Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function \( F(x) \) and probability density function (pdf) \( f(x) \). Let \( R_{nl} \) be the largest observation after observing \( n \)th record and \( R_{(ns)} \) be the smallest observation after observing the \( n \)th record. Then we say \( W_{nr} = R_{nl} - R_{(ns)}, n > 1 \), as the \( n \)th record range. We will consider some distributional properties of \( W_{nr} \) when \( f(x) = 1, 0 \leq x \leq 1 \).

1 Introduction

Let \( \{X_i, i = 1, 2, \ldots\} \) be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with pdf \( f(x) \). Let \( R_{U(1)} = X_1, R_{U(2)}, \ldots \) be the upper records and \( R_{L(1)}, R_{L(2)}, \ldots \) be lower records of \( \{X_i, i = 1, 2, \ldots\} \). For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose \( R_{nl} \) is the largest observation after observing \( n \)th record and \( R_{(ns)} \) is the smallest observation after observing the \( n \)th record. Then we say \( W_{nr} = R_{nl} - R_{(ns)}, n > 1 \), as the \( n \)th record range. The

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joint pdf of \( f_{(nl,(ns))} \) of \( R_{nl} \) and \( R_{(ns)} \) is given by (see Arnold et al. 1998, p. 275) as

\[
f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} \left[ -\ln(F(y) + F(x)) \right]^{n-2} f(x) f(y), \quad (1.1)
\]

\(-\infty < x < y < \infty\)

The pdf of \( f_{wnr} \) of \( W_{nr} \) is given by

\[
f_{wnr}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} \left[ -\ln(F(w+u)+F(u)) \right]^{n-2} f(w+u) f(u) \, du
\]

\( (1.2) \)

Suppose \( X_i's \) are distributed as uniform with

\[
f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
\]

\( (1.3) \)

Using (1.3) in (1.2), we obtain

\[
f_{wnr}(w) = \begin{cases} \frac{2^{n-1}}{\Gamma(n-1)} \left[ -\ln(1-w) \right]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases}
\]

\( (1.4) \)

Figure 1.1 gives the pdf of \( W_{nr} \) for \( n = 10 \) when \( X_i's \) are distributed as uniform.

In this paper we will consider distributional properties of \( W_{nr} \) for the case \( X_i's \) are distributed as uniform distribution.

2 Main Results

Lemma 2.1. For \( n \geq 2 \) and \( 0 < x < 1 \),

\[
F_{wnr}(x) = \Gamma_{-2\ln(1-x)}(n-1),
\]

where

\[
\Gamma_x(r) = \int_{0}^{x} \frac{1}{\Gamma(r)} a^{r-1} e^{-a} \, da
\]

Proof.

\[
F_{wnr}(x) = \int_{0}^{x} \frac{2^{n-1}}{\Gamma(n-1)} \left[ -\ln(1-u) \right]^{n-2} du
\]

\[
= \int_{0}^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t+n-2} \, dt
\]

\[
= \Gamma_{-2\ln(1-x)}(n-1). \quad \Box
\]
Figure 1.1: $f(x) = \frac{2^9(1-x)}{\Gamma(9)} (-\ln(1-x))^8$

Remark 2.1.

$$1 - F_{W_{nr}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2 \ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

Lemma 2.2.

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)}$$

Proof.

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{\Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n)}{\Gamma(n)}$$

$$= \frac{1}{\Gamma(n)} (-2 \ln(1-x))^{n-1} e^{-2\ln(1-x)}$$

$$= \frac{(1-x)^2}{\Gamma(n)} (-2 \ln(1-x))^{n-1}$$

$$= \frac{f_{W_{n+1r}}(x)}{2(1-x)}.$$
\[ \mu_{nr}^{p} = E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p(1-w) [-\ln(1-w)]^{n-2} dw \]

\[ = \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1-w)^p w [-\ln(w)]^{n-2} dw \]

\[ = 2^{n-1} \left( \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(2+k)^{n-1}} \right) \quad (2.1) \]

Using \( p = 1 \) and \( p = 2 \), we can get the mean and variance of \( W_{nr} \) as

\[ E(W_{nr}) = 1 - \left( \frac{2}{3} \right)^{n-1} \]

and

\[ \text{Var}(W_{nr}) = \left( \frac{1}{2} \right)^{n-1} - \left( \frac{4}{9} \right)^{n-1}. \quad \square \]

