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On the Complete Convergence of Weighted Sums for Dependent Random Variables

V. Fakoor, H. A. Azarnoosh

Department of Statistics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Iran. (fakoor@math.um.ac.ir,azarnush@math.um.ac.ir)

Abstract. We study the limiting behavior of weighted sums for negatively associated (NA) random variables. We extend results in Wu (1999) and a theorem in Chow and Lai (1973) for NA random variables.

1 Introduction

Wu (1999) proved the equivalence of the almost sure and complete convergence of a particular weighted sum of iid random variables. In section 2 we extend this result to NA random variables. In section 3 we prove a theorem about almost sure convergence of the weighted sum

$$n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} a_{nk} X_k, \quad 1 \leq \alpha \leq 2,$$

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for negatively dependent random variables under the condition

$$\limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2 < \infty,$$

for double arrays $\{a_{nk} : n \geq 1,\ 1 \leq k \leq n\}$. Basic definitions and properties will be briefly listed in the remainder of this section for later reference while obtaining the major results.

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise negatively quadratic dependent (NQD) if

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j), \tag{1.1}$$

for all $x_i, x_j \in \mathbb{R}$ and for all $i, j \geq 1, i \neq j$.

It is easy to show that definition (1.1) implies

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j), \tag{1.2}$$

for all $x_i, x_j \in \mathbb{R}$ and for all $i, j \geq 1, i \neq j$. Also we can show that (1.2) implies that (1.1). Therefore (1.1) and (1.2) are equivalent.

Definition 1.2. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$

$$COV\{f_1(X_i, i \in A), f_2(X_j, j \in B)\} \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

This dependence structure was first introduced by Alam and Saxena (1981) and carefully studied by Joag-Dov and Proschan (1983). Joag-Dov and Proschan (1983) showed that negatively correlated normal random variables are NA, also permutation distribution are NA. Readers can refer to Joag-Dov and Proschan (1983) for other interesting results.

Remark 1.1. Every sequence of NA random variables is also pairwise NQD.

Remark 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables. Then for any disjoint A, B and positive λ_j's, $\Sigma_{k\in A}\lambda_k X_k$ and $\Sigma_{l\in B}\lambda_l X_k$ are NQD.

Remark 1.3. Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables. Then the sequence of $\{-X_n, n \geq 1\}$ is also NA random variables.

For a sequence of NA random variables and positive λ_j's, we have

Proposition 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables, then we have the following inequality*

$$E\exp(\Sigma_{k\in A}\lambda_k X_k) \leq \prod_{k\in A} E\exp(\lambda_k X_k). \tag{1.3}$$

Proof. See (P2), page 46 in Mari et al. (2001).

Definition 1.3. *A random variable X is said to be a generalized Gaussian(GG) random variable if there exists a nonnegative real number α, such that for each real number t,*

$$Ee^{tX} \leq e^{\frac{t^2\alpha^2}{2}}. \tag{1.4}$$

The minimum of the α's satisfying (1.4) will be denoted by $\tau(X)$. (cf. Chow, 1966.)

Definition 1.4. *A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to the random variable X if $\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty$ for each $\epsilon > 0$.*
This definition is due to Hsu and Robbins(1947).

Throughout this note, let $\{X_n, n \geq 1\}$ be a sequence of NA identically distributed random variables with $EX_1 = 0$.

2 Main result

Let

$$\mu_n := \frac{1}{\log\log n}\sum_{j=2}^{n}\frac{X_{n+2-j}}{j\log j},$$

and it is assumed hereafter that $n > n_0 = [e^{20}]$, with [.] denoting the integer part as usual.

Wu (1999) proved the following theorem for iid random variables. We extend it for NA identically distibuted random variables. The constant C in the following may denote different quantities at different appearances.

Theorem 2.1. We have $\mu_n \to 0$ a.s. if and only if $\mu_n \to 0$ completely.

Proof. We know that if $\mu_n \to 0$ completely, then the Borel-Canteli lemma trivially implies that, $\mu_n \to 0$ a.s. The converse follows from lemmas (2.2) and (2.3) below.

Lemma 2.1. *We have*

$$\sum_{j=[\frac{1}{2}\log n]}^{n} \log\left(\ 1 + C\left(\ \frac{\log n}{j\log j}\ \right)^2\ \right) < C\log n.$$

For proof see Wu (1999).

