Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1$, $0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ with pdf $f(x)$. Let $R_{U(1)} = X_1, R_{U(2)}, \ldots$, be the upper records and $R_{L(1)}, R_{L(2)}, \ldots$, be lower records of $\{X_i, i = 1, 2, \ldots\}$. For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose $R_{nl}$ is the largest observation after observing $n$th record and $R_{(ns)}$ is the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range.

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joint pdf of \( f_{(nl,(ns))} \) of \( R_{nl} \) and \( R_{(ns)} \) is given by (see Arnold et. al. 1998, p. 275) as

\[
f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} [-\ln(F(y) + F(x))]^{n-2} f(x) f(y), \quad (1.1)
\]

\(-\infty < x < y < \infty\)

The pdf of \( f_{wnr} \) of \( W_{nr} \) is given by

\[
f_{wnr}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} [-\ln(F(w+u)+F(u))]^{n-2} f(w+u) f(u) \, du
\]

Suppose \( X_i's \) are distributed as uniform with

\[
f(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases}
\]

Using (1.3) in (1.2), we obtain

\[
f_{wnr}(w) = \begin{cases} 
\frac{2^{n-1}(1-w)}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, & 0 < w < 1, n \geq 2 \\
0, & \text{otherwise}
\end{cases}
\]

Figure 1.1 gives the pdf of \( W_{nr} \) for \( n = 10 \) when \( X_i's \) are distributed as uniform.

In this paper we will consider distributional properties of \( W_{nr} \) for the case \( X_i's \) are distributed as uniform distribution.

2 Main Results

Lemma 2.1. For \( n \geq 2 \) and \( 0 < x < 1 \),

\[
F_{wnr}(x) = \Gamma_{-2\ln(1-x)}(n-1),
\]

where

\[
\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du
\]

Proof. \[
F_{wnr}(x) = \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du
\]

\[= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt
\]

\[= \Gamma_{-2\ln(1-x)}(n-1). \square
\]
Remark 2.1.

$$1 - F_{W_{nr}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2 \ln(1 - x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

Lemma 2.2.

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1 - x)}$$

Proof.

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{\Gamma_{-2 \ln(1-x)}(n-1) - \Gamma_{-2 \ln(1-x)}(n)}{\Gamma(n)}$$

$$= \frac{1}{\Gamma(n)}(-2 \ln(1 - x))^{n-1}e^{-2 \ln(1 - x)}$$

$$= \frac{(1 - x)^2}{\Gamma(n)}(-2 \ln(1 - x))^{n-1}$$

$$= \frac{f_{W_{n+1r}}(x)}{2(1 - x)}.$$
\[ \mu_{nr}^p = E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p (1 - w) \left( -\ln(1 - w) \right)^{n-2} dw \]

\[ = \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1 - w)^p w^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(2 + k)^{n-1}} \]  \hspace{1cm} (2.1)

Using \( p = 1 \) and \( p = 2 \), we can get the mean and variance of \( W_{nr} \) as

\[ E(W_{nr}) = 1 - \left( \frac{2}{3} \right)^{n-1} \]

and

\[ \text{Var}(W_{nr}) = \left( \frac{1}{2} \right)^{n-1} - \left( \frac{4}{9} \right)^{n-1}. \]

**Theorem 2.1.** Let \( \mu_n^r = E(W_{nr}^r) \), then for \( n \geq 2 \) and \( r = 1, 2, \ldots \)

\[ (r + 2) \mu_n^r - r \mu_n^{r-1} = 2 \mu_{n-1}^r. \]  \hspace{1cm} (2.2)

**Proof.**

\[ r(\mu_n^{r-1} - \mu_n^r) \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 [(1 - w)^2 w^{r-1} \left( -\ln(1 - w) \right)^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r \left[ 2(1 - w) \left( -\ln(1 - w) \right)^{n-2} dw \right. \]

\[ - \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1 - w)^2 w^{r-1} \left[ -\ln(1 - w) \right]^{n-3} \frac{1}{1 - w} dw \]

\[ = 2 \mu_n^r - 2 \mu_{n-1}^r. \]