**Theorem 2.1.** Let \( \mu_n^r = E(W_{nr}^r) \), then for \( n \geq 2 \) and \( r = 1, 2, \ldots \)

\[ (r+2)\mu_n^r - r\mu_{n-1}^r = 2\mu_{n-1}^r. \quad (2.2) \]

**Proof.**

\[ r(\mu_n^{r-1} - \mu_n^r) \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)\theta^2} \int_0^1 [(1-w)^2rw^{r-1}[-\ln(1-w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r[2(1-w)[-\ln(1-w)]^{n-2} dw \]

\[ - \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2w^{r-1}[-\ln(1-w)]^{n-3} \frac{1}{1-w} dw \]

\[ = 2\mu_n^r - 2\mu_{n-1}^r. \]

On simplification we get the result. \( \square \)

**Theorem 2.2.** For \( n \geq 2, p > 0, \)

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \]
Proof.

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1}) (1 - w) [-\ln(1 - w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p (1 - w)^2 [-\ln(1 - w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^p w^2 [-\ln(w)]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \quad (2.3) \]

Theorem 2.3. For \( n \geq 1 \),

\[ 1 - W_{n+1} \overset{d}{=} \prod_{j=1}^{n} V_j, \quad (2.4) \]

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2 \), \( 0 < v < 1 \).

Proof. We will show first that

\[ 1 - W_{n+1} \overset{d}{=} (1 - W_{nr}) V_n, \]

where \( V_n \) is independent of \( 1 - W_{nr} \) and is distributed with pdf as \( f_V(v) = 2v \), \( 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_{nr}) V_n \), \( n \geq 2 \), and \( f_n \) be the pdf of \( 1 - W_{nr} \), then

\[ f_n = \frac{2^{n-1} w}{\Gamma(n-1)} [-\ln(1 - w)]^{n-2}, \quad 0 < w < 1, \]

\[ F(y) = P(Y_{n+1} \leq y) = P(1 - W_{nr}) V_n \leq y \]

\[ = y^2 + \int_y^1 F_n(y) \left( \frac{v}{y} \right)^2 dv, \text{ where } F_n \text{ is the df of } Y(n) \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^3} dt \]

\[ = y^2 + y^2 \left[ F_n(t) \frac{1}{t^2} \right]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \quad (2.5) \]
Differentiating both sides of (2.5), with respect to y, we obtain
\[ f(y) = f_{n}(y) - f_{n}(y) + 2y \int_{y}^{1} f_{n}(t) \frac{1}{t^{2}} dt \]
\[ = 2y \int_{y}^{1} f_{n}(t) \frac{1}{t^{2}} dt \]
i.e.
\[ \frac{f(y)}{y} = 2 \int_{y}^{1} f_{n}(t) \frac{1}{t^{2}} dt \]
\[ = 2 \int_{y}^{1} \frac{2^{n-1}t}{\Gamma(n-1)} [-\ln(1-t)^{n-1}]^{\frac{1}{2}} dt \]
\[ = \left[ \frac{2^{n}}{\Gamma(n-1)} [-\ln t]^{n-1} - \frac{1}{n-1} \right]^{y} \]
\[ = \frac{2^{n}}{\Gamma(n)} [-\ln y]^{n-1} \quad (2.6) \]

Hence
\[ f(y) = \frac{2^{n}}{\Gamma(n-1)} y[-\ln y]^{n-1} \quad (2.7) \]
which is the pdf of \( Y = 1 - W_{n+1r} \).

Note that the sequence \( Y_{2}, Y_{3}, ... \) forms a Markov chain. □

Using (2.6), we have the following representation of \( W_{nr} \) for
\[ 1 - W_{n+1r} \overset{d}{=} \prod_{j=1}^{n} V_{j}, \quad n \geq 1 \quad (2.8) \]
where \( V_{1}, V_{2}, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^{2}, \quad 0 < v < 1. \)