Lemma 2.2. *If* $\mu_n \to 0$ *a.s. then*

$$\mu_n{}^{(1)} := \frac{1}{\log\log n}\sum_{j=[\frac{1}{2}\log n]}^{n} \frac{X_{n+2-j}}{j\log j} \to 0, \qquad \textit{completely},$$

and

$$\frac{1}{\log\log n}\sum_{j=[\frac{1}{2}\log n]+1}^{[\frac{1}{2}\log[n\log n]]} \frac{X_{n+2-j}}{j\log j} \to 0, \qquad \textit{completely}.$$

Proof. For $\theta \in [0,1]$, we have $|e^{\theta X} - 1 - \theta X| \le \theta^2 e^{|X|}$. So $Ee^{\theta X} \le 1 + \theta^2 Ee^{|X|} = 1 + C\theta^2$. Note that if $n > n_0$ and $n \ge j \ge [\frac{1}{2}\log n]$, then $(\log n)/(j\log j) < 1$. Hence, for any $\epsilon > 0$, by proposition (1.1) and lemma (2.1). We have

$$\begin{aligned}
\sum_{n=n_0}^{\infty} P(\mu_n{}^{(1)} > \epsilon) &\le \sum_{n=n_0}^{\infty} n^{-\epsilon\log\log n} E n^{\sum_{j=[\frac{1}{2}\log n]}^{n} \frac{X_{n+2-j}}{j\log j}} \\
&\le \sum_{n=n_0}^{\infty} n^{-\epsilon\log\log n} \prod_{j=[\frac{1}{2}\log n]}^{n} Ee^{(\frac{\log n}{j\log j})X_{n+2-j}}
\end{aligned}$$

$$\leq \sum_{n=n_0}^{\infty} n^{-\epsilon \log\log n} \prod_{j=[\frac{1}{2}\log n]}^{n} \left(1 + C\left(\frac{\log n}{j \log j} \right)^2 \right)$$

$$\leq \sum_{n=n_0}^{\infty} n^{-\epsilon \log\log n} e^{C \log n} < \infty.$$

Similarly we can obtain $\sum_{n=n_0}^{\infty} P({\mu_n}^{(1)} < -\epsilon) < \infty$. The first statement now follows if we combine the two inequalities together. The same technique yields the second statement.

Lemma 2.3. *If*

$${\mu_n}^{(2)} := \frac{1}{\log\log n} \sum_{j=2}^{[(\frac{1}{2})\log n]} \frac{X_{n+2-j}}{j \log j} \to 0, \qquad a.s.$$

then

$${\mu_n}^{(2)} \to 0, \qquad completely.$$

Proof. For any $\epsilon > 0$, we have

$$0 = P(\limsup_{n\to\infty} |{\mu_n}^{(2)}| > \epsilon) \geq P(\limsup_{m\to\infty} |{\mu_{n(m)}}^{(2)}| > \epsilon), \qquad (1.5)$$

where $n(m) = [m \log m],\ m \in N$. Since

$$n(m+1) + 2 - (1/2)\log n(m+1) > n(m),$$

for $m > 3$, by remark (1.2), the random variables ${\mu_{n(m)}}^{(2)}, m \geq 3$ are pairwise NQD. By (1.5), ${\mu_{n(m)}}^{(2)} \to 0$ a.s. therefore $({\mu_{n(m)}}^{(2)})^+ \to 0$ a.s. and $({\mu_{n(m)}}^{(2)})^- \to 0$ a.s. (where $X^+ = \max(0, X)$ and $X^- = \max(0, -X)$). It is easy to verify that $\{({\mu_{n(m)}}^{(2)})^+\}_{m\geq 3}$ and $\{({\mu_{n(m)}}^{(2)})^-\}_{m\geq 3}$ are pairwise NQD.
Defining the following events,

$$A_m = [({\mu_{n(m)}}^{(2)})^+ > \epsilon/3], \qquad B_m = [({\mu_{n(m)}}^{(2)})^- > \epsilon/3],$$

$m > 3$, we have

$$P(A_k \cap A_l) \leq P(A_k)P(A_l), \qquad P(B_k \cap B_l) \leq P(B_k)P(B_l), \ \ for\ \ k \neq l.$$