On simplification we get the result. □

**Theorem 2.2.** For \( n \geq 2, p > 0 \),

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3 + k)^{n-1}} \]
Proof.

\[
\begin{align*}
\mu_{nr}^p - \mu_{nr}^{p+1} &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1 - w) [-\ln(1 - w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p(1-w)^2 [-\ln(1 - w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1-w)^p w^2 [-\ln(w)]^{n-2} dw \\
&= 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}}
\end{align*}
\]  

(2.3)

Theorem 2.3. For \(n \geq 1\),

\[
1 - W_{n+1r} = \prod_{j=1}^n V_j,
\]

(2.4)

where \(V_1, V_2, ..., V_{n-1}\) are i.i.d. with \(F(v) = v^2, \ 0 < v < 1\).

Proof. We will show first that

\[
1 - W_{n+1r} = (1 - W_{nr})V_n,
\]

where \(V_n\) is independent of \(1 - W_{nr}\) and is distributed with pdf as \(f_V(v) = 2v, \ 0 < v < 1\).

Let \(Y_{n-1} = (1 - W_{nr})V_n, \ n \geq 2\), and \(f_n\) be the pdf of \(1 - W_{nr}\), then

\[
f_n = \frac{\prod_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}}}{\Gamma(n-1)} [-\ln(1 - w)]^{n-2}, \ 0 < w < 1,
\]

\[
\begin{align*}
F(y) &= P(Y_{n+1} \leq y) = P(1 - W_{nr})V_n \leq y \\
&= y^2 + \int_y^1 F_n(t) \frac{t^2}{t^3} dt, \text{ where } F_{n-1} \text{ is the df of } Y_{(n)} \\
&= y^2 + \int_y^1 F_n(t) \frac{2}{t^2} dt \\
&= y^2 + \int_y^1 F_n(t) \frac{1}{t^2} dt + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt
\end{align*}
\]  

(2.5)
Differentiating both sides of (2.5), with respect to y, we obtain

\[ f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt \]

i.e.

\[ \frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = 2 \int_y^1 \frac{2^{n-1}t}{\Gamma(n-1)} [-\ln(1-t)^{n-2} \frac{1}{t^2} dt] \]

\[ = \frac{2^n}{\Gamma(n)} [-\ln t]^{n-1} \frac{1}{n-1} y \]

\[ = \frac{2^n}{\Gamma(n)} [-\ln y]^{n-1} \]

(2.6)

Hence

\[ f(y) = \frac{2^n}{\Gamma(n-1)} y [-\ln y]^{n-1} \]

(2.7)

which is the pdf of \( Y = 1 - W_{n+1r} \).

Note that the sequence \( Y_2, Y_3, \ldots \) forms a Markov chain. □

Using (2.6), we have the following representation of \( W_{nr} \) for

\[ 1 - W_{n+1r} \overset{d}{=} \prod_{j=1}^n V_j, \quad n \geq 1 \]

(2.8)

where \( V_1, V_2, \ldots, V_{n-1} \) are i.i.d. with \( F(v) = v^2, \quad 0 < v < 1 \).

The conditional expectation of

\[ 1 - W_{nr}|1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1, \]

is

\[ E(1 - W_{nr}|1 - W_{mr} = x) = x \left( \frac{2}{3} \right)^{n-m}. \]

Thus

\[ Cov(W_{nr}, W_{mr}) = \left( \frac{2}{3} \right)^{n-m} Var(W_{mr}) \]

\[ = \left( \frac{2}{3} \right)^{n-m} \left[ \frac{1}{2}^{m-1} - \left( \frac{4}{9} \right)^{m-1} \right]. \]
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{(\frac{2}{3})^{m-n} \sqrt{[\frac{1}{2})^{m-1} - (\frac{3}{8})^{m-1}]} \approx \frac{\sqrt{[(\frac{2}{3})^{m-1} - 1]} \sqrt{[(\frac{3}{8})^{n-1} - 1]} - \frac{m}{\sqrt{[(\frac{2}{3})^{n-1} - 1]}}}{0}$$

for any fixed $m$ as $n \to \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{1}{81}$</td>
<td>$\frac{2}{243}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{17}{324}$</td>
<td>$\frac{17}{386}$</td>
<td>$\frac{17}{729}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{81}$</td>
<td>$\frac{17}{386}$</td>
<td>$\frac{217}{3862}$</td>
<td>$\frac{217}{8748}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{4}{243}$</td>
<td>$\frac{17}{729}$</td>
<td>$\frac{217}{8748}$</td>
<td>$\frac{2465}{104976}$</td>
</tr>
</tbody>
</table>