The conditional expectation of
\[ 1 - W_{nr} | 1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1, \]
is
\[ E(1 - W_{nr} | 1 - W_{mr} = x) = x(\frac{2}{3})^{n-m}. \]
Thus
\[ Cov(W_{nr}W_{mr}) = (\frac{2}{3})^{n-m} Var(W_{mr}) \]
\[ = (\frac{2}{3})^{n-m} \left[ (\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1} \right]. \]
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{\left(\frac{2}{3}\right)^{n-m} \sqrt{\left[\left(\frac{2}{3}\right)^{m-1} - \left(\frac{2}{3}\right)^{m-1}\right]} - \left(\frac{2}{3}\right)^{n-1}} {\sqrt{\left[\left(\frac{2}{3}\right)^{n-1} - \left(\frac{2}{3}\right)^{n-1}\right]} - \left(\frac{2}{3}\right)^{n-1}} \rightarrow 0$$

for any fixed $m$ as $n \rightarrow \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

<table>
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**Theorem 2.4.** The joint pdf $f_{m,n}^*$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$f_{m,n}^*(x, y) = \frac{1}{\Gamma(m - 1)\Gamma(n - m)} 2^{n-1}(-\ln(1-x))^{m-2}$$

$$\times \left[-\ln(1 - x) + \ln(1 - y)\right]^{n-m-1} \frac{1 - y}{1 - x},$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \prod_{j=1}^{m} V_j$ and $U_2 = \prod_{j=1}^{n-m} V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m - 1)}(-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n - m)}(-2 \ln u_2)^{n-m-1}.$$
Sustituing $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$f^*_{m,n}(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1-x))^{m-2} \times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{1-x},$$

$$0 < x < y < 1. \quad \Box$$

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3}E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3}E(W_{mr})^{p+q+1}$$

**Proof.**

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]$$

$$= \int_0^1 \int_0^1 [(x)^p ((y)^q (1-y))] f^*_{m,m+1}(x,y) dy dx$$

$$= \int_0^1 (x)^p \frac{1}{\Gamma(m-1)} 2^{n-1}(-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, \quad (2.9)$$

where

$$H(x) = \int_x^1 (y)^q (1-y) \frac{(1-y)^2}{2} dy$$

$$= \int_x^1 (y)^q (1-y)^2 dy$$

$$= \frac{y^{q+1}}{q+1} (1-y)^2 \bigg|_x^1 + \int_x^1 \frac{2y^{q+1}}{(q+1)(1-y)} \, dy$$

$$= -\frac{x^{q+1}}{q+1} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)(1-y)} \, dy$$

Substituting in (2.9), we obtain

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]$$

$$= \int_0^1 [(x)^p (1-y)] \frac{1}{\Gamma(m-1)} 2^{n-1}(-\ln(1-x))^{m-2} \frac{1}{(1-x)}$$

$$\cdot \left[ -\frac{2x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)(1-y)} \, dy \right] dx$$

$$= -\frac{2}{q+1}E((W_{mr})^{p+q+1}) + \frac{2}{q+1}E(W_{mr})^p(W_{m+1r})^{q+1}$$
Thus
\[ E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E((W_{mr})^p(W_{m+1r})^{q}) + \frac{2}{q + 3} E(W_{mr})^{p+q+1}. \]

**Theorem 2.6.** For \( m \geq 2, n > m \geq 2 \), \( p \geq 0 \) and \( q \geq 0 \),
\[ E(W_{mr})^p(W_{nr})^{q+1} = \frac{q + 1}{q + 3} E((W_{mr})^p(W_{nr})^{q+1}) + \frac{2}{q + 3} E((W_{mr})^p(W_{n-1r})^{q}). \]

**Proof.**
\[
E[(W_{mr})^p(W_{nr})^{q} - (W_{mr})^p(W_{n+1r})^{q+1}]
= \int_0^1 \int_x \frac{1}{\Gamma(n-m)\Gamma(n-m)2^{m+k-1}}
\times (1-y)^{m-2} \frac{1}{(1-x)^{m+k-1}} \frac{1}{(1-x)H(x)dx,}
\]
where
\[
\frac{1}{(1-x)^{n-m}H(x)}
= \int_x [y^q(1-y)][-\ln(1-x) + \ln(1-y)]^{n-m-1}
\times (1-y)^dy
= \int_x \frac{y^{q+1}}{(q+1)}[\ln(1-x) - \ln(1-y)]^{n-m-1}(1-y)^2dy
= \frac{y^{q+1}}{(q+1)}[\ln(1-x) - \ln(1-y)]^{n-m-1}(1-y)^2|_0^{1}
- \int_x \frac{y^{q+1}}{q+1} \frac{d}{dy}[-\ln(1-x) + \ln(1-y)]^{n-m-1}
\times (1-y)^2dy
\]
\[
= - \int_0^1 \frac{y^{q+1}}{q+1} \frac{d}{dy} \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-1} \times (1-y)^2 \] \] 
\[
= 2 \int_0^1 \frac{y^{q+1}}{q+1} \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-1}(1-y) \] \] 
\[
+ \int_0^1 \frac{y^{q+1}}{\theta^q(q+1)} (n-m-1) \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-2} \times (1-y) \theta \] \] 
\[
\text{Substituting } H(x) \text{ in (2.10), we obtain}
\[
E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{nr})^{q+1}] = \frac{2}{q+1} E((W_{mr})^p(W_{nr})^{q+1}) \] 
\[
- \frac{1}{q+1} E((W_{mr})^p(W_{n-1r})^{q+1}) \] 
\[
\text{On simplification, we obtain}
\[
E((W_{mr})^p(W_{nr})^{q+1}) = \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^{q+1}) \quad \square
\]

