zhuang, 2005a, Lemma 2.1). Using these observations, the first and the second terms in (4.5) and (4.7), respectively, can be written as

$$\mathbb{E}\left[E^{\lambda,\widetilde{R}}_{(r_3-r_2:n-r_2:N-\sum_{l=1}^{r_2} R_l - r_2)}\right] = \sum_{k=1}^{r_3-r_2} \frac{1}{\lambda\gamma_{k,n-r_2}}, \quad (4.8)$$

and

$$\mathbb{E}\left[E^{\lambda,\widetilde{R}}_{(r_2-r_1:n-r_1:N-\sum_{l=1}^{r_1} R_l - r_1)}\right] = \sum_{k=1}^{r_2-r_1} \frac{1}{\lambda\gamma_{k,n-r_1}}. \quad (4.9)$$

On the other hand,

$$\begin{aligned}
\int_{t_1}^{\infty} \frac{u f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(u)}{\bar{F}_{E^{\lambda,\widetilde{R}}_{r_1:n:N}(t_1)}} du &= E\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right] \\
&= t_1 + E\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} - t_1 | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right] \\
&= t_1 + \int_0^{\infty} \frac{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{(r_1:n:N)} > x + t_1\right]}{\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1\right]} dx \\
&= t_1 + \int_0^{\infty} \frac{\sum_{i=1}^{r} a_{i,r_1,n} \gamma_{i,n}^{-1} e^{-\lambda\gamma_{i,n}(x+t_1)}}{\sum_{i=1}^{r_1} a_{i,r_1,n} \gamma_{i,n}^{-1} e^{-\lambda\gamma_{i,n} t_1}} dx \\
&= t_1 + \frac{\sum_{i=1}^{r_1} a_{i,r_1,n} \gamma_{i,n}^{-2} \lambda^{-1} e^{-\lambda\gamma_{i,n} t_1}}{\sum_{i=1}^{r_1} a_{i,r_1,n} \gamma_{i,n}^{-1} e^{-\lambda\gamma_{i,n} t_1}}, \quad (4.10)
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right] \\
&= \int_{t_1}^{\infty}\int_{t_2}^{\infty} f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}, E^{\lambda,\widetilde{R}}_{r_2:n:N}}(u,v)\,dvdu \\
&= \int_{t_1}^{\infty}\int_{t_2}^{\infty} f_{E^{\lambda,\widetilde{R}}_{r_2:n:N}|E^{\lambda,\widetilde{R}}_{r_1:n:N}}(v|u) f_{E^{\lambda,\widetilde{R}}_{r_1:n:N}}(u)\,dvdu \\
&= \int_{t_1}^{\infty}\int_{t_2}^{\infty} c_{r_2-r_1-1,n-r_1}\sum_{j=1}^{r_2-r_1} a_{j,r_2-r_1,n-r_1} \\
&\quad\times \left[\frac{\bar{F}(v)}{\bar{F}(u)}\right]^{\gamma_{j,n-r_1}-1} \frac{f(v)}{\bar{F}(u)} c_{r_1-1,n}\sum_{i=1}^{r_1} a_{i,r_1,n}\left[\bar{F}(u)\right]^{\gamma_{i,n}-1} f(u)\,dvdu \\
&= c_{r_2-r_1-1,n}c_{r_1-1,n}\sum_{i=1}^{r_1}\sum_{j=1}^{r_2-r_1} a_{i,r_1}a_{j,r_2-r_1} \\
&\quad\times \int_{t_1}^{\infty}\int_{t_2}^{\infty} e^{-\lambda(v-u)(\gamma_{j,n-r_1}-1)}\lambda e^{\lambda(v-u)}e^{-\lambda u(\gamma_{i,n}-1)}\lambda e^{-\lambda u}\,dvdu \\
&= c_{r_2-r_1-1,n-r_1}c_{r_1-1,n}\sum_{i=1}^{r_1}\sum_{j=1}^{r_2-r_1} a_{i,r_1,n}a_{j,r_2-r_1,n-r_1} \\
&\quad\times \int_{t_1}^{\infty}\int_{t_2}^{\infty} \lambda^2 e^{-\lambda\gamma_j v}e^{\lambda(\gamma_{i,n}-\gamma_{j,n-r_1})u}\,dvdu \\
&= c_{r_2-r_1-1,n-r_1}c_{r_1-1,n}e^{-\lambda t_2}\sum_{i=1}^{r_1}\sum_{j=1}^{r_2-r_1} a_{i,r_1,n}a_{j,r_2-r_1,n-r_1} \\
&\quad\times \frac{e^{-\lambda\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)t_1}}{\gamma_{j,n-r_1}\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)} \qquad (4.11)
\end{aligned}
$$

and

$$
\mathbb{P}\left[E^{\lambda,\widetilde{R}}_{r_1:n:N:} > t_1\right] = c_{r_1-1,n}\sum_{i=1}^{r_1}\frac{a_{i,r_1,n}}{\gamma_{i,n}}e^{-(\lambda\gamma_{i,n})t_1}. \qquad (4.12)
$$