**Theorem 2.4.** The joint pdf $f_{m,n}^*(x,y)$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$f_{m,n}^*(x,y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1} (1 - \ln(1 - x))^{m-2} \times [-\ln(1 - x) + \ln(1 - y)]^{n-m-1} \frac{1 - y}{1 - x},$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \Pi_{j=1}^m V_j$ and $U_2 = \Pi_{j=m+1}^n V_{n-j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1,u_2) = \frac{2u_1}{\Gamma(m-1)} (-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)} (-2 \ln u_2)^{n-m-1}.$$

Let $T_1 = U_1$ and $T_2 = U_1 U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$f_{T_1T_2}(t_1,t_2) = \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)} (-2 \ln t_1)^{m-2} \frac{t_2}{t_1} (-2 \ln t_2)^{n-m-1}$$

$$= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)} (-\ln t_1)^{m-2} t_2 \frac{t_2}{t_1} (-\ln t_2 - \ln t_1)^{n-m-1}$$
Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, we obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$f_{m,n}^*(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1-x))^{m-2} \times [\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{1-x},$$

$$0 < x < y < 1.$$  
\[\square\]

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3} E(W_{mr})^p(W_{m+1r})^q + \frac{2}{q+3} E(W_{mr})^{p+q+1}$$

**Proof.**

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]$$

$$= \int_0^1 \int_x^1 [(x)^p((y)^q(1-y))] f_{m,m+1}^*(x,y) dy dx$$

$$= \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)}2^{n-1}(-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x)] dx, \quad (2.9)$$

where

$$H(x) = \int_x^1 (y)^q(1-y) dy$$

$$= \int_x^1 (y)^q(1-y)^2 dy$$

$$= \frac{y^{q+1}}{(q+1)} (1-y)^2 \bigg|_x^1 + \int_x^1 \frac{2y^{q+1}}{(q+1)} (1-y)$$

$$= -\frac{x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)} (1-y) dy$$

Substituting in (2.9), we obtain

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]$$

$$= \int_0^1 [(x)^p(1-y)] \frac{1}{\Gamma(m-1)}2^{n-1}(-\ln(1-x))^{m-2} \frac{1}{(1-x)}$$

$$\cdot \left[-\frac{2x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)} (1-y) dy \right] dx$$

$$= -\frac{2}{q+1} E((W_{mr})^{p+q+1}) + \frac{2}{q+1} E(W_{mr})^p(W_{m+1r})^{q+1}$$
Thus

\[
E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1}.
\]

\[\square\]

**Theorem 2.6.** For \(m \geq 2\), \(n > m \geq 2\), \(p \geq 0\) and \(q \geq 0\),

\[
E(W_{mr})^p(W_{nr})^{q+1} = \frac{q+1}{q+3} E((W_{mr})^p(W_{nr})^q) + \frac{2}{q+3} E((W_{mr})^{p+q+1})
\]

\[
E((W_{mr})^p(W_{nr})^{q+1}) = \frac{q+1}{q+3} E([W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^q)
\]

**Proof.**

\[
E([W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] = \int_0^1 \int_x^1 [(y)^p\{(y)^q(1-y)\}] f_{m,n}^*(x, y) dy dx
\]

\[
= \int_0^1 [(x)^p \Gamma(m-1)\Gamma(n-m)^{-2m+k-1}]
\]

\[
(\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, \quad (2.10)
\]

where

\[
\frac{1}{\Gamma(n-m)} H(x)
\]

\[
= \int_x^1 [(y)^q(1-y)] [-\ln(1-x) + \ln(1-y)]^{n-m-1}
\]

\[
\times (1-y) dy
\]

\[
= \int_x^1 \frac{y^q}{q} [-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2 dy
\]

\[
= \frac{y^{q+1}}{(q+1)} [-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2
\]

\[
- \int_x^1 \frac{y^{q+1}}{q+1} \frac{d}{dy} [-\ln(1-x) + \ln(1-y)]^{n-m-1}
\]

\[
\times (1-y)^2 dy
\]
\[ E \left[ (W_{mr})^p (W_{nr})^q - (W_{mr})^p (W_{nr})^{q+1} \right] = \frac{2}{q+1} E((W_{mr})^p (W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^p (W_{n-1r})^{q+1}) \]