**Entropy of \( W_{nr} \).**
The entropy of \( W_{nr} \) is given in the following theorem.

**Theorem 2.7.** *The entropy, \( I_n \) of \( W_{nr} \), \( n \geq 2 \), is given by*
\[
I_n = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2,
\]
*where* \( \Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)} \).

**Proof.**
\[
I_n = E(- \ln f_{W_{nr}})
\]
\[
= \int_0^1 \left[ \ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \right]
\]
\[
\ln(- \ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} \left[ - \ln(1-u) \right]^{n-2} du
\]
\[
= \ln \Gamma(n-1) - (n-1) \ln 2 - H_1 - H_2, \quad (2.11)
\]
Record Range of Uniform Distribution

where

\[
H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2},
\]

\[
H_2 = \int_0^1 (n-2) \ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [\ln(1-u)]^{n-2} du
\]

Substituting \(-\ln(1-u) = t\), we obtain

\[
H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt
\]

\[
= \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-t} dt - (n-2) \ln 2
\]

\[
= (n-2)[\Psi(n-1) - \ln 2].
\]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[
I_n = \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2}
\]

\[
- (n-2)[\Psi(n-1) - \ln 2].
\]

\[
= \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square(2.12)
\]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
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</thead>
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</tr>
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<td>9</td>
<td>2.2775</td>
</tr>
<tr>
<td>10</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range, when \(X_1, X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \ 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) of \(\theta\) is

\[
\hat{\theta} = \frac{1}{3(2^n - 1)} [3.2^{n-1} W_{n+1r} - 2^{n-2} W_{nr} - 2^{n-3} W_{n-1r} - \ldots - W_{2r}]
\]

and

\[
\text{Var}(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}.
\]
Proof. Let
\[ Z_1 = d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \]
\[ Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), d_n = 3.2^{\frac{n}{2}} \]
\[ Z' = (Z_1, Z_2, ..., Z_n), \]

then \( E(Z') = A\theta \), where
\[ A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{n}{2}}). A'A = 2(2^n - 1). \]

Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is
\[
\hat{\theta} = (A'A)^{-1} A'Z
= \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2 Z_2 + ... + 2^{\frac{n}{2}} Z_n]
= \frac{1}{2(2^n - 1)} [3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2 W_{2r}]. \quad (2.13)
\]

\[ Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square \]

For example, if \( n = 4 \), then
\[
\hat{\theta} = \frac{1}{15} [24 W_{5r} - 4 W_{4r} - 2 W_{3r} - W_{2r}]
\]

and
\[ Var(\hat{\theta}) = \frac{\theta^2}{30}. \]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c - 1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2(2n+1)} + (c - 1)^2\theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2n+1-2}{2n+1-1}$.

The bias of $\tilde{\theta} = (c - 1)\theta = \frac{1}{2} - \frac{1}{3}$ and $\text{MSE}(\tilde{\theta}) = \frac{1}{2n+1-1}$.

### Prediction of $W_{n+sr}$

We consider the prediction of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

**Theorem 2.5.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.

**Proof.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$W^*_{n+sr} = E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n)$$

$$= E(W_{n+sr}|W_{nr} = x_n), \text{by Markov property of } W_{2r}, W_{3r}, \ldots$$

$$= \theta[1 - (\frac{2}{3})^s] + x_n(\frac{2}{3})^s.$$

### References