Now, substituting(4.8), (4.9), (4.10), (4.11) into (4.5), the required result follows. □

Theorem 4.15. *Let $\{X^{\widetilde{R}}_{i:n:N}, i = 1,\ldots,n\}$ be n progressively type II censored order statistics based on distribution function F with non-increasing hazard rate function r_F such that for a positive constant λ*

and each $x > 0$, $r_F(x) \geq \lambda$. *Let also*

$$\widetilde{R} = (\overbrace{R_1, R_1, \cdots, R_1}^{r_1-1}, R^{(r_1)}, \overbrace{R_2, R_2, \cdots, R_2}^{r_2-r_1-1}, R^{(r_2)}, \cdots,$$
$$\overbrace{R_{i-1}, R_{i-1}, \cdots, R_{i-1}}^{r_1-r_{i-1}-1}, R^{(r_i)}, \overbrace{R_{i+1}, \cdots, R_n}^{n-r_i-1}).$$

Then, for $t_1 \leq t_2$,

$$E\left[X^{\widetilde{R}}_{r_3:n:N} - t_2 | X^{\widetilde{R}}_{r_1:n:N} > t_1, X^{\widetilde{R}}_{r_2:n:N} > t_2\right] \leq$$
$$\sum_{k=1}^{r_3-r_2} \frac{1}{\lambda\gamma_{k,n-r_2}} + \left[\sum_{k=1}^{r_2-r_1} \frac{1}{\lambda\gamma_{k,n-r_1}} + t_1 + \frac{\sum_{i=1}^{r_1} a_{i,r_1,n}\gamma_{i,n}^{-2}\lambda^{-1}e^{-\lambda\gamma_{i,n}t_1}}{\sum_{i=1}^{r_1} a_{i,r_1,n}\gamma_{i,n}^{-1}e^{-\lambda\gamma_{i,n}t_1}}\right]$$
$$\times \{\frac{e^{\lambda t_2}\sum_{i=1}^{r_1}\frac{a_{i,r_1,n}}{\gamma_{i,n}}e^{-(\lambda\gamma_{i,n})t_1}}{c_{r_2-r_1,n-r_1}\sum_{i=1}^{r_1}\sum_{j=1}^{r_2-r_1} a_{i,r_1,n}a_{j,r_2-r_1,n-r_1}\frac{e^{-\lambda\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)t_1}}{\gamma_{j,n-r_1}\left(\gamma_{i,n}-\gamma_{j,n-r_1}\right)}}\} \tag{4.13}$$

Proof. Let $\{E^{\lambda,\widetilde{R}}_{i:n:N}\}$ be n progressive type II censored order statistics based on an exponential distribution with the hazard rate λ. The assumptions that $r_F(x)$ is non-increasing in x and $r_F(x) \geq \lambda$ implies that $X \leq_{\mathrm{lr}} E^{\lambda}$ (cf. Lemma 3.5 in Belzunce et al., (2001)). Using this and the sublattice $L = [(x_{(1)}, \ldots, x_{(n)}) \in \mathbb{R}^n | x_{(i)} > t_1, x_{(j)} > t_2]$ in Theorem 2.8, where $x_{(1)} \leq \ldots \leq x_{(n)}$, $t_1, t_2 \in \mathbb{R}$ and $t_1 \leq t_2$, we obtain that

$$E\left[X^{\widetilde{R}}_{r_3:n:N} - t_2 | X^{\widetilde{R}}_{r_1:n:N} > t_1, X^{\widetilde{R}}_{r_2:n:N} > t_2\right] \leq$$
$$E\left[E^{\lambda,\widetilde{R}}_{r_3:n:N} - t_2 | E^{\lambda,\widetilde{R}}_{r_1:n:N} > t_1, E^{\lambda,\widetilde{R}}_{r_2:n:N} > t_2\right].$$

Now the required results follows from Proposition 4.14. □

Moreover, the joint density function of every subset from $X^{\widetilde{R}}_{i:n:N}$'s can be derived from Lemma (2.5).

Remark 4.16. In particular, the result of Theorem 4.15 can also be applied to the case when

$$\widetilde{R} = (\overbrace{R_1, R_1, \cdots, R_1}^{r_1}, \overbrace{R_2, R_2, \cdots, R_2}^{r_2-r_1}, \cdots, \overbrace{R_{i-1}, R_{i-1}, \cdots, R_{i-1}}^{r_1-r_{i-1}}, \overbrace{R_{i+1}, \cdots, R_n}^{n-r_i}),$$

which is of practical importance.

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