By Lemma 1 in Matula (1992), if $\sum_{m=1}^{\infty} P(A_m)$ diverges, then

$$P(A_n \ i.o.\ ) = 1$$

contrary to the almost sure convergence of $({\mu_{n(m)}}^{(2)})^+$ to zero. Therefore $\sum_{m=1}^{\infty} P(A_m) < \infty$.
The same argument for $({\mu_{n(m)}}^{(2)})^-$ yields $\sum_{m=1}^{\infty} P(B_m) < \infty$. Thus,

$$\begin{aligned}\sum_{n=n_0}^{\infty} P(|{\mu_{n(m)}}^{(2)}| > \epsilon) \quad &\leq \quad \sum_{n=n_0}^{\infty} P(({\mu_{n(m)}}^{(2)})^+ > \epsilon/3) \\ &\quad + \sum_{n=n_0}^{\infty} P(({\mu_{n(m)}}^{(2)})^- > \epsilon/3) < \infty.\end{aligned}$$

Now by the second statement of lemma (2.2) it follows that $\mu_n^{(2)} \to 0$ completely.

The result below extends a corollary in Wu (1999) to NA random variables.

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of NA identically distributed random variables with $EX_1 = 0$, if $\mu_n \to 0$ a.e. Then*

$$\max_{2\leq j\leq n} \left(\frac{S_j}{\log\log n} \right) \to 0, \qquad \textit{completely.}$$

Proof. It is easy to see that $ES_n^2 \leq EX_1^2 \sum_{j=2}^{n} \frac{1}{(j\log j)^2} \leq C < \infty$. Now we apply Kolmogorov-type inequality (cf. Matula, 1999.) by taking $\eta = (\log\log n)\epsilon$, for $\epsilon > 0$,

$$\sum_{n=n_0}^{\infty} P(\max_{2\leq j\leq n} |S_j| > \eta) \leq C \sum_{n=n_0}^{\infty} \eta^{-2} \sum_{j=2}^{n} \frac{1}{(j\log j)^2} < \infty,$$

and to get the result.

3 Almost sure convergence for NAGG random variables

In this section we impose a GG condition on the NA random variables and get a result for a weighted sum.

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated generalized Gaussian (NAGG) random variables with $EX_n = 0$ for all n. Assume there exists $\alpha \in \mathbb{R}$ such that for all n, $\tau(X_n) \leq \alpha$. Let*

$\{a_n\}$ be a sequence of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} a_n^2 = A < \infty$, and $Y = \sum_{n=1}^{\infty} a_n X_n$, then for any $c > 0$,

$$P(Y \geq c\alpha) \leq \exp\Big(-\tfrac{c^2}{2A} \Big).$$

Proof. For all $t > 0$ and $n \geq 1$, $Ee^{tX_n} \leq e^{\frac{t^2\alpha^2}{2}}$. By Fatou's lemma and proposition (1.1) we have

$$\begin{aligned} P(Y \geq c\alpha) &\leq e^{-c\alpha t} Ee^{tY} \\ &\leq e^{-c\alpha t}. \liminf_{n\to\infty} Ee^{t\sum_{k=1}^{n} a_k X_k} \\ &\leq e^{-c\alpha t}. \liminf_{n\to\infty} \prod_{k=1}^{n} Ee^{ta_k X_k} \\ &\leq e^{-c\alpha t}. \liminf_{n\to\infty} \prod_{k=1}^{n} e^{\frac{t^2\alpha^2 {a_k}^2}{2}} \\ &= e^{-c\alpha t}.e^{\frac{t^2\alpha^2 A^2}{2}}, \end{aligned}$$

setting $t = \frac{c}{\alpha A}$, we obtain the desired inequality.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of NAGG random variables with $EX_n = 0$ and suppose there exists $\alpha \in \mathbb{R}$ such that for all n, $\tau(X_n) \leq \alpha$, then for $1 \leq \alpha \leq 2$, and for every nonnegative array $\{a_{nk} : n \geq 1, 1 \leq k \leq n\}$ of real numbers such that*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2 < \infty,$$

then we have

$$n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} a_{nk} X_k \to 0, \qquad \textit{completely}. \tag{3.1}$$

Proof. Suppose a_{nk} is a double array of nonnegative real numbers such that

$$\limsup_{n\to\infty} A_n = A < \infty,$$

where $A_n = \sum_{k=1}^{n} a_{nk}^2$. Define $T_n = n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} a_{nk} X_k$, from lemma (3.1) for every $\epsilon > 0$

$$P(T_n \geq \epsilon) \leq \exp\Big(-\tfrac{\epsilon^2}{2A_n\alpha^2} n^{\frac{2}{\alpha}} \Big),$$

therefore $\sum_{n=1}^{\infty} P(T_n \geq \epsilon) < \infty$. Replacing X_n by $-X_n$, we obtain (3.1) since ϵ is arbitrary.

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