On simplification, we obtain
\[ E((W_{mr})^p (W_{nr})^{q+1}) = \frac{q+1}{q+3} E((W_{mr})^p (W_{nr})^q) + \frac{2}{q+3} E((W_{mr})^p (W_{n-1r})^{q+1}) \]

\[ \square \]

Entropy of \( W_{nr} \).
The entropy of \( W_{nr} \) is given in the following theorem.

**Theorem 2.7.** The entropy, \( I_n \) of \( W_{nr}, n \geq 2 \), is given by
\[ I_n = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2) \Psi(n-1) - \ln 2, \]
where \( \Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)} \).

**Proof.**
\[ I_n = E(-\ln f_{W_{nr}}) \]
\[ = \int_0^1 \ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du \]
\[ = - \ln \Gamma(n-1) - (n-1) \ln 2 - H_1 - H_2, \quad (2.11) \]
where

\[ H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2}, \]

\[ H_2 = \int_0^1 (n-2)\ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du \]

Substituting \(-\ln(1-u) = t\), we obtain

\[ H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^{\infty} t^{n-2} \ln t e^{-2t} dt \]

\[ = \frac{n-2}{\Gamma(n-1)} \int_0^{\infty} t^{n-2} \ln t e^{-t} dt - (n-2) \ln 2 \]

\[ = (n-2)[\Psi(n-1) - \ln 2]. \]

Substituting \( H_1 \) and \( H_2 \) in (2.11), we obtain

\[ I_n = \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2} \]

\[ - (n-2)[\Psi(n-1) - \ln 2]. \]

\[ = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square (2.12) \]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
<thead>
<tr>
<th>n</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-I</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range, when \(X_1,X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \ 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) of \(\theta\) is

\[ \hat{\theta} = \frac{1}{3(2^n - 1)} [3.2^{n-1}W_{n+1} - 2^{n-2}W_n - 2^{n-3}W_{n-1} - \ldots - W_2] \]

and

\[ Var(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}. \]
Proof. Let
\[ Z_1 = d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \]
\[ Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), d_n = 3.2^{\frac{n}{2}} \]
\[ Z' = (Z_1, Z_2, \ldots, Z_n) \]
then \( E(Z') = A\theta \), where
\[ A' = (2^{\frac{1}{2}}, 2, \ldots, 2^{\frac{n}{2}}). A' A = 2(2^n - 1). \]

Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is
\[
\hat{\theta} = (A'A)^{-1} A' Z
= \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2 Z_2 + \ldots + 2^{\frac{n}{2}} Z_n]
= \frac{1}{2(2^n - 1)} [3.2^2 W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2 W_{2r}]. \quad (2.13)
\]

\[ \text{Var}(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square \]

For example, if \( n = 4 \), then
\[
\hat{\theta} = \frac{1}{15} [24 W_{5r} - 4 W_{4r} - 2 W_{3r} - W_{2r}]
\]
and
\[ \text{Var}(\hat{\theta}) = \frac{\theta^2}{30}. \]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c - 1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2n+1-1} + (c - 1)^2 \theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2n+1-2}{2n+1-1}$.

The bias of $\tilde{\theta} = (c - 1)\theta \frac{1}{2n+1-1}$ and $\text{MSE}(\tilde{\theta}) = \frac{1}{2n+1-1}$.

**Prediction of $W_{n+sr}$**

We consider the prediction of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

**Theorem 2.5.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.

**Proof.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$W^*_{n+sr} = E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n)$$

$$= E(W_{n+sr}|W_{nr} = x_n),$$

by Markov property of $W_{2r}, W_{3r}, \ldots$

$$= \theta[1 - (\frac{2}{3})^s] + x_n(\frac{2}{3})^s.